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Anisotropic Hardy inequalities

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We study some Hardy-type inequalities involving a general norm in $\mathbb{R}^n$ and an anisotropic distance function to the boundary. The case of the optimality of the constants is also addressed.

Keywords: Hardy inequalities; anisotropic operators; optimal constants

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1. Introduction

Let F be a smooth norm of $\mathbb{R}^n$. In this paper we investigate the validity of Hardy-type inequalities

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x \geqslant C_F(\Omega)\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x \quad \forall u \in H_0^1(\Omega), \tag{1.1}$$

where Ω is a domain of $\mathbb{R}^n$ and d_F is the anisotropic distance to the boundary with respect to the dual norm (see § 2 for the precise assumptions and definitions). We aim to study the best possible constant for which (1.1) holds, in the sense that

$$C_F(\Omega) = \inf_{u\in H_0^1(\Omega)} \frac{\int_\Omega F^2(\nabla u)\,\mathrm{d}x}{\int_\Omega (u^2/d_F^2)\,\mathrm{d}x}.$$

In the case of the Euclidean norm, that is, when $F = \mathcal{E} = |\cdot|$, inequality (1.1) reduces to

$$\int_{\Omega} |\nabla u|^2 \,\mathrm{d}x \geqslant C_{\mathcal{E}}(\Omega) \int_{\Omega} \frac{u^2}{d_{\mathcal{E}}^2} \,\mathrm{d}x \quad \forall u \in H_0^1(\Omega), \tag{1.2}$$

where

$$d_{\mathcal{E}}(x) = \inf_{y \in \partial\Omega} |x - y|, \quad x \in \Omega,$$

is the usual distance function from the boundary of Ω, and $C_{\mathcal{E}}$ is the best possible constant.

Inequality (1.2) has been studied by many authors, from several points of view. For example, it is known that for any bounded domain with Lipschitz boundary Ω of $\mathbb{R}^n$, $0 < C_{\mathcal{E}}(\Omega) \leqslant \frac{1}{4}$ (see [10,13,26]). In particular, if Ω is a convex domain of $\mathbb{R}^n$, the optimal constant $C_{\mathcal{E}}$ in (1.2) is independent of Ω and its value is $C_{\mathcal{E}} = \frac{1}{4}$, but there are smooth bounded domains such that $C_{\mathcal{E}}(\Omega) < \frac{1}{4}$ (see [26,27]). Furthermore, in [26] it was proved that $C_{\mathcal{E}}$ is achieved if and only if it is strictly smaller than $\frac{1}{4}$.

Actually, the value of the best constant $C_{\mathcal{E}}(\Omega)$ is still $\frac{1}{4}$ for a more general class of domains. This has been shown, for example, in [6], under the assumption that $d_{\mathcal{E}}$ is superharmonic in Ω, in the sense that

$$\Delta d_{\mathcal{E}} \leqslant 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{1.3}$$

As a matter of fact, when $\partial\Omega$ is sufficiently smooth, condition (1.3) is equivalent to the requirement that $\partial\Omega$ is weakly mean convex, that is, its mean curvature is non-negative at any point. This equivalence goes back to Gromov [22], and it has been established independently in [21, 25, 28].

The fact that the constant $C_{\mathcal{E}}(\Omega) = \frac{1}{4}$ is not achieved has lead to an interest in studying 'improved' versions of (1.2) by adding remainder terms that depend, in general, on suitable norms of u. For instance, when Ω satisfies condition (1.3), several improved versions of (1.2) can be found, for example, in [5, 6, 10, 19]. More precisely, in [10] (for bounded convex domains) and [6] it was proved that

$$\int_{\Omega} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{4} \int_{\Omega} \frac{u^2}{d_{\mathcal{E}}^2} \,\mathrm{d}x \geqslant \frac{1}{4} \int_{\Omega} \frac{u^2}{d_{\mathcal{E}}^2} \left(\log \frac{D_0}{d_{\mathcal{E}}} \right)^{-2} \mathrm{d}x \tag{1.4}$$

for a suitable $D_0 \geqslant \mathrm{e} \sup\{d_{\mathcal{E}}(x, \partial\Omega)\}$ and $u \in H_0^1(\Omega)$.

As a matter of fact, for Ω a bounded and convex set, in [10] it was deduced from (1.4) that

$$\int_{\Omega} |\nabla u|^2 \,\mathrm{d}x \geqslant \frac{1}{4} \int_{\Omega} \frac{u^2}{d_{\mathcal{E}}^2} \,\mathrm{d}x + \frac{1}{4L^2} \int_{\Omega} u^2 \,\mathrm{d}x \quad \forall u \in H_0^1(\Omega), \tag{1.5}$$

where L is the diameter of Ω. Actually, in [24] the authors showed that the value $1/4L^2$ can be replaced by a constant that depends on the volume of Ω, namely $c(n)|\Omega|^{-2/n}$; here $c(n)$ is a suitable constant depending only on the dimension of the space (see also [19]).

The aim of this paper is to study Hardy inequalities of type (1.1), and to show improved versions in the anisotropic setting given by means of the norm F in the spirit of (1.4). For example, one of our main results states that for suitable domains Ω of $\mathbb{R}^n$, and for every function $u \in H_0^1(\Omega)$, it holds that

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x \geqslant \frac{1}{4}\int_\Omega \frac{u^2}{d_F^2}\left(\log\frac{D}{d_F}\right)^{-2}\mathrm{d}x, \tag{1.6}$$

where $D = \mathrm{e}\sup\{d_F(x,\partial\Omega),\ x \in \Omega\}$.

