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SET-VALUED CASH SUB-ADDITIVE RISK MEASURES

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In this paper, we introduce a new class of set-valued risk measures which satisfies cash sub-additivity. Dual representation for them is provided. Moreover, we also investigate dynamic set-valued cash sub-additive risk measures and discuss the corresponding multi-portfolio time consistency. The equivalent characterization of the multi-portfolio time consistency is given. Finally, an example is also given to illustrate the introduction of set-valued cash sub-additive risk measures. The present paper can be considered as a set-valued extension of scalar cash sub-additive risk measures introduced by El Karouii and Ravanelli [8].

Keywords: cash additive; cash sub-additive; dynamic; risk measure; set-valued

1. INTRODUCTION

In their seminal paper, Artzner et al. [2,3] firstly introduced the class of coherent risk measures, by proposing four basic properties to be satisfied by every sound financial risk measure. Further, Föllmer and Schied [13], and independently, Frittelli and Rosazza Gianin [14] introduced the broader class, named convex risk measures, by dropping one of the coherency axioms.

In the past decade, to evaluate the risk of a portfolio consisting of several financial positions, Jouini et al. [19] firstly introduced the class of set-valued coherent risk measures by proposing some axioms. Hamel [15] introduced set-valued convex risk measures by an axiomatic approach. For more studies on set-valued risk measures, see Cascos and Molchanov [4], Hamel and Heyde [16], Hamel et al. [17], Hamel et al. [18], Labuschagne and Offwood-Le Roux [20], Ararat et al. [1], Tahar and Lépinette [26], Farkas et al. [9], Molchanov and Cascos [23], Lepinette and Molchanov [21] and the references therein.

In all the above-mentioned works on set-valued risk measures, an axiom of translation invariance, which is also called cash additivity, is employed. However, as pointed out by El Karouii and Ravanelli [8], the cash additive axiom may fail once there is any form of uncertainty about interest rates because the money is of time value. For example, when m dollars are added to a future position X_T, the capital requirement at time $t = 0$ is reduced by less than m dollars because the value of the money may grow as the time goes by. Therefore, it is more appropriate to use cash sub-additivity to replace cash additivity. This observation motivated us to study the set-valued cash sub-additive risk measures.

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In this paper, first, we will introduce a new class of set-valued risk measures, which is called set-valued cash sub-additive risk measures. Dual representation for them is provided. Second, we will also introduce the dynamic set-valued cash sub-additive risk measures, and discuss the issue of the so-called multi-portfolio time consistency. The equivalent characterization of the multi-portfolio time consistency is given. These newly introduced set-valued cash sub-additive risk measures can be considered as a set-valued extension of scalar cash sub-additive risk measures introduced by El Karoui and Ravanelli [8].

The rest of the paper is organized as follows. In Section 2, we will briefly introduce preliminaries, including notations. In Section 3, we will state the definition of set-valued cash sub-additive risk measures, and provide the dual representation. An example will also be given in this section. Section 4 is devoted to dynamic set-valued cash sub-additive risk measures, where the corresponding dual representation is given. Finally, in Section 5, the multi-portfolio time consistency is discussed.

2. PRELIMINARIES

In this section, we will briefly introduce the preliminaries. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space and $\mathbb{R}^d$ be the d-dimensional Euclidean space, $d \geq 1$. Denote by $L_d^p := L_d^p(\Omega, \mathcal{F}, \mathbb{P})$ with $p \in [1, \infty]$ the linear space of $\mathcal{F}$-measurable functions $X : \Omega \to \mathbb{R}^d$ such that $\|X\|_p := \int_\Omega |X|^p d\mathbb{P} < \infty$ for $p \in [1, \infty)$ and $\|X\|_p := \text{esssup}|X| < \infty$ for $p = \infty$, where $|\cdot|$ is an arbitrary fixed norm on $\mathbb{R}^d$. Then $(L_d^p, \|\cdot\|_p)$ is a Banach space. For $X, Y \in L_d^p$, we will identify X with Y if $\mathbb{P}(X = Y) = 1$. The space L_d^p represents the set of financial positions. Positive values of $X \in L_d^p$ correspond to gains, while negative values correspond to losses. From now on, we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ with $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_T := \mathcal{F}$. Denote $L_d^p(\mathcal{F}_t) := L_d^p(\Omega, \mathcal{F}_t, \mathbb{P})$. Note that $L_d^p = L_d^p(\mathcal{F}_T)$.

The d-dimensional financial positions in $L_d^p(\mathcal{F}_t)$ have a strong realistic interpretation. This is indeed the case if we consider the situations where the investors have accesses to different markets and form multi-asset portfolios in the presence of frictions such as transaction costs, liquidity problems, irreversible transfers, etc.

Let K be a closed convex polyhedral cone of $\mathbb{R}^d$ with $K \supseteq \mathbb{R}_+^d := \{(x_1, \ldots, x_d) \in \mathbb{R}^d; x_i \geq 0, 1 \leq i \leq d\}$. For any $X = (X^1, \ldots, X^d)$, $Y = (Y^1, \ldots, Y^d) \in L_d^p$, $X + Y$ stands for $(X^1 + Y^1, \ldots, X^d + Y^d)$ and aX stands for $(aX^1, \ldots, aX^d)$ for $a \in \mathbb{R}$. The partial order with respect to K is denoted by $X \leq_K Y$, which means $Y - X \in K$. Let $L_d^p(K) := \{X \in L_d^p : X \in K\}$ be a closed convex cone in L_d^p and $M := \mathbb{R}^m \times \{0\}^{d-m}$ be the linear subspace of $\mathbb{R}^d$ for $1 \leq m \leq d$. The introduction of M was considered by Jouini et al. [19] and Hamel [15], which means that a regulator could only accept security deposits in the first m reference instruments. We denote by $K_M := K \cap M$ the closed convex polyhedral cone in M, $M^\perp := \{0\}^m \times \mathbb{R}^{d-m}$, $K_M^* := \{u \in M : u^{tr} z \geq 0 \text{for any} z \in K_M\}$ the positive dual cone of K_M in M, where u^{tr} means the transpose of u, and by $\text{ri}K_M$ the relative interior of K_M. Given a set Υ, we denote $Q_\Upsilon^t := \{A \subset \mathbb{R}^d : A = clco(A + \Upsilon)\}$ and $Q_M^t := Q_{K_M}^t = \{A \subset M : A = clco(A + K_M)\}$, where $clco(A)$ represents the closed convex hull of A. Given a set $Z \subset \mathbb{R}$, I_Z stands for the indicator function.

The cone K models proportional frictions between the markets and contains those reference vectors which can be transferred (with paying transaction costs) into positions in $\mathbb{R}_+^d$, see Hamel [15]. The cone K is also introduced to play the role of the solvency set of all positions that can be liquidated without any debt, or equivalently, it allows to define a liquidation value function as we need it to take into account the interdependencies between currencies, for example with respect to transaction costs.

We denote by $\mathcal{M}_{1,d}^{\mathbb{P}} := \mathcal{M}_{1,d}^{\mathbb{P}}(\mathcal{F}_T)$ the set of all vector-valued probability measures whose components are absolutely continuous with respect to $\mathbb{P}$, that is $\mathbb{Q} \in \mathcal{M}_{1,d}^{\mathbb{P}}$ with component $\mathbb{Q}_i : \mathcal{F} \to [0, 1]$ being a probability measure such that $d\mathbb{Q}_i/d\mathbb{P} \in L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq i \leq d$. Denote by $E^{\mathbb{Q}}[X] := (E^{\mathbb{Q}_1}[X^1], \ldots, \mathbb{E}^{\mathbb{Q}_d}[X^d])^{tr}$ the vectorial expectation of $X := (X^1, \ldots, X^d)$ with respect to vector-valued probability measure $\mathbb{Q}$, where $\mathbb{E}^{\mathbb{Q}_i}[X^i]$ means the expectation of X^i with respect to the probability measure $\mathbb{Q}_i$. $\mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}_t]$ denotes the $\mathbb{R}^d$-valued $\mathbb{Q}$-conditional expectation of X under component-wise sense. Let $\mathcal{M}_{s,f}^d$ denote the set of all finite additive vector sub-probability measures, that is $\mathcal{M}_{s,f}^d := \{u = (u_1, \ldots, u_d)^{tr} | u_i : \mathcal{F} \to [0, 1]$ is finite additive and $0 \leq u_i(\Omega) \leq 1, 1 \leq i \leq d\}$. We will also denote by $diag(\upsilon)$ the diagonal matrix with the elements of a vector υ as the main diagonal.

