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LIMIT THEOREMS FOR RANDOM POLYTOPES WITH VERTICES ON CONVEX SURFACES

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Abstract

We consider the random polytope K_n, defined as the convex hull of n points chosen independently and uniformly at random on the boundary of a smooth convex body in $\mathbb{R}^d$. We present both lower and upper variance bounds, a strong law of large numbers, and a central limit theorem for the intrinsic volumes of K_n. A normal approximation bound from Stein's method and estimates for surface bodies are among the tools involved.

Keywords: Central limit theorem; intrinsic volume; random polytope; stochastic geometry; surface body; variance

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Secondary 60D05; 60F05

1. Introduction and main results

For fixed $d \geq 2$, let $\mathcal{K}_+^2$ be the set of convex bodies in $\mathbb{R}^d$ which have a twice differentiable boundary with everywhere positive Gaussian curvature. Given some $K \in \mathcal{K}_+^2$, we denote by $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure on ∂K, normalized such that $\mathcal{H}^{d-1}(\partial K) = 1$. For $n \geq d+1$, we choose independently random points $X_1, \ldots, X_n$ from ∂K, according to $\mathcal{H}^{d-1}$. We denote by K_n the convex hull of $X_1, \ldots, X_n$. This means that K_n is a random polytope having its vertices on the boundary of K. The interest of this paper is in the intrinsic volumes $V_\ell(K_n)$ of K_n, $\ell \in \{1, \ldots, d\}$, which are related to familiar quantities like the volume, surface area, and mean width of K_n. The importance of these functionals is well known and arises from convex and integral geometry. Indeed, as Hadwiger's theorem states, they form (together with the Euler characteristic) a basis of the vector space of all motion invariant and continuous valuations on convex bodies. In this paper we provide lower and upper variance bounds, a strong law of large numbers, and a central limit theorem for $V_\ell(K_n)$, $\ell \in \{1, \ldots, d\}$, closing some gaps that remain in the study of these objects.

Intrinsic volumes have been investigated extensively in the alternative setting of random polytopes that arise as convex hulls of points chosen uniformly at random *inside* a fixed convex body. Results concerning the expectations of $V_\ell(K_n)$, $\ell \in \{1, \ldots, d\}$, have been studied, for example, in [14], variance bounds can be found in [2] and [4], and central limit theorems were treated in [11], [15], [22], and [24]. More details can be found in the references therein.

On the other hand, the approximation of a convex body K by means of a sequence of random polytopes K_n is improved whenever the vertices of K_n are restricted to lie on the boundary of K, therefore making it a model of particular interest. Indeed, in this framework the expectations of

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$V_\ell(K_n)$, $\ell \in \{1, \dots, d\}$, have been studied, for example, in [3], [5], [12], and [19]. However, more detailed information is only known about the distribution of the volume $V_d(K_n)$. In particular, an upper variance bound was found in [13] and a lower variance bound together with concentration inequalities in [16]. Only recently, Thäle [21] obtained a quantitative central limit theorem for $V_d(K_n)$ based on Stein's method.

Our first aim is to generalize the results obtained in [13] and [16] to the full regime of intrinsic volumes $V_\ell(K_n)$, $\ell \in \{1, \dots, d\}$. In fact, we prove a lower variance bound following the ideas of [2], [15], and [16], and an upper variance bound in the manner of [2], making use of a version of the Efron–Stein jackknife inequality formulated in [13]. In particular, the upper variance bound implies a strong law of large numbers as in [2]. Secondly, we prove a quantitative central limit theorem for $V_\ell(K_n)$, $\ell \in \{1, \dots, d\}$, using a normal approximation bound obtained in [10], extending the result of [21].

We now introduce some notation in order to present our results. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of positive real numbers. We write $a_n \ll b_n$ (or $a_n \gg b_n$) if there exist a constant $c \in (0, \infty)$ and a positive number n_0 such that $a_n \le c\, b_n$ (or $a_n \ge c\, b_n$) for all $n \ge n_0$. Furthermore, $a_n = \Theta(b_n)$ means that $b_n \ll a_n \ll b_n$.

Our first result concerns the asymptotic lower and upper bounds for the variances of the intrinsic volumes.

Theorem 1. *Let $K \in \mathcal{K}^2_+$. Choose n independent random points on ∂K according to the probability distribution $\mathcal{H}^{d-1}$, and let K_n be their convex hull. Then, for all $\ell \in \{1, \dots, d\}$,*

$$\operatorname{var}[V_\ell(K_n)] = \Theta(n^{-(d+3)/(d-1)}).$$

Based on a result stated in [12, Theorem 1] concerning the behaviour of $V_\ell(K) - \mathbb{E}[V_\ell(K_n)]$, the upper variance bound of Theorem 1 implies a strong law of large numbers.

Theorem 2. *In the setup of Theorem 1 and, for all $\ell \in \{1, \dots, d\}$, it holds that*

$$\mathbb{P}\Big(\lim_{n\to\infty} (V_\ell(K) - V_\ell(K_n))n^{2/(d-1)} = c_{K,\ell}\Big) = 1$$

for some constants $c_{K,\ell} \in (0, \infty)$ that depend on K and ℓ.

The constants $c_{K,\ell}$ appear in an explicit form in [12, Theorem 1] and can be expressed in the form of integrals of the principal curvatures of K.

Next we introduce the standardized intrinsic volume functionals, defined by

$$W_\ell(K_n) := \frac{V_\ell(K_n) - \mathbb{E}[V_\ell(K_n)]}{\sqrt{\operatorname{var}[V_\ell(K_n)]}}, \qquad \ell \in \{1, \dots, d\}.$$

We prove the following central limit theorem for such functionals.

Theorem 3. *In the setup of Theorem 1 and, for all $\ell \in \{1, \dots, d\}$, it holds that*

$$\sup_{u\in\mathbb{R}} |\mathbb{P}(W_\ell(K_n) \le u) - \mathbb{P}(N \le u)| \ll n^{-1/2}(\log n)^{3+6/(d-1)},$$

where N is a standard Gaussian random variable. In particular, $W_\ell(K_n)$ converges in distribution to N as $n \to \infty$.

Note that the rate of convergence in Theorem 3 does not depend on ℓ. Moreover, the same rate of convergence was already obtained in [21] for the case in which $\ell = d$.

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Remark. In the literature there exist results concerning central limit theorems and variance asymptotics for geometric functionals of random polytopes in multiple settings, which make use of stabilization techniques. This has been done in the Poisson framework (see [6]–[9] and [20]), and in [11] for a fixed number of random points in the interior of K. When possible, the use of stabilization techniques results in optimal rates of convergence. However, none of the aforementioned papers deal at the same time with nonaffine-invariant geometric functionals (as the intrinsic volumes V_ℓ, $\ell \in \{1, \dots, d-1\}$, are) of random polytopes with vertices on the boundary of a convex body in $\mathcal{K}^2_+$. Extending the existing techniques to include our setting would allow an improvement on the rate of convergence in Theorem 3 by a logarithmic factor. Although this may be possible, it could result in a very long and technical process, and it is ultimately beyond the scope of this paper. Indeed, our focus is giving a short argument, specific to our problem. Moreover, every approach needs the strict positivity of the constant in the lower variance bound and this has always been proved separately using other methods.

The paper is organised as follows. In Section 2 we introduce the notation and recall some background material from convex geometry, results concerning the surface and floating bodies and the normal bound from [10] that will be used in the proof of Theorem 3. In Section 3 we present the geometric construction needed for the proof of the lower bound of Theorem 1 and the proof itself. In Section 4 we prove the upper bound of Theorem 1 by means of the Efron–Stein jackknife inequality and we also prove Theorem 2, which directly follows from the former. Finally, in Section 5 we give the proof of Theorem 3.

2. Background material

2.1. General notation

The closed Euclidean ball of radius r centred at $x \in \mathbb{R}^d$ is denoted by $B^d(x, r)$, and $B^d = B^d(0, 1)$ stands for the centred Euclidean unit ball. The boundary of B^d is indicated with $\mathbb{S}^{d-1}$. Moreover, the volume of B^d is denoted by $\kappa_d = \pi^{d/2}\Gamma(1 + d/2)^{-1}$. For a finite set $A = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$, the convex hull of A is denoted by $[x_1, \dots, x_n]$. The vectors $e_1, \dots, e_d$ represent the standard orthonormal basis of $\mathbb{R}^d$. We indicate with $\sphericalangle(u, v)$ the angle between two vectors $u, v \in \mathbb{R}^d$. For a linear subspace V of $\mathbb{R}^d$, we define $\sphericalangle(u, V) := \inf\{\sphericalangle(u, v) : v \in V\}$. Given a subset $U \subseteq \mathbb{R}^d$, its projection onto $\mathbb{R}^{d-1}$ is denoted by $\operatorname{proj}_{\mathbb{R}^{d-1}} U = \{x \in \mathbb{R}^{d-1} : (x, y) \in U \text{ for some } y \in \mathbb{R}\}$. For a function $f : \mathbb{R}^d \to \mathbb{R}$, we say that $f \in \mathcal{C}^2$ if it is twice differentiable with continuous second-order partial derivatives.

