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On the Poincaré problem and Liouvillian integrability of quadratic Liénard differential equations

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We present the complete classification of irreducible invariant algebraic curves of quadratic Liénard differential equations. We prove that these equations have irreducible invariant algebraic curves of unbounded degrees, in contrast with what is wrongly claimed in the literature. In addition, we classify all the quadratic Liénard differential equations that admit a Liouvillian first integral.

Keywords: invariant algebraic curves; Darboux polynomials; Liouvillian first integrals; Liénard systems; Poincaré problem

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1. Introduction

Planar quadratic dynamical systems are used to describe different phenomena in various fields of physics, economics, biology, chemistry, etc. Quadratic systems seem to be the most studied ones. Dynamical properties, phase portraits, existence of invariant algebraic curves, limit cycles and first integrals have attracted much attention in recent years. Despite the fact that quadratic systems are the most simple non-linear systems, the classification of integrable quadratic systems is far from being complete. In this paper, we focus on quadratic Liénard differential equations, that is, equations of the form

$$x' = y, \quad y' = -f(x)y - g(x) \tag{1.1}$$

where the prime means the derivative with respect to the independent variable t and $f(x), g(x) \in \mathbb{C}[x]$. The degree of the system is two. More concretely we consider the system

$$x' = y, \quad y' = \alpha + \beta x + \gamma y + mxy + nx^2, \tag{1.2}$$

where $\alpha, \beta, \gamma, m, n \in \mathbb{C}$.

Our first aim is to solve completely the hard problem of characterizing all systems (1.2) that admit a Liouvillian first integral. We recall that Liouvillian functions are functions built up from rational functions using exponentiation, integration and algebraic functions. To do that, we begin by characterizing all irreducible invariant algebraic curves of system (1.2), since the existence of invariant algebraic curves is a central object in the theory of integrability. It turns out that when the degree of $f(x)$ is one and the degree of $g(x)$ is two, that is, $mn \neq 0$ (known as Liénard dynamical systems of type $(1, 2)$) these systems have invariant algebraic curves of unbounded degrees provided that we impose no restrictions on the coefficients of the systems (see theorem 1.3 (a.3)). Hence, in particular, a uniform upper bound that depends entirely on the degree of the dynamical system under consideration does not exist for the Liénard dynamical systems of type $(1, 2)$.

We recall that the problem of finding a bound on the degree of irreducible invariant algebraic curves of a single polynomial differential system goes back to Poincaré [**9**] and, if the bound exists, such a problem is known as the *Poincaré problem*. More precisely, the Poincaré problem can be simply stated as the problem of recognizing when a single polynomial differential system admits a first integral which is rational. If such a system admits a rational first integral then Poincaré says that the system is algebraically integrable. The earliest publication of Poincaré in which he refers to this problem appeared in April 1891 in *Comptes Rendus de l'Académie des Sciences* t.112, pp. 761–764 where Poincaré began the paper with the following phrase: 'The question of algebraic integrability of differential equations of the first order and of the first degree did not attract the attention of the geometers as much as it deserved…'. The next paper in which Poincaré discusses this problem is in [**9**] in which he begins by saying: 'To recognize if a differential equation of the first order and the first degree is algebraically integrable, it evidently suffices to find a superior limit of the degree of the integral; it only remains afterwards to perform purely algebraic calculations'. Assuming we know what the maximum degree of an algebraic invariant curve of a system is, then in principle we can perform the calculations and find the curves and determine a first integral in case it exists. However in practice, even with our most powerful computers these calculations are not easy. So that reducing the problem to finding the bound for the degrees of these curves is not exactly solving the problem of Poincaré but is a very good beginning especially if such a bound is low.

As a somewhat extension of the Poincaré problem to a family of differential equations (and not a single one) one can also asks whether there exists a uniform upper bound that depends entirely on the degree of a family of differential equations and not on the coefficients of such a family. This is precisely one of the main issues that is studied in this paper for the family of Liénard dynamical systems of type $(1, 2)$. Being more precise, we shall prove in theorem 1.1 that such an upper bound does not exist for the family of differential equations given by (1.2). Although there are available some examples of families of dynamical systems with irreducible invariant algebraic curves of unbounded degrees assuming that no restrictions are imposed on the coefficients of such families, the case of Liénard dynamical systems of type $(1, 2)$ seems to be the first example of a natural (i.e. not artificially constructed) family of models with this property. Liénard differential systems (1.2) were previously considered by Chavarriga, García, and Sorolla [**3**]. The authors claimed that in fact the extension of the Poincaré problem to a family of dynamical systems was solved for the Liénard dynamical systems of type $(1, 2)$ by stating that any irreducible invariant algebraic curve of such systems has at most degree 3. The results in [**3**] are not on solid ground because they are based on results of Żołądek [**12**], which turned out to be wrong for the case of Liénard dynamical systems (and in particular for Liénard dynamical systems of type $(1, 2)$).

To state our first main result we introduce some auxiliary notation:

$$\begin{aligned}
\sigma_0 &= -\frac{m(\beta m - 2\gamma n)}{4n^2}, \quad \delta_0 = \frac{m^2(\beta\gamma m - \alpha m^2 - \gamma^2 n)}{16n^3}, \\
b_0 &= \frac{m^2}{4n}, \quad b_1 = \frac{\gamma}{n}, \quad b_2 = -\frac{m^3}{8n^2}, \quad b_3 = -\frac{m}{2n}, \\
s &= b_0(x + b_1), \quad z = b_2 y + b_0^2(x + b_1)^2 + b_0(x + b_1), \\
\sigma_1 &= -\sqrt{\beta^2 - 4\alpha n}, \quad X = \frac{n}{6}\left(x + \frac{\beta + \sigma_1}{2n}\right), \quad Y = \frac{n}{6} y.
\end{aligned} \tag{1.3}$$

THEOREM 1.1. *The following holds for systems (1.2) satisfying $|m|^2 + |n|^2 \neq 0$.*

(a) *If $mn \neq 0$, the unique invariant algebraic curves are*
  (a.1) *$z - \delta_0 = 0$ with cofactor $K = b_3$ whenever $\sigma_0 = -1$;*

  (a.2) *$s^2 + 3s/2 - z - 2\delta_0 = 0$ with cofactor $K = b_3(-2s - 1/2)$ whenever $\sigma_0 = 4\delta_0 - 1/4$;*