Let 1_t denote one unit cash available at time t with $0 \leq t \leq T$. Denote $D_T := (D_T^1, \ldots, D_T^d)$, where D_T^i is the stochastic discount factor for certain currency, for reference see Ng et al. [24]. Throughout this paper, we assume that $D_T^i \in [0, 1]$ for $1 \leq i \leq d$. Without loss of generality, we assume that the i^{th} component of D_T corresponds to the same currency as that of the i^{th} component of X_T.

3. SET-VALUED CASH SUB-ADDITIVE RISK MEASURES

In this section, we will introduce the definition of set-valued cash sub-additive risk measures and discuss the relation between set-valued cash additive risk measures and set-valued cash sub-additive risk measures. Using this relation, we will provide the dual representation for set-valued cash sub-additive risk measures.

3.1. Definition

We begin with recalling some properties related to the set-valued mapping $R : L_d^p(\mathcal{F}_T) \to Q_M^t$.

- A0 Normalization: $K_M \subseteq R(0)$ and $R(0) \cap -\text{ri}K_M = \phi$;
- A1 Cash additivity (or cash invariance at first m currencies): for any $X \in L_d^p(\mathcal{F}_T)$ and $b \in M$, $R(X + b) = R(X) - b$;
- A2 Monotonicity: for any $X,Y \in L_d^p(\mathcal{F}_T)$, $X - Y \in L_d^p(K)$ implies that $R(X) \supseteq R(Y)$;
- A3 Convexity: for any $\lambda \in [0, 1]$ and $X,Y \in L_d^p(\mathcal{F}_T)$, $R(\lambda X + (1 - \lambda)Y) \supseteq \lambda R(X) + (1 - \lambda)R(Y)$;
- A4 Proper: for any $X \in L_d^p(\mathcal{F}_T)$, $\text{dom}R := \{X \in L_d^p(\mathcal{F}_T) : R(X) \neq \emptyset\} \neq \emptyset$ and $R(X) \neq M$;
- A5 Closed: for any $X \in L_d^p(\mathcal{F}_T)$, $\text{graph}R := \{(X, u) \in L_d^p(\mathcal{F}_T) \times M : u \in R(X)\}$ is closed.

Remark 3.1: As introduced by Hamel [15], a set-valued cash additive risk measure is a mapping on $L_d^p(\mathcal{F}_T)$ which satisfies $\mathbf{A0} - \mathbf{A3}$.

Now we define a set-valued risk measure on the discounted position $D_T X_T := (D_T^1 X_T^1, \ldots, D_T^d X_T^d)$ for $D_T = (D_T^1, \ldots, D_T^d)$ and $X_T = (X_T^1, \ldots, X_T^d)$.

DEFINITION 3.1: *Let D_T be a $\mathcal{F}_T$-measurable discount factor. A set-valued spot risk measure, say ϱ_0, is a cash additive risk measure defined on the discounted factor $D_T X_T$ with $X_T \in L_d^p(\mathcal{F}_T)$.*

Given a (stochastic) discount factor $D_T \in [0, 1]$ and a set-valued spot risk measure ϱ_0, we can define a convex set-valued function on $L_d^p(\mathcal{F}_T)$ by $R(X_T) := \varrho_0(D_T X_T)$. As shown by El Karoui and Ravanelli (2009), for any $z \in K_M$, we have

$$R(X_T + z1_T) = \varrho_0(D_T X_T + D_T z) \subseteq \varrho_0(D_T X_T + z) = \varrho_0(D_T X_T) - z = R(X_T) - z.$$

This property of R is called cash sub-additivity. The fact $D_T z \leq_K z$ can also be understood as the time value of the money. That is to say that R is expressed in terms of the current numéraire but defined on the future financial positions with the future numéraire.

It is worth mentioning that cash sub-additivity does have a great meaning for quasi-convex risk measures. As pointed out by Cerreia-Vioglio et al. [5], when there is uncertainty about interest rates, the cash additivity assumption on risk measures becomes problematic. Hence, under the cash sub-additivity assumption, the equivalence between convexity and the diversification principle no longer holds. In fact, this diversification principle only implies quasiconvexity.

Next, we will introduce the definition of set-valued cash sub-additive risk measures.

Definition 3.2: *A set-valued cash sub-additive risk measure* $R : L_d^p(\mathcal{F}_T) \to Q_M^t$ *is a set-valued mapping which satisfies* **A0**, **A2**, **A3** *and the following property:*

A6 Cash sub-additivity: for any $X_T \in L_d^p(\mathcal{F}_T)$, $z_1, z_2 \in M$ *and* $z_1 \leq_K z_2$,

$$R(X_T + z_1 1_T) + z_1 \supseteq R(X_T + z_2 1_T) + z_2.$$

Remark 3.2: Cash sub-additivity **A6** can also be expressed as follows. For any $X_T \in L_d^p(\mathcal{F}_T)$, $z \in K_M$,

$$R(X_T + z1_T) \subseteq R(X_T) - z \quad \text{or} \quad R(X_T - z1_T) \supseteq R(X_T) + z.$$

Proof: We first show that **A6** is equivalent to $R(X_T + z1_T) \subseteq R(X_T) - z$. Suppose that **A6** holds. Let $z_1 = z \in K_M$ and $z_2 = 0$. The implication that **A6** implies

$$R(X_T + z1_T) \subseteq R(X_T) - z \tag{3.1}$$

is straightforward. Now we show the reverse implication. For any $X_T \in L_d^p(\mathcal{F}_T)$ and $z_1, z_2 \in M$ with $z_1 \leq_K z_2$, we know that $X_T + z_1 1_T \in L_d^p(\mathcal{F}_T)$. From (3.1) it follows that

$$R(X_T + z_1 1_T + (z_2 - z_1)1_T) \subseteq R(X_T + z_1 1_T) - (z_2 - z_1),$$

which is equivalent to

$$R(X_T + z_2 1_T) \subseteq R(X_T + z_1 1_T) - (z_2 - z_1),$$

which is nothing else but

$$R(X_T + z_1 1_T) + z_1 \supseteq R(X_T + z_2 1_T) + z_2,$$

which is exactly **A6**. The equivalence between **A6** and $R(X_T - z1_T) \supseteq R(X_T) + z$ can be shown similarly. The proof is completed. ■

We will end this subsection with a special class of set-valued cash sub-additive risk measures. This special class consists of set-valued convex loss-based risk measures, see Sun et al. [25]. Note that the scalar case of convex loss-based risk measures was studied by Cont et al. [7].

DEFINITION 3.3: *A set-valued convex loss-based risk measure is a proper closed* ($\sigma(L_d^\infty, L_d^1)$-*closed if* $p = \infty$) *mapping* $\varrho : L_d^p(\mathcal{F}_T) \to Q_{M+}^t := \{A \subset K_M : A = clco(A + K_M)\}$ *which satisfies the following properties:*

- *R0 Normalization:* $K_M \subseteq \varrho(0)$ *and* $\varrho(0) \cap -riK_M = \phi$*;*
- *R1 Cash losses: for any* $z \in K_M$, $z \in \varrho(-z)$*;*
- *R2 Monotonicity: for any* $X, Y \in L_d^p(\mathcal{F}_T)$, $X - Y \in L_d^p(K)$ *implies* $\varrho(X) \supseteq \varrho(Y)$*;*
- *R3 Loss-dependence: for any* $X \in L_d^p(\mathcal{F}_T)$, $\varrho(X) = \varrho(X \wedge 0)$, *where* $X \wedge 0 := (X^1 \wedge 0, \ldots, X^d \wedge 0)$*;*
- *R4 Convexity: for any* $\lambda \in [0, 1]$ *and* $X, Y \in L_d^p(\mathcal{F}_T)$, $\varrho(\lambda X + (1-\lambda)Y) \supseteq \lambda\varrho(X) + (1-\lambda)\varrho(Y)$.