Let $u \in \mathbb{R}^d$ and $h \in \mathbb{R}$. We denote by $H(u, h)$ the hyperplane $\{x \in \mathbb{R}^d : \langle x, u\rangle = h\}$. The corresponding half-space $\{x \in \mathbb{R}^d : \langle x, u\rangle \geq h\}$ is denoted by $H^+(u, h)$. Often one describes a convex body by its support function. The support function of K is defined by

$$h_K(u) = \sup\{\langle x, u\rangle : x \in K\}, \qquad u \in \mathbb{S}^{d-1}.$$

Since $K \in \mathcal{K}^2_+$, there exists a unique unit outward normal u_x for each $x \in \partial K$. The intersection of K with $H^+(u_x, h_K(u_x) - h)$ is denoted by $C^K(x, h)$. We call $C^K(x, h)$ a cap of K at x of height h. A cap C^K is called an ε-cap if $V_d(C^K) = \varepsilon$, where $V_d(\cdot)$ denotes the d-dimensional volume. Analogously, a cap C^K with $\mathcal{H}^{d-1}(C^K \cap \partial K) = \varepsilon$ is called an ε-boundary cap. For the cap $C^{B^d}(x, h)$, the central angle is defined as

$$\alpha(h) := \max\{\sphericalangle(x, y) : y \in C^{B^d}(x, h)\}.$$

Let $\ell \in \{0, \dots, d\}$. We denote by $G(d, \ell)$ the Grassmannian of all ℓ-dimensional linear subspaces of $\mathbb{R}^d$, which is supplied with the unique Haar probability measure ν_ℓ; see [17].

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For $L \in G(d,\ell)$, we write $\mathrm{vol}_\ell(K \mid L)$ to indicate the ℓ-dimensional Lebesgue measure of the orthogonal projection of K onto L. Then the ℓth intrinsic volume of a convex body K can be defined as

$$V_\ell(K) := \binom{d}{\ell} \frac{\kappa_d}{\kappa_\ell \kappa_{d-\ell}} \int_{G(d,\ell)} \mathrm{vol}_\ell(K \mid L) \nu_\ell(\mathrm{d}L); \tag{1}$$

see [18, Equations (5.5) and (6.11)]. In particular, $V_d(K)$ is the ordinary volume (Lebesgue measure), $V_{d-1}(K)$ is half of the surface area, $V_1(K)$ is a constant multiple of the mean width, and $V_0(K)$ is the Euler characteristic of K.

We define the function $v\colon K \to \mathbb{R}$ by

$$v(x) := \min\{V_d(K \cap H)\colon H \text{ is a half-space in } \mathbb{R}^d \text{ containing } x\}.$$

Then the set

$$K(v \geq t) := \{x \in K : v(x) \geq t\}$$

is called the floating body of K with parameter $t > 0$. The wet part of K is defined by

$$K(t) = K(v \leq t) := \{x \in K : v(x) \leq t\}.$$

In a similar way, we define the function $s\colon K \to \mathbb{R}$ by

$$s(x) := \min\{\mathcal{H}^{d-1}(\partial K \cap H)\colon H \text{ is a half-space in } \mathbb{R}^d \text{ containing } x\}.$$

The surface body of K with parameter $t > 0$ is defined by

$$K(s \geq t) := \{x \in K : s(x) \geq t\}.$$

Analogously, we set

$$K(s \leq t) := \{x \in K : s(x) \leq t\}.$$

We define the visibility region (with respect to s) of a point $z \in \partial K$ with parameter $t > 0$ as

$$\mathrm{Vis}_z(t) := \{x \in K(s \leq t)\colon [x,z] \cap K(s \geq t) = \varnothing\},$$

where $[x,z]$ denotes the closed line segment which connects x and z.

We use the convention that constants with the same subscript may differ from section to section.

2.2. Geometric tools

The concept of the surface body is convenient in view of Lemma 1, which clarifies its connection with the random polytope K_n.

Lemma 1. ([16, Lemma 4.2].) *For all $\alpha \in (0,\infty)$, there exists a constant $c_\alpha \in (0,\infty)$ depending only on α such that*

$$\mathbb{P}(K(s \geq \tau_n) \not\subseteq K_n) \leq n^{-\alpha},$$

where

$$\tau_n := c_\alpha \frac{\log n}{n}.$$

In the following, we present some well-known geometric results in order to keep our presentation reasonably self-contained. For every point $x \in \partial K$, there exists a paraboloid Q_x, given by a quadratic form b_{Q_x}, osculating at x. The following precise description of the local behaviour of the boundary of a convex body $K \in \mathcal{K}^2_+$ is due to Reitzner [12].

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Lemma 2. ([12, Lemma 6].) *Let $K \in \mathcal{K}_+^2$, and choose $\delta > 0$ sufficiently small. Then there exists a $\lambda > 0$, depending only on δ and K, such that, for each $x \in \partial K$, the following holds. Identify the hyperplane tangent to K at x with $\mathbb{R}^{d-1}$ and x with the origin. The λ-neighbourhood U^λ of x in ∂K defined by $\mathrm{proj}_{\mathbb{R}^{d-1}} U^\lambda = \lambda B^{d-1}$ can be represented by a convex function $f^{(x)}(y) \in \mathcal{C}^2$, i.e. $(y, f^{(x)}(y)) \in \partial K$ for $y \in \lambda B^{d-1}$. Denote by $f_{ij}^{(x)}(0)$ the second-order partial derivatives of $f^{(x)}$ at the origin. Then*

$$b_{Q_x}(y) = \frac{1}{2} \sum_{i,j} f_{ij}^{(x)}(0) y_i y_j,$$

and it holds that

$$(1+\delta)^{-1} b_{Q_x}(y) \leq f^{(x)}(y) \leq (1+\delta) b_{Q_x}(y) \quad \text{for } y \in \lambda B^{d-1}.$$

In the next lemma we state two well-known relations regarding ε-caps and ε-boundary caps.

Lemma 3. ([16, Lemma 6.2].) *For a given $K \in \mathcal{K}_+^2$, there exist constants $\varepsilon_0, c_1, c_2 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, we have, for any ε-cap C^K of K,*

$$c_1^{-1} \varepsilon^{(d-1)/(d+1)} \leq \mathcal{H}^{d-1}(C^K \cap \partial K) \leq c_1 \varepsilon^{(d-1)/(d+1)},$$

and, for any ε-boundary cap $\widetilde{C}^K$ of K,

$$c_2^{-1} \varepsilon^{(d+1)/(d-1)} \leq V_d(\widetilde{C}^K) \leq c_2 \varepsilon^{(d+1)/(d-1)}.$$

This result will be used to relate Lemma 1 in terms of the classic floating body.

For the next geometrical lemma, we assume that ε is sufficiently small.

Lemma 4. ([23, Lemma 6.2].) *Let x be a point on the boundary of K, and let $D(x, \varepsilon)$ be the set of all points on the boundary which are of distance at most ε to x. Then the convex hull of $D(x, \varepsilon)$ has volume at most $c_3 \varepsilon^{d+1}$, where $c_3 > 0$ is a constant.*

The following result is known as the economic cap covering theorem; see [1] and [2].

Proposition 1. ([2, Theorem 4].) *Assume that K is a convex body with unit volume, and let $0 < t < t_0 = (2d)^{-2d}$. Then there are caps $C_1, \ldots, C_m$ and pairwise disjoint convex sets $C_1', \ldots, C_m'$ such that $C_i' \subset C_i$ for each i, and*

1. $\bigcup_{i=1}^m C_i' \subset K(t) \subset \bigcup_{i=1}^m C_i$,
2. *$V_d(C_i') \gg t$ and $V_d(C_i) \ll t$ for each i,*
3. *for each cap C with $C \cap K(v > t) = \varnothing$, there is a C_i containing C.*

We conclude this section with a statement about the measure of the set of linear subspaces of $\mathbb{R}^d$ that form a small angle with a fixed vector, which will be useful later.

Lemma 5. ([2, Lemma 1].) *For fixed $z \in \mathbb{S}^{d-1}$ and small $a > 0$,*

$$\nu_\ell(\{L \in G(d, \ell) \colon \sphericalangle(z, L) \leq a\}) = \Theta(a^{d-\ell}), \qquad \ell \in \{1, \ldots, d\}.$$

2.3. Bound for the normal approximation

Let X and Y be two random variables with cumulative distribution functions $F_X(u) = \mathbb{P}(X \leq u)$ and $F_Y(u) = \mathbb{P}(Y \leq u)$, respectively. Note that X and Y need not be defined on a common probability space. Thus, we interpret $\mathbb{P}$ on the appropriate probability space in each case. The Kolmogorov distance between the random variables X and Y is defined by

$$d_K(X, Y) = \sup_{u \in \mathbb{R}} |F_X(u) - F_Y(u)|.$$

It is important to recall that the Kolmogorov distance is a metrization of the convergence in distribution, i.e. given a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ and another random variable Y such that $\lim_{n\to\infty} d_K(X_n, Y) = 0$, then $(X_n)_{n\in\mathbb{N}}$ converges in distribution to Y.

Let S be a Polish space. Consider a function $f\colon \bigcup_{k=1}^{n} S^k \to \mathbb{R}$ that acts on the point configurations of at most $n \in \mathbb{N}$ points of S. Let f be measurable and symmetric, i.e. invariant under permutations of the arguments. In the setting of this paper, S is the boundary of a smooth convex body, while f is an intrinsic volume of the convex hull of its arguments. Given a point $x = (x_1, \dots, x_h) \in \bigcup_{k=1}^{n} S^k$, we indicate with x^i the vector obtained from x by removing its ith coordinate, namely, $x^i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_h)$. Analogously, we define $x^{ij} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_h)$.