The condition we will impose on Ω in order to have (1.6) will involve the sign of an anisotropic Laplacian of d_F (see §§ 2 and 3). We will also show that such a condition is, in general, not equivalent to (1.3).

Actually, we will also deal with the optimality of the involved constants. Moreover, we will show that (1.6) implies an improved version of (1.1) in terms of the L^2-norm of u in the spirit of (1.5). More precisely, we will show that if Ω is a convex bounded open set, then

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{|u|^2}{d_F^2}\,\mathrm{d}x \geqslant C(n)|\Omega|^{-2/n}\int_\Omega |u|^2\,\mathrm{d}x.$$

Finally, the series expansion of the anisotropic Hardy inequality will be considered (see theorem 4.5).

We emphasize that Hardy-type inequalities in anisotropic settings have been studied, for example, in [9, 16, 30], where, instead of considering the weight d_F^{-2}, a function of the distance from the origin is considered (see, for example, [2, 6, 8, 11, 20, 23, 31] and references therein for the Euclidean case).

The structure of the paper is as follows. In § 2 we fix the necessary notation and provide some preliminary results that will be needed later. Moreover, we discuss in some detail the condition that we impose on Ω in order for (1.6) and (1.1) to be true. In § 3 we study inequality (1.1) and give some applications. In § 4 the improved versions of (1.1) are investigated, and, finally, § 5 is devoted to the study of the optimality of the constants in (1.6).

2. Notation and preliminaries

Throughout the paper we will consider a convex even 1-homogeneous function

$$\xi \in \mathbb{R}^n \mapsto F(\xi) \in [0,+\infty[,$$

that is, a convex function such that

$$F(t\xi) = |t|F(\xi), \quad t \in \mathbb{R},\ \xi \in \mathbb{R}^n, \tag{2.1}$$

and such that

$$\alpha_1|\xi| \leqslant F(\xi), \quad \xi \in \mathbb{R}^n, \tag{2.2}$$

for some constant $0 < \alpha_1$. Under this hypothesis it is easy to see that there exists an $\alpha_2 \geqslant \alpha_1$ such that

$$H(\xi) \leqslant \alpha_2|\xi|, \quad \xi \in \mathbb{R}^n.$$

Furthermore, we suppose that F^2 is strongly convex in the sense that $F \in C^2(\mathbb{R}^n \setminus \{0\})$ and

$$\nabla_\xi^2 F^2 > 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

In this context, an important role is played by the polar function of F, namely, the function F° defined as

$$x \in \mathbb{R}^n \mapsto F^\circ(x) = \sup_{\xi \neq 0} \frac{\xi \cdot x}{F(\xi)}.$$

It is not difficult to verify that F° is a convex, 1-homogeneous function that satisfies

$$\frac{1}{\alpha_2}|\xi| \leqslant F^\circ(\xi) \leqslant \frac{1}{\alpha_1}|\xi| \quad \forall \xi \in \mathbb{R}^n. \tag{2.3}$$

Moreover, the hypotheses on F ensure that, for $F^\circ \in C^2(\mathbb{R}^n \setminus \{0\})$ (see, for instance, [29]),

$$F(x) = (F^\circ)^o(x) = \sup_{\xi \neq 0} \frac{\xi \cdot x}{F^\circ(\xi)}.$$

The following well-known properties hold true:

$$F_\xi(\xi) \cdot \xi = F(\xi), \qquad \xi \neq 0, \tag{2.4}$$

$$F_\xi(t\xi) = \operatorname{sign} t \cdot F_\xi(\xi), \quad \xi \neq 0,\ t \neq 0, \tag{2.5}$$

$$\nabla^2_\xi F(t\xi) = \frac{1}{|t|}\nabla^2_\xi F(\xi) \qquad \xi \neq 0,\ t \neq 0, \tag{2.6}$$

$$F(F^o_\xi(\xi)) = 1 \qquad \forall \xi \neq 0, \tag{2.7}$$

$$F^\circ(\xi) F_\xi(F^\circ_\xi(\xi)) = \xi \qquad \forall \xi \neq 0. \tag{2.8}$$

Analogous properties hold by interchanging the roles of F and F°.

The open set

$$\mathcal{W} = \{\xi \in \mathbb{R}^n \colon F^\circ(\xi) < 1\}$$

is the so-called Wulff shape centred at the origin. More generally, we define

$$\mathcal{W}_r(x_0) = r\mathcal{W} + x_0 = \{x \in \mathbb{R}^2 \colon F^\circ(x - x_0) < r\},$$

and $\mathcal{W}_r(0) = \mathcal{W}_r$.

We recall the definition and some properties of anisotropic curvature for a smooth set. For further details we refer the reader to, for example, [1, 7].

Definition 2.1. Let $A \subset \mathbb{R}^n$ be an open set with smooth boundary. The anisotropic outer normal n_A is defined as

$$n_A(x) = \nabla_\xi F(\nu_A(x)), \quad x \in \partial A,$$

where ν_A is the unit Euclidean outer normal to ∂A.

Remark 2.2. We stress that if $A = \mathcal{W}_r(x_0)$, by the properties of F it follows that

$$n_A(x) = \nabla_\xi F(\nabla_\xi F^\circ(x - x_0)) = \frac{1}{r}(x - x_0), \quad x \in \partial A.$$

Finally, let us recall the definition of the Finsler Laplacian of a function u:

$$\Delta_F u = \operatorname{div}(F(\nabla u) F_\xi(\nabla u)). \tag{2.9}$$

Several properties of Δ_F are listed in [18].

2.1. Anisotropic distance function

Due to the nature of the problem, it seems natural to consider a suitable notion of distance to the boundary.

Let us consider a domain Ω, that is, a connected open set of $\mathbb{R}^n$, with non-empty boundary.