We claim that the set-valued convex loss-based risk measures are the special cases of set-valued cash sub-additive risk measures. Indeed, for any $X \in L_d^p(\mathcal{F}_T)$, $z \in K_M$ and $\varepsilon \in (0, 1)$, we have

$$\begin{aligned}\varrho\left((1-\varepsilon)X - z\right) &= \varrho\left((1-\varepsilon)X + \varepsilon\left(-\frac{z}{\varepsilon}\right)\right)\\ &\supseteq (1-\varepsilon)\varrho(X) + \varepsilon\varrho\left(-\frac{z}{\varepsilon}\right)\\ &\supseteq (1-\varepsilon)\varrho(X) + z,\end{aligned}$$

where the last inclusion is due to the property of cash losses. Since ϱ is a proper closed mapping, it is lower semi-continuous, that is, if $\{X^k; k \geq 1\} \subseteq L_d^p(\mathcal{F}_T)$ is a sequence with $X^k \to X$ $\mathbb{P}$-almost surely, then

$$\varrho(X) \supseteq \liminf_{k\to\infty} \varrho(X^k) = \left\{u \in M : \forall k \geq 1, \exists u^k \in \varrho(X^k) \text{ such that } \lim_{k\to\infty} u^k = u\right\}.$$

[Note that, in the terminology of Theorem 6.2 of Hamel and Heyde [16], the lower semi-continuity of ϱ is called the Fatou property when $p = \infty$.] Thus, by the arbitrariness of ε, we conclude that

$$\varrho(X - z) \supseteq \varrho(X) + z,$$

which means ϱ is cash sub-additive.

Next, we will give an example of set-valued convex loss-based risk measure called $AV@R^{loss}$.

Example 3.1: (Loss-based average value at risk)

For any $X := (X^1, \cdots, X^d) \in L_d^p(\mathcal{F}_T)$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $0 < \alpha_i < 1$, $i = 1, \ldots, d$,

$$\begin{aligned}AV@R_\alpha^{loss}(X) := \Bigg(&\inf_{z_1\in\mathbb{R}}\left\{\frac{1}{\alpha_1}\mathbb{E}[(-(X^1\wedge 0) + z_1)^+] - z_1\right\}, \ldots,\\ &\inf_{z_m\in\mathbb{R}}\left\{\frac{1}{\alpha_m}\mathbb{E}[(-(X^m\wedge 0) + z_m)^+] - z_m\right\}\Bigg) + \mathbb{R}_+^m.\end{aligned}$$

It is not hard to check that $AV@R^{loss}$ satisfies all the properties of Definition 3.3. So $AV@R^{loss}$ is a set-valued convex loss-based risk measure, and hence it is also cash sub-additive.

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3.2. Dual representation

In order to get the dual representation, we enlarge the space of financial positions. Denote $\Omega^* := \{0, 1\}$. Any pair (X_T, a), where $X_T \in L_d^p(\mathcal{F}_T)$ and $a \in \mathbb{R}^d$, can be viewed as the coordinates of a function $\widehat{X}_T$ defined on the enlarged space $\widehat{\Omega} := \Omega \times \Omega^*$ with the element (ω, θ),

$$\widehat{X}_T(\omega, \theta) := X_T(\omega) I_{\{1\}}(\theta) + a I_{\{0\}}(\theta). \tag{3.2}$$

We endow $\widehat{\Omega}$ with the σ-algebra $\widehat{\mathcal{F}}_T$, generated by all the random variables $\widehat{X}_T$ defined above. We denote by $\mathcal{X}$ the linear space of all random variables $\widehat{X}_T$ defined as in (3.2). The constant random variable in $\mathcal{X}$ is denoted by $b := bI_{\{1\}} + bI_{\{0\}} = b$. Note that the event $\{\theta = 0\}$ is atomic and all $\widehat{\mathcal{F}}_T$-measurable random variables are constant on this event.

Let $\mathcal{F}^* := \{\emptyset, \Omega^*, \{0\}, \{1\}\}$ and $\mathbb{P}^*$ be a probability measure on the measurable space $(\Omega^*, \mathcal{F}^*)$. Denote by $(\widehat{\Omega}, \widetilde{\mathcal{F}}_T, \widetilde{\mathbb{P}})$ the product probability space, where $\widetilde{\mathcal{F}}_T := \mathcal{F}_T \times \mathcal{F}^*$, the product σ-algebra of $\mathcal{F}_T$ and $\mathcal{F}^*$, $\widetilde{\mathbb{P}} := \mathbb{P} \times \mathbb{P}^*$, the product probability of $\mathbb{P}$ and $\mathbb{P}^*$. It is not hard to check that $\widehat{\mathcal{F}}_T \subseteq \widetilde{\mathcal{F}}_T$. Thus, we denote by $\widehat{\mathbb{P}}$ the restriction of $\widetilde{\mathbb{P}}$ to $\widehat{\mathcal{F}}_T$. Note that $(\widehat{\Omega}, \widehat{\mathcal{F}}_T, \widehat{\mathbb{P}})$ is a probability space and we denote by $L_d^p(\widehat{\mathcal{F}}_T) := L_d^p(\widehat{\Omega}, \widehat{\mathcal{F}}_T, \widehat{\mathbb{P}})$ the linear space of $\widehat{\mathcal{F}}_T$-measurable functions $\widetilde{X} : \widehat{\Omega} \to \mathbb{R}^d$ such that $\|\widetilde{X}\|_p := \int_\Omega |\widetilde{X}|^p d\mathbb{P} < \infty$ for $p \in [1, \infty)$ and $\|\widetilde{X}\|_p := \text{esssup}|\widetilde{X}| < \infty$ for $p = \infty$. It is not hard to check that $\mathcal{X}$ is a linear subspace of $L_d^p(\widehat{\mathcal{F}}_T)$. We endow $\mathcal{X}$ with the weak topology $\sigma(\mathcal{X}, L_d^q(\widehat{\mathcal{F}}_T))$, which is the coarsest topology on $\mathcal{X}$ such that for all $v \in L_d^q(\widehat{\mathcal{F}}_T)$, $u \to \langle u, v\rangle := \mathbb{E}[u^{tr} v]$ is a continuous linear function on $\mathcal{X}$. Hence, the topological dual space of $(\mathcal{X}, \sigma(\mathcal{X}, L_d^q(\widehat{\mathcal{F}}_T)))$ is $L_d^q(\widehat{\mathcal{F}}_T)$, that is

$$\left(\mathcal{X}, \sigma\left(\mathcal{X}, L_d^q(\widehat{\mathcal{F}}_T)\right)\right)^* \cong L_d^q(\widehat{\mathcal{F}}_T). \tag{3.3}$$

Moreover, $(\mathcal{X}, \sigma(\mathcal{X}, L_d^q(\widehat{\mathcal{F}}_T)))$ is a separated, locally convex topological linear space.

We denote by $\widehat{\mathcal{M}_{1,d}^{\mathbb{P}}} := \mathcal{M}_{1,d}^{\widehat{\mathbb{P}}}(\widehat{\mathcal{F}}_T)$ the set of all vector-valued probability measures whose components are absolutely continuous with respect to $\widehat{\mathbb{P}}$, that is $\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}_{1,d}^{\mathbb{P}}}$ with component $\widehat{\mathbb{Q}}_i : \widehat{\mathcal{F}}_T \to [0, 1]$ being a probability measure such that $d\widehat{\mathbb{Q}}_i / d\widehat{\mathbb{P}} \in L^q(\widehat{\mathcal{F}}_T)$. Let $K^+ := \{u \in \mathbb{R}^d : u^{tr}(vI_{\{1\}} + aI_{\{0\}}) \geq 0 \text{ for any } v, a \in K\}$. Then, we denote by $\mathcal{X}(K) := \{\widehat{X}_T \in \mathcal{X} : \widehat{X}_T \in K \quad \widehat{\mathbb{P}} - a.s.\}$ and $L_d^q(\widehat{\mathcal{F}}_T, K^+) := \{\widetilde{X} \in L_d^q(\widehat{\mathcal{F}}_T) : \widetilde{X} \in K^+ \quad \widetilde{\mathbb{P}} - a.s.\}$. It is not hard to check that $L_d^q(\widehat{\mathcal{F}}_T, K^+)$ is the positive dual cone of $\mathcal{X}(K)$. For any $\widehat{X}_T, \widehat{Y}_T \in \mathcal{X}$ with $\widehat{X}_T = X_T I_{\{1\}} + a_1 I_{\{0\}}$ and $\widehat{Y}_T = Y_T I_{\{1\}} + a_2 I_{\{0\}}$, where $X_T, Y_T \in L_d^p(\mathcal{F}_T)$, $a_1, a_2 \in \mathbb{R}^d$, we define the order in $\mathcal{X}$ as $\widehat{X}_T - \widehat{Y}_T \in \mathcal{X}(K)$ if and only if $Y_T \leq_K X_T$ and $a_2 \leq_K a_1$. For any $a := (a_1, \ldots, a_d) \in \mathbb{R}^d$, $a|_M$ denotes the the vector $(a_1, \ldots, a_m, 0, \ldots, 0) \in M$.