  (a.3) *$g_2(z)s^2 + g_1(z)s + g_0(z) = 0$ with cofactor $K = b_3(-2s + M - 3/2)$ whenever $\sigma_0 = -1, \delta_0 = (2M - 3)(2M + 1)/16$, and $M \in \mathbb{N} \setminus \{1\}$,*

  where $g_0(z), g_1(z)$, and $g_0(z)$ are polynomials given by the relations

$$\begin{aligned}
g_0(z) &= \frac{1}{2} Z^2 g_{2,z,z} - \left(M - \frac{1}{2}\right) Z g_{2,z} + \left(\frac{1}{16}(2M + 1)^2 - Z\right) g_2, \\
g_1(z) &= -Z g_{2,z} + \left(M + \frac{1}{2}\right) g_2, \\
g_2(z) &= \sum_{m=0}^{M-1} \frac{(-1)^m (M + m - 1)!(2m - 1)!! Z^{M-m-1}}{8^m m!(M - m - 1)!},
\end{aligned} \tag{1.4}$$

  the variable Z reads as

$$Z = z - \frac{(2M - 3)(2M + 1)}{16} \quad \text{and} \quad (-1)!! = 1. \tag{1.5}$$

(b) *If $n \neq 0$ and $m = 0$, the unique invariant algebraic curves are*
  (b.1) *$y^2/2 - \alpha x - \beta x^2/2 - nx^3/3 = 0$ with zero cofactor whenever $\gamma = 0$;*

  (b.2) *$Y^2 - 4\gamma/5(X + \gamma^2/25)Y - 4X^3 - (8\gamma^2/25)X^2 - (4\gamma^4/625)X = 0$ with cofactor $K = 6\gamma/5$ whenever $\sigma_1 = -6\gamma^2/25$;*

  (b.3) *$Y^2 - (4\gamma/5)XY - 4X^3 + (4\gamma^2/25)X^2 = 0$ with cofactor $K = 6\gamma/5$ whenever $\sigma_1 = 6\gamma^2/25$.*

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(c) *If* $n = 0$ *and* $m \neq 0$, *the unique invariant algebraic curves are*
(c.1) $y + \beta/m = 0$ *with cofactor* $K = mx + \gamma$ *whenever* $\alpha = \beta\gamma/m$ *and* $\beta \neq 0$;

(c.2) $y - mx^2/2 - \gamma x - c_0 = 0, c_0 \in \mathbb{C}$ *with zero cofactor whenever* $\alpha = 0$ *and* $\beta = 0$.

The proof of statement (a) of theorem 1.1 is given in § 3 while the proof of statements (b) and (c) are given in §§ 4 and 5, respectively. This problem was also considered in [**11**] with an extra assumption on the coefficients that in this paper is removed. However, in the current paper we provide a different proof for the cases already proved in [**11**] and a new one for the cases not proved in [**11**].

Finally, we provide the complete characterization of systems (1.1), which admit a Liouvillian first integral.

THEOREM 1.2. *The following holds for systems* (1.2) *satisfying* $|m|^2 + |n|^2 \neq 0$.

(a) *If* $mn \neq 0$, *it has a Liouvillian first integral if and only if* $\sigma_0 = -1$ *and* $\delta_0 = (2M - 3)(2M + 1)/16$. *A Liouvillian first integral is*

$$H = \frac{2(z - \delta_0)^{M-1/2}}{\sqrt{-g_1(z)^2 + 4g_0(z)g_2(z)}} \arctan\left(\frac{g_1(z) + 2g_2(z)s}{\sqrt{-g_1(z)^2 + 4g_0(z)g_2(z)}}\right), \tag{1.6}$$

where $g_i(z), z$ *and* δ_0 *are given in the statement of theorem* 1.1 *and in* (1.3), *respectively*;

(b) *If* $n \neq 0$ *and* $m = 0$ *it has a Liouvillian first integral if and only if*
(b.1) $\gamma = 0$ *and a Liouvillian first integral is* $H = y^2/2 - \alpha x - \beta x^2/2 - nx^3/3$;

(b.2) $\sigma_1 = -6\gamma^2/25$ *and a Liouvillian first integral is*

$$\begin{aligned} H = {} & 125 \left(-4\gamma^4 X - 20\gamma^3 Y - 200\gamma^2 X^2 - 500\gamma XY + 625(Y^2 - 4X^3)\right)^{1/6} \\ & - \frac{5\sqrt[3]{2}\gamma(\gamma^2 + 25X)^{-1/4}(-2\gamma^3 + 2(\gamma^2 + 25X)^{3/2} - 50\gamma X + 125Y)^{5/6} h_2 h_3}{(-4\gamma^4 X - 20\gamma^3 Y - 200\gamma^2 X^2 - 500\gamma XY + 625(Y^2 - 4X^3))^{5/6}}, \end{aligned} \tag{1.7}$$

where h_2 *and* h_3 *are given by*

$$h_2 = 2\gamma^3 + 2\sqrt{(\gamma^2 + 25X)^3} + 50\gamma X - 125Y$$
$$h_3 = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{7}{6}; \frac{h_2}{4\left(\gamma^2 + 25x\right)^{3/2}}\right),$$

and ${}_2F_1$ *stands for the hypergeometric function;*

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(b.3) $\sigma_1 = 6\gamma^2/25$ *and a Liouvillian first integral is*

$$H = \frac{(-2\gamma X + 5Y - 10X^{3/2})(-2\gamma X + 5Y + 10X^{3/2})^{5/6} h_1}{(4X^2(\gamma^2 - 25X) - 20\gamma XY + 25Y^2)^{5/6}} \tag{1.8}$$

where h_1 *is given by*

$$h_1 = 2^{1/3}5^{1/6}X^{-1/4}\,{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{7}{6}; \frac{2\gamma X - 5Y + 10X^{3/2}}{20X^{3/2}}\right) - 5(-2\gamma X + 5Y + 10X^{3/2})^{1/6};$$

(c) *If* $n = 0$ *and* $m \neq 0$, *it has a Liouvillian first integral if and only if*
(c.1) $\alpha = \beta\gamma/m$ *with* $\beta \neq 0$ *and a Liouvillian first integral is*

$$(\beta + my)^{\beta/m} \exp\left(\frac{mx^2}{2} + \gamma x - y\right); \tag{1.9}$$

(c.2) $\alpha = \beta = 0$ *and a Liouvillian first integral is* $H = y - mx^2/2 - \gamma x$.

The proof of theorem 1.2(a) is given in § 6, the proof of statement (b) is presented in § 7, while the proof of statement (c) can be found in § 8. We have included § 2 where we give some definitions and present a piece of theory that we shall use in order to establish our results.

2. Preliminary definitions and results

Consider a two-dimensional polynomial differential system of degree $d \in \mathbb{N}$

$$\dot{x} = P_1(x, y), \quad \dot{y} = P_2(x, y) \tag{2.1}$$

where $(x, y) \in \mathbb{C}^2$, $P_1, P_2 \in \mathbb{C}[x, y]$, $d = \max\{\deg P_1, \deg P_2\}$ and the dot denotes derivative with respect to the independent variable t.