We now define the first- and second-order difference operators, applied to f, as

$$D_i f(x) := f(x) - f(x^i) \quad \text{and} \quad D_{i,j} f(x) := f(x) - f(x^i) - f(x^j) + f(x^{ij}),$$

respectively. We indicate with $X = (X_1, \dots, X_n)$ a random vector of elements of S. Let X' and $\tilde{X}$ be independent copies of X. A vector $Z = (Z_1, \dots, Z_n)$ is called a recombination of $\{X, X', \tilde{X}\}$, whenever $Z_i \in \{X_i, X'_i, \tilde{X}_i\}$ for every $i \in \{1, \dots, n\}$. For a subset $A \subseteq \{1, \dots, n\}$ of the index set, we write $X^A = (X_1^A, \dots, X_n^A)$ with

$$X_i^A := \begin{cases} X_i, & i \notin A, \\ X'_i, & i \in A. \end{cases}$$

In order to rephrase the normal approximation bound from [10], it is convenient to define the following quantities:

$$\begin{aligned}
\gamma_1 &:= \sup_{(Y,Y',Z,Z')} \mathbb{E}[\mathbf{1}\{D_{1,2} f(Y) \neq 0\}\, \mathbf{1}\{D_{1,3} f(Y') \neq 0\} D_2 f(Z)^2 D_3 f(Z')^2],\\
\gamma_2 &:= \sup_{(Y,Z,Z')} \mathbb{E}[\mathbf{1}\{D_{1,2} f(Y) \neq 0\} D_1 f(Z)^2 D_2 f(Z')^2],\\
\gamma_3 &:= \mathbb{E}[|D_1 f(X)|^4],\\
\gamma_4 &:= \mathbb{E}[|D_1 f(X)|^3],\\
\gamma_5 &:= \sup_{A\subseteq\{1,\dots,n\}} \mathbb{E}[|f(X) D_1 f(X^A)^3|].
\end{aligned}$$

The suprema in the definitions of γ_1 and γ_2 run over all combinations of vectors (Y, Y', Z, Z') or (Y, Z, Z') that are recombinations of $\{X, X', \tilde{X}\}$.

Proposition 2. ([10, Theorem 5.1].) *Let $W := f(X_1, \dots, X_n)$, and assume that $\mathbb{E}[W] = 0$ and $0 < \mathbb{E}[W^2] < \infty$. Moreover, let N be a standard Gaussian random variable. Then the following bound for the normal approximation holds:*

$$d_K\left(\frac{W}{\sqrt{\operatorname{var}[W]}}, N\right) \ll \frac{\sqrt{n}}{\operatorname{var}[W]}(\sqrt{n^2\gamma_1} + \sqrt{n\gamma_2} + \sqrt{\gamma_3}) + \frac{n}{(\operatorname{var}[W])^{3/2}}\gamma_4 + \frac{n}{(\operatorname{var}[W])^2}\gamma_5.$$

3. Lower variance bound

In order to prove a lower variance bound, we first introduce in Section 3.1 a geometrical construction taken from [16, Section 3.1]. More precisely, for $x \in \partial K$ and sufficiently small h, we define $d+1$ disjoint subsets of $C^K(x, h) \cap \partial K$ which are denoted by $D'_i(x)$, $i = 0, \dots, d$. In Section 3.2 we fix some particular points $y_1, \dots, y_n \in \partial K$ and h_n. The event that exactly one random point is contained in each $D'_i(y_j)$, $i \in \{0, \dots, d\}$, and every other point is outside

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of $C^K(y_j, h_n) \cap \partial K$ is indicated by A_j, $j \in \{1, \dots, n\}$. Then our strategy is as follows. By conditioning on the σ-field $\mathcal{F}$ generated by the positions of all $X_1, \dots, X_n$ except those which are contained in $D'_0(y_j)$ with $\mathbf{1}_{A_j} = 1$, it will turn out that

$$\operatorname{var}[V_\ell(K_n)] \geq \mathbb{E}[\operatorname{var}[V_\ell(K_n) \mid \mathcal{F}]] = \mathbb{E}\Bigg[\sum_{j=1}^{n} \operatorname{var}_j[V_\ell(K_n)]\,\mathbf{1}_{A_j}\Bigg],$$

where the variances $\operatorname{var}_j[\cdot]$ are taken over $X_j \in D'_0(y_j)$. Finally, it remains to determine the behaviour of $\operatorname{var}_j[\cdot]$ and $\mathbb{P}(A_j)$, $j \in \{1, \dots, n\}$. This way we bound the variance from below by a quantity that is asymptotically of the desired order.

3.1. Auxiliary geometric construction

Let E be the standard paraboloid given by

$$E = \{z \in \mathbb{R}^d : z_d \geq z_1^2 + \cdots + z_{d-1}^2\}.$$

We construct a simplex S in $C^E(0, 1)$ in the following way. The base is a regular simplex whose vertices $v_1, \dots, v_d$ lie on $\partial E \cap H(e_d, 1/(3(d-1)^2))$, while $v_0 = (0, \dots, 0)$ is the apex of S. Note that $2E \cap H(e_d, 1)$ has radius $\sqrt{2}$, while the inradius of the base of the simplex is $1/(\sqrt{3}(d-1)^2)$ and, therefore, $\{\lambda z \in \mathbb{R}^d : \lambda \geq 0,\ z \in S\} \cap H(e_d, 1)$ has inradius $3(d-1)^2/(\sqrt{3}(d-1)^2) = \sqrt{3}$. In particular, this implies that

$$\{\lambda z \in \mathbb{R}^d : \lambda \geq 0,\ z \in S\} \supseteq 2E \cap H(e_d, 1);$$

see Figure 1 for the construction of S. For $i \in \{0, 1, \dots, d\}$, let v'_i be the orthogonal projection of v_i onto $\operatorname{span}\{e_1, \dots, e_{d-1}\}$. Consider $B_0 := B^{d-1}(v'_0, r) \subseteq \mathbb{R}^{d-1}$ and $B_i := B^{d-1}(v'_i, r') \subseteq \mathbb{R}^{d-1}$, $i \in \{1, \dots, d\}$, for some radii r and r' to be chosen later. Let b_E be the quadratic form associated with E, i.e. $b_E(y) = \|y\|^2$ for $y \in \mathbb{R}^{d-1}$. For $i \in \{0, \dots, d\}$, we define the lift $B'_i := \tilde{b}(B_i)$ on ∂E of the sets B_i, where $\tilde{b}$ indicates the mapping

$$\tilde{b} : \mathbb{R}^{d-1} \to \partial E, \qquad y \mapsto (y, b_E(y)).$$

Note that, if r and r' are small enough, then, by continuity, for any $(d+1)$-tuple of points $x_i \in B'_i$, the following still holds:

$$\{\lambda z \in \mathbb{R}^d : \lambda \geq 0,\ z \in [x_0, \dots, x_d]\} \supseteq 2E \cap H(e_d, 1). \tag{2}$$

Then we extend the aforementioned argument to arbitrary caps of ∂K. For each point $x \in \partial K$, we consider the approximating paraboloid Q_x of K at x. Let $T_x(K)$ be the tangent space of K at the point x. The space $T_x(K)$ can be identified with $\mathbb{R}^{d-1}$ having x as its origin.

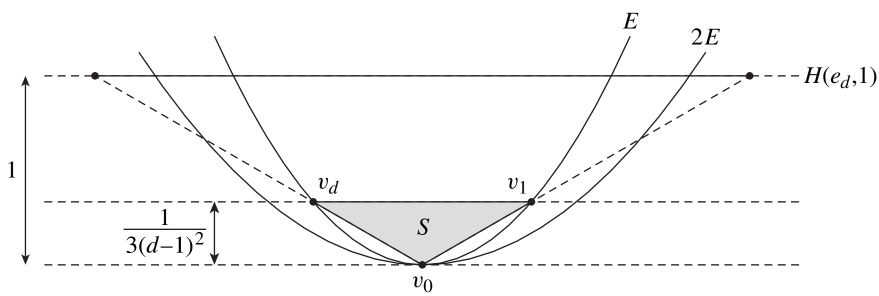

FIGURE 1: Construction of the simplex S.

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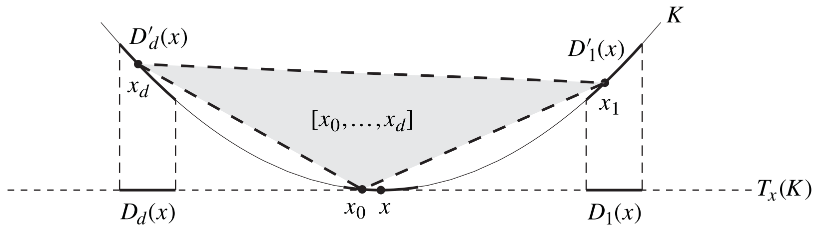

FIGURE 2: Example of a simplex $[x_0, \dots, x_d]$.

Then there exists a unique affine map A_x such that $A_x(C^E(0,1)) = C^{Q_x}(x,h)$, while mapping the coordinate axes onto the coordinate axes of $T_x(K) \times \mathbb{R}$. We define $D_i(x) := A_x(B_i)$, $i \in \{0, \dots, d\}$. Then it is true that $\mathrm{vol}_{d-1}(D_i(x)) = c_1 h^{(d-1)/2}$ for a constant $c_1 > 0$. We define now $D'_i(x) := \tilde{f}^{(x)}(D_i(x))$, where

$$\tilde{f}^{(x)}\colon U \to \partial K, \qquad y \mapsto (y, f^{(x)}(y))$$

for a neighbourhood $U \subseteq T_x(K)$ of x. Since $K \in \mathcal{K}^2_+$, there exist positive lower and upper bounds for the curvature. Thus, due to the curvature bounds of K, it holds that

$$c_K h^{(d-1)/2} \le \mathcal{H}^{d-1}(D'_i(x)) \le C_K h^{(d-1)/2}, \tag{3}$$

where c_K and C_K are positive constants depending only on K.