The anisotropic distance of $x \in \overline{\Omega}$ to the boundary $\partial\Omega$ is the function

$$d_F(x) = \inf_{y\in\partial\Omega} F^\circ(x-y), \quad x \in \overline{\Omega}. \tag{2.10}$$

We stress that when $F = |\cdot|$, we have $d_F = d_{\mathcal{E}}$, the Euclidean distance function from the boundary.

It is not difficult to prove that d_F is a uniform Lipschitz function in $\bar{\Omega}$ and, using the property (2.7),

$$F(\nabla d_F(x)) = 1 \quad \text{almost everywhere in } \Omega. \tag{2.11}$$

Obviously, assuming that $\sup_\Omega d_F < +\infty$, $d_F \in W_0^{1,\infty}(\Omega)$ and the quantity

$$r_F = \sup\{d_F(x),\ x \in \Omega\} \tag{2.12}$$

is called the anisotropic inradius of Ω.

For further properties of the anisotropic distance function we refer the reader to [12].

The main assumption in this paper will be that d_F is Δ_F-superharmonic, that is,

$$-\Delta_F d_F \geqslant 0 \quad \text{in } \mathcal{D}', \tag{C_H}$$

which means that

$$\int_\Omega F(\nabla u)F_\xi(\nabla u)\cdot\nabla\varphi\,\mathrm{d}x \geqslant 0 \quad \forall\varphi \in C_0^\infty(\Omega),\ \varphi \geqslant 0.$$

Similarly, in the case of the Euclidean norm we will write that $d_{\mathcal{E}}$ is Δ-superharmonic if $d_{\mathcal{E}}$ is superharmonic in the usual sense, that is, $-\Delta d_{\mathcal{E}} \geqslant 0$ in $\mathcal{D}'$.

Remark 2.3. We emphasize that if Ω is a convex set, the functions $d_{\mathcal{E}}$ and d_F are respectively Δ- and Δ_F-superharmonic. This can be easily proved by using the concavity of $d_{\mathcal{E}}$ and d_F in Ω (see, for instance, [17], and [15] for the anisotropic case).

Actually, in the Euclidean case there exist non-convex sets for which $d_{\mathcal{E}}$ is still Δ-superharmonic. An example can be obtained, for instance, in dimension $n = 3$, taking the standard torus (see [4]). Similarly, in the following example we show that there exists a non-convex set such that the anisotropic distance function d_F is Δ_F-superharmonic.

Example 2.4. Let us consider the following Finsler norm in $\mathbb{R}^3$,

$$F(x_1, x_2, x_3) = (x_1^2 + x_2^2 + a^2x_3^2)^{1/2},$$

with $a > 0$; then

$$F^\circ(x_1, x_2, x_3) = \left(x_1^2 + x_2^2 + \frac{x_3^2}{a^2}\right)^{1/2}.$$

We consider the set $\Omega \subset \mathbb{R}^3$ obtained by rotating the ellipse

$$\gamma = \left\{(0, x_2, x_3)\colon (x_2 - R)^2 + \frac{x_3^2}{a^2} < r^2\right\} \quad \text{with } R > r > 0$$

about the x_3-axis. Obviously, Ω is not convex. In order to show that d_F verifies (C_H), we first observe that if we fix a generic point $x \in \gamma$, then, given that F is isotropic with respect to the first two components, the anisotropic distance is achieved at a point $\bar{x}$ of the boundary of γ. Moreover, it is not difficult to show that the vector $\bar{x} - x$ has the same direction of the anisotropic normal n_Ω (see definition 2.1, and also [14]). Hence, by remark 2.2,

$$d_F(x) = r - F^\circ(x - x_0),$$

where $x_0 = (0, R, 0)$ is the centre of the ellipse.

Now let us introduce plane polar coordinates (ρ, ϑ) such that with a generic point $Q = (x_1, x_2, x_3) \in \mathbb{R}^3$ is associated the point $Q' = (\rho\cos\vartheta, \rho\sin\vartheta, x_3)$, where $\rho = \sqrt{x_1^2 + x_2^2}$ and $\vartheta \in [0, 2\pi]$. Then, by construction,

$$\Omega = \{Q' \in \mathbb{R}^3\colon F^\circ(Q' - C) < r\},$$

where $C = (R\cos\vartheta, R\sin\vartheta, 0)$ and $F^\circ(Q' - C)^2 = (R - \rho)^2 + x_3^2/a^2$.

Then as observed before, for fixed $Q' \in \Omega$,

$$d_F(Q') = r - F^\circ(Q' - C) = r - \sqrt{(R-\rho)^2 + \frac{x_3^2}{a^2}} = r - \sqrt{(R - \sqrt{x_1^2 + x_2^2})^2 + \frac{x_3^2}{a^2}}.$$

Now we are in position to prove that d_F verifies (C_H). We note that, for all $Q' \neq C$,

$$\begin{aligned}
\Delta_F d_F(Q') &= \operatorname{div}(F(\nabla d_F)F_\xi(\nabla d_F)) \\
&= \frac{\partial^2 d_F}{\partial x_1^2} + \frac{\partial^2 d_F}{\partial x_2^2} + a^2\frac{\partial^2 d_F}{\partial x_3^2} \\
&= \frac{1}{\rho}\frac{\partial d_F}{\partial \rho} + \frac{\partial^2 d_F}{\partial \rho^2} + a^2\frac{\partial^2 d_F}{\partial x_3^2} \\
&= \frac{R - 2\rho}{\rho F^\circ(Q' - C)}.
\end{aligned} \tag{2.13}$$

Given that $\varrho > R - r$, we get that d_F is Δ_F-superharmonic in Ω if $R > 2r$ for all $a > 0$.