Before we state the main result, we first show a one to one relation between a set-valued cash additive risk measure and a set-valued cash sub-additive risk measure.

Proposition 3.1: *Given a set-valued cash sub-additive risk measure R on $L_d^p(\mathcal{F}_T)$ with $0 \in R(0)$, we define a set-valued risk measure $\widehat{\varrho}$ as follows. For any $\widehat{X}_T \in \mathcal{X}$ where $\widehat{X}_T(\omega, \theta) = X_T(\omega) I_{\{1\}}(\theta) + a I_{\{0\}}(\theta)$ with $X_T \in L_d^p(\mathcal{F}_T)$, $a \in \mathbb{R}^d$,*

$$\widehat{\varrho}(\widehat{X}_T) := R(X_T - a 1_T) - a|_M. \tag{3.4}$$

Then $\widehat{\varrho}$ is a cash additive risk measure with $\widehat{\varrho}(0) = 0$ and $\widehat{\varrho}(X_T I_{\{1\}}) = R(X_T)$.

PROOF: It is not hard to check that $\widehat{\varrho}(0) = 0$, $\widehat{\varrho}(X_T I_{\{1\}}) = R(X_T)$ and $\widehat{\varrho}$ satisfies the property of **A0**. Next, we will show that $\widehat{\varrho}$ satisfies properties of **A1**, **A2** and **A3**.

A1. Cash additivity: for any $b \in M$ and $\widehat{X}_T \in \mathcal{X}$ with $\widehat{X}_T = X_T I_{\{1\}} + aI_{\{0\}}$ where $X_T \in L_d^p(\mathcal{F}_T)$,

$$\begin{aligned}\widehat{\varrho}(\widehat{X}_T + b) &= \widehat{\varrho}\left((X_T + b)I_{\{1\}} + (a+b)I_{\{0\}}\right)\\ &= R\left(X_T + b1_T - (a+b)1_T\right) - a|_M - b\\ &= R(X_T - a1_T) - a|_M - b\\ &= \widehat{\varrho}(\widehat{X}_T) - b,\end{aligned}$$

which shows that $\widehat{\varrho}$ is cash additive.

A2. Monotonicity: for any $\widehat{X}_T$, $\widehat{Y}_T \in \mathcal{X}$ with $\widehat{X}_T = X_T I_{\{1\}} + a_1 I_{\{0\}}$, $\widehat{Y}_T = Y_T I_{\{1\}} + a_2 I_{\{0\}}$, where X_T, $Y_T \in L_d^p(\mathcal{F}_T)$, a_1, $a_2 \in \mathbb{R}^d$ with $\widehat{X}_T - \widehat{Y}_T \in \mathcal{X}(K)$, then

$$\begin{aligned}\widehat{\varrho}(\widehat{X}_T) = R(X_T - a_1 1_T) - a_1|_M &\supseteq R(X_T - a_2 1_T) - a_2|_M\\ &\supseteq R(Y_T - a_2 1_T) - a_2|_M = \widehat{\varrho}(\widehat{Y}_T),\end{aligned}$$

which shows that $\widehat{\varrho}$ is monotone.

A3. Convexity: for any $\lambda \in (0, 1)$, $\widehat{X}_T$, $\widehat{Y}_T \in \mathcal{X}$ with $\widehat{X}_T = X_T I_{\{1\}} + a_1 I_{\{0\}}$, $\widehat{Y}_T = Y_T I_{\{1\}} + a_2 I_{\{0\}}$, where X_T, $Y_T \in L_d^p(\mathcal{F}_T)$, a_1, $a_2 \in \mathbb{R}^d$,

$$\begin{aligned}&\widehat{\varrho}(\lambda\widehat{X}_T + (1-\lambda)\widehat{Y}_T)\\ &= \widehat{\varrho}\Big(\big(\lambda X_T + (1-\lambda)Y_T\big)I_{\{1\}} + \big(\lambda a_1 + (1-\lambda)a_2\big)I_{\{0\}}\Big)\\ &= R\Big(\big(\lambda X_T + (1-\lambda)Y_T\big) - \big(\lambda a_1 + (1-\lambda)a_2\big)1_T\Big) - \lambda a_1|_M - (1-\lambda)a_2|_M\\ &= R\Big(\lambda(X_T - a_1 1_T) + (1-\lambda)(Y_T - a_2 1_T)\Big) - \lambda a_1|_M - (1-\lambda)a_2|_M\\ &\supseteq \lambda R(X_T - a_1 1_T) + (1-\lambda)R(Y_T - a_2 1_T) - \lambda a_1|_M - (1-\lambda)a_2|_M\\ &= \lambda\widehat{\varrho}(\widehat{X}_T) + (1-\lambda)\widehat{\varrho}(\widehat{Y}_T),\end{aligned}$$

which shows that $\widehat{\varrho}$ is convex. The proof is completed.

■

Remark 3.3: Taking (3.3) into account, the topological dual space of $(\mathcal{X}, \sigma(\mathcal{X}, L_d^q(\widehat{\mathcal{F}}_T)))$ is $L_d^q(\widehat{\mathcal{F}}_T)$. Then, we can applying the set-valued Fenchel–Moreau theorem, that is Theorem 2 of Hamel [15], to the case where the linear space X is specified to $\mathcal{X}$. If f is a proper closed convex function on $\mathcal{X}$, for any $\widehat{X}_T \in \mathcal{X}$,

$$f(\widehat{X}_T) = f^{**}(\widehat{X}_T) := \bigcap_{(\widehat{Y},u)\in L_d^q(\widehat{\mathcal{F}}_T)\times K_M^*\backslash\{0\}} \Big\{ - f^*(\widehat{Y}, u) + S_{(\widehat{Y},u)}(\widehat{X}_T)\Big\}, \tag{3.5}$$

where

$$S_{(\widehat{Y},u)}(\widehat{X}_T) := \{z \in M : \mathbb{E}[\widehat{X}_T^{tr}\widehat{Y}] + u^{tr}z \geq 0\}$$

and

$$-f^*(\widehat{Y}, u) := cl \bigcup_{\widehat{X}_T\in\mathcal{X}} \Big(f(\widehat{X}_T) + S_{(\widehat{Y},u)}(-\widehat{X}_T)\Big).$$

The main purpose of this section is to derive the dual representation for set-valued cash sub-additive risk measures on $L_d^p(\mathcal{F}_T)$. To reach the purpose, we will first derive the dual representation of set-valued cash additive risk measures on $\mathcal{X}$. Then, by applying the one–one relation between the set-valued cash additive risk measures on $\mathcal{X}$ and the set-valued cash sub-additive risk measures on $L_d^p(\mathcal{F}_T)$ established in Proposition 3.1, we will derive the dual representation of set-valued cash sub-additive risk measures on $L_d^p(\mathcal{F}_T)$. To this end, two propositions are needed. Proposition 3.2 below shows the conjugate function of set-valued cash additive risk measures. Then, by (3.5), Proposition 3.3 below gives the dual representation for set-valued cash additive risk measures on $\mathcal{X}$.