By continuity, if every x_i belongs to a ball $B^d(v_i, \eta)$, (2) is preserved whenever $\eta > 0$ is small enough. Moreover, we can choose r and r' to be small enough such that, for every $x \in \partial K$ and every $i \in \{0, \dots, d\}$, $D'_i(x) \subseteq A_x(B^d(v_i, \eta))$. Indeed, define, for $\varepsilon > 0$ and every $i \in \{0, \dots, d\}$, the set $U_i = \{(x, y) \in \mathbb{R}^d : x \in B^{d-1}(\mathrm{proj}_{\mathbb{R}^{d-1}} v_i, \eta/2),\ y \in [(1+\varepsilon)^{-1} b_E(x), (1+\varepsilon) b_E(x)]\}$. If ε is small enough then $U_i \subseteq B^d(v_i, \eta)$. Using Lemma 2, we can take h small enough such that $(1+\varepsilon)^{-1} b_{Q_x}(y) \le f^{(x)}(y) \le (1+\varepsilon) b_{Q_x}(y)$. In particular, if we choose $r, r' < \eta/2$ then $D'_i(x) \subseteq A_x(U_i) \subseteq A_x(B^d(v_i, \eta))$. We can choose a point $x_i \in D'_i(x)$ for any $i \in \{0, \dots, d\}$, as in Figure 2. As a consequence of the previous inclusion, we have

$$\begin{aligned} \{\lambda z \in \mathbb{R}^d : \lambda \ge 0, z \in [x_0, \dots, x_d]\} &\supseteq 2Q_x \cap H(u_x, h_K(u_x) - h) \\ &\supseteq K \cap H(u_x, h_K(u_x) - h), \end{aligned} \tag{4}$$

where the last inclusion holds whenever $h \le h_0$ for sufficiently small h_0. Therefore, from now on, r, r', and h_0 are chosen such that the previous argument holds.

3.2. Proof of the lower bound

In this section we combine tools from [2], [15], and [16]. Let $K \in \mathcal{K}^2_+$ and $X_1, \dots, X_n$ be independent random points that are chosen from ∂K according to the probability distribution $\mathcal{H}^{d-1}$. Due to [15, Lemma 13], we can choose n points $y_1, \dots, y_n \in \partial K$ and corresponding disjoint caps of K, namely, $C^K(y_j, h_n)$ for $j \in \{1, \dots, n\}$, with $h_n = \Theta(n^{-2/(d-1)})$. For all $i \in \{0, \dots, d\}$ and $j \in \{1, \dots, n\}$, we define the sets $\{D_i(y_j)\}$ and $\{D'_i(y_j)\}$ as in Section 3.1. Let A_j, $j \in \{1, \dots, n\}$, be the event that exactly one random point is contained in each $D'_i(y_j)$, $i \in \{0, \dots, d\}$, and every other point is outside of $C^K(y_j, h_n) \cap \partial K$.

Lemma 6. ([16, Section 3.2].) *For large enough n, and all $j \in \{1, \dots, n\}$, there exists a constant $c \in (0, 1)$ such that $\mathbb{P}(A_j) \ge c$.*

Proof. The probability of the event A_j is

$$\begin{aligned}\mathbb{P}(A_j) &= n(n-1)\cdots(n-d)\mathbb{P}(X_{i+1} \in D_i'(y_j),\ i \in \{0,\dots,d\}) \\ &\quad \times \mathbb{P}(X_{i+1} \notin C^K(y_j,h_n) \cap \partial K,\ i \in \{d+1,\dots,n-1\}) \\ &= n(n-1)\cdots(n-d) \prod_{i=0}^{d} \mathcal{H}^{d-1}(D_i'(y_j)) \prod_{k=d+1}^{n-1} (1 - \mathcal{H}^{d-1}(C^K(y_j,h_n) \cap \partial K)).\end{aligned}$$

Combining Lemma 3, [15, Lemma 13], and (3), we obtain

$$\mathbb{P}(A_j) \ge c_2 n^{d+1} n^{-d-1} (1 - c_3 n^{-1})^{n-d-1} \ge c > 0,$$

where all constants are positive. □

Let $\mathcal{F}$ be the σ-field generated by the positions of all $X_1,\dots,X_n$ except those which are contained in $D_0'(y_j)$ with $\mathbf{1}_{A_j} = 1$. Assume that $\mathbf{1}_{A_j} = \mathbf{1}_{A_k} = 1$ for some $j,k \in \{1,\dots,n\}$ and, without loss of generality, that X_j and X_k are the points in $D_0'(y_j)$ and $D_0'(y_k)$. By (4), it is not possible that there is an edge between X_j and X_k. Therefore, the change of the intrinsic volume affected by moving X_j within $D_0'(y_j)$ is independent of the change of the intrinsic volume of moving X_k within $D_0'(y_k)$. As a consequence, we obtain

$$\operatorname{var}[V_\ell(K_n) \mid \mathcal{F}] = \sum_{j=1}^{n} \operatorname{var}_j[V_\ell(K_n)]\, \mathbf{1}_{A_j},$$

where the variances $\operatorname{var}_j[\cdot]$ are taken over $X_j \in D_0'(y_j)$; compare with [2].

For $j \in \{1,\dots,n\}$ and $i \in \{0,\dots,d\}$, let z_j^i be an arbitrary point in $D_i'(y_j)$. We indicate with N_j the normal cone of the simplex $[z_j^0,\dots,z_j^d]$ at vertex z_j^0. Let S_j be the cone with base $H(u_{z_j^0}, h_K(u_{z_j^0}) - h_n) \cap 2Q_x$ and vertex z_j^0. Note that $u_{z_j^0}$ is the unique unit outer normal of K at z_j^0. The corresponding normal cone of S_j at z_j^0 is denoted by $\bar{N}_j$. Moreover, the angular aperture of S_j at z_j^0 is at most $c_K'\sqrt{h_n}$, where $c_K' > 0$ is a constant that depends on K. Because of this and (4), we can find sets Σ_j such that

$$\mathbb{S}^{d-1} \cap N_j \subset \mathbb{S}^{d-1} \cap \bar{N}_j \subset \mathbb{S}^{d-1} \cap (u_{z_j^0} + c_K'\sqrt{h_n} B^d) =: \Sigma_j. \tag{5}$$

We fix $j \in \{1,\dots,n\}$ and $z_j^i \in D_i'(y_j)$ for all $i \in \{1,\dots,d\}$. Let $F_j := [z_j^1,\dots,z_j^d]$ and define

$$\widetilde{V}_\ell(z; F_j) := \binom{d}{\ell} \frac{\kappa_d}{\kappa_\ell \kappa_{d-\ell}} \int_{G(d,\ell)} \mathbf{1}_{\{L \cap \Sigma_j \ne \varnothing\}} \operatorname{vol}_\ell([z, F_j] \mid L) \nu_\ell(\mathrm{d}L)$$

for $z \in D_0'(y_j)$ and $\ell \in \{1,\dots,d\}$.

Lemma 7. *Let $j \in \{1,\dots,n\}$, and let X_j be a point chosen with respect to the normalized Hausdorff measure restricted to $D_0'(y_j)$. Then*

$$\operatorname{var}_j[\widetilde{V}_\ell(X_j; F_j)] = \Theta(n^{-2(d+1)/(d-1)}), \qquad \ell \in \{1,\dots,d\}.$$

Proof. Note that $[X_j, F_j] \mid L$ is a simplex in $L \in G(d,\ell)$ with base $F_j \mid L$ and additional point $X_j \mid L$. As a consequence, the height of $[X_j, F_j] \mid L$ is proportional to h_n and

$$\operatorname{vol}_{\ell-1}(F_j \mid L) = \Theta(h_n^{(\ell-1)/2}),$$

where $L \in G(d,\ell)$ with $L \cap \Sigma_j \ne \varnothing$. Thus,

$$\operatorname{vol}_\ell([X_j, F_j] \mid L) = \Theta(h_n^{(\ell+1)/2}).$$

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Due to Lemma 5 and (5), it follows that

$$\int_{G(d,\ell)} \mathbf{1}_{\{L\cap\Sigma_j\neq\varnothing\}}\,\nu_\ell(\mathrm{d}L) = \nu_\ell(\{L \in G(d,\ell)\colon L\cap\Sigma_j \neq \varnothing\}) = \Theta(h_n^{(d-\ell)/2}).$$

Therefore, we obtain

$$\widetilde{V}_\ell(X_j; F_j) = \Theta(h_n^{(d+1)/2}).$$

Let X_j^1 and X_j^2 be independent copies of X_j. Then

$$|\widetilde{V}_\ell(X_j^1; F_j) - \widetilde{V}_\ell(X_j^2; F_j)| = \Theta(h_n^{(d+1)/2}),$$

since the heights of $X_j^1 \mid L$ and $X_j^2 \mid L$ are different with probability 1. Using the fact that $h_n = \Theta(n^{-2/(d-1)})$, we obtain

$$\mathrm{var}_j[\widetilde{V}_\ell(X_j; F_j)] = \tfrac{1}{2}\mathbb{E}[|\widetilde{V}_\ell(X_j^1; F_j) - \widetilde{V}_\ell(X_j^2; F_j)|^2] = \Theta(n^{-2(d+1)/(d-1)}),$$

as claimed. □

We can now proceed with the proof of the lower variance bound.