Remark 2.5. In general, if Ω is not convex, requiring that d_F is Δ_F-superharmonic does not ensure that $d_\mathcal{E}$ is Δ-superharmonic. Indeed, let Ω be as in example 2.4; if we take $R \geqslant 2r$, then, as shown before, $-\Delta_F d_F \geqslant 0$. On the other hand, it is possible to choose $a > 0$ such that $d_\mathcal{E}$ is not Δ-superharmonic. To do that, it is enough to prove that the mean curvature of Ω is negative at some points of the boundary for a suitable choice of a. Indeed, in [25] it was proved that $d_\mathcal{E}$ is Δ-superharmonic on Ω if and only if the mean curvature $H_\Omega(y) \geqslant 0$ for all $y \in \partial\Omega$.

We denote the parametric equations of $\partial\Omega$ by $\varphi(t,\vartheta)$:

$$x_1 = (R + r\cos\vartheta)\cos t = \phi(\vartheta)\cos t,$$
$$x_2 = (R + r\cos\vartheta)\sin t = \phi(\vartheta)\sin t,$$
$$x_3 = ar\sin\vartheta = \psi(\vartheta),$$

where $t, \vartheta \in [0, 2\pi]$.

Then for $y = \varphi(t,\vartheta) \in \partial\Omega$ we have

$$\begin{aligned} H_\Omega(y) &= -\frac{\phi(\phi''\psi' - \phi'\psi'') - \psi'((\phi')^2 + (\psi')^2)}{2|\phi|((\phi')^2 + (\psi')^2)^{3/2}} \\ &= \frac{ar^2(R + 2r\cos\vartheta + r\cos^3\vartheta(a^2-1))}{2|R + r\cos\vartheta|(r^2\sin^2\vartheta + a^2r^2\cos^2\vartheta)^{3/2}}. \end{aligned} \tag{2.14}$$

Finally, we observe that if $\vartheta = \pi$, then $H_\Omega(y) < 0$ if $a > 1$.

3. Anisotropic Hardy inequality

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and suppose that d_F satisfies condition (C_H). Then for every function $u \in H_0^1(\Omega)$ the following anisotropic Hardy inequality holds:*

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x \geqslant \frac{1}{4}\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x, \tag{3.1}$$

where d_F is the anisotropic distance function from the boundary of Ω defined in (2.10).

Proof. We prove inequality (3.1) for $u \in C_0^\infty(\Omega)$. Given that F^2 is convex, we have that

$$F^2(\xi_1) \geqslant F^2(\xi_2) + 2F(\xi_2)F_\xi(\xi_2)\cdot(\xi_1 - \xi_2).$$

Hence, putting $\xi_1 = \nabla u$, $\xi_2 = Au(\nabla d_F/d_F)$, with A a positive constant, and recalling that $F(\nabla d_F) = 1$, by the homogeneity of F we get

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x \geqslant -A^2\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x + A\int_\Omega \frac{2u}{d_F}F_\xi(\nabla d_F)\cdot\nabla u\,\mathrm{d}x.$$

By the divergence theorem (in a general setting; see, for example, [3]) we have

$$\begin{aligned} \int_\Omega F^2(\nabla u)\,\mathrm{d}x &\geqslant -A^2\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x + A\int_\Omega \frac{F_\xi(\nabla d_F)}{d_F}\cdot\nabla(u^2)\,\mathrm{d}x \\ &\geqslant -A^2\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x - A\int_\Omega u^2\frac{\Delta_F d_F}{d_F}\,\mathrm{d}x + A\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x. \end{aligned}$$

Given that $-\Delta_F d_F \geqslant 0$, we get

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x \geqslant (A - A^2)\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x.$$

Then, maximizing with respect to A we obtain that $A = \frac{1}{2}$, and (3.1) follows. □

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Remark 3.2. We observe that if Ω is a convex domain in $\mathbb{R}^n$, an inequality of type (3.1) can be immediately obtained by using the following classical Hardy inequality involving the Euclidean distance function $d_{\mathcal{E}}$:

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x \geqslant \frac{1}{4}\int_\Omega \frac{u^2}{d_{\mathcal{E}}^2}\,\mathrm{d}x. \tag{3.2}$$

By (2.2) and (2.3) we easily get

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x \geqslant \alpha_1^2 \int_\Omega |\nabla u|^2\,\mathrm{d}x \geqslant \frac{\alpha_1^2}{4}\int_\Omega \frac{u^2}{d_{\mathcal{E}}^2}\,\mathrm{d}x \geqslant \frac{1}{4}\frac{\alpha_1^2}{\alpha_2^2}\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x, \tag{3.3}$$

where the constant on the right-hand side is smaller than $\frac{1}{4}$ since $\alpha_1 < \alpha_2$. We emphasize that if Ω is not convex, inequality (3.3) holds under the assumption that $d_{\mathcal{E}}$ is $\mathcal{E}$-superharmonic, since (3.2) is in force. On the other hand, the assumption on $d_{\mathcal{E}}$ is not related to the hypothesis required about d_F in theorem 3.1, as observed in remark 2.5.

Using theorem 3.1, it is not difficult to obtain a lower bound for the first eigenvalue of Δ_F defined in (2.9).

Corollary 3.3. *Let Ω be a bounded domain of $\mathbb{R}^n$ and suppose that d_F satisfies condition (C_H). Let $\lambda_1(\Omega)$ be the first Dirichlet eigenvalue of the Finsler Laplacian, that is,*

$$\lambda_1(\Omega) = \min_{\substack{u\in H_0^1(\Omega)\\ u\neq 0}} \frac{\int_\Omega F^2(\nabla u)\,\mathrm{d}x}{\int_\Omega u^2\,\mathrm{d}x}. \tag{3.4}$$

Then

$$\lambda_1(\Omega) \geqslant \frac{1}{4r_F^2},$$

where r_F is the anisotropic inradius of Ω defined in (2.12).