Proposition 3.2: *Let $\widehat{\varrho}: \mathcal{X} \to Q_M^t$ be a proper closed cash additive risk measure with $\widehat{Y} \in L_d^q(\widehat{\mathcal{F}}_T)$ and $u \in K_M^* \backslash \{0\}$. Then*

$$-\widehat{\varrho}^*(\widehat{Y}, u) = \begin{cases} cl \bigcup_{\widehat{X}_T \in \mathcal{A}_{\widehat{\varrho}}} S_{(\widehat{Y},u)}(-\widehat{X}_T), & \widehat{Y} \in L_d^q(\widehat{\mathcal{F}}_T, K^+),\ u \in (\mathbb{E}[\widehat{Y}] + M^\perp) \cap K_M^* \backslash \{0\}, \\ M, & \textit{elsewhere}, \end{cases} \tag{3.6}$$

where

$$\mathcal{A}_{\widehat{\varrho}} := \{\widehat{X}_T \in \mathcal{X} : 0 \in \widehat{\varrho}(\widehat{X}_T)\}.$$

Proof: For any $\widehat{X}_T \in \mathcal{X}$ and $v \in M$, we have

$$\begin{aligned} S_{(\widehat{Y},u)}(-\widehat{X}_T - v) &= \{z \in M : \mathbb{E}[-\widehat{X}_T^{tr}\widehat{Y}] \geq -u^{tr}z + \mathbb{E}[\widehat{Y}]^{tr}v\} \\ &= \{z - v \in M : \mathbb{E}[-\widehat{X}_T^{tr}\widehat{Y}] \geq -u^{tr}(z - v) + (\mathbb{E}[\widehat{Y}] - u)^{tr}v\} + v \\ &= \{z \in M : \mathbb{E}[-\widehat{X}_T^{tr}\widehat{Y}] \geq -u^{tr}z + (\mathbb{E}[\widehat{Y}] - u)^{tr}v\} + v. \end{aligned}$$

When $\mathbb{E}[\widehat{Y}] - u \in M^\perp$, we have $S_{(\widehat{Y},u)}(-\widehat{X}_T - v) = S_{(\widehat{Y},u)}(-\widehat{X}_T) + v$. However, when $u \notin (\mathbb{E}[\widehat{Y}] + M^\perp)$ that is $\mathbb{E}[\widehat{Y}] - u \notin M^\perp$, we can find a $v \in M$, such that for any $z \in M$,

$$\mathbb{E}[-\widehat{X}_T^{tr}\widehat{Y}] \geq -u^{tr}z + (\mathbb{E}[\widehat{Y}] - u)^{tr}v.$$

Thus, we have

$$z + v \in S_{(\widehat{Y},u)}(-\widehat{X}_T - v).$$

Therefore,

$$\bigcup_{z,v \in M} (z + v) \subseteq \bigcup_{v \in M} S_{(\widehat{Y},u)}(-\widehat{X}_T - v),$$

which yields

$$M \subseteq \bigcup_{v \in M} S_{(\widehat{Y},u)}(-\widehat{X}_T - v).$$

By the definition of $S_{(\widehat{Y},u)}$, the inverse inclusion is always true. So we conclude that

$$M = \bigcup_{v \in M} S_{(\widehat{Y},u)}(-\widehat{X}_T - v).$$

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It is not hard to check that

$$
\begin{aligned}
-\widehat{\varrho}^*(\widehat{Y},u) &= cl \bigcup_{\widehat{X}_T\in\mathcal{X}, v\in M} \Big(\widehat{\varrho}(\widehat{X}_T+v)+S_{(\widehat{Y},u)}(-\widehat{X}_T-v)\Big)\\
&= cl \bigcup_{\widehat{X}_T\in\mathcal{X}, v\in M} \Big(\widehat{\varrho}(\widehat{X}_T+v)+M\Big)\\
&= M,
\end{aligned}
$$

where the last equality is because that the M is a linear space and $\widehat{\varrho}(\widehat{X}_T)\subseteq M$. If $\widehat{Y}\notin L^q_d(\widehat{\mathcal{F}}_T,\, K^+)$, then there is an $\bar{X}\in\mathcal{X}(K)$ such that $\mathbb{E}[\bar{X}^{tr}\widehat{Y}]<0$. Since $\mathcal{X}(K)\subseteq\mathcal{A}_{\widehat{\varrho}}$, then by the definition of $S_{(\widehat{Y},\,u)}$, we have $S_{(\widehat{Y},\,u)}(-t\bar{X})=\{z\in M:\mathbb{E}[-t\bar{X}^{tr}\widehat{Y}]+u^{tr}z\geq 0\}$ for $t>0$. Thus,

$$
cl \bigcup_{\widehat{X}_T\in\mathcal{A}_{\widehat{\varrho}}} S_{(\widehat{Y},u)}(-\widehat{X}_T) \supseteq cl \bigcup_{\widehat{X}_T\in\mathcal{X}(K)} S_{(\widehat{Y},u)}(-\widehat{X}_T) \supseteq \bigcup_{t>0} S_{(\widehat{Y},u)}(-t\bar{X}) = M.
$$

The last equality is due to $\mathbb{E}[-t\bar{X}^{tr}\widehat{Y}]\to+\infty$ when $t\to+\infty$. By the definition of $S_{(\widehat{Y},\,u)}$, we conclude that $cl \bigcup_{\widehat{X}_T\in\mathcal{A}_{\widehat{\varrho}}} S_{(\widehat{Y},\,u)}(-\widehat{X}_T)\subseteq M$. Hence,

$$
cl \bigcup_{\widehat{X}_T\in\mathcal{A}_{\widehat{\varrho}}} S_{(\widehat{Y},u)}(-\widehat{X}_T) = M \qquad \text{whenever} \qquad \widehat{Y}\notin L^q_d(\widehat{\mathcal{F}}_T, K^+).
$$

Since $-\widehat{\varrho}^*(\widehat{Y},\,u):=cl \bigcup_{\widehat{X}_T\in\mathcal{X}} (\widehat{\varrho}(\widehat{X}_T)+S_{(\widehat{Y},\,u)}(-\widehat{X}_T))$, we know that

$$
-\widehat{\varrho}^*(\widehat{Y},u) \supseteq cl \bigcup_{\widehat{X}_T\in\mathcal{A}_{\widehat{\varrho}}} \Big(\widehat{\varrho}(\widehat{X}_T)+S_{(\widehat{Y},u)}(-\widehat{X}_T)\Big) \supseteq cl \bigcup_{\widehat{X}_T\in\mathcal{A}_{\widehat{\varrho}}} S_{(\widehat{Y},u)}(-\widehat{X}_T).
$$

Hence,

$$
-\widehat{\varrho}^*(\widehat{Y},u) \supseteq cl \bigcup_{\widehat{X}_T\in\mathcal{A}_{\widehat{\varrho}}} S_{(\widehat{Y},u)}(-\widehat{X}_T).
$$

Now, we only need to show $-\widehat{\varrho}^*(\widehat{Y},\,u)\subseteq cl \bigcup_{\widehat{X}_T\in\mathcal{A}_{\widehat{\varrho}}} S_{(\widehat{Y},\,u)}(-\widehat{X}_T)$. In fact, for any $z\in\widehat{\varrho}(\widehat{X}_T)$ and $\widehat{X}_T\in\mathcal{X}$, we have $\widehat{X}_T+z\in\mathcal{A}_{\widehat{\varrho}}$. Thus

$$
cl \bigcup_{\widehat{X}_T\in\mathcal{A}_{\widehat{\varrho}}} S_{(\widehat{Y},u)}(-\widehat{X}_T) \supseteq S_{(\widehat{Y},u)}(-\widehat{X}_T-z) = S_{(\widehat{Y},u)}(-\widehat{X}_T)+z.
$$

By the arbitrariness of z, we have

$$
\widehat{\varrho}(\widehat{X}_T)+S_{(\widehat{Y},u)}(-\widehat{X}_T) \subseteq cl \bigcup_{\widehat{X}_T\in\mathcal{A}_{\widehat{\varrho}}} S_{(\widehat{Y},u)}(-\widehat{X}_T),
$$

which yields

$$
-\widehat{\varrho}^*(\widehat{Y},u) \subseteq cl \bigcup_{\widehat{X}_T\in\mathcal{A}_{\widehat{\varrho}}} S_{(\widehat{Y},u)}(-\widehat{X}_T).
$$

The proof is completed. ■

Now, with the conjugate function $-\widehat{\varrho}^*$ of $\widehat{\varrho}$ in Proposition 3.2, we can derive the dual representation for the set-valued cash additive risk measures $\widehat{\varrho}$ on $\mathcal{X}$.