Proof of the lower bound of Theorem 1. Let $\mathcal{F}$ be the σ-field defined as above. The conditional variance formula implies that

$$\mathrm{var}[V_\ell(K_n)] = \mathbb{E}[\mathrm{var}[V_\ell(K_n) \mid \mathcal{F}]] + \mathrm{var}[\mathbb{E}[V_\ell(K_n) \mid \mathcal{F}]] \geq \mathbb{E}[\mathrm{var}[V_\ell(K_n) \mid \mathcal{F}]].$$

As already mentioned, $\mathcal{F}$ induces an independence property. Therefore, we obtain

$$\mathrm{var}[V_\ell(K_n) \mid \mathcal{F}] = \sum_{j=1}^{n} \mathrm{var}_j[V_\ell(K_n)]\,\mathbf{1}_{A_j} = \sum_{j=1}^{n} \mathrm{var}_j[\widetilde{V}_\ell(X_j; F_j)]\,\mathbf{1}_{A_j}.$$

Finally, applying Lemma 6 and Lemma 7, and taking the expectation yields

$$\mathrm{var}[V_\ell(K_n)] \gg n^{-2(d+1)/(d-1)} \sum_{j=1}^{n} \mathbb{P}(A_j) \gg n^{-2(d+1)/(d-1)} n = n^{-(d+3)/(d-1)},$$

which concludes the proof. □

4. Upper variance bound

In the following we find an upper bound for $\mathrm{var}[V_\ell(K_n)]$, $\ell \in \{1,\dots,d\}$. The proof is based on the Efron–Stein jackknife inequality and follows the ideas of [2]. In contrast to [2], we use the concept of surface body, in particular, Lemma 1, which states that the surface body is contained in the random polytope K_n with high probability. Moreover, we make use of Lemma 3 for our estimates. The proof is given in full details for the case $K = B^d$. From a geometric point of view, this case is easier to handle. However, the general case is also related to this basis case. The corresponding arguments are stated at the end of the proof.

Proof of the upper bound of Theorem 1. First, let $K = B^d$. We indicate with T_n the event that the surface body $K(s \geq \tau_n)$ is contained in K_n. Let $\ell \in \{1,\dots,d\}$. Applying the Efron–Stein jackknife inequality yields

$$\begin{aligned}\mathrm{var}[V_\ell(K_n)] &\ll n\mathbb{E}[(V_\ell(K_{n+1}) - V_\ell(K_n))^2] \\ &= n\mathbb{E}[(V_\ell(K_{n+1}) - V_\ell(K_n))^2\,\mathbf{1}_{T_n}] + n\mathbb{E}[(V_\ell(K_{n+1}) - V_\ell(K_n))^2\,\mathbf{1}_{T_n^{\mathrm{c}}}]. \qquad (6)\end{aligned}$$

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It is obvious that $(V_\ell(K_{n+1})-V_\ell(K_n))^2 \le V_\ell(K)^2$ and $\mathbb{E}[\mathbf{1}_{T_n^c}] = \mathbb{P}(T_n^c)$. Since the parameter α can be chosen arbitrarily large in Lemma 1, the second term in (6) is negligible in the asymptotic analysis. By (1), we obtain

$$\begin{aligned}\operatorname{var}[V_\ell(K_n)] &\ll n\mathbb{E}[(V_\ell(K_{n+1}) - V_\ell(K_n))^2 \,\mathbf{1}_{T_n}] \\ &\ll n\mathbb{E}\bigg[\int_{G(d,\ell)} \operatorname{vol}_\ell((K_{n+1} \mid A) \setminus (K_n \mid A))\nu_\ell(\mathrm{d}A) \\ &\qquad \times \int_{G(d,\ell)} \operatorname{vol}_\ell((K_{n+1} \mid B) \setminus (K_n \mid B))\nu_\ell(\mathrm{d}B)\,\mathbf{1}_{T_n}\bigg] \\ &\ll n\mathbb{E}\bigg[\int_{G(d,\ell)}\int_{G(d,\ell)} \operatorname{vol}_\ell((K_{n+1} \mid A) \setminus (K_n \mid A)) \operatorname{vol}_\ell((K_{n+1} \mid B) \setminus (K_n \mid B)) \\ &\qquad \times \mathbf{1}_{T_n}\, \nu_\ell(\mathrm{d}A)\nu_\ell(\mathrm{d}B)\bigg]. \end{aligned} \tag{7}$$

If $X_{n+1} \mid A \in K_n \mid A$ then the set $(K_{n+1} \mid A) \setminus (K_n \mid A)$ is clearly empty. Otherwise, $(K_{n+1} \mid A) \setminus (K_n \mid A)$ consists of several disjoint simplices which are the convex hull of $X_{n+1} \mid A$ and those facets of $K_n \mid A$ that can be 'seen' from $X_{n+1} \mid A$. For $I = \{i_1, \dots, i_\ell\} \subset \{1, \dots, n\}$, we indicate with F_I the convex hull of $X_{i_1}, \dots, X_{i_\ell}$. Note that F_I and $F_I \mid A$ are $(\ell-1)$-dimensional simplices with probability 1. The closed half-space in $\mathbb{R}^d$ which is determined by the hyperplane $A^\perp + \operatorname{aff} F_I$ and contains the origin is denoted by $H_0(F_I, A)$. The other half-space is $H_+(F_I, A)$. The corresponding ℓ-dimensional half-spaces in A are denoted by $H_0(F_I \mid A)$ and $H_+(F_I \mid A)$. Let $\tilde{\mathcal{F}}(A)$ be the set of $(\ell-1)$-dimensional facets of $K_n \mid A$ that can be seen from $X_{n+1} \mid A$. It is defined by

$$\begin{aligned}\tilde{\mathcal{F}}(A) = \{F_I \mid A \colon K_n \mid A \subset H_0(F_I \mid A),\ X_{n+1} \mid A \in H_+(F_I \mid A),\\ I = \{i_1, \dots, i_\ell\} \subset \{1, \dots, n\}\}.\end{aligned}$$

Note that $\tilde{\mathcal{F}}(A)$ is random since it depends on the points $X_1, \dots, X_n$. In the following we use a deterministic version of it for fixed points $x_1, \dots, x_n$. The deterministic version is denoted by $\mathcal{F}(A)$. Therefore,

$$\begin{aligned}(7) \ll n\int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} \int_{G(d,\ell)}\int_{G(d,\ell)} \bigg(\sum_{F\in\mathcal{F}(A)} \operatorname{vol}_\ell([x_{n+1} \mid A, F])\bigg) \\ \times \bigg(\sum_{F'\in\mathcal{F}(B)} \operatorname{vol}_\ell([x_{n+1} \mid B, F'])\,\mathbf{1}_{T_n}\bigg)\nu_\ell(\mathrm{d}A)\nu_\ell(\mathrm{d}B) \\ \times \mathcal{H}^{d-1}(\mathrm{d}x_1)\cdots\mathcal{H}^{d-1}(\mathrm{d}x_{n+1}).\end{aligned} \tag{8}$$

Next, the integration is extended over all possible index sets I and J, and the order of integration is changed. As a consequence, we obtain

$$\begin{aligned}(8) \ll n\int_{G(d,\ell)}\int_{G(d,\ell)}\int_{(\mathbb{S}^{d-1})^{n+1}} \bigg(\sum_I \mathbf{1}\{F_I \mid A \in \mathcal{F}(A)\} \operatorname{vol}_\ell([F_I, x_{n+1}] \mid A)\bigg) \\ \times \bigg(\sum_J \mathbf{1}\{F_J \mid B \in \mathcal{F}(B)\} \operatorname{vol}_\ell([F_J, x_{n+1}] \mid B)\,\mathbf{1}_{T_n}\bigg) \\ \times \mathcal{H}^{d-1}(\mathrm{d}x_1)\cdots\mathcal{H}^{d-1}(\mathrm{d}x_{n+1})\nu_\ell(\mathrm{d}A)\nu_\ell(\mathrm{d}B).\end{aligned}$$

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Note that $[F_I, X_{n+1}] \mid A$ and $[F_J, X_{n+1}] \mid B$ are contained in the associated caps $C_\ell(I, A) := H_+(F_I, A) \cap B^d$ and $C_\ell(J, B)$. Moreover, we use the abbreviation $C_d(I, A) = (H_+(F_I \mid A) + A^\perp) \cap B^d$. We indicate with $V_\ell(I, A) = \mathrm{vol}_\ell(C_\ell(I, A))$ and $V_d(I, A) = V_d(C_d(I, A))$ the volumes of these caps. Therefore, the variance is bounded by

$$\begin{aligned}\mathrm{var}[V_\ell(K_n)] &\\ \ll n \sum_I \sum_J \int_{G(d,\ell)} \int_{G(d,\ell)} \int_{(\mathbb{S}^{d-1})^{n+1}} & \mathbf{1}\{F_I \mid A \in \mathcal{F}(A)\} V_\ell(I, A) \\ &\times \mathbf{1}\{F_J \mid B \in \mathcal{F}(B)\} V_\ell(J, B)\, \mathbf{1}_{T_n} \\ &\times \mathcal{H}^{d-1}(\mathrm{d}x_1) \cdots \mathcal{H}^{d-1}(\mathrm{d}x_{n+1}) \nu_\ell(\mathrm{d}A) \nu_\ell(\mathrm{d}B),\end{aligned}$$

where the summation extends over all ℓ-tuples I and J. Of course, these tuples may have a nonempty intersection. However, if the size of $I \cap J$ is fixed to be k then the corresponding terms in the sum are independent of the choices of $i_1, \dots, i_\ell$ and $j_1, \dots, j_\ell$. For any $k \in \{0, 1, \dots, \ell\}$, we indicate with F the convex hull of $X_1, \dots, X_\ell$ and by G the convex hull of $X_{\ell-k+1}, \dots, X_{2\ell-k}$. As in [2], we obtain

$$\begin{aligned}\mathrm{var}[V_\ell(K_n)] &\\ \ll n \sum_{k=0}^{\ell} \binom{n}{\ell}\binom{\ell}{k}\binom{n-\ell}{\ell-k} \int_{G(d,\ell)} \int_{G(d,\ell)} \int_{(\mathbb{S}^{d-1})^{n+1}} & \mathbf{1}\{F_I \mid A \in \mathcal{F}(A)\} V_\ell(I, A) \\ &\times \mathbf{1}\{F_J \mid B \in \mathcal{F}(B)\} V_\ell(J, B)\, \mathbf{1}_{T_n} \\ &\times \mathcal{H}^{d-1}(\mathrm{d}x_1) \cdots \mathcal{H}^{d-1}(\mathrm{d}x_{n+1}) \\ &\times \nu_\ell(\mathrm{d}A) \nu_\ell(\mathrm{d}B). \qquad (9)\end{aligned}$$