A function $H(x, y)$ is a *first integral* of system (2.1) if it is continuous on a full Lebesgue measure subset $\Omega \subseteq \mathbb{C}^2$, is not locally constant on any positive Lebesgue measure subset of Ω and moreover is constant along each orbit in Ω of system (2.1). If $\mathcal{X}$ is the vector field associated with the system (2.1), i.e. $\mathcal{X} = (P_1(x, y), P_2(x, y))$, and H is $\mathcal{C}^1$, then we have

$$\mathcal{X}(H) = P_1\frac{\partial H}{\partial x} + P_2\frac{\partial H}{\partial y} = 0.$$

A *Darboux polynomial* of (2.1) is a polynomial $f \in \mathbb{C}[x, y]$ such that

$$\mathcal{X}(f) = P_1\frac{\partial f}{\partial x} + P_1\frac{\partial f}{\partial y} = Kf,$$

where $K \in \mathbb{C}[x, y]$. The polynomial K is called the *cofactor* of f and has degree at most $d - 1$. An *invariant algebraic curve* is a curve $f = 0$, where f is a Darboux polynomial. Note that the curve is invariant by the dynamics in the sense that if a trajectory starts on the curve it does not leave it.

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An *exponential factor* of (2.1) is a function $F = \exp(g/f)$, with coprime $f, g \in \mathbb{C}[x, y]$ such that

$$\mathcal{X}(F) = P_1 \frac{\partial F}{\partial x} + P_2 \frac{\partial F}{\partial y} = LF,$$

where $L \in \mathbb{C}[x, y]$ is called the *cofactor* of F and has degree at most $d - 1$. It is widely known that either f is constant or f is a Darboux polynomial of (2.1). In the latter case the following relation is valid: $\mathcal{X}(g) = Kg + Lf$, where K and L are as defined above.

An *inverse integrating factor* of (2.1) is a function $V(x, y)$ such that

$$\mathcal{X}(F) = P_1 \frac{\partial V}{\partial x} + P_2 \frac{\partial V}{\partial y} = \left(\frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y} \right) V.$$

In other words $V(x, y)$ satisfies

$$\operatorname{div} \left(\frac{P_1}{V}, \frac{P_2}{V} \right) = 0,$$

where div stands for the divergence of the system. Hence, the system

$$\dot{x} = \frac{P_1}{V}, \quad \dot{y} = \frac{P_2}{V}$$

is Hamiltonian and a first integral $H(x, y)$ can be obtained solving the system

$$\frac{P_1}{V} = -\frac{\partial H}{\partial y}, \quad \frac{P_2}{V} = \frac{\partial H}{\partial x}. \tag{2.2}$$

If $V(x, y)$ is a polynomial, then it is a Darboux polynomial whose cofactor equals the divergence of the system.

The following results, proved in [7], explain how to find Darboux and Liouvillian first integrals.

THEOREM 2.1. *Assume that a polynomial differential system* $\mathcal{X}$ *of degree* d *defined in* $\mathbb{C}^2$ *admits* p *Darboux polynomials* f_i *with cofactors* $K_i, i = 1, \ldots, p$, *and* q *exponential factors* $F_j = \exp(g_j/h_j)$ *with cofactors* $L_j, j = 1, \ldots, q$. *Then, the following statements hold*:

(a) *There exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$$

if and only if the function

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}, \tag{2.3}$$

is a first integral of $\mathcal{X}$. *Such a function is called a* Darboux function.

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(b) *There exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = div(\mathcal{X})$$

if and only if the Darboux function (2.3) *is an inverse integrating factor of* $\mathcal{X}$. *Here* $div(\mathcal{X})$ *stands for the divergence of the system.*

To prove the results related with the Liouvillian first integrals we used the following result proved in [**4**, **10**]. We recall that system (2.1) is said to be *Liouvillian integrable* if it has a Liouvillian first integral.

THEOREM 2.2. *The polynomial differential system* (2.1) *has a Liouvillian first integral if and only if it has an integrating factor which is a Darboux function.*

In what follows we shall use some facts from the theory of Puiseux series. A Puiseux series in a neighbourhood of the point x_0 reads as

$$y(x) = \sum_{l=0}^{+\infty} b_l (x - x_0)^{(l_0+l)/n_0}, \tag{2.4}$$

where $l_0 \in \mathbb{Z}$, $n_0 \in \mathbb{N}$. In its turn a Puiseux series in a neighbourhood of the point $x = \infty$ is given by

$$y(x) = \sum_{l=0}^{+\infty} c_l x^{(l_0-l)/n_0}, \tag{2.5}$$

where again $l_0 \in \mathbb{Z}$, $n_0 \in \mathbb{N}$.

It is known [**8**, § 7.8, p. 136] that a Puiseux series of the form (2.4) that satisfy equation $G(x, y) = 0$, where $G(x, y)$ is an element of the ring $\mathbb{C}[x, y]$, is convergent in a neighbourhood of the point x_0 (the point x_0 is excluded from domain of convergence whenever $l_0 < 0$). Analogously, a Puiseux series of the form (2.5) that satisfy equation $G(x, y) = 0$ is convergent in a neighbourhood of the point $x = \infty$ (the point $x = \infty$ is excluded from domain of convergence whenever $l_0 > 0$). If $n_0 > 1$ then the convergence of the corresponding series is understood in the sense that a certain branch of the n_0th root is chosen and a cut forbidding going around the branch point is introduced. The set of all Puiseux series of the form (2.4) or (2.5) forms an algebraically closed field.

Let us privilege the variable y with respect to the variable x and consider y as a function of the variable x. It is straightforward to prove [**6**, Lemma 2.1] that invariant algebraic curves of the vector field $\mathcal{X}$ and related dynamical system (2.1) capture Puiseux series satisfying the first-order ordinary differential equation

$$P_1(x, y) y_x - P_2(x, y) = 0, \tag{2.6}$$

where $y_x = \mathrm{d}y/\mathrm{d}x$ and $P_1(x, y)$, $P_2(x, y) \in \mathbb{C}[x, y]$ are the polynomials appearing in the dynamical system under consideration.

All the Puiseux series that solve equation (2.6) can be obtained with the help of the Painlevé methods. Let us present a brief description of such a method. A first-order algebraic ordinary differential equation can be represented as $W(x, y, y_x) = 0$, where $W(x, y, y_x)$ can be regarded as the sum of differential monomials given by

$$M[x, y(x)] = Cx^l y^{j_0} \left\{\frac{\mathrm{d}y}{\mathrm{d}x}\right\}^{j_1}, \quad C \in \mathbb{C} \setminus \{0\}, \quad l, j_0, j_1 \in \mathbb{N}_0. \tag{2.7}$$

The set of all the differential monomials of the form (2.7) will be denoted as $\mathbb{M}$. In addition in order to simplify notation the expression $E[x, y(x)]$ will stand for a polynomial in x, $y(x)$ and $y_x(x)$ with coefficients in the field $\mathbb{C}$.