Proof. Let v be the first eigenfunction related to $\lambda_1(\Omega)$ such that $\|v\|_{L^2} = 1$. Then (3.4) and inequality (3.1) imply that

$$\lambda_1(\Omega) = \int_\Omega F^2(\nabla v)\,\mathrm{d}x \geqslant \frac{1}{4}\int_\Omega \frac{v^2}{d_F^2}\,\mathrm{d}x \geqslant \frac{1}{4r_F^2},$$

which is the claim. □

4. Hardy inequality with remainder terms

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let us suppose also that d_F satisfies condition (C_H), and $r_F < +\infty$, where r_F is the anisotropic inradius of Ω defined in (2.12). Then for every function $u \in H_0^1(\Omega)$ the following inequality holds:*

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x \geqslant \frac{1}{4}\int_\Omega \frac{u^2}{d_F^2}\left(\log\frac{D}{d_F}\right)^{-2}\mathrm{d}x, \tag{4.1}$$

where $D = \mathrm{e}\, r_F$.

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Proof. We will use the following notation:

$$X(t) = -\frac{1}{\log t}, \quad t \in \left]0, \frac{1}{\mathrm{e}}\right[.$$

Let us explicitly observe that $X(t) \in]0, 1[$.

Given that F^2 is convex, we have that

$$F^2(\xi_1) \geqslant F^2(\xi_2) + 2F(\xi_2)F_\xi(\xi_2) \cdot (\xi_1 - \xi_2).$$

Let us consider

$$\xi_1 = \nabla u, \qquad \xi_2 = \frac{u}{2}\frac{\nabla d_F}{d_F}\left[1 - X\left(\frac{d_F}{D}\right)\right].$$

Given that $d_F(x) \leqslant D/\mathrm{e}$, by the 1-homogeneity of F we get

$$\begin{aligned} F^2(\nabla u) & \\ &\geqslant \frac{1}{4}\frac{u^2}{d_F^2}F^2(\nabla d_F)\left[1 - X\left(\frac{d_F}{D}\right)\right]^2 \\ &\quad + \frac{u}{d_F}\left[1 - X\left(\frac{d_F}{D}\right)\right]F(\nabla d_F)F_\xi(\nabla d_F) \cdot \left(\nabla u - \tfrac{1}{2}u\frac{\nabla d_F}{d_F}\left[1 - X\left(\frac{d_F}{D}\right)\right]\right) \\ &= -\frac{1}{4}\frac{u^2}{d_F^2}\left[1 - X\left(\frac{d_F}{D}\right)\right]^2 + \frac{u}{d_F}\left[1 - X\left(\frac{d_F}{D}\right)\right]F_\xi(\nabla d_F) \cdot \nabla u, \end{aligned} \tag{4.2}$$

where last equality follows from $F(\nabla d_F) = 1$, the 1-homogeneity of F and property (2.4). Let us observe that, using the divergence theorem (in a general setting; see, for example, [3]), we have

$$\begin{aligned} \int_\Omega \frac{u}{d_F}&\left[1 - X\left(\frac{d_F}{D}\right)\right]F_\xi(\nabla d_F) \cdot \nabla u \,\mathrm{d}x \\ &= -\int_\Omega \frac{u^2}{2}\operatorname{div}\left(\left[1 - X\left(\frac{d_F}{D}\right)\right]\frac{F_\xi(\nabla d_F)}{d_F}\right)\mathrm{d}x \\ &= \int_\Omega \frac{u^2}{2}\left\{\left[1 - X\left(\frac{d_F}{D}\right) + X^2\left(\frac{d_F}{D}\right)\right]\frac{F_\xi(\nabla d_F) \cdot \nabla d_F}{d_F^2}\right. \\ &\qquad\qquad \left. - \left[1 - X\left(\frac{d_F}{D}\right)\right]\frac{\Delta_F d_F}{d_F}\right\}\mathrm{d}x \\ &\geqslant \int_\Omega \frac{1}{2}\frac{u^2}{d_F^2}\left[1 - X\left(\frac{d_F}{D}\right) + X^2\left(\frac{d_F}{D}\right)\right]\mathrm{d}x, \end{aligned} \tag{4.3}$$

where the last inequality follows by using the condition $-\Delta_F d_F \geqslant 0$.

Integrating (4.2) and using (4.3), we easily get

$$\begin{aligned} \int_\Omega F^2(\nabla u)\,\mathrm{d}x &\geqslant \int_\Omega \frac{1}{4}\frac{u^2}{d_F^2}\left\{-\left[1 - X\left(\frac{d_F}{D}\right)\right]^2 + 2 - 2X\left(\frac{d_F}{D}\right) + 2X^2\left(\frac{d_F}{D}\right)\right\}\mathrm{d}x \\ &= \frac{1}{4}\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x + \frac{1}{4}\int_\Omega \frac{u^2}{d_F^2}X^2\left(\frac{d_F}{D}\right)\mathrm{d}x, \end{aligned}$$

and the proof is complete. □

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Remark 4.2. We observe that if Ω is a convex domain in $\mathbb{R}^n$, arguing as in remark 3.2, an inequality of type (4.1) can be immediately obtained by using the following improved Hardy inequality involving $d_{\mathcal{E}}$, contained in [6]:

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{|u|^2}{d_{\mathcal{E}}^2}\,\mathrm{d}x \geqslant \frac{1}{4}\int_\Omega \frac{|u|^2}{d_{\mathcal{E}}^2}\left(\log\frac{D_0}{d_{\mathcal{E}}}\right)^{-2}\mathrm{d}x, \tag{4.4}$$

where $D_0 \geqslant \mathrm{e}\sup d_{\mathcal{E}}(x,\partial\Omega)$ and $u \in H_0^1(\Omega)$. Obviously, also in this case it is not possible to obtain the optimal constants.