PROPOSITION 3.3: *If $\widehat{\varrho} : \mathcal{X} \to Q_M^t$ is a proper closed cash additive risk measure, then there is a $-\widehat{\alpha} : \widehat{\mathcal{M}}_{1,d}^{\mathbb{P}} \times K^+ \backslash M^\perp \to Q_M^t$, that is not identically M on the set*

$$\widehat{\mathcal{W}} := \left\{ (\widehat{\mathbb{Q}}, v) \in \widehat{\mathcal{M}}_{1,d}^{\mathbb{P}} \times K^+ \backslash M^\perp : diag(v) \frac{d\widehat{\mathbb{Q}}}{d\widehat{\mathbb{P}}} \in L_d^q(\widehat{\mathcal{F}}_T, K^+) \right\},$$

such that for any $\widehat{X}_T \in \mathcal{X}$,

$$\widehat{\varrho}(\widehat{X}_T) = \bigcap_{(\widehat{\mathbb{Q}}, v) \in \widehat{\mathcal{W}}} \left\{ -\widehat{\alpha}(\widehat{\mathbb{Q}}, v) + \left(\mathbb{E}^{\widehat{\mathbb{Q}}}[-\widehat{X}_T] + G(v) \right) \cap M \right\}, \tag{3.7}$$

where

$$G(v) := \{u \in \mathbb{R}^d : u^{tr} v \geq 0\}.$$

Moreover, the $-\widehat{\alpha}(\widehat{\mathbb{Q}}, v)$ can be replaced by the minimal penalty function $-\widehat{\alpha}_{\min}(\widehat{\mathbb{Q}}, v)$, which is defined as

$$-\widehat{\alpha}_{\min}(\widehat{\mathbb{Q}}, v) := cl \bigcup_{\widehat{Z}_T \in \mathcal{X}} \left(\widehat{\varrho}(\widehat{Z}_T) + \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{Z}_T] + G(v) \right) \cap M.$$

PROOF: By Remark 3.3, we can apply the set-valued Fenchel-Moreau theorem, that is Theorem 2 of Hamel [15], to the case where the linear space X is specified to $\mathcal{X}$ and the function f is specified to the proper closed cash additive risk measure $\widehat{\varrho}$. That is

$$\widehat{\varrho}(\widehat{X}_T) = \widehat{\varrho}^{**}(\widehat{X}_T) := \bigcap_{(\widehat{Y}, u) \in L_d^q(\widehat{\mathcal{F}}_T) \times K_M^* \backslash \{0\}} \left\{ -\widehat{\varrho}^*(\widehat{Y}, u) + S_{(\widehat{Y}, u)}(\widehat{X}_T) \right\}.$$

Then, from Proposition 3.2 it follows that

$$\widehat{\varrho}(\widehat{X}_T) := \bigcap_{(\widehat{Y}, u) \in L_d^q(\widehat{\mathcal{F}}_T, K^+) \times (\mathbb{E}[\widehat{Y}] + M^\perp) \cap K_M^* \backslash \{0\}} \left\{ cl \bigcup_{\widehat{X}_T \in \mathcal{A}_{\widehat{\varrho}}} S_{(\widehat{Y}, u)}(-\widehat{X}_T) + S_{(\widehat{Y}, u)}(\widehat{X}_T) \right\}.$$

Take $\widehat{Y} \in L_d^q(\widehat{\mathcal{F}}_T, K^+)$ and set $v = \mathbb{E}[\widehat{Y}] \in K^+$. Since $u \in (\mathbb{E}[\widehat{Y}] + M^\perp) \cap K_M^* \backslash \{0\}$, it is not hard to check that $u \notin M^\perp$, which makes $v \in K^+ \backslash M^\perp$. Now, we choose $Z_i = \frac{1}{v_i} \widehat{Y}_i$ when $v_i > 0$ and $\mathbb{E}[Z_i] = 1$ when $v_i = 0$, where $i \in \{1, \cdots, d\}$. We define $\widehat{\mathbb{Q}}$ by $d\widehat{\mathbb{Q}}/d\widehat{\mathbb{P}} = Z$, which makes $\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}_{1,d}^{\mathbb{P}}$. Then $\widehat{Y} = \text{diag}(v) d\widehat{\mathbb{Q}}/d\widehat{\mathbb{P}} \in L_d^q(\widehat{\mathcal{F}}_T, K^+)$. So we have $\mathbb{E}[\widehat{X}_T^{tr} \widehat{Y}] = \mathbb{E}[\widehat{X}_T^{tr} \text{diag}(v) d\widehat{\mathbb{Q}}/d\widehat{\mathbb{P}}] = v^{tr} \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{X}_T]$. Since $u \in v + M^\perp$, we have $u^{tr} z = v^{tr} z$ for any $z \in M$.

Hence,

$$S_{(\widehat{Y},u)}(\widehat{X}_T) = \{z \in M : v^{tr}\mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{X}_T] + v^{tr}z \geq 0\}$$
$$= \left(\mathbb{E}^{\widehat{\mathbb{Q}}}[-\widehat{X}_T] + G(v)\right) \cap M,$$

where

$$G(v) = \{z \in \mathbb{R}^d : v^{tr}z \geq 0\}.$$

Hence,

$$\widehat{\varrho}(\widehat{X}_T) = \bigcap_{(\widehat{\mathbb{Q}},v)\in\widehat{\mathcal{W}}} \left\{ -\widehat{\alpha}(\widehat{\mathbb{Q}}, v) + \left(\mathbb{E}^{\widehat{\mathbb{Q}}}[-\widehat{X}_T] + G(v)\right) \cap M \right\},$$

with

$$\widehat{\mathcal{W}} := \left\{(\widehat{\mathbb{Q}}, v) \in \widehat{\mathcal{M}}^{\mathbb{P}}_{1,d} \times K^+ \backslash M^\perp : \mathrm{diag}(v)\frac{d\widehat{\mathbb{Q}}}{d\widehat{\mathbb{P}}} \in L^q_d(\widehat{\mathcal{F}}_T, K^+)\right\},$$

where the $-\widehat{\alpha}(\widehat{\mathbb{Q}}, v)$ can be replaced by the minimal penalty function $-\widehat{\alpha}_{\min}(\widehat{\mathbb{Q}}, v)$, which is

$$-\widehat{\alpha}_{\min}(\widehat{\mathbb{Q}}, v) := cl \bigcup_{\widehat{Z}_T\in\mathcal{X}} \left(\widehat{\varrho}(\widehat{Z}_T) + \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{Z}_T] + G(v)\right) \cap M.$$

The proof is completed. ∎

Now, with the help of Propositions 3.1 and 3.3, we are ready to state the main result of this section.

THEOREM 3.1: *Any proper closed ($\sigma(L^\infty_d, L^1_d)$-closed if $p = \infty$) cash sub-additive risk measure R on $L^p_d(\mathcal{F}_T)$ is of the following form. For any $X_T \in L^p_d(\mathcal{F}_T)$,*

$$R(X_T) = \bigcap_{(\mu,v)\in\mathcal{T}} \left\{ -\alpha(\mu, v) + \left(\mu[-X_T] + G(v)\right) \cap M \right\}, \tag{3.8}$$

where

$$\mathcal{T} = \left\{(\mu, v) \in \mathcal{M}^d_{s,f} \times K^+ \backslash M^\perp : diag(v)\frac{d\mu}{d\mathbb{P}} \in L^q_d(K^+)\right\},$$

and

$$-\alpha_{\min}(\mu, v) = cl \bigcup_{Z_T\in L^p_d(\mathcal{F}_T)} \left(R(Z_T) + \mu[Z_T] + G(v)\right) \cap M.$$

PROOF: From Proposition 3.1, we can define a set-valued cash additive risk measure $\widehat{\varrho}$ on $\mathcal{X}$ by R, such that $\widehat{\varrho}(X_T I_{\{1\}}) = R(X_T)$ for any $X_T \in L^p_d(\mathcal{F}_T)$. Indeed, since $0 \in \mathbb{R}^d$, we have

$X_T I_{\{1\}} \in \mathcal{X}$. Thus, by Proposition 3.3,

$$R(X_T) = \widehat{\varrho}(X_T I_{\{1\}}) = \bigcap_{(\widehat{\mathbb{Q}},v)\in\widehat{\mathcal{W}}} \Big\{ -\widehat{\alpha}(\widehat{\mathbb{Q}},v) + \Big(\mathbb{E}^{\widehat{\mathbb{Q}}}[X_T I_{\{1\}}] + G(v)\Big) \cap M \Big\},$$

where

$$\widehat{\mathcal{W}} = \Big\{ (\widehat{\mathbb{Q}},v) \in \widehat{\mathcal{M}}^{\mathbb{P}}_{1,d} \times K^+ \backslash M^\perp : \operatorname{diag}(v)\frac{d\widehat{\mathbb{Q}}}{d\widehat{\mathbb{P}}} \in L^q_d(\widehat{\mathcal{F}}_T, K^+) \Big\}.$$