We indicate with Σ_k the kth term in the previous sum. By symmetry, we can restrict the summation to those tuples where $V_d(I, A) \geq V_d(J, B)$. In addition, we multiply the integrand by $\mathbf{1}\{C_d(I, A) \cap C_d(J, B) \neq \varnothing\}$. This is indeed possible because the caps have at least the point X_{n+1} in common. It follows immediately that

$$\begin{aligned}\Sigma_k \ll n^{2\ell-k+1} \int_{G(d,\ell)} \int_{G(d,\ell)} \int_{(\mathbb{S}^{d-1})^{n+1}} & \mathbf{1}\{F \mid A \in \mathcal{F}(A)\} V_\ell(I, A) \\ &\times \mathbf{1}\{C_d(I, A) \cap C_d(J, B) \neq \varnothing\} \\ &\times \mathbf{1}\{G \mid B \in \mathcal{F}(B)\} V_\ell(J, B) \\ &\times \mathbf{1}\{V_d(I, A) \geq V_d(J, B)\} \\ &\times \mathbf{1}_{T_n}\, \mathcal{H}^{d-1}(\mathrm{d}x_1) \cdots \mathcal{H}^{d-1}(\mathrm{d}x_{n+1}) \nu_\ell(\mathrm{d}A) \nu_\ell(\mathrm{d}B).\end{aligned}$$

Next we integrate with respect to $x_{2\ell-k+1}, \dots, x_n, x_{n+1}$. Due to the condition $F \mid A \in \mathcal{F}(A)$, the points $X_{2\ell-k+1}, \dots, X_n$ are contained in $H_0(F, A)$ and X_{n+1} is in $H_+(F, A)$. Therefore,

$$\begin{aligned}\Sigma_k \ll n^{2\ell-k+1} \int_{G(d,\ell)} \int_{G(d,\ell)} \int_{(\mathbb{S}^{d-1})^{2\ell-k}} & (1 - \mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1}))^{n-2\ell+k} \\ &\times \mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1}) V_\ell(I, A) \\ &\times \mathbf{1}\{C_d(I, A) \cap C_d(J, B) \neq \varnothing\} V_\ell(J, B) \\ &\times \mathbf{1}\{V_d(I, A) \geq V_d(J, B)\}\, \mathbf{1}_{T_n} \\ &\times \mathcal{H}^{d-1}(\mathrm{d}x_1) \cdots \mathcal{H}^{d-1}(\mathrm{d}x_{2\ell-k}) \nu_\ell(\mathrm{d}A) \nu_\ell(\mathrm{d}B).\end{aligned}$$

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The assumption that $V_d(I, A) \geq V_d(J, B)$ implies that the height of the cap $C_d(I, A)$ is at least the height of $C_d(J, B)$. Due to $C_d(I, A) \cap C_d(J, B) \neq \varnothing$, we find a constant β such that $C_d(J, B)$ is contained in $\beta C_d(I, A)$. More precisely, $\beta C_d(I, A)$ is an enlarged homothetic copy of $C_d(I, A)$, where the center of homothety $z \in \mathbb{S}^{d-1}$ coincides with the center of the cap $C_d(I, A)$. It follows from the homogeneity that the Hausdorff measure (restricted to $\beta\mathbb{S}^{d-1}$) of $\beta C_d(I, A)$ is up to a constant $\mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1})$. Therefore,

$$\begin{aligned}\int_{(\mathbb{S}^{d-1})^{\ell-k}} &\mathbf{1}\{C_d(I, A) \cap C_d(J, B) \neq \varnothing\}\mathbf{1}\{V_d(I, A) \geq V_d(J, B)\}\\ &\times V_\ell(J, B)\mathcal{H}^{d-1}(\mathrm{d}x_{\ell+1}) \cdots \mathcal{H}^{d-1}(\mathrm{d}x_{2\ell-k})\\ &\ll \mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1})^{\ell-k} V_\ell(I, A).\end{aligned}$$

As in [2], the conditions $C_d(I, A) \cap C_d(J, B) \neq \varnothing$ and $V_d(I, A) \geq V_d(J, B)$ are satisfied only if the angle between z and the subspace B is not larger than twice the central angle δ of the cap $C_d(I, A)$. Moreover, δ is bounded by

$$\delta \ll V_d(I, A)^{1/(d+1)}. \tag{10}$$

Thus,

$$\begin{aligned}\Sigma_k \ll n^{2\ell-k+1} \int_{G(d,\ell)} \int_{G(d,\ell)} \int_{(\mathbb{S}^{d-1})^{\ell}} &(1 - \mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1}))^{n-2\ell+k}\\ &\times \mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1})^{\ell-k+1} V_\ell(I, A)^2\\ &\times \mathbf{1}\{\sphericalangle(z, B) \ll V_d(I, A)^{1/(d+1)}\}\\ &\times \mathbf{1}_{T_n} \mathcal{H}^{d-1}(\mathrm{d}x_1) \cdots \mathcal{H}^{d-1}(\mathrm{d}x_\ell)\nu_\ell(\mathrm{d}A)\nu_\ell(\mathrm{d}B).\end{aligned}$$

Due to Lemma 3, the condition T_n can be replaced by the condition

$$V_d(I, A) \leq c_1\left(\frac{\log n}{n}\right)^{(d+1)/(d-1)}$$

for some constant $c_1 > 0$. In the following, the economic cap covering theorem is used; recall Proposition 1. Let h be a positive integer such that $2^{-h} \leq \log n/n$. Note that the smallest possible value of h is $h_0 = -\lfloor \log_2(\log n/n) \rfloor$. According to the economic cap covering theorem, we find, for each h, a collection of caps $\{C_1, \dots, C_{m(h)}\}$ which cover the wet part of $B^d \mid A$ with parameter $(2^{-h})^{(\ell+1)/(d-1)}$. This collection of caps is denoted by $\mathcal{M}_h$. Each cap C_i can be viewed as a projection of a d-dimensional cap $C_i(A)$ from B^d to A. Now we consider an arbitrary tuple $(X_1, \dots, X_\ell)$ which has a corresponding cap $C_d(I, A)$ having volume at most $c_1(\log n/n)^{(d+1)/(d-1)}$. We relate to $(X_1, \dots, X_\ell)$ the maximal h such that $C_\ell(I, A) \subset C_i$ for some $C_i \in \mathcal{M}_h$. This is indeed possible since at least 2^{-h_0} is roughly $\log n/n$ and the volume of the caps in $\mathcal{M}_h$ tends to 0 as $h \to \infty$. As a consequence, we obtain

$$V_\ell(I, A) \leq \mathrm{vol}_\ell(C_i) \ll 2^{-h(\ell+1)/(d-1)}$$

and

$$V_d(I, A) \leq V_d(C_i(A)) \ll 2^{-h(d+1)/(d-1)}.$$

According to Lemma 3, $\mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1}) \leq \mathcal{H}^{d-1}(C_i(A) \cap \mathbb{S}^{d-1}) \ll 2^{-h}$. Due to the maximality of h, it holds that $V_d(I, A) \geq 2^{-(h+1)(d+1)/(d-1)}$. In addition, it follows from

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Lemma 3 that $\mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1}) \geq c_2 2^{-(h+1)}$ for some constant $c_2 > 0$. Therefore, we obtain

$$(1 - \mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1}))^{n-2\ell+k} \mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1})^{\ell-k+1} V_\ell(I, A)^2 \\ \ll (1 - c_2 2^{-(h+1)})^{n-2\ell+k} 2^{-h(\ell-k+1)} 2^{-2h(\ell+1)/(d-1)}.$$

Then we integrate each $(X_1, \dots, X_\ell)$ on $(C_i(A))^\ell$ and we use the fact that $1 - x \leq \exp(-x)$ to obtain

$$\exp(-c_2(n - 2\ell + k)2^{-h-1}) 2^{-h(\ell-k+1)} 2^{-2h(\ell+1)/(d-1)} \mathcal{H}^{d-1}(C_i(A) \cap \mathbb{S}^{d-1})^\ell \\ \ll \exp(-c_2(n - 2\ell + k)2^{-h-1}) 2^{-h(\ell-k+1)} 2^{-2h(\ell+1)/(d-1)} 2^{-h\ell}.$$

Since the volume of the wet part of B^ℓ with parameter $2^{-h(\ell+1)/(d-1)}$ is $\Theta(2^{-2h/(d-1)})$ (note that $h \to \infty$ as $n \to \infty$), we obtain

$$|\mathcal{M}_h| \ll \frac{2^{-2h/(d-1)}}{2^{-h(\ell+1)/(d-1)}} = 2^{h(\ell-1)/(d-1)}. \tag{11}$$

Finally, this results in

$$\int_{G(d,\ell)} \int_{(\mathbb{S}^{d-1})^\ell} (1 - \mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1}))^{n-2\ell+k} \mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1})^{\ell-k+1} V_\ell(I, A)^2 \\ \times \mathbf{1}\{\sphericalangle(z, B) \ll V_d(I, A)^{1/(d+1)}\} \mathbf{1}_{T_n} \mathcal{H}^{d-1}(\mathrm{d}x_1) \cdots \mathcal{H}^{d-1}(\mathrm{d}x_\ell) \nu_\ell(\mathrm{d}B) \\ \ll \sum_{h=h_0}^{\infty} \exp(-c_2(n - 2\ell + k)2^{-h-1}) 2^{-h(\ell-k+1)} 2^{-2h(\ell+1)/(d-1)} 2^{-h\ell} \\ \times |\mathcal{M}_h| \nu_\ell(\{\sphericalangle(z, B) \ll V_d(I, A)^{1/(d+1)}\}) \\ \ll \sum_{h=h_0}^{\infty} \exp(-c_2(n - 2\ell + k)2^{-h-1}) 2^{-h[(2\ell-k+1)+(d+3)/(d-1)]}.$$

