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# A 2-ARC TRANSITIVE PENTAVALENT CAYLEY GRAPH OF $A_{39}$

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**Abstract**

Zhou and Feng [‘On symmetric graphs of valency five’, *Discrete Math.* **310** (2010), 1725–1732] proved that all connected pentavalent 1-transitive Cayley graphs of finite nonabelian simple groups are normal. We construct an example of a nonnormal 2-arc transitive pentavalent symmetric Cayley graph on the alternating group $A_{39}$. Furthermore, we show that the full automorphism group of this graph is isomorphic to the alternating group $A_{40}$.

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## 1. Introduction

For a graph $\Gamma$, we use $V\Gamma$, $E\Gamma$ and $\mathsf{Aut}\Gamma$ to denote the vertex set, edge set and full automorphism group of $\Gamma$, respectively. An *arc* in $\Gamma$ is an ordered pair of two adjacent vertices. A graph $\Gamma$ is said to be *symmetric* if $\mathsf{Aut}\Gamma$ acts transitively on the set of all arcs of $\Gamma$.

Let $G$ be a finite group with identity 1 and let $S$ be a subset of $G$ such that $1 \notin S$ and $S = S^{-1} := \{x^{-1} \mid x \in S\}$. The Cayley graph of $G$ with respect to $S$, denoted by $\mathsf{Cay}(G, S)$, is defined on $G$ such that $g, h \in G$ are adjacent if and only if $hg^{-1} \in S$. Then $\mathsf{Cay}(G, S)$ is a regular undirected graph of valency $|S|$. It is well known that $\Gamma$ is connected if and only if $\langle S \rangle = G$, that is, $S$ is a generating set of the group $G$. For a Cayley graph $\mathsf{Cay}(G, S)$, the underlying group $G$ can be viewed as a regular subgroup of $\mathsf{AutCay}(G, S)$ which acts on $G$ by right multiplication. Conversely, a graph $\Gamma$ is isomorphic to a Cayley graph of a group $G$ if and only if $\mathsf{Aut}\Gamma$ contains a subgroup which is regular on $V\Gamma$ and isomorphic to $G$ (see [10]). A Cayley graph $\Gamma = \mathsf{Cay}(G, S)$ is said to be *normal* if $G$ is normal in $\mathsf{Aut}\Gamma$; otherwise, $\Gamma$ is called *nonnormal*.

Cayley graphs of finite simple groups have received much attention. Let $T$ be a finite nonabelian simple group and let $\Gamma = \mathsf{Cay}(T, S)$ be a connected symmetric Cayley

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graph of $T$. In the case where $\Gamma$ is cubic, Li [6] proved that $\Gamma$ must be normal except for seven finite nonabelian simple groups. On the basis of Li's result, Xu *et al.* [11, 12] proved that a nonnormal $\Gamma$ must have automorphism group $A_{48}$ and be isomorphic to one of two Cayley graphs of $A_{47}$. In the case where $\Gamma$ is pentavalent, Zhou and Feng [13] proved that $\Gamma$ is normal when $\Gamma$ is 1-transitive. But there are no known examples of connected pentavalent symmetric Cayley graphs of finite simple groups which are nonnormal. In this paper, we construct a 2-arc transitive pentavalent nonnormal Cayley graph of a finite simple group.

THEOREM 1.1. *There exists a nonnormal connected pentavalent Cayley graph on the alternating group* $A_{39}$ *with full automorphism group* $A_{40}$.

## 2. Preliminaries

In this section, we give some necessary preliminary results.

First we introduce the definition of a coset graph. Let $G$ be a finite group and let $H$ be a core-free subgroup of $G$. Define the *coset graph* $\mathsf{Cos}(G, H, g)$ of $G$ with respect to $H$ as the graph with vertex set $[G : H]$ such that $Hx$, $Hy$ are adjacent if and only if $yx^{-1} \in HgH$. The following lemma about coset graphs is well known.