Corollary 4.3. *Under the same assumptions as theorem 4.1, the following anisotropic improved Hardy inequality holds:*

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x \geqslant \frac{1}{4r_F^2}\int_\Omega u^2\,\mathrm{d}x, \tag{4.5}$$

where r_F is the anisotropic inradius defined in (2.12).

Proof. By theorem 4.1, to prove (4.5) it is sufficient to show that

$$\int_\Omega \frac{u^2}{d_F^2}\left(\log\frac{D}{d_F}\right)^{-2}\mathrm{d}x \geqslant \frac{1}{r_F^2}\int_\Omega u^2\,\mathrm{d}x.$$

This is a consequence of the monotonicity of the following function

$$f(t) = -t\log\left(\frac{t}{\mathrm{e}r_F}\right), \quad 0 < t \leqslant r_F.$$

Indeed, f is strictly increasing and its maximum is r_F. This concludes the proof. □

An immediate consequence of the previous result is contained in the following remark.

Remark 4.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Then there exists a positive constant $C(n) > 0$ such that for any $u \in H_0^1(\Omega)$ we have

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x \geqslant C(n)|\Omega|^{-2/n}\int_\Omega u^2\,\mathrm{d}x.$$

The final theorem concerns the series expansion of the anisotropic Hardy inequality, which generalizes the one in the Euclidean case contained in [5].

To this end, we introduce the function

$$X_1(t) = \frac{1}{1-\log t}, \quad t \in \,]0,1[,$$

and recursively (observe that $X_1(t) \in \,]0,1[$)

$$X_{k+1}(t) = X_k(X_1(t)), \quad k \in \mathbb{N}.$$

THEOREM 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let us suppose also that d_F verifies condition (C_H), and the anisotropic inradius r_F is finite. Then for every function $u \in H_0^1(\Omega)$ the following inequality holds:*

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x \geqslant \frac{1}{4}\sum_{k=1}^{\infty}\int_\Omega \frac{u^2}{d_F^2} X_1^2\left(\frac{d_F}{r_F}\right) X_2^2\left(\frac{d_F}{r_F}\right)\cdots X_k^2\left(\frac{d_F}{r_F}\right)\mathrm{d}x. \tag{4.6}$$

Proof. The proof runs similarly to the one contained in [5] in the Euclidean case.

For any $m \in \mathbb{N}$, define

$$\eta(t) = \sum_{k=1}^{m} X_1(t)\cdots X_k(t), \quad t \in \,]0,1[.$$

Let us observe that

$$\begin{aligned}\eta'(t) &= \frac{1}{t}[X_1^2 + (X_1^2X_2 + X_1^2X_2^2) + \cdots + (X_1^2X_2\cdots X_m + \cdots + X_1^2X_2^2\cdots X_m^2)] \\ &= \frac{1}{2}\frac{\eta^2(t)}{t} + \frac{1}{2t}\sum_{k=1}^{m} X_1^2(t)\cdots X_k^2(t).\end{aligned}$$

Choosing

$$\xi_1 = \nabla u, \qquad \xi_2 = \frac{u}{2}\frac{\nabla d_F}{d_F}\left[1 - \eta\left(\frac{d_F}{r_F}\right)\right]$$

and proceeding as in theorems 3.1 and 4.1, we obtain the result. □

5. Optimality of the constants

Here we prove the optimality of the constants and of the exponent that appear in the Hardy inequality (4.1). More precisely, we prove the following result.

THEOREM 5.1. *Let Ω be a piecewise C^2 domain of $\mathbb{R}^n$. Suppose that $r_F < +\infty$, and that the following Hardy inequality holds:*

$$\int_\Omega F^2(\nabla u)\,\mathrm{d}x - A\int_\Omega \frac{u^2}{d_F^2}\,\mathrm{d}x \geqslant B\int_\Omega \frac{u^2}{d_F^2}\left(\log\frac{D}{d_F}\right)^{-\gamma}\mathrm{d}x \quad \forall u \in H_0^1(\Omega), \tag{5.1}$$

for some constants $A > 0$, $B \geqslant 0$, $\gamma > 0$, where $D = \mathrm{e}\,r_F$. Then

(T_1) *$A \leqslant \frac{1}{4}$;*

(T_2) *if $A = \frac{1}{4}$ and $B > 0$, then $\gamma \geqslant 2$;*

(T_3) *if $A = \frac{1}{4}$ and $\gamma = 2$, then $B \leqslant \frac{1}{4}$.*

Proof. The proof is similar to the one obtained in the Euclidean case in [6]. For the sake of completeness, we describe it in detail. As before, let us define

$$X(t) = -\frac{1}{\log t}, \quad t \in \left]0, \frac{1}{\mathrm{e}}\right[.$$

In order to prove the results we will provide a local analysis. Hence, we fix a Wulff shape $\mathcal{W}_\delta$ of radius δ centred at a point $x_0 \in \partial\Omega$. Given that $\partial\Omega$ is piecewise smooth, we may suppose that, for a sufficiently small δ, $\partial\Omega \cap \mathcal{W}_\delta$ is C^2. Now let φ be a non-negative cut-off function in $C_0^\infty(\mathcal{W}_\delta \cap \Omega)$ such that $\varphi(x) = 1$ for $x \in \mathcal{W}_{\delta/2}$. First of all, we prove some technical estimates that will be useful in the following. For $\varepsilon > 0$ and $\beta \in \mathbb{R}$ let us consider

$$J_\beta(\varepsilon) = \int_\Omega \varphi^2 d_F^{-1+2\varepsilon} X^{-\beta}\left(\frac{d_F}{D}\right) \mathrm{d}x. \tag{5.2}$$

We split the proof into several claims.