Note that we used Lemma 5 and (11) in the last step. As in [2], we divide the previous sum into two parts in order to see the magnitude of the variance. The integer h_1 is defined by

$$2^{-h_1} \leq \frac{1}{n} < 2^{-h_1+1}.$$

On the one hand, we have

$$\sum_{h=h_1}^{\infty} \exp(-c_2(n - 2\ell + k)2^{-h-1}) 2^{-h[(2\ell-k+1)+(d+3)/(d-1)]} \leq \sum_{h=h_1}^{\infty} 2^{-h[(2\ell-k+1)+(d+3)/(d-1)]} \\ \ll n^{-(2\ell-k+1)} n^{-(d+3)/(d-1)}.$$

On the other hand, let $i = h_1 - h$. Then we can perform the following estimate:

$$\sum_{h=h_0}^{h_1-1} \exp(-c_2(n - 2\ell + k)2^{-h-1}) 2^{-h[(2\ell-k+1)+(d+3)/(d-1)]} \\ \leq \sum_{i=1}^{h_1-h_0} \exp(-c_2(n - 2\ell + k)2^{-h_1+i-1}) 2^{-(h_1-i)[(2\ell-k+1)+(d+3)/(d-1)]} \\ \ll \sum_{i=1}^{h_1-h_0} \exp(-c_2(n - 2\ell + k)2^{-h_1+i-1}) n^{-(2\ell-k+1)} n^{-(d+3)/(d-1)} 2^{i[(2\ell-k+1)+(d+3)/(d-1)]}$$

$$\ll n^{-(2\ell-k+1)} n^{-(d+3)/(d-1)} \sum_{i=1}^{\infty} \exp(-c_2 2^i) 2^{i[(2\ell-k+1)+(d+3)/(d-1)]}$$
$$\ll n^{-(2\ell-k+1)} n^{-(d+3)/(d-1)} \sum_{j=1}^{\infty} \exp(-c_2 j) j^{5d}$$
$$\ll n^{-(2\ell-k+1)} n^{-(d+3)/(d-1)}.$$

As a consequence, it holds that

$$\Sigma_k \ll n^{2\ell-k+1} \int_{G(d,\ell)} n^{-(2\ell-k+1)} n^{-(d+3)/(d-1)} \nu_\ell(\mathrm{d}A) \ll n^{-(d+3)/(d-1)}.$$

Finally, the upper bound is proven by summing up all Σ_k, $k = 0, \dots, \ell$, in (9).

In order to extend the proof to the case of a convex body $K \in \mathcal{K}_+^2$, we follow the ideas presented in [2, Section 6]. By the compactness of ∂K, there exist $\gamma > 0$ and $\Gamma > 0$, the global upper and the global lower bounds on the principal curvatures of ∂K, respectively. In our setting, all projected images of ∂K also have a boundary with the same properties as ∂K; see, for example, [17, Remark 5, p. 126]. Without loss of generality, we can choose γ and Γ to also be bounds on the principal curvatures of the boundaries of all ℓ-dimensional projections of K. Hence, we can locally approximate ∂K with affine images of balls, and the volume of an ℓ-dimensional cap with small height $h > 0$ has order $h^{(\ell+1)/2}$. Note that $C^K(x, h)$ is the intersection of K with the hyperplane $\tilde{H}(x, h) = \{y \in \mathbb{R}^d : \langle x - y, u_x \rangle = h\}$. As in [2, Equation (27)], it holds that

$$((x - hu_x) + \gamma_1 \sqrt{h} B^d) \cap \tilde{H}(x, h) \subset K \cap \tilde{H}(x, h)$$
$$\subset ((x - hu_x) + \gamma_2 \sqrt{h} B^d) \cap \tilde{H}(x, h),$$

where the constants γ_1 and γ_2 depend on γ and Γ. The last equation ensures that (10) still holds. □

In the fashion of [2, Section 7], we derive a strong law of large numbers from the upper variance bound together with the following result from [12].

Proposition 3. ([12, Theorem 1].) *Let $K \in \mathcal{K}_+^2$, and choose n random points on ∂K independently and according to the probability distribution $\mathcal{H}^{d-1}$. Then there exist positive constants $c_{K,\ell}$ depending on ℓ and the principal curvatures of K such that*

$$\lim_{n\to\infty} (V_\ell(K) - \mathbb{E}[V_\ell(K_n)]) n^{2/(d-1)} = c_{K,\ell}, \qquad \ell \in \{1, \dots, d\}. \tag{12}$$

For the sake of brevity, the explicit expression of $c_{K,\ell}$ is omitted here. It can be found in [12, Equation (2)].

Proof of Theorem 2. Let $\ell \in \{1, \dots, d\}$. Chebyshev's inequality and the variance upper bound yield

$$\mathbb{P}(|V_\ell(K) - V_\ell(K_n) - \mathbb{E}[V_\ell(K) - V_\ell(K_n)]| n^{2/(d-1)} \geq \varepsilon) \leq \varepsilon^{-2} n^{4/(d-1)} \operatorname{var}[V_\ell(K_n)] \ll n^{-1}.$$

Now select the subsequence of indices $n_k = k^2$. Then it follows that

$$\sum_{k=1}^{\infty} \mathbb{P}(|V_\ell(K) - V_\ell(K_{n_k}) - \mathbb{E}[V_\ell(K) - V_\ell(K_{n_k})]| n_k^{2/(d-1)} \geq \varepsilon) \ll \sum_{k=1}^{\infty} k^{-2} < \infty.$$

Applying the Borel–Cantelli lemma together with (12), it follows that

$$\lim_{k\to\infty}(V_\ell(K)-V_\ell(K_{n_k}))n_k^{2/(d-1)}=c_{K,\ell}$$

holds with probability 1. Note that $V_\ell(K)-V_\ell(K_n)$ is a decreasing and positive sequence. Therefore, this gives

$$(V_\ell(K)-V_\ell(K_{n_k}))n_{k-1}^{2/(d-1)}\le(V_\ell(K)-V_\ell(K_n))n^{2/(d-1)}\le(V_\ell(K)-V_\ell(K_{n_{k-1}}))n_k^{2/(d-1)},$$

whenever $n_{k-1}\le n\le n_k$. Taking the limit as $k\to\infty$, $n_{k-1}/n_k\to 1$, which allows us to conclude that the desired limit is reached by the whole sequence with probability 1. □

5. Central limit theorem

In this last section we prove the central limit theorem. In contrast to [22], where floating bodies were used, here we work with surface bodies, as was already done in [21] for the case of the volume. In addition, we make use of the normal approximation bound of Proposition 2. Since the arguments are naturally easier to follow for $K=B^d$, the details are given in this particular setting and the arguments for the general case are stated at the end of the proof.

Proof of Theorem 3. First, we prove the central limit theorem for $K=B^d$. To this end, let us introduce the two events B_1 and B_2. The event that the random polytope $[X_2,\dots,X_n]$ contains the surface body $K(s\ge\tau_n)$ is denoted by B_1. Due to the definition of B_1, it follows by Lemma 1 that

$$\mathbb{P}(B_1^{\mathrm{c}})\le c_1n^{-\alpha},$$

where $c_1\in(0,\infty)$ is independent of n.

We denote by B_2 the event that the random polytope $\bigcap_{W\in\{Y,Y',Z,Z'\}}[W_4,\dots,W_n]$ contains the surface body $K(s\ge\tau_n)$, where Y, Y', Z, and Z' are recombinations of the random vector $X=(X_1,\dots,X_n)$. By taking the union bound, we obtain

$$\mathbb{P}(B_2^{\mathrm{c}})\le c_2n^{-\alpha},$$

where $c_2\in(0,\infty)$ is again independent of n. Next, for any $\ell\in\{1,\dots,d\}$, we apply the bound in Proposition 2 to the random variables

$$W=f(X_1,\dots,X_n):=V_\ell([X_1,\dots,X_n])-\mathbb{E}[V_\ell(K_n)].$$

Note that $D_iW=D_iV_\ell(K_n)$ and $D_{i_1,i_2}W=D_{i_1,i_2}V_\ell(K_n)$ for $i,i_1,i_2\in\{1,\dots,n\}$. Conditioned on the event B_1, we obtain, from (1),

$$D_1V_\ell(K_n)=\binom{d}{\ell}\frac{\kappa_d}{\kappa_\ell\kappa_{d-\ell}}\int_{G(d,\ell)}\mathrm{vol}_\ell((K_n\mid L)\setminus([X_2,\dots,X_n]\mid L))\nu_\ell(\mathrm{d}L). \tag{13}$$

We now define a full-dimensional cap C in such a way that $K_n\setminus[X_2,\dots,X_n]$ is contained in C. Consider now the visibility region $\mathrm{Vis}_{X_1}(\tau_n)$ of X_1. By the definition of the surface body and by Lemma 3, the diameter of this visibility region is at most $c_3\tau_n^{1/(d-1)}$, where $c_3>0$. We now indicate with $D(X_1,c_3\tau_n^{1/(d-1)})$ the points on ∂K with distance at most $c_3\tau_n^{1/(d-1)}$ from X_1. Then $C:=\mathrm{conv}\{D(X_1,c_3\tau_n^{1/(d-1)})\}$ is a spherical cap and it follows from Lemma 4 that C has volume of order at most $\tau_n^{(d+1)/(d-1)}$. We call α the central angle of C. For any subspace $L\in G(d,\ell)$, it holds that $(K_n\mid L)\setminus([X_2,\dots,X_n]\mid L)\subseteq(C\mid L)$. We obtain $\mathrm{vol}_\ell(C\mid L)\ll\tau_n^{(\ell+1)/(d-1)}$. Indeed, the height of $C\mid L$ has the same order as the height of C,