LEMMA 2.1. *A graph* $\Gamma$ *is* $G$*-arc transitive for some* $G \le \mathsf{Aut}\Gamma$ *if and only if* $\Gamma \cong \mathsf{Cos}(G, H, g)$*, where* $H = G_\alpha$ *for some* $\alpha \in V\Gamma$*,* $g \in N_G(G_{\alpha\beta})\backslash G_\alpha$ *is a 2-element such that* $g^2 \in H$ *and* $\beta$ *is adjacent to* $\alpha$*.*

*In particular, for the coset graph* $\Gamma = \mathsf{Cos}(G, H, g)$*, the following statements hold:*

(1) *the valency,* $\mathsf{val}\Gamma$*, of* $\Gamma$ *is given by* $\mathsf{val}\Gamma = |H : H \cap H^g|$*;*
(2) $\Gamma$ *is connected if and only if* $\langle H, g\rangle = G$*;*
(3) *if* $G$ *has a subgroup* $R$ *acting regularly on the vertices of* $\Gamma$*, then* $\mathsf{Cos}(G, H, g) \cong \mathsf{Cay}(R, S)$*, where* $S = R \cap HgH$*.*

Denote by $F_{20}$ the Frobenius group of order 20. The next lemma gives the structure of the vertex stabilisers of pentavalent symmetric graphs, as determined in [4, 13].

LEMMA 2.2. *Let* $\Gamma$ *be a pentavalent* $(X, s)$*-transitive graph for some* $X \le \mathsf{Aut}\Gamma$ *and* $s \ge 1$*. Let* $v \in V\Gamma$*. If* $X_v$ *is soluble, then* $|X_v| \mid 80$ *and* $s \le 3$*. If* $X_v$ *is insoluble, then* $|X_v| \mid 2^9 \cdot 3^2 \cdot 5$ *and* $2 \le s \le 5$*. Furthermore, one of the following holds:*

(1) $s = 1$, $X_v \cong \mathbb{Z}_5$, $D_{10}$ *or* $D_{20}$*;*
(2) $s = 2$, $X_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$, $A_5$ *or* $S_5$*;*
(3) $s = 3$, $X_v \cong F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $(A_4 \times A_5) : \mathbb{Z}_2$ *or* $S_4 \times S_5$*;*
(4) $s = 4$, $X_v \cong \mathrm{ASL}(2, 4)$, $\mathrm{AGL}(2, 4)$, $\mathrm{A\Sigma L}(2, 4)$ *or* $\mathrm{A\Gamma L}(2, 4)$*;*
(5) $s = 5$, $X_v \cong \mathbb{Z}_2^6 : \Gamma\mathrm{L}(2, 4)$.

Let $G$ be a finite group and let $H$ be a subgroup of $G$. Denote by $C_G(H)$ the centraliser of $H$ in $G$ and by $N_G(H)$ the normaliser of $H$ in $G$. Then we have the following lemma (see [5, Ch. I, Theorem 4.5]).

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LEMMA 2.3. *The quotient group* $N_G(H)/C_G(H)$ *is isomorphic to a subgroup of the automorphism group* $\mathsf{Aut}(H)$ *of* $H$.

Simple groups which have subgroups of index dividing $2^6 \cdot 3^2$ are given in the following lemma (see [2, Lemma 2.4]).

LEMMA 2.4. *Let* $T$ *be a nonabelian simple group which has a subgroup* $L$ *of index dividing* $2^6 \cdot 3^2$. *Then* $T$, $L$ *and* $n := |T : L|$ *are given in the following table.*

| $T$ | $L$ | $n$ | *Remark* |
|---|---|---|---|
| $A_n$ | $A_{n-1}$ | $n$ | $n \mid 2^6 \cdot 3^2$ |
| $M_{11}$ | PSL(2, 11) | 12 | |
| $M_{12}$ | $M_{11}$ | 12 | |
| $M_{24}$ | $M_{23}$ | 24 | |

The following proposition, from [7, Proposition 3.2] plays an important role in the proof of Theorem 1.1.