At *the first step* one needs to find the so-called dominant balances of equation (2.6), see definitions below. A direct and simple way to find all the dominant balances is to use the Newton polygon of the algebraic ordinary differential equation under consideration. This technique now known as the power geometry was developed by Bruno [**1**, **2**].

Define the map $q : \mathbb{M} \to \mathbb{R}^2$ by the following rules:

$$Cx^{q_1} y^{q_2} \mapsto q = (q_1, q_2), \quad \frac{\mathrm{d}^k y}{\mathrm{d}x^k} \mapsto q = (-k, 1), \quad q(M_1 M_2) = q(M_1) + q(M_2),$$

where $C \in \mathbb{C} \setminus \{0\}$ is a constant, M_1 and M_2 are differential monomials. We denote the set of all points $p \in \mathbb{R}^2$ corresponding to the monomials of equation (2.6) as $S(W)$. The convex hull of $S(W)$ is called *the Newton polygon* of equation (2.6).

The boundary of the Newton polygon consists of vertices and edges. Selecting all the differential monomials of the original equation that generate the vertices and the edges of the Newton polygon, we obtain a number of balances. *The balance* for a vertex is defined as the sum of those differential monomials in $W(x, y, y_x)$ that are mapped into the vertex. *The balance* for an edge is defined as the sum of differential monomials in $W(x, y, y_x)$ whose images belong to the edge. If solutions of equation (2.6) possess an asymptotic of the form $y(x) = c_0 x^r$ with $x \to 0$ or $x \to \infty$, then there exists a balance $E[x, y(x)]$ such that the function $y(x) = c_0 x^r$ satisfies the equation $E[x, y(x)] = 0$. Conversely, the function $y(x) = c_0 x^r$ solving equation $E[x, y(x)] = 0$, where $E[x, y(x)]$ is a balance is an asymptotics at $x \to 0$ (or $x \to \infty$) for solutions of equation (2.6) whenever for all the differential monomials $M[x, y(x)]$ of the original equation not involved into $E[x, y(x)] = 0$ we have $\operatorname{Re} \varkappa > \operatorname{Re} \varkappa_0$ (or $\operatorname{Re} \varkappa < \operatorname{Re} \varkappa_0$), where $M[x, c_0 x^r] = Bx^{\varkappa}$ and $M_0[x, c_0 x^r] = B_0 x^{\varkappa_0}$ with $M_0[x, y(x)]$ being a differential monomial of the balance $E[x, y(x)]$.

Consequently, having found all the power solutions $y(x) = c_0 x^r$ for all the balances, one needs to select those that give asymptotics at $x \to 0$ or $x \to \infty$. Using power asymptotics it is possible to derive asymptotic series possessing these asymptotics as leading-order terms.

In what follows we shall need Puiseux series near $x = \infty$ that satisfy equation (2.6), therefore we shall focus at the case $x \to \infty$. Let us suppose that a balance $E_0[y(x), x]$ of equation (2.6) has a solution $y(x) = c_0 x^r$, which is an asymptotics at $x \to \infty$.

In *the second step* one calculates the formal Gâteaux derivative of the balance $E_0[y(x), x]$ at the solution $y(x) = c_0 x^r$:

$$\frac{\delta E_0}{\delta y}[c_0 x^r] = \lim_{s \to 0} \frac{E_0[c_0 x^r + s x^{r-j}, x] - E_0[c_0 x^r, x]}{s} = V(j) x^{\tilde{r}}.$$

In this expression $V(j)$ is a first-degree polynomial with respect to j. Note that the coefficients of this polynomial depend on c_0 and on the parameters (if any) of the original equation involved into the balance $E_0[y(x), x]$. The zero j_0 of $V(j)$ is called *the Fuchs index* (or *the resonance*) of the balance $E_0[y(x), x]$ and its power solution $y(x) = c_0 x^r$. If the Fuchs index j_0 is not a positive rational number, then $n_0 = r_2$ where r_2 is defined as $r = r_1/r_2$, $r_1 \in \mathbb{Z}$, $r_2 \in \mathbb{N}$, and $(r_1, r_2) = 1$. Otherwise we obtain $n_0 = \operatorname{lcm}(p_2, r_2)$, where r_2 was defined previously and p_2 is given by $j_0 = p_1/p_2$ where p_1, $p_2 \in \mathbb{N}$, $(p_1, p_2) = 1$. By lcm we denote the lowest common multiple.

In *the third step* one verifies the existence of the Puiseux series of the form (2.5) with $l_0 = r n_0$. If the balance $E_0[y(x), x]$ corresponds to a vertex of the Newton polygon, then the Puiseux series always exists and possess an arbitrary coefficient c_0. Note that in this case the Fuchs index is equal to zero. Now let us suppose that the balance $E_0[y(x), x]$ corresponds to an edge of the Newton polygon. Substituting series (2.5) into equation (2.6) one can find the recurrence relation for its coefficients. This relation takes the form

$$V\left(\frac{k}{n_0}\right) c_k = U_k(c_0, \ldots, c_{k-1}), \quad k \in \mathbb{N},$$

where U_k is a polynomial of its arguments. Note that U_k can also depend on the parameters (if any) of the original equation. The equation $U_{n_0 j_0} = 0$ is called *the compatibility condition*. If the compatibility condition is not satisfied, then the Puiseux series under consideration does not exist. Otherwise the corresponding Puiseux series exists and possesses an arbitrary coefficient $c_{n_0 j_0}$.

The Puiseux series under consideration has uniquely determined coefficients whenever there are no non-negative rational Fuchs indices.

We note that if one wishes to find all the Puiseux series of the form (2.5) that satisfy the original equation, then it is necessary to implement the procedure described above for all the dominant balances and for all their power solutions.

We also observe that there may exist balances and their power solutions such that $V(j) \equiv 0$. If $V(j)$ is identically zero, then one should make the substitution $y(x) = c_0 x^r + w(x)$ in equation (2.6) and find all the Puiseux series $w(x) = c_1 x^{r_1} + \cdots$ of the latter such that $r_1 < r$ and $x \to \infty$.

The following theorem was proved in article [**6**].

THEOREM 2.3. *Let $F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}, F_y \not\equiv 0$ be an irreducible invariant algebraic curve of polynomial vector field $\mathcal{X}$ and related dynamical system* (2.1)*. Then $F(x, y)$ takes the form*

$$F(x, y) = \left\{ \mu(x) \prod_{j=1}^{N} \{y - y_j(x)\} \right\}_+, \quad N \in \mathbb{N},$$

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where $\mu(x) \in \mathbb{C}[x]$ *and* $y_1(x), \ldots, y_N(x)$ *are pairwise distinct Puiseux series in a neighbourhood of the point* $x = \infty$ *that satisfy equation* (2.6). *The symbol* $\{W(x,y)\}_+$ *means that we take the polynomial part of the expression* $W(x,y)$. *Moreover, the degree of* $F(x,y)$ *with respect to* y *does not exceed the number of distinct Puiseux series of the from* (2.5) *satisfying equation* (2.6) *whenever the latter is finite.*

Clearly, if there are no Puiseux series in a neighbourhood of the point $x = \infty$ that satisfy equation (2.6), then polynomial vector field $\mathcal{X}$ and related dynamical system (2.1) do not have invariant algebraic curves such that $F_y \not\equiv 0$.