Claim 5.2. *The following estimates hold:*

(i) $c_1\varepsilon^{-1-\beta} \leqslant J_\beta(\varepsilon) \leqslant c_2\varepsilon^{-1-\beta}$ *for* $\beta > -1$, *where* c_1, c_2 *are positive constants independent of* ε*;*

(ii) $J_\beta(\varepsilon) = \dfrac{2\varepsilon}{\beta+1} J_{\beta+1}(\varepsilon) + O_\varepsilon(1)$ *for* $\beta > -1$*;*

(iii) $J_\beta(\varepsilon) = O_\varepsilon(1)$ *for* $\beta < -1$.

Proof of claim 5.2. By the coarea formula,

$$J_\beta(\varepsilon) = \int_0^\delta r^{-1+2\varepsilon} X^{-\beta}(r/D)\left(\int_{d_F=r} \frac{\varphi^2}{|\nabla d_F|}\, \mathrm{d}H^{n-1}\right) \mathrm{d}r.$$

Given that $F(\nabla d_F) = 1$, by (2.2), $0 < \alpha_1 \leqslant |\nabla d_F|^{-1} \leqslant \alpha_2$ and

$$0 < C_1 \leqslant \int_{d_F=r} \frac{\varphi^2}{|\nabla d_F|}\, \mathrm{d}H^{n-1} \leqslant C_2.$$

Then if $\beta < -1$, (iii) easily follows. Moreover, if $\beta > -1$, performing the change of variables $r = Ds^{1/\varepsilon}$, (i) holds.

As regards (ii), let us observe that

$$\frac{\mathrm{d}}{\mathrm{d}r} X^\beta = \beta \frac{X^{\beta+1}}{r}.$$

Recalling that $1 = F(\nabla d_F) F_\xi(\nabla d_F) \cdot \nabla d_F$, we have

$$\begin{aligned}
(\beta+1)J_\beta(\varepsilon) &= -\int_\Omega \varphi^2 d_F^{2\varepsilon} F(\nabla d_F)F_\xi(\nabla d_F)\cdot\nabla\left[X^{-\beta-1}\left(\frac{d_F}{D}\right)\right]\mathrm{d}x \\
&= \int_\Omega \operatorname{div}(\varphi^2 d^{2\varepsilon} F(\nabla d_F)F_\xi(\nabla d_F)) X^{-\beta-1}\left(\frac{d_F}{D}\right)\mathrm{d}x \\
&= 2\int_\Omega \varphi d_F^{2\varepsilon} X^{-\beta-1}\left(\frac{d_F}{D}\right) F(\nabla d_F)F_\xi(\nabla d_F)\cdot\nabla\varphi\,\mathrm{d}x \\
&\quad + 2\varepsilon\int_\Omega \varphi^2 d_F^{2\varepsilon-1} X^{-\beta-1}\left(\frac{d_F}{D}\right)\mathrm{d}x \\
&\quad + \int_\Omega \varphi^2 d_F^{2\varepsilon} X^{-\beta-1}\left(\frac{d_F}{D}\right)\Delta_F d_F\,\mathrm{d}x \\
&= O_\varepsilon(1) + 2\varepsilon J_{\beta+1}(\varepsilon).
\end{aligned}$$

We explicitly observe that

$$\int_\Omega \varphi^2 d_F^{2\varepsilon} X^{-\beta-1}\left(\frac{d_F}{D}\right)\Delta_F d_F \,\mathrm{d}x = O_\varepsilon(1)$$

since d_F is a C^2 function in a neighbourhood of the boundary (see [12]). Then (ii) holds. □

In the next claim we estimate the left-hand side of (5.1) when $u = U_\varepsilon$, with

$$U_\varepsilon(x) = \varphi(x) w_\varepsilon(x), \quad w_\varepsilon(x) = d_F^{1/2+\varepsilon} X^{-\theta}(d_F(x)/D), \quad \tfrac{1}{2} < \theta < 1.$$

Let us define

$$\mathcal{Q}[U_\varepsilon] := \int_\Omega \left(F^2(\nabla U_\varepsilon) - \frac{1}{4}\frac{U_\varepsilon^2}{d_F^2}\right)\mathrm{d}x.$$

Claim 5.3. *The following estimates hold:*

$$\mathcal{Q}[U_\varepsilon] \leqslant \tfrac{1}{2}\theta J_{2\theta-2}(\varepsilon) + O_\varepsilon(1) \quad \textit{as } \varepsilon \to 0, \tag{5.3}$$

$$\int_{W_\delta\cap\Omega} F^2(\nabla U_\varepsilon)\,\mathrm{d}x \leqslant \tfrac{1}{4} J_{2\theta}(\varepsilon) + O_\varepsilon(\varepsilon^{1-2\theta}) \quad \textit{as } \varepsilon \to 0. \tag{5.4}$$

Proof of claim 5.3. The convexity of F implies that

$$F^2(\xi+\eta) \leqslant F^2(\xi) + 2F(\xi)F(\eta) + F^2(\eta) \quad \forall \xi,\eta \in \mathbb{R}^n.$$