PROPOSITION 2.5. *Let* $\Gamma = \mathsf{Cay}(G, S)$ *be a connected* $X$*-arc-transitive Cayley graph, where* $G \le X \le \mathsf{Aut}\Gamma$. *Let* $H$ *be the stabiliser of* $1 \in V\Gamma$ *in* $X$. *If* $S$ *contains an involution* $z$, *then* $z \in N_{S_{|H|}}(H \cap H^z) \backslash (\bigcup_{1 \ne K \trianglelefteq H} N_{S_{|H|}}(K))$, $\Gamma \cong \mathsf{Cos}(X, H, z)$, $X = \langle z, H \rangle$, $G = \{\sigma \in X \mid 1^\sigma = 1\}$ *and* $S = \{\sigma \in HzH \mid 1^\sigma = 1\}$.

## 3. Construction

CONSTRUCTION 3.1. Let $G$ be the alternating group $A_{39}$ and $X$ the alternating group $A_{40}$ and let $H = \langle a, b, c \rangle < A_{40}$, where

$$a = (1\ 21\ 11\ 31)(2\ 22\ 12\ 32)(3\ 25\ 19\ 37)(4\ 26\ 20\ 38)(5\ 29\ 17\ 33)$$
$$(6\ 30\ 18\ 34)(7\ 23\ 15\ 39)(8\ 24\ 16\ 40)(9\ 27\ 13\ 35)(10\ 28\ 14\ 36),$$
$$b = (1\ 3\ 5\ 7\ 9)(2\ 4\ 6\ 8\ 10)(11\ 13\ 15\ 17\ 19)(12\ 14\ 16\ 18\ 20)$$
$$(21\ 23\ 25\ 27\ 29)(22\ 24\ 26\ 28\ 30)(31\ 33\ 35\ 37\ 39)(32\ 34\ 36\ 38\ 40),$$
$$c = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)(13\ 14)(15\ 16)(17\ 18)(19\ 20)$$
$$(21\ 22)(23\ 24)(25\ 26)(27\ 28)(29\ 30)(31\ 32)(33\ 34)(35\ 36)(37\ 38)(39\ 40).$$

Define an involution $x_1 \in G$ by

$$x_1 = (2\ 12)(3\ 34)(4\ 29)(5\ 38)(6\ 25)(7\ 14)(8\ 9)(10\ 15)(13\ 16)(17\ 26)$$
$$(18\ 37)(19\ 30)(20\ 33)(22\ 32)(23\ 36)(24\ 27)(28\ 39)(35\ 40).$$

Define $\Gamma = \mathsf{Cos}(X, H, x_1)$.

LEMMA 3.2. *The graph* $\Gamma = \mathsf{Cos}(X, H, x_1)$ *from Construction 3.1 is connected, symmetric and isomorphic to the nonnormal Cayley graph* $\mathsf{Cay}(G, S)$ *of* $G$, *determined by* $S = \{x_1, x_2, x_2^{-1}, x_3, x_3^{-1}\}$ *with*

$$x_2 = (2\ 20\ 32\ 40\ 10\ 4\ 15\ 19\ 29\ 27\ 9\ 17\ 3\ 6\ 24\ 35\ 13\ 16)$$
$$(5\ 37\ 33\ 36\ 8\ 30\ 34\ 25\ 39\ 11\ 23\ 26\ 22\ 14)(7\ 31\ 21)(12\ 38\ 28),$$
$$x_3 = (2\ 18\ 29\ 7\ 30\ 12\ 34\ 28\ 9\ 3\ 33\ 31\ 5\ 21\ 27\ 20\ 22\ 32\ 14)$$
$$(4\ 26\ 36\ 8\ 39\ 35\ 11\ 25\ 23\ 13\ 17\ 40\ 24\ 6\ 38\ 19\ 37\ 10\ 16).$$

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Proof. Let $\Omega = \{1, 2, \ldots, 40\}$ and consider the natural action of $X$ on $\Omega$. By Magma [1], $\langle H, x_1 \rangle = X$ and so $\Gamma$ is connected by Lemma 2.1(2). Note that $b^a = b^2$ and $c$ centralises $\langle a, b\rangle$. It follows that $H = \langle a, b, c\rangle = \langle a, b\rangle \times \langle c\rangle \cong (\mathbb{Z}_5 : \mathbb{Z}_4) \times \mathbb{Z}_2$. It is easy to see that $H$ is transitive on $\Omega$ and so is regular on $\Omega$. Hence, $X$ has a factorisation $X = GH = HG$ with $G \cap H = 1$. Therefore, $\Gamma$ is isomorphic to a Cayley graph of $G = \mathrm{A}_{39}$. Further computation shows that $|H|/|H \cap H^{x_1}| = 5$. By Lemma 2.1(1), $\Gamma$ is pentavalent. Let