3. Proof of theorem 1.1(a)

Introducing the change of variables and reparameterization of the independent variable t given by

$$x_1 = b_0(x + b_1), \quad y_1 = b_2 y \quad \text{and} \quad t = b_3\tau$$

where b_i are presented in (1.3), the Liénard dynamical systems (1.2) with $mn \neq 0$ can be written as

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = -2x_1y_1 + x_1^2 - \sigma_0 x_1 - \delta_0.$$

Now the dot means the derivative with respect to the new independent variable τ and the parameters σ_0, δ_0 are also given in (1.3).

Introducing the invertible change of variables $x_1 = s$, $y_1 = z - s^2 - s$ (or $s = x_1$, $z = y_1 + x_1^2 + x_1$, see (1.3)) yields the following dynamical system

$$\dot{s} = z - s^2 - s, \quad \dot{z} = z - (1 + \sigma_0)s - \delta_0. \tag{3.1}$$

There exists the one-to-one correspondence between irreducible invariant algebraic curves $f(x,y) = 0$ of Liénard dynamical system (1.2) and irreducible invariant algebraic curves $G(s,z)$ of system (3.1). The following theorem was proved in article [**5**].

THEOREM 3.1. *Let* $G(s,z) \in \mathbb{C}[s,z] \setminus \mathbb{C}$ *be an irreducible invariant algebraic curve of dynamical system* (3.1). *Then the degree of* $G(s,z)$ *with respect to* s *is either* 0 *or* 2.

It is also clear that the cofactor K of the invariant algebraic curve must be of the form $K = A_0 + A_1 s$ with $A_0, A_1 \in \mathbb{C}$.

The proof of theorem 1.1(a) will be an immediate consequence of the proof of the following theorem.

THEOREM 3.2. *The unique irreducible invariant algebraic curves* $G(s,z) = 0$ *of dynamical systems* (3.1) *have cofactor* $K = A_0 + A_1 s$ *and take the form*

(i) $G(s,z) = z - \delta_0$ *with* $A_1 = 0$ *and* $A_0 = 1$ *whenever* $\sigma_0 = -1$;

(ii) $G(s,z) = s^2 + 3s/2 - z - 2\delta_0$ *with* $A_1 = -2, A_0 = -1/2$ *whenever* $\sigma_0 = 4\delta_0 - 1/4$;

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(iii) $G(s,z) = g_2(z)s^2 + g_1(z)s + g_0(z)$ *with* $A_1 = -2$ *and* $A_0 = M - 3/2$ *whenever* $\sigma_0 = -1$ *and* $\delta_0 = (2M-3)(2M+1)/16$ *with* $M \in \mathbb{N} \setminus \{1\}$,

where $g_0(z), g_1(z)$ *and* $g_0(z)$ *are the polynomials given by the relations in* (1.4) (*see also* (1.5)).

We recall that the value $M = 1$ is excluded in the last sequence of invariant algebraic curves since the latter is a partial case of the algebraic curve given in (*ii*).

Proof of theorem 3.2. Invariant algebraic curves of system (3.1) satisfy the following equation

$$G_s(z - s^2 - s) + G_z(z - (1+\sigma_0)s - \delta_0) = (A_0 + A_1 s)G. \tag{3.2}$$

We begin by classifying irreducible invariant algebraic curves that do not depend on s.

Case 1. $G(s,z)$ has degree 0 in s. Substituting $G(s,z) = z - z_0$ with $z_0 \in \mathbb{C}$ into equation (3.2), we obtain the following relation

$$z - (1+\sigma_0)s - \delta_0 = (A_0 + A_1 s)(z - z_0).$$

Setting to zero the coefficients at different powers of s and z, we find $z_0 = \delta_0$, $A_1 = 0$, $A_0 = 1$ and $\sigma_0 = -1$. This result is given in (*i*).

Case 2. $G(s,z)$ has degree 2 in s. Representing irreducible invariant algebraic curves $G(s,z)$ as follows:

$$G(s,z) = g_2(z)s^2 + g_1(z)s + g_0(z), \tag{3.3}$$

where $g_0(z)$, $g_1(z)$, $g_2(z) \in \mathbb{C}[z]$ and $g_2(z) \not\equiv 0$, we substitute this representation into equation (3.2) and set to zero the coefficients at different powers of s. The equation resulting from the coefficient at s^3 reads as

$$(\sigma_0 + 1)g_{2,z} + (A_1 + 2)g_2 = 0. \tag{3.4}$$

If $\sigma_0 \neq -1$, then $g_2(z)$ takes the form

$$g_2(z) = C_0 \exp\left[-\frac{(A_1+2)z}{\sigma_0 + 1}\right], \quad C_0 \in \mathbb{C} \setminus \{0\}.$$

Consequently, $g_2(z)$ is a polynomial if and only if $A_1 = -2$. In this case we get $g_2(z) = C_0$. Without loss of generality, we take $C_0 = 1$. Further, we substitute relation (3.3) with $g_2(z) = 1$ into equation (3.2) and set to zero the coefficients at different powers of s. The coefficient at s^2 gives the following ordinary differential equation

$$(\sigma_0 + 1)g_{1,z} - g_1 + A_0 + 2 = 0.$$

The unique polynomial solution of this equation is $g_1(z) = A_0 + 2$. Considering the coefficient at s^1, we obtain the following ordinary differential equation

$$(\sigma_0 + 1)g_{0,z} - 2g_0 - 2z + (A_0 + 2)(A_0 + 1) = 0$$

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with the unique polynomial solution given by

$$g_0(z) = -z + \tfrac{1}{2}\left(A_0^2 + 3A_0 - \sigma_0 + 1\right).$$

Finally, setting to zero the coefficient at s^0, we find

$$\sigma_0 = 4\delta_0 - \tfrac{1}{4}, \quad A_0 = -\tfrac{1}{2}.$$

The resulting irreducible invariant algebraic curve is given in (ii).