Write $\mu(\cdot) := \widehat{\mathbb{Q}}(\cdot I_{\{1\}})$. It is not hard to check that μ is a sub-probability measure and $\operatorname{diag}(v)\frac{d\mu}{d\mathbb{P}} \in L^q_d(K^+)$. Because $\widehat{\varrho}(X_T I_{\{1\}}) = R(X_T)$, we can express R as

$$R(X_T) = \bigcap_{(\mu,v)\in\mathcal{T}} \Big\{ -\alpha(\mu,v) + \Big(\mu[-X_T] + G(v)\Big) \cap M \Big\},$$

where $\alpha(\mu,\, v) = \widehat{\alpha}(\widehat{\mathbb{Q}},\, v)$ and

$$\mathcal{T} = \Big\{ (\mu,v) \in \mathcal{M}^d_{s,f} \times K^+ \backslash M^\perp : \operatorname{diag}(v)\frac{d\mu}{d\mathbb{P}} \in L^q_d(K^+) \Big\}.$$

Next, we will show the minimum penalty function $-\alpha_{\min}(\mu,\, v)$ of R. Since $-\widehat{\alpha}_{\min}(\widehat{\mathbb{Q}},\, v)$ is the minimum penalty function of $\widehat{\varrho}$,

$$\begin{aligned}
-\widehat{\alpha}_{\min}(\widehat{\mathbb{Q}},v) &= cl \bigcup_{\widehat{X}_T\in\mathcal{X}} \Big(\widehat{\varrho}(\widehat{X}_T) + \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{X}_T] + G(v)\Big) \cap M \\
&= cl \bigcup_{\widehat{X}_T\in\mathcal{X}} \Big(R(X_T - a1_T) - a|_M + \mathbb{E}^{\widehat{\mathbb{Q}}}\big[X_T I_{\{1\}} + a I_{\{0\}}\big] + G(v)\Big) \cap M \\
&= cl \bigcup_{X_T\in L^p_d(\mathcal{F}_T)} \Big(R(X_T - a1_T) + \mathbb{E}^{\widehat{\mathbb{Q}}}\big[(X_T - a1_T) I_{\{1\}}\big] + G(v)\Big) \cap M \\
&= cl \bigcup_{Z_T\in L^p_d(\mathcal{F}_T)} \Big(R(Z_T) + \mathbb{E}^{\widehat{\mathbb{Q}}}\big[Z_T I_{\{1\}}\big] + G(v)\Big) \cap M \\
&= cl \bigcup_{Z_T\in L^p_d(\mathcal{F}_T)} \Big(R(Z_T) + \mu[Z_T] + G(v)\Big) \cap M.
\end{aligned}$$

Hence,

$$-\alpha_{\min}(\mu,v) := -\widehat{\alpha}_{\min}(\widehat{\mathbb{Q}},v) = cl \bigcup_{Z_T\in L^p_d(\mathcal{F}_T)} \Big(R(Z_T) + \mu[Z_T] + G(v)\Big) \cap M.$$

The proof is completed. ■

4. DYNAMIC CASH SUB-ADDITIVITY RISK MEASURES

In this section, we will study the set-valued dynamic cash sub-additive risk measures. First, we will introduce the definition of set-valued dynamic cash sub-additive risk measures. Second, we will provide the dual representation for set-valued dynamic cash sub-additive risk measures.

4.1. Notions and definition

We denote by $L_d^p(\mathcal{F}_t)_+ := \{X \in L_d^p(\mathcal{F}_t) : X \in \mathbb{R}_+^d \quad \mathbb{P}-a.s.\}$ the convex cone of $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable random vectors. For any set D, denote $L_d^p(\mathcal{F}_t; D) := \{X_t \in L_d^p(\mathcal{F}_t) : X_t \in D \quad \mathbb{P}-a.s.\}$. Denote by $M_t := L_d^p(\mathcal{F}_t; M)$ the closed ($weak^*$-closed if $p = \infty$) linear subspace of $L_d^p(\mathcal{F}_t)$. We also denote $M_{t,+} := M_t \cap L_d^p(\mathcal{F}_t)_+,\ M_{t,-} := -M_{t,+},\ M_t^\perp := \{u \in L_d^q(\mathcal{F}_t) : \mathbb{E}[u^{tr}v] = 0 \text{ for any } v \in M_t\}$. Denote $M_{t,+}^+ := \{u \in L_d^q(\mathcal{F}_t) : \mathbb{E}[u^{tr}(vI_{\{1\}} + cI_{\{0\}})] \geq 0 \text{ for any } v \in M_{t,+},\ c \in \mathbb{R}_+^d \cap M\}$ and $\mathcal{G}(M_t; M_{t,+}) := \{D \subseteq M_t : D = clco(D + M_{t,+})\}$.

We will begin with recalling some properties related to the set-valued mapping R_t: $L_d^p(\mathcal{F}_T) \to \mathcal{G}(M_t; M_{t,+})$ at time t.

- B1 M_t-translation: for any $m_t \in M_t$, $R_t(X + m_t) = R_t(X) - m_t$;
- B2 $L_d^p(\mathcal{F}_T)_+$-monotonicity: for any $Y - X \in L_d^p(\mathcal{F}_T)_+$, $R_t(Y) \supseteq R_t(X)$;
- B3 Finite at zero: $\emptyset \neq R_t(0) \neq M_t$;
- B4 Normalization: for any $X \in L_d^p(\mathcal{F}_t)$, $R_t(X) = R_t(X) + R_t(0)$;
- B5 (B5') (Conditionally) Convexity: for any $\lambda \in [0, 1]$ ($\lambda \in L_d^\infty(\mathcal{F}_t; \mathbb{R})$ such that $\lambda \in [0, 1]$), $R_t(\lambda X + (1-\lambda)Y) \supseteq \lambda R_t(X) + (1-\lambda)R_t(Y)$.

Remark 4.1: As introduced by Feinstein and Rudloff [11,12], a set-valued (conditionally) cash additive risk measure at time t is a mapping $\varrho_t : L_d^p(\mathcal{F}_T) \to \mathcal{G}(M_t; M_{t,+})$ which satisfies $\mathbf{B1}-\mathbf{B5}(\mathbf{B1}-\mathbf{B4}, \mathbf{B5}')$. The acceptance set related to ϱ_t is defined by $\mathcal{A}_t := \{X \in L_d^p(\mathcal{F}_T) : 0 \in \varrho_t(X)\}$.

Next, we will introduce the definition of cash sub-additive risk measures at time t.

Definition 4.1: *A mapping $R_t : L_d^p(\mathcal{F}_T) \to \mathcal{G}(M_t; M_{t,+})$ at time t is called cash sub-additive if it satisfies*

$$B6 \qquad R_t(X_T + z1_T) \subseteq R_t(X_T) - z \quad or \quad R_t(X_T - z1_T) \supseteq R_t(X_T) + z$$

for any $X \in L_d^p(\mathcal{F}_T)$, $z \in K_M$.

Definition 4.2: *A mapping $R_t : L_d^p(\mathcal{F}_T) \to \mathcal{G}(M_t; M_{t,+})$ is called (conditionally) cash sub-additive risk measure at time t, if it satisfies* **B2**–**B6**(**B2**, **B3**, **B4**, **B5**′, **B6**).

Definition 4.3: *We call $(R_t)_{t=0}^T$ dynamic (conditionally) cash sub-additive risk measures if R_t is a (conditionally) cash sub-additive risk measure at time t.*

4.2. Dual representation

In order to get the dual representation, we still enlarge the space of financial positions. By the same arguments as in Section 3.2, let $\mathcal{X}$ be again the linear space of all random variables $\widehat{X}_T$ defined as in (3.2). Then taking Remark 3.3, Proposition 3.2 and Proposition 3.3 into

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account, we can get the dual representations for set-valued (conditionally) cash additive risk measures at time t on $\mathcal{X}$.

We first show a one–one relation between a (conditionally) cash additive risk measure and a (conditionally) cash sub-additive risk measure at time t. The definition of (conditionally) cash additive risk measure on the enlarged space was also motivated by Cheridito and Kupper [6].