$$\begin{aligned} x_2 = {} & (2\ 20\ 32\ 40\ 10\ 4\ 15\ 19\ 29\ 27\ 9\ 17\ 3\ 6\ 24\ 35\ 13\ 16) \\ & (5\ 37\ 33\ 36\ 8\ 30\ 34\ 25\ 39\ 11\ 23\ 26\ 22\ 14)(7\ 31\ 21)(12\ 38\ 28), \\ x_3 = {} & (2\ 18\ 29\ 7\ 30\ 12\ 34\ 28\ 9\ 3\ 33\ 31\ 5\ 21\ 27\ 20\ 22\ 32\ 14) \\ & (4\ 26\ 36\ 8\ 39\ 35\ 11\ 25\ 23\ 13\ 17\ 40\ 24\ 6\ 38\ 19\ 37\ 10\ 16) \end{aligned}$$

and $S = \{x_1, x_2, x_2^{-1}, x_3, x_3^{-1}\}$. Computation shows that $G \cap (Hx_1H) = S$. Then $\Gamma \cong \mathsf{Cay}(G, S)$ by Lemma 2.1(3). Obviously, $G$ is not normal in $X$ and so is in $\mathsf{Aut}\Gamma$. Thus, $\Gamma$ is nonnormal. □

For convenience, we recall some definitions here. A transitive permutation group $G$ is *quasiprimitive* if each nontrivial normal subgroup of $G$ is transitive. Praeger [8] extended the O'Nan–Scott theorem for primitive groups to quasiprimitive groups, and divided quasiprimitive groups into eight O'Nan–Scott types, namely HA, AS, HS, HC, SD, CD, TW and PA. Further details can be found in [3].

The next lemma completes the proof of Theorem 1.1.

Lemma 3.3. *Let $\Gamma = \mathsf{Cos}(X, H, x_1)$ as in Construction 3.1 and let $A = \mathsf{Aut}\Gamma$. Then $A$ acts quasiprimitively on $V\Gamma$ and $A = \mathrm{A}_{40}$ acts 2-arc transitively on $\Gamma$.*

Proof. Suppose, on the contrary, that $A$ is not quasiprimitive on $V\Gamma$. Let $N$ be a minimal normal subgroup of $A$ which is not transitive on $V\Gamma$. Then $N \cap X \trianglelefteq X$. It follows that $N \cap X = 1$ or $\mathrm{A}_{40}$. If $N \cap X = \mathrm{A}_{40}$, then $X \leq N \trianglelefteq A$. This implies that $N$ is transitive on $V\Gamma$, which is a contradiction. If $N \cap X = 1$, then $|N|$ divides $|A|/|X|$. Let $v$ be a vertex of $\Gamma$. It is easy to see that $X_v = H \cong \mathrm{F}_{20} \times \mathbb{Z}_2$. Then $|A_v|/|X_v| \mid 2^6 \cdot 3^2$ by Lemma 2.2. Since $|A|/|X| = |A_v|/|X_v|$, it follows that $|N|$ divides $2^6 \cdot 3^2$. Thus, $N \cong \mathbb{Z}_2^r$ or $\mathbb{Z}_3^l$, where $1 \leq r \leq 6$ and $1 \leq l \leq 2$. Let $F = NX$. Then $F = N : X$. By Lemma 2.3, $F/\mathrm{C}_F(N) \lesssim \mathsf{Aut}(N) \cong \mathrm{GL}(r, 2)$ or $\mathrm{GL}(l, 3)$. Note that $N \leq \mathrm{C}_F(N)$. If $N = \mathrm{C}_F(N)$, then $F/\mathrm{C}_F(N) = F/N = X = \mathrm{A}_{40}$. However, by Magma [1], $\mathrm{GL}(2, r)$ and $\mathrm{GL}(3, l)$ have no subgroup isomorphic to $\mathrm{A}_{40}$ for $1 \leq r \leq 6$ and $1 \leq l \leq 2$. Hence, we have $N < \mathrm{C}_F(N)$ and $1 \neq \mathrm{C}_F(N)/N \trianglelefteq F/N = X = \mathrm{A}_{40}$. Thus, $\mathrm{C}_F(N)/N = \mathrm{A}_{40}$, that is, $X$ centralises $N$. Hence, $F = N \times X = N \times \mathrm{A}_{40}$ and $F_v/X_v \cong F/X = N$. This implies that $F_v$ is soluble. Since $X_v = \mathrm{F}_{20} \times \mathbb{Z}_2$, it follows from Lemma 2.2 that $F_v = \mathrm{F}_{20} \times \mathbb{Z}_4$ and $N \cong \mathbb{Z}_2$. So, $F = \mathbb{Z}_2 \times \mathrm{A}_{40}$.