Now let us consider the case $\sigma_0 = -1$. From equation (3.4) we obtain $A_1 = -2$. Considering equations resulting from the coefficient at s^2 and s, we can express g_1 and g_0 via g_2 and its derivatives. Thus we get

$$\begin{aligned} g_1 &= -(z-\delta_0)g_{2,z} + (A_0+2)g_2, \\ g_0 &= \frac{1}{2}(z-\delta_0)^2 g_{2,zz} - (A_0+1)(z-\delta_0)g_{2,z} + \left(1 - z + \frac{1}{2}A_0(A_0+3)\right) g_2. \end{aligned} \tag{3.5}$$

Substituting these expressions into the equation resulting from the coefficient at s^0 yields the following third order linear ordinary differential equation

$$\begin{aligned} &Z^3 g_{2,ZZZ} - 3A_0 Z^2 g_{2,ZZ} - (4Z + 4\delta_0 - 3A_0(A_0+1))Z g_{2,Z} \\ &\quad + (2(2A_0+1)Z + (A_0+1)(4\delta_0 - A_0(A_0+2)))\, g_2 = 0 \end{aligned} \tag{3.6}$$

Note that for simplicity we introduce the new variable $Z = z - \delta_0$. Further, using relations (3.5) we verify that if $z = \delta_0$ is a zero of $g_2(z)$, then it is also a zero of $g_1(z)$ and $g_2(z)$. If such a situation occurs, then $G(s, z)$ is reducible. In view of this let us suppose that $g_2(z)$ takes the form

$$g_2(z) = Z^{M-1} + B_1 Z^{M-2} + \cdots + B_{M-1}, \quad Z = z - \delta_0, \quad M \in \mathbb{N}, \tag{3.7}$$

where $B_{M-1} \neq 0$. Note that if $g_2(z)$ is of degree $M-1$ with respect to z, then $g_0(z)$ is of degree M with respect to z and so does $G(s, z)$. Substituting representation (3.7) into equation (3.6) and setting to zero the coefficients at Z^M and Z^0, we find the following necessary conditions for polynomial solutions to exist:

$$2A_0 - 2M + 3 = 0, \quad (A_0+1)(A_0(A_0+2) - 4\delta_0)B_{M-1} = 0. \tag{3.8}$$

Recalling the fact that $B_{M-1} \neq 0$, we solve system (3.8). This gives

$$A_0 = M - \frac{3}{2}, \quad \delta_0 = \frac{(2M-3)(2M+1)}{16}. \tag{3.9}$$

Further, we find the recurrence relation for the coefficients of $g_2(z)$. The result takes the form

$$(M-m-1)(2m+1)B_{M-m-1} + 8(m+1)B_{M-m-2} = 0, \quad m \in \mathbb{Z}, \tag{3.10}$$

where $B_0 = 1$ and $B_m = 0$ whenever $m \geqslant M$ and $m < 0$. Analysing this relation, we conclude that equation (3.6) possesses the unique polynomial solution of degree

$M-1$ provided that conditions (3.9) are satisfied. Finally, relation (3.10) and the normalization $B_0 = 1$ give the following explicit representation

$$B_m = \frac{(-1)^m (M+m-1)!(2m-1)!!}{8^m m!(M-m-1)!}.$$

This completes the proof. □

4. Proof of theorem 1.1(b)

If $\gamma = 0$ then system (1.2) becomes

$$x' = y, \quad y' = \alpha + \beta x + nx^2.$$

This system has the first integral

$$H(x, y) = y^2/2 - \alpha x - \beta x^2/2 - nx^3/3$$

and so $y^2/2 - \alpha x - \beta x^2/2 - nx^3/3 = 0$ is an invariant algebraic curve with zero cofactor. This proves statement (b.1).

Now assume that $\gamma \neq 0$. Introducing the change of variables

$$X = \frac{n}{6}\left(x + \frac{\beta - \sigma_1}{2n}\right), \quad Y = \frac{n}{6}y$$

system (1.2) with $m = 0$ and $n \neq 0$ becomes

$$X' = Y, \quad Y' = \gamma Y + 6X^2 - \sigma_1 X. \tag{4.1}$$

Our aim is to provide the complete classification of irreducible invariant algebraic curves of Liénard dynamical systems (4.1). We shall use the method of Puiseux series introduced in articles [**5**, **6**] and briefly described in § 2.

The proof of theorem 1.1(b) will be an immediate consequence of the proof of the following theorem.

THEOREM 4.1. *The unique irreducible invariant algebraic curves $G(X, Y) = 0$ of dynamical systems (4.1) with $\gamma \neq 0$ have cofactor $K = 6\gamma/5$ and take the form*

(i) $G(X, Y) = Y^2 - (4\gamma/5)(X + \gamma^2/25)Y - 4X^3 - (8\gamma^2/25)X^2 - (4\gamma^4/625)X$ *whenever $\sigma_1 = -6\gamma^2/25$;*

(ii) $G(X, Y) = Y^2 - (4\gamma/5)XY - 4X^3 + (4\gamma^2/25)X^2$ *whenever $\sigma_1 = 6\gamma^2/25$.*

Proof. Invariant algebraic curves of dynamical system (4.1) satisfy the equation

$$YG_X - (-\gamma Y - 6X^2 + \sigma_1 X)G_Y = K(X, Y)G. \tag{4.2}$$

Balancing higher-order terms with respect to y, we see that $K(X, Y) = K(X)$, $\mu(X) = \mu_0$, where $K(X) \in \mathbb{C}[X]$ and $\mu_0 \in \mathbb{C}$. Without loss of generality, we set $\mu_0 = 1$. Substituting $G(X, Y) = G(X)$ into equation (4.2), we show that there are no invariant algebraic curves that do not depend on Y.

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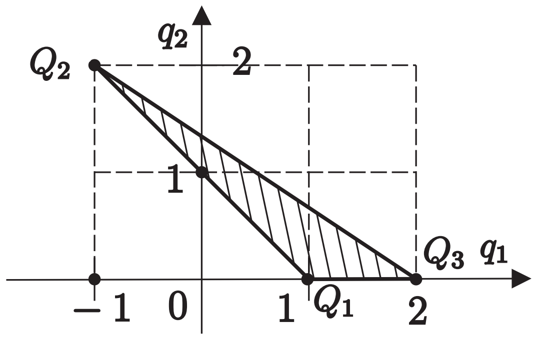

Figure 1. The Newton polygon of equation (4.3) with $\sigma_1 \neq 0$.