Proposition 4.1: *Given a set-valued (conditionally) cash sub-additive risk measure R_t at time t on $L^p_d(\mathcal{F}_T)$ with $0 \in R_t(0)$, we define a set-valued risk measure $\widehat{\varrho}_t$ at time t on $\mathcal{X}$ as follows. For any $\widehat{X}_T \in \mathcal{X}$ where $\widehat{X}_T(\omega, \theta) = X_T(\omega) I_{\{1\}}(\theta) + a I_{\{0\}}(\theta)$ with $X_T \in L^p_d(\mathcal{F}_T)$, $a \in \mathbb{R}^d$,*

$$\widehat{\varrho}_t(\widehat{X}_T) := R_t(X_T - a 1_T) - a|_M. \tag{4.1}$$

Then $\widehat{\varrho}_t$ is a (conditionally) cash additive risk measure at time t with $\widehat{\varrho}_t(0) = 0$ and $\widehat{\varrho}_t(X_T I_{\{1\}}) = R_t(X_T)$.

Proof: By the same arguments as in the proof of Proposition 3.1, one can steadily show Proposition 4.1. The proof is completed. ∎

Next, we will introduce some notions under Proposition 4.1. Denote $\mu(\cdot) := \widehat{\mathbb{Q}}(\cdot I_{\{1\}})$, then $\mu \in \mathcal{M}^d_{s,f}$. For any $X_T \in L^p_d(\mathcal{F}_T)$, write

$$\mu[X_T|\mathcal{F}_t] := \mathbb{E}[\delta_{t,T}(\mu) X_T|\mathcal{F}_t],$$

where

$$\delta_{t,T}(\mu) = (\delta_{t,T}(\mu_1), \ldots, \delta_{t,T}(\mu_d))^{tr},$$

with

$$\delta_{t,T}(\mu_i)[\omega] := \begin{cases} \frac{\mathbb{E}[\frac{d\mu_i}{d\mathbb{P}}|\mathcal{F}_T](\omega)}{\mathbb{E}[\frac{d\mu_i}{d\mathbb{P}}|\mathcal{F}_t](\omega)}, & \mathrm{E}[\frac{d\mu_i}{d\mathbb{P}}|\mathcal{F}_t](\omega) > 0, \\ 1, & \text{else}, \end{cases}$$

for each $\omega \in \Omega$. Then we denote by $\mathcal{Y}_t$ the set of dual variables,

$$\mathcal{Y}_t := \left\{(\mu, \tau) \in \mathcal{M}^d_{s,f} \times (M^+_{t,+} \backslash M^\perp_t) : Y^T_t(\mu, \tau) \in L^q_d(\mathcal{F}_T)_+, \mu = \mathbb{P}|_{\mathcal{F}_t}\right\}$$

where $Y^T_t(\mu, \tau) := \tau \delta_{t,T}(\mu)$.

Now, we provide the dual representation for the dynamic set-valued (conditionally) cash sub-additive risk measures.

Theorem 4.1: *Any proper closed ($\sigma(L^\infty_d, L^1_d)$-closed if $p = \infty$) cash sub-additive risk measure R_t at time t on $L^p_d(\mathcal{F}_T)$ is of the following form. For any $X_T \in L^p_d(\mathcal{F}_T)$,*

$$R_t(X_T) = \bigcap_{(\mu,\tau) \in \mathcal{Y}_t} \left\{ -\beta^{\min}_t(\mu, \tau) + \Big(\mu[-X_T|\mathcal{F}_t] + \Gamma_t(\tau)\Big) \cap M_t \right\},$$

where

$$\Gamma_t(\tau) := \{u \in L^q_d(\mathcal{F}_t) : 0 \le \mathbb{E}[\tau^{tr} u]\}$$

and

$$-\beta^{\min}_t(\mu, \tau) = cl \bigcup_{Z_T \in L^p_d(\mathcal{F}_T)} \Big(R_t(Z_T) + \mu[Z_T|\mathcal{F}_t] + \Gamma_t(\tau)\Big) \cap M_t.$$

PROOF: By Proposition 4.1 and the same arguments as in the proof of Theorem 3.1, one can steadily show Theorem 4.1. The proof is completed. ∎

THEOREM 4.2: *Any proper closed ($\sigma(L_d^\infty, L_d^1)$-closed if $p = \infty$) conditionally cash sub-additive risk measure R_t^c at time t on $L_d^p(\mathcal{F}_T)$ is of the following form. For any $X_T \in L_d^p(\mathcal{F}_T)$,*

$$R_t^c(X_T) = \bigcap_{(\mu,\tau)\in\mathcal{Y}_t} \Big\{ -\alpha_t^{\min}(\mu,\tau) + \Big(\mu[-X_T|\mathcal{F}_t] + G_t(\tau)\Big) \cap M_t \Big\},$$

where

$$G_t(\tau) := \{u \in L_d^p(\mathcal{F}_t) : 0 \le \tau^{tr} u \quad \mathbb{P} - a.s.\}$$

and

$$-\alpha_t^{\min}(\mu,\tau) = cl \bigcup_{Z_T \in L_d^p(\mathcal{F}_T)} \Big(R_t^c(Z_T) + \mu[Z_T|\mathcal{F}_t] + G_t(\tau)\Big) \cap M_t.$$

PROOF: By Proposition 4.1 and the same arguments as in the proof of Theorem 3.1, one can steadily show Theorem 4.2. The proof is completed. ∎

5. MULTI-PORTFOLIO TIME CONSISTENCY

The time consistency for cash sub-additive risk measures was first studied by Mastrogiacomo and Rosazza Gianin [22] for scalar case. While the multi-portfolio time consistency was studied in detail by Feinstein and Rudloff [10] for set-valued dynamic risk measures.

In this section, we will study the multi-portfolio time consistency for set-valued dynamic (conditionally) cash sub-additive risk measures, which were introduced in Section 4.

Firstly, we will state the definition of the multi-portfolio time consistency for set-valued dynamic cash sub-additive risk measures.

DEFINITION 5.1: *A set-valued dynamic (conditionally) cash sub-additive risk measure $(R_t)_{t=0}^T$ is called multi-portfolio time consistent if for all time $0 \le t < s \le T$, portfolios $X_T \in L_d^p(\mathcal{F}_T)$ and sets $\mathcal{X}$, we have*

$$R_s(X_T) \subseteq \bigcup_{Y\in\mathcal{X}} R_s(Y) \Rightarrow R_t(X_T) \subseteq \bigcup_{Y\in\mathcal{X}} R_t(Y).$$

Remark 5.1: Multi-portfolio time consistency means that if at some time s, any risk compensation portfolio for X could compensate the risk of some portfolio Y in the set $\mathcal{X}$, then at any prior time t, the same relation should hold true.

Secondly, we will show the equivalent condition for multi-portfolio time consistency of dynamic (conditionally) cash sub-additive risk measures.

THEOREM 5.1: *For a normalized dynamic (conditionally) cash sub-additive risk measure $(R_t)_{t=0}^T$, the following are equivalent:*

(1) *$(R_t)_{t=0}^T$ is multi-portfolio time consistent;*

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(2) R_t *is recursive, that is, for all times* $0 \le t < s \le T$,

$$R_t(X_T) = \bigcup_{Y \in R_s(X_T)} R_t(-Y) := R_t(-R_s(X_T)). \tag{5.1}$$

PROOF: (1) ⇒ (2). Since $(R_t)_{t=0}^T$ is a normalized dynamic (conditionally) cash sub-additive risk measure, for every $X_T \in L_d^p(\mathcal{F}_T)$ and $t \in \{0, 1, \ldots, T\}$,

$$\bigcup_{Y \in R_s(X_T)} R_s(-Y) = \bigcup_{Y \in R_s(X_T)} (R_s(0) + Y) = R_s(0) + R_s(X_T) = R_s(X_T).$$

Thus by the multi-portfolio time consistency of $(R_t)_{t=0}^T$ with $\mathcal{X} := -R_s(X_T)$, we have

$$R_s(X_T) = \bigcup_{Y \in R_s(X_T)} R_s(-Y) \Rightarrow R_t(X_T) = \bigcup_{Y \in R_s(X_T)} R_t(-Y).$$

(2) ⇒ (1). For any $\mathcal{X} \subseteq L_d^p(\mathcal{F}_T)$ with $R_s(X_T) \subseteq \bigcup_{Y \in \mathcal{X}} R_s(Y)$, by (5.1), we have

$$R_t(X_T) = \bigcup_{Z \in R_s(X_T)} R_t(-Z) \subseteq \bigcup_{Z \in \cup_{Y \in \mathcal{X}} R_s(Y)} R_t(-Z) = \bigcup_{Y \in \mathcal{X}} R_t(Y).$$

The proof is completed. ■

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