Let $\Delta = [F : G]$, the set of right cosets of $G$ in $F$. Since $G = \mathrm{A}_{39}$, the core of $G$ in $F$ is $\mathsf{Core}_F(G) := \bigcap_{x \in F} G^x = 1$. Thus, $F$ may be viewed as a subgroup of the symmetric group $S_{|\Delta|} \cong \mathrm{S}_{80}$ by considering the right multiplication action of $F$ on $\Delta$. For convenience, we identify $\Delta = [F : G]$ with $\Omega = \{1, 2, \ldots, 80\}$. Then the action of $F$ on $\Delta$ is equivalent to the natural action of $F$ on $\Omega$. Now $F_v$ is a regular subgroup of $\mathrm{S}_{80}$ and $G$ is a stabiliser of $i \in \{1, 2, \ldots, 80\}$ in $F$. Without loss of generality, we may assume that $G$ fixes 1. Since $F$ is transitive on the set of arcs of $\Gamma$, by Lemma 2.1,

$\Gamma$ can be represented as a coset graph $\mathsf{Cos}(F, F_v, \tau)$, where $\tau \in \mathrm{N}_F(F_{vw})$ is a 2-element such that $\tau^2 \in F_v$, $v \in V\Gamma$ and $w \in \Gamma(v)$. Note that $F_v$ is a regular subgroup of $\mathrm{S}_{80}$ and all isomorphic regular subgroups of $\mathrm{S}_{80}$ are conjugate in $\mathrm{S}_{80}$ (see, for example, [12, Lemma 4.6]). Thus, we may assume that $F_v = \langle a, b, c\rangle$, where

$$\begin{aligned} a = {} & (1\ 16\ 11\ 6)(2\ 17\ 12\ 7)(3\ 18\ 13\ 8)(4\ 19\ 14\ 9)(5\ 20\ 15\ 10)\\ & (21\ 36\ 31\ 26)(22\ 37\ 32\ 27)(23\ 38\ 33\ 28)(24\ 39\ 34\ 29)(25\ 40\ 35\ 30)\\ & (41\ 56\ 51\ 46)(42\ 57\ 52\ 47)(43\ 58\ 53\ 48)(44\ 59\ 54\ 49)(45\ 60\ 55\ 50)\\ & (61\ 76\ 71\ 66)(62\ 77\ 72\ 67)(63\ 78\ 73\ 68)(64\ 79\ 74\ 69)(65\ 80\ 75\ 70),\\ b = {} & (1\ 46\ 77\ 35)(2\ 43\ 66\ 28)(3\ 60\ 75\ 21)(4\ 57\ 64\ 34)(5\ 54\ 73\ 27)\\ & (6\ 51\ 62\ 40)(7\ 48\ 71\ 33)(8\ 45\ 80\ 26)(9\ 42\ 69\ 39)(10\ 59\ 78\ 32)\\ & (11\ 56\ 67\ 25)(12\ 53\ 76\ 38)(13\ 50\ 65\ 31)(14\ 47\ 74\ 24)(15\ 44\ 63\ 37)\\ & (16\ 41\ 72\ 30)(17\ 58\ 61\ 23)(18\ 55\ 70\ 36)(19\ 52\ 79\ 29)(20\ 49\ 68\ 22),\\ c = {} & (1\ 17\ 13\ 9\ 5)(2\ 18\ 14\ 10\ 6)(3\ 19\ 15\ 11\ 7)(4\ 20\ 16\ 12\ 8)\\ & (21\ 37\ 33\ 29\ 25)(22\ 38\ 34\ 30\ 26)(23\ 39\ 35\ 31\ 27)(24\ 40\ 36\ 32\ 28)\\ & (41\ 57\ 53\ 49\ 45)(42\ 58\ 54\ 50\ 46)(43\ 59\ 55\ 51\ 47)(44\ 60\ 56\ 52\ 48)\\ & (61\ 77\ 73\ 69\ 65)(62\ 78\ 74\ 70\ 66)(63\ 79\ 75\ 71\ 67)(64\ 80\ 76\ 72\ 68). \end{aligned}$$