Let us find irreducible invariant algebraic curves $G(X, Y)$ such that $G_Y \not\equiv 0$. Supposing that the variable Y is dependent and the variable X is independent, we see that the function $Y = Y(X)$ satisfies the following first-order ordinary differential equation

$$YY_X - \gamma Y - 6X^2 + \sigma_1 X = 0. \tag{4.3}$$

Our aim is to find Puiseux series near the point $X = \infty$ that satisfy equation (4.3). The Newton polygon of equation (4.3) is presented in figure 1. Analysing the Newton polygon we find that there exists only one dominant balance related to the point $X = \infty$. This balance and the power solutions take the form

$$YY_X - 6X^2 = 0: \quad y^{(1,2)}(X) = c_0^{(1,2)} X^{3/2}, \quad c_0^{(1,2)} = \pm 2.$$

The Fuchs index is $j_0 = 3$ and the corresponding Puiseux series take the form

$$y^{(1,2)}(X) = \sum_{k=0}^{+\infty} c_k^{(1,2)} X^{3/2-k/2}. \tag{4.4}$$

The compatibility condition for both series to exist resulting from the presence of the Fuchs index $j_0 = 3$ reads as

$$(6\gamma^2 - 25\sigma_1)(6\gamma^2 + 25\sigma_1) = 0. \tag{4.5}$$

Consequently series (4.4) possess arbitrary coefficients $c_6^{(1,2)}$ provided that condition (4.5) holds. Using the consequence to theorem 2.3 and the fact that there are no Puiseux series near $X = \infty$ satisfying equation (4.3) with $(6\gamma^2 - 25\sigma_1)(6\gamma^2 + 25\sigma_1) \neq 0$, we conclude that the corresponding dynamical system does not have invariant algebraic curves whenever condition (4.5) is not satisfied.

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Further, we represent invariant algebraic curves via the Puiseux series as follows

$$G(X,Y) = \left\{ \prod_{j=1}^{N_1} \{Y - Y_j^{(1)}(X)\} \prod_{j=1}^{N_2} \{Y - Y_j^{(2)}(X)\} \right\}_+, \tag{4.6}$$

where N_1, $N_2 \in \mathbb{N} \cup \{0\}$ and each series $Y_j^{(1,2)}(X)$ contains an arbitrary coefficient $c_{6,j}^{(1,2)}$, arbitrary in the sense that it is not provided by equation (4.3). These coefficients with the same upper index should be pairwise distinct. Further, we require that the non-polynomial part of expression in brackets in (4.6) equals zero. Setting to zero the coefficient at $Y^{N_1+N_2-1}X^{3/2}$ gives the following relation $N_2 = N_1$. The equation resulting from the coefficient at $Y^{N_1+N_2-1}X^{1/2}$ is automatically satisfied. Further, we consider the algebraic system

$$\sum_{j=1}^{N_1} \left\{ c_{k,j}^{(1)} + c_{k,j}^{(2)} \right\} = 0, \quad k = 4, 5, \ldots \tag{4.7}$$

resulting from the coefficient at $Y^{N_1+N_2-1}X^{(3-k)/2}$. For convenience we introduce the following notation

$$C_m^{(1)} = \sum_{j=1}^{N_1} \left\{ c_{6,j}^{(1)} \right\}^m, \quad C_m^{(2)} = \sum_{j=1}^{N_1} \left\{ c_{6,j}^{(2)} \right\}^m, \quad m \in \mathbb{N}.$$

Solving system (4.7) accompanied by equation (4.5) we obtain the following results

$$\sigma_1 = \frac{6}{25}\gamma^2, \quad C_m^{(1)} = 0, \quad C_m^{(2)} = 0, \quad m \in \mathbb{N};$$
$$\sigma_1 = -\frac{6}{25}\gamma^2, \quad C_m^{(1)} = \frac{(-1)^m}{2^m 250^{2m}}\gamma^{6m} N_1, \quad C_m^{(2)} = \frac{(-1)^{m+1}}{2^m 250^{2m}}\gamma^{6m} N_1, \quad N_1, m \in \mathbb{N}.$$

Requiring that the resulting invariant algebraic curve is irreducible, we should set $N_1 = 1$

$$\sigma_1 = \frac{6}{25}\gamma^2, \quad c_{6,1}^{(1)} = 0, \quad c_{6,1}^{(2)} = 0, \quad N_1 = 1;$$
$$\sigma_1 = -\frac{6}{25}\gamma^2, \quad c_{6,1}^{(1)} = -\frac{1}{125000}\gamma^6, \quad c_{6,1}^{(2)} = \frac{1}{125000}\gamma^6, \quad N_1 = 1.$$

Substituting these expressions into representation (4.6), we obtain the corresponding irreducible invariant algebraic curves. Their cofactors we find by direct substitution of the polynomials representing the algebraic curves under consideration into equation (4.2). The results are given in (i) and (ii). □

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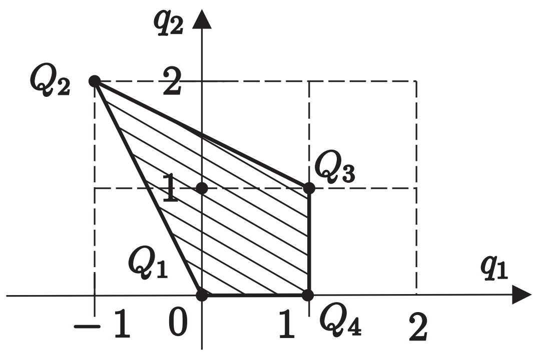

Figure 2. The Newton polygon of equation (5.2) with $\beta\alpha \neq 0$.

5. Proof of theorem 1.1(c)

Proof. Invariant algebraic curves of dynamical system (1.2) with $n = 0$ and $m \neq 0$ satisfy the following partial differential equation

$$yF_x + (mxy + \gamma y + \beta x + \alpha)F_y = K(x, y)F. \tag{5.1}$$

Balancing higher-order terms with respect to y, we find $K(x, y) = K(x)$, $\mu(x) = \mu_0$, where $K(x) \in \mathbb{C}[x]$ and $\mu_0 \in \mathbb{C}$. Without loss of generality, we set $\mu_0 = 1$. Substituting $F(x, y) = F(x)$ into equation (5.1), we see that there are no invariant algebraic curves that do not depend on y.

Let us find other irreducible invariant algebraic curves. The function $y = y(x)$ satisfies the first-order ordinary differential equation

$$yy_x - mxy - \gamma y - \beta x - \alpha = 0 \tag{5.2}$$

with the Newton polygon given in figure 2.

The dominant balances giving power asymptotics at $x = \infty$ correspond to the edges $[Q_2, Q_3]$ and $[Q_3, Q_4]$. These balances and the power solutions take the form

$$(I): \quad yy_x - mxy = 0: \quad y(x) = \frac{m}{2}x^2;$$
$$(II): \quad mxy + \beta x = 0: \quad y(x) = -\frac{\beta}{m}$$

The Fuchs index of the first series is $j_0 = 2$. This series takes the form

$$(I): \quad y(x) = \sum_{k=0}^{+\infty} c_k x^{2-k}, \quad c_0 = \frac{m}{2}$$

and exists whenever $\beta = 0$. This equality results from the compatibility condition for the Fuchs index $j_0 = 2$. Note that the coefficient c_2 is arbitrary. The second series does not have Fuchs indices and consequently possesses uniquely determined

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coefficients

$$(II): \quad y(x) = \sum_{k=0}^{+\infty} a_k x^{-k}, \quad a_0 = -\frac{\beta}{m}.$$

It follows from theorem 2.3 that if $\beta \neq 0$, then the unique irreducible invariant algebraic curve exists whenever series of type (II) terminates at zero term. This gives $m\alpha - \gamma\beta = 0$. The cofactor we calculate by direct substitution of the relation $F(x, y) = y + \beta/m$ into equation (5.2).