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namely, $\tau_n^{2/(d-1)}$, while the order of its base changes from $((\tau_n)^{1/(d-1)})^{d-1}$ to $((\tau_n)^{1/(d-1)})^{\ell-1}$, since the dimension of L is ℓ. By the construction of C, it now follows that if $\sphericalangle(X_1, L)$, the angle between X_1 and L, is too wide compared to α, then $C \mid L \subseteq K_n \mid L$ for sufficiently large n. Whenever this occurs, it also holds in particular that $(K_n \setminus [X_2, \dots, X_n]) \mid L \subseteq K_n \mid L$, i.e. $K_n \mid L = [X_2, \dots, X_n] \mid L$. In fact, we can check that the integrand in (13) can only be nonzero if $\sphericalangle(X_1, L) \ll \alpha$. Therefore, we can restrict the integration to the set $\{L \in G(d,\ell)\colon \sphericalangle(X_1, L) \ll \alpha\}$. Moreover, it holds that $\alpha \ll V_d(C)^{1/(d+1)}$; see, e.g. [2, Equation (21)]. According to Lemma 5, this gives

$$\nu_\ell(\{L \in G(d,\ell)\colon \sphericalangle(X_1, L) \ll V_d(C)^{1/(d+1)}\}) \ll \tau_n^{(d-\ell)/(d-1)}.$$

Putting everything together, we see that

$$D_1 V_\ell(K_n) \ll \tau_n^{(\ell+1)/(d-1)} \tau_n^{(d-\ell)/(d-1)} \ll \left(\frac{\log n}{n}\right)^{(d+1)/(d-1)}. \tag{14}$$

On the complement B_1^{c} of B_1 we use the trivial estimate $D_1 V_\ell(K_n) \le V_\ell(K)$. Since $\mathbb{P}(B_1^{\mathrm{c}}) \ll n^{-\alpha}$, we obtain

$$\mathbb{E}[(D_1 V_\ell(K_n))^p] = \mathbb{E}[(D_1 V_\ell(K_n))^p \, \mathbf{1}_{B_1}] + \mathbb{E}[(D_1 V_\ell(K_n))^p \, \mathbf{1}_{B_1^{\mathrm{c}}}] \ll \left(\frac{\log n}{n}\right)^{p(d+1)/(d-1)}$$

for all $p \ge 1$. As a consequence, we can bound the terms in the normal approximation bound which involve γ_3 and γ_4. Thus,

$$\frac{\sqrt{n}}{\mathrm{var}[V_\ell(K_n)]}\sqrt{\gamma_3} \ll \frac{\sqrt{n}}{n^{-(d+3)/(d-1)}}\left(\frac{\log n}{n}\right)^{2(d+1)/(d-1)} = n^{-1/2}(\log n)^{2+4/(d-1)},$$

$$\frac{n}{(\mathrm{var}[V_\ell(K_n)])^{3/2}}\gamma_4 \ll \frac{n}{n^{-3(d+3)/2(d-1)}}\left(\frac{\log n}{n}\right)^{3(d+1)/(d-1)} = n^{-1/2}(\log n)^{3+6/(d-1)}.$$

By using the Cauchy–Schwarz inequality, we can estimate γ_5 as well. Namely,

$$\gamma_5 \le \sqrt{\mathrm{var}[V_\ell(K_n)]} \sup_{A \subseteq \{1,\dots,n\}} \sqrt{\mathbb{E}[|D_1 f(X^A)|]^6} \ll n^{-(d+3)/2(d-1)}\left(\frac{\log n}{n}\right)^{3(d+1)/(d-1)}.$$

Thus, we obtain

$$\begin{aligned}\frac{n}{(\mathrm{var}[V_\ell(K_n)])^2}\gamma_5 &\ll \frac{n}{n^{-2(d+3)/(d-1)}} n^{-(d+3)/2(d-1)}\left(\frac{\log n}{n}\right)^{3(d+1)/(d-1)} \\ &= n^{-1/2}(\log n)^{3+6/(d-1)}.\end{aligned}$$

In the next step we consider the terms involving the second-order difference operator. On the event B_2, it may be concluded from (14) that $D_i f(V)^2 \ll (\log n/n)^{2(d+1)/(d-1)}$ for all $i \in \{1,2,3\}$ and $V \in \{Z, Z'\}$. Moreover, we note that on B_2 the following inclusions hold:

$$\{D_{1,2} f(Y) \ne 0\} \subseteq \{\mathrm{Vis}_{Y_1}(\tau_n) \cap \mathrm{Vis}_{Y_2}(\tau_n) \ne \varnothing\} \subseteq \left\{Y_2 \in \bigcup_{x \in \mathrm{Vis}_{Y_1}(\tau_n)} \mathrm{Vis}_x(\tau_n)\right\}.$$

The same applies to $D_{1,3}f(Y')$. Thus,

$$\mathbb{E}[\mathbf{1}\{D_{1,2}f(Y)\neq 0\}\mathbf{1}_{B_2}] \leq \sup_{z\in\partial K}\mathbb{P}\bigg(Y_2\in\bigcup_{x\in\mathrm{Vis}_z(\tau_n)}\mathrm{Vis}_x(\tau_n)\bigg).$$

We note that the diameter of the previous union is at most $c_4\tau_n^{1/(d-1)}$, where $c_4>0$. As before, we define the spherical cap $C':=\mathrm{conv}\{D(z,c_4\tau_n^{1/(d-1)})\}$. It follows from Lemma 4 that C' has volume of order at most $\tau_n^{(d+1)/(d-1)}$. We obtain

$$\begin{aligned}\sup_{z\in\partial K}\mathbb{P}\bigg(Y_2\in\bigcup_{x\in\mathrm{Vis}_z(\tau_n)}\mathrm{Vis}_x(\tau_n)\bigg) &= \sup_{z\in\partial K}\mathcal{H}^{d-1}\bigg(\bigg(\bigcup_{x\in\mathrm{Vis}_z(\tau_n)}\mathrm{Vis}_x(\tau_n)\bigg)\cap\partial K\bigg)\\ &\leq \sup_{z\in\partial K}\mathcal{H}^{d-1}(C'\cap\partial K)\\ &\ll \tau_n,\end{aligned}$$

where, for the last inequality, we used Lemma 3. On the event B_2^{c}, we use the trivial estimate $V_\ell(K)$ for all difference operators and estimate all indicators by one. Since $\mathbb{P}(B_2^{\mathrm{c}})\ll n^{-\alpha}$, we obtain

$$\gamma_2\ll\bigg(\frac{\log n}{n}\bigg)^{1+4(d+1)/(d-1)}.$$

Analogously, we can bound γ_1. Indeed, suppose that $Y_1=Y_1'$ (by independence, $Y_1\neq Y_1'$ gives a smaller order). Then

$$\{D_{1,2}f(Y)\neq 0\}\cap\{D_{1,3}f(Y')\neq 0\}\subseteq\bigg\{\{Y_2,Y_3'\}\subseteq\bigcup_{x\in\mathrm{Vis}_{Y_1}(\tau_n)}\mathrm{Vis}_x(\tau_n)\bigg\},$$

and we obtain

$$\mathbb{E}[\mathbf{1}\{D_{1,2}f(Y)\neq 0\}\mathbf{1}\{D_{1,3}f(Y')\neq 0\}]\ll\bigg(\frac{\log n}{n}\bigg)^2.$$

Thus,

$$\gamma_1\ll\bigg(\frac{\log n}{n}\bigg)^{2+4(d+1)/(d-1)}.$$

Finally,

$$\begin{aligned}\frac{\sqrt{n}}{\mathrm{var}[V_\ell(K_n)]}\sqrt{n^2\gamma_1}&\ll\frac{\sqrt{n}}{n^{-(d+3)/(d-1)}}\sqrt{n^2\bigg(\frac{\log n}{n}\bigg)^{2+4(d+1)/(d-1)}}=n^{-1/2}(\log n)^{3+4/(d-1)},\\ \frac{\sqrt{n}}{\mathrm{var}[V_\ell(K_n)]}\sqrt{n\gamma_2}&\ll\frac{\sqrt{n}}{n^{-(d+3)/(d-1)}}\sqrt{n\bigg(\frac{\log n}{n}\bigg)^{1+4(d+1)/(d-1)}}=n^{-1/2}(\log n)^{5/2+4/(d-1)}.\end{aligned}$$

Considering all the estimates together, we obtain, by Proposition 2,

$$\begin{aligned}d_K(W_\ell(K_n),N)&\ll n^{-1/2}((\log n)^{3+4/(d-1)}+(\log n)^{5/2+4/(d-1)}+(\log n)^{2+4/(d-1)}\\&\qquad+(\log n)^{3+6/(d-1)}+(\log n)^{3+6/(d-1)})\\&\ll n^{-1/2}(\log n)^{3+6/(d-1)}.\end{aligned}$$

For the case of a generic $K\in\mathcal{K}_+^2$, we argue as at the end of the proof of the upper bound of Theorem 1. Because of the global bounds on the principal curvatures and the local approximation of ∂K with affine images of balls, the construction of C and the relations regarding its

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volume, its central angle, and the subspaces L which ensure $C \mid L \subseteq K_n \mid L$ are not afflicted. In particular, the asymptotic bounds $\mathrm{vol}_\ell(C \mid L) \ll \tau_n^{(\ell+1)/(d-1)}$, $\alpha \ll V_d(C)^{1/(d+1)} \ll \tau_n^{1/(d-1)}$, and $\sphericalangle(X_1, L) \ll \alpha$ stated above still hold, with the difference that the implicit constants depend on γ and Γ, the bounds on the principal curvatures of ∂K. The proof can be completed as in the case of the ball. □

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