By Lemma 3.2, $\Gamma \cong \mathsf{Cay}(G, S)$. Then the 2-element $\tau$ is an involution by Proposition 2.5 and $\tau \in \mathrm{N}_{\mathrm{S}_{80}}(F_v \cap F_v^\tau)\backslash(\bigcup_{1\neq K\trianglelefteq F_v} \mathrm{N}_{\mathrm{S}_{80}}(K))$. Since $\Gamma$ is pentavalent, we have $|F_v : F_v \cap F_v^\tau| = 5$. Thus, $F_v \cap F_v^\tau$ is a Sylow 2-subgroup of $F_v$. Since all Sylow 2-subgroups of $F_v$ are conjugate in $F_v$, we may assume that $F_v \cap F_v^\tau = \langle a, b\rangle$. Then $\tau \in \mathrm{N}_{\mathrm{S}_{80}}(\langle a, b\rangle)\backslash(\bigcup_{1\neq K\trianglelefteq F_v} \mathrm{N}_{\mathrm{S}_{80}}(K))$ is such that $1^\tau = 1$ and $\langle F_v, \tau\rangle = F \cong \mathbb{Z}_2 \times \mathrm{A}_{40}$. However, by computing with Magma [1], such $\tau$ does not exist.

Hence, $A$ is quasiprimitive on $V\Gamma$. Let $S = \mathsf{soc}(A)$, the socle of $A$. Then $S \cong T^d$ is transitive on $V\Gamma$, where $T$ is simple and the integer $d \geq 1$. Since $|V\Gamma| = |G| = |\mathrm{A}_{39}|$, $A$ is obviously not of type HA. Note that $|A_v| \mid 2^9 \cdot 3^2 \cdot 5$. It follows that $|G| \mid |S| \mid 2^9 \cdot 3^2 \cdot 5 \cdot |G|$. So, there must be a prime $p$ such that $p \mid |S|$ and $p^2 \nmid |S|$. Consequently, $d = 1$, that is, $S = \mathsf{soc}(A) = T$ is a nonabelian simple group. It follows that $A$ is not of type HS, HC, CD, SD, TW or PA. Hence, $A$ is almost simple. Since $S \cap X \trianglelefteq X \cong \mathrm{A}_{40}$, it follows that $S \cap X = 1$ or $\mathrm{A}_{40}$. If $S \cap X = 1$, then $|S| \mid |A|/|X| \mid 2^6 \cdot 3^2$. By the Burnside $p$–$q$ theorem (see [9, page 240]), $S$ is soluble, which is not possible. Thus, $S \cap X = X$ and so $X \leq S$. It follows that $|S : X| \mid |A : X| \mid 2^6 \cdot 3^2$. By Lemma 2.4, we can conclude that $S = X \cong \mathrm{A}_{40}$. Thus, $A \leq \mathsf{Aut}(S) \cong \mathrm{S}_{40}$. If $A \cong \mathrm{S}_{40}$, then $|A_v| = |A|/|G| = 80$. By Lemma 2.2, $A_v \cong \mathrm{F}_{20} \times \mathbb{Z}_4$. This also leads to a contradiction by arguments similar to those used for $F_v$ in the previous paragraph. Hence, $A \cong \mathrm{A}_{40}$ and so $A_v \cong \mathrm{F}_{20} \times \mathbb{Z}_2$. By Lemma 2.2, $\Gamma$ is 2-arc transitive. This completes the proof of the lemma. □

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