Hence, by the homogeneity of F,

$$\begin{aligned}\int_\Omega F^2(\nabla U_\varepsilon)\,\mathrm{d}x &\leqslant \int_{\mathcal{W}_\delta\cap\Omega} \varphi^2 F^2(\nabla w_\varepsilon)\,\mathrm{d}x + \int_{\mathcal{W}_\delta\cap\Omega} w_\varepsilon^2 F^2(\nabla\varphi)\,\mathrm{d}x \\ &\quad + \int_{\mathcal{W}_\delta\cap\Omega} 2\varphi w_\varepsilon F(\nabla\varphi)F(\nabla w_\varepsilon)\,\mathrm{d}x \\ &= \int_{W_\delta\cap\Omega} \varphi^2 d_F^{2\varepsilon-1} X^{-2\theta}\left(\frac{d_F}{D}\right)\left(\varepsilon + \frac{1}{2} - \theta X\left(\frac{d_F}{D}\right)\right)^2 \mathrm{d}x + I_1 + I_2.\end{aligned}$$

As a matter of fact,

$$I_2 \leqslant C\int_{W_\delta\cap\Omega} d_F^{2\varepsilon} X^{-2\theta}\left(\frac{d_F}{D}\right)\mathrm{d}x = O_\varepsilon(1);$$

similarly, $I_1 = O_\varepsilon(1)$. Then

$$\begin{aligned}\mathcal{Q}[U_\varepsilon] &\leqslant \int_{W_\delta\cap\Omega} \varphi^2 d_F^{2\varepsilon-1} X^{-2\theta}\left(\frac{d_F}{D}\right)\left[\left(\varepsilon + \frac{1}{2} - \theta X\left(\frac{d_F}{D}\right)\right)^2 - \frac{1}{4}\right]\mathrm{d}x + O_\varepsilon(1) \\ &\leqslant \int_{W_\delta\cap\Omega} \varphi^2 d_F^{2\varepsilon-1} X^{-2\theta}\left(\frac{d_F}{D}\right)\left(\varepsilon - \theta X\left(\frac{d_F}{D}\right)\right)^2 \mathrm{d}x \\ &\quad + \int_{W_\delta\cap\Omega} \varphi^2 d_F^{2\varepsilon-1} X^{-2\theta}\left(\frac{d_F}{D}\right)\left(\varepsilon - \theta X\left(\frac{d_F}{D}\right)\right) + O_\varepsilon(1) \\ &= a_1 + a_2 + O_\varepsilon(1).\end{aligned} \tag{5.5}$$

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Using claim 5.2(ii) with $\beta = 2\theta - 1$, we get

$$a_2 = O_\varepsilon(1), \quad \varepsilon \to 0. \tag{5.6}$$

Similarly, as regards a_1, applying claim 5.2(ii) first with $\beta = 2\theta - 1$ and second with $\beta = 2\theta - 2$ we obtain

$$a_1 = \frac{\theta}{2} \int_{\mathcal{W}_\delta} \varphi^2 d_F^{2\varepsilon-1} X^{2-2\theta}\left(\frac{d_F}{D}\right) \mathrm{d}x + O_\varepsilon(1). \tag{5.7}$$

Then (5.3) follows by (5.5)–(5.7) and (5.2). Finally, observing that

$$\int_{\mathcal{W}_\delta \cap \Omega} F^2(\nabla U_\varepsilon)\, \mathrm{d}x = \mathcal{Q}[U_\varepsilon] + \tfrac{1}{4} J_{2\theta}(\varepsilon), \tag{5.8}$$

inequality (5.4) follows from (5.3) and claim 5.2(i). □

Now we are in a position to conclude the proof of the theorem.

Since inequality (5.1) holds for any $u \in H_0^1(\Omega)$, we take as a test function U_ε. Then, by (5.4) and claim 5.2(i), we have

$$A \leqslant \frac{1}{J_{2\theta}(\varepsilon)} \int_{\mathcal{W}_\delta \cap \Omega} F^2(\nabla U_\varepsilon)\, \mathrm{d}x \leqslant \tfrac{1}{4} + O_\varepsilon(\varepsilon).$$

Letting $\varepsilon \to 0$, we obtain (T_1).

In order to prove (T_2) we put $A = \frac{1}{4}$, and reasoning by contradiction we assume that $\gamma < 2$. As before, by (5.3) and claim 5.2(i) we have

$$0 < B \leqslant \frac{Q[U_\varepsilon]}{J_{2\theta-\gamma}(\varepsilon)} \leqslant C \frac{\varepsilon^{1-2\theta}}{\varepsilon^{\gamma-1-2\theta}} = C\varepsilon^{2-\gamma} \to 0 \quad \text{as } \varepsilon \to 0,$$

which is a contradiction, and then $\gamma \geqslant 2$.

To conclude the proof of the theorem we just have to prove (T_3).

If $A = \frac{1}{4}$ and $\gamma = 2$, then by (5.3) we have

$$B \leqslant \frac{Q[U_\varepsilon]}{J_{2\theta-2}(\varepsilon)} \leqslant \frac{\frac{1}{2}\theta J_{2\theta-2}(\varepsilon) + O_\varepsilon(1)}{J_{2\theta-2}(\varepsilon)}.$$

Then, by assumption on θ and claim 5.2(i), letting $\varepsilon \to 0$ we get

$$B \leqslant \frac{\theta}{2}.$$

Hence (T_3) follows by letting $\theta \to \frac{1}{2}$. □

Remark 5.4. We stress that theorem 5.1 ensures that the involved constants in (4.1), and also in the anisotropic Hardy inequality (3.1), are optimal. Actually, the presence of the remainder term in (4.1) guarantees that the constant $\frac{1}{4}$ in (3.1) is not achieved.

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