Now let us consider the case $\beta = 0$. Irreducible invariant algebraic curves can be represented via the Puiseux series as follows:

$$G(X,Y) = \left\{ \prod_{j=1}^{N-k} \left(y - \frac{m}{2}x^2 - c_1 x - c_{0,\,j} - \cdots \right) (y - a_0 - \cdots)^k \right\}_+,$$

where $N \in \mathbb{N}$ and $k = 0$ or $k = 1$. Setting to zero the coefficients at $y^{N-1}x^l$ with $l < 0$, we see that the unique possibility is $k = 0$ and $\alpha = 0$. The resulting algebraic curve and its cofactor are $F(x, y) = y - mx^2/2 - \gamma x - c_0$ and $K = 0$. Note that c_0 is arbitrary and in fact $F(x, y)$ gives the polynomial first integral. □

6. Proof of theorem 1.2(a)

The proof of statement (a) in theorem 1.2 will be a direct consequence of the following theorem.

THEOREM 6.1. *The unique dynamical systems* (3.1), *which admit a Liouvillian first integral are the ones with the conditions*

$$\sigma_0 = -1, \quad \delta_0 = \frac{(2M-3)(2M+1)}{16}, \quad M \in \mathbb{N}.$$

Proof. Note that if $\sigma_0 = -1$ and $\delta_0 = (2M-3)(2M+1)/16$, in view of theorem 3.2 there exist two distinct irreducible invariant algebraic curves $f_1 = z - \delta_0$ and $f_2 = G(x, y + x^2 + x)$, where the bivariate polynomial $G(s, z)$ is given in the statement of theorem 1.1 and in (1.4). The cofactors take the form $K_1 = 1$ and $K_2 = -2s + M - 3/2$ accordingly. Clearly,

$$V = f_1^{3/2-M} f_2$$

is an inverse integrating function and by theorem 2.2 system (3.1) admits a Liouvillian first integral. The first integral H can be obtained solving (2.2). Doing so, we get that a Liouvillian first integral reads as (1.6).

Now we shall show that in any other case systems (3.1) do not admit a Liouvillian first integral.

Case 1. $\sigma_0 = -1$ and $\delta_0 \neq (2M-3)(2M+1)/16$ for any $M \in \mathbb{N}$. In this case systems (3.1) can have an exponential factor of the form $E = \exp(f/(z - \delta_0)^n)$ with $n > 0$ and f coprime with $z - \delta_0$. We will show that this is not possible.

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Assume that it is the case, then simplifying the exponential term E, we see that f satisfies the following equation

$$(z - s^2 - s)\frac{\partial f}{\partial s} + (z - \delta_0)\frac{\partial f}{\partial z} - nf = (\beta_0 + \beta_1 s + \beta_2 z)(z - \delta_0)^n. \tag{6.1}$$

Set now $\bar{f} = f|_{z=\delta_0}$. Then $\bar{f} \neq 0$ (otherwise f would not be coprime with $z - \delta_0$) and it satisfies equation (6.1) restricted to $z = \delta_0$, i.e.

$$(\delta - s^2 - s)\frac{\mathrm{d}\bar{f}}{\mathrm{d}s} = n\bar{f}.$$

Solving this linear differential equation we conclude that: if $\delta_0 \neq -1/4$ then

$$\bar{f} = \alpha \exp\Big(\frac{-2n}{\sqrt{-1-4\delta_0}} \arctan\Big(\frac{2s+1}{\sqrt{-1-4\delta_0}}\Big)\Big), \quad \alpha \in \mathbb{C} \setminus \{0\}$$

and if $\delta_0 = -1/4$ then

$$\bar{f} = \alpha e^{2n/(1+2s)}, \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

Since $\bar{f}$ must be a polynomial, and $n\alpha \neq 0$ we get a contradiction. So, this case is not possible. In short, the unique possible exponential factors must be of the form $E = \exp(g)$ with $g \in \mathbb{C}[x, y]$. Moreover, by theorems 2.1 and 2.2, if there is a Liouvillian first integral then the cofactor must be of the form $2s - \lambda$ with $\lambda \in \mathbb{C}$. So,

$$(z - s^2 - s)\frac{\partial g}{\partial s} + (z - \delta_0)\frac{\partial g}{\partial z} = 2s - \lambda.$$

Evaluating it on $z = \delta_0$ yields

$$(\delta - s^2 - s)\frac{\partial g}{\partial s}\Big|_{z=\delta} = 2s - \lambda$$

which is not possible because g is a polynomial. This concludes the proof in this case.

Case 2. $\sigma_0 = 4\delta - 1/4$ *with* $\delta_0 \neq -3/16$. In this case system (3.1) can have an exponential factor of the form $E = \exp(f/(s^2 + 3s/2 - z - 2\delta_0)^n)$ with $n > 0$ and f coprime with $s^2 + (3/2)s - z - 2\delta_0$. If it exists, then f satisfies, after simplifying the exponential term E, the equation

$$\begin{aligned}(z - s^2 - s)\frac{\partial f}{\partial s} + \Big(z - \Big(4\delta_0 + \frac{3}{4}\Big)s - \delta_0\Big)\frac{\partial f}{\partial z} - n\Big(2s + \frac{1}{2}\Big)f \\ = (\beta_0 + \beta_1 s + \beta_2 z)\Big(s^2 + \frac{3}{2}s - z - 2\delta_0\Big)^n.\end{aligned}$$

Set now $\bar{f} = f|_{z=s^2+(3/2)s-2\delta_0}$. Then $\bar{f} \neq 0$ and it satisfies

$$\Big(\frac{1}{2}s - 2\delta_0\Big)\frac{\mathrm{d}\bar{f}}{\mathrm{d}s} = -n\Big(2s + \frac{1}{2}\Big)\bar{f}.$$

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Solving this linear differential equation we get

$$\bar{f} = \alpha(s - 4\delta_0)^{-n(1+16\delta_0)} \exp(-4ns), \quad \alpha \in \mathbb{C} \setminus \{0\},$$

what is not possible because $\bar{f}$ is a polynomial and $n\alpha \neq 0$. So, the unique possible exponential factors must be of the form $E = \exp(g)$ with $g \in \mathbb{C}[x, y]$. Moreover, by theorems 2.1 and 2.2, if there is a Liouvillian first integral then the cofactor must be of the form $2s - \lambda$ with $\lambda \in \mathbb{C}$. Hence,

$$(z - s^2 - s)\frac{\partial g}{\partial s} + \left(z - \left(4\delta_0 + \frac{3}{4}\right) s - \delta_0\right)\frac{\partial g}{\partial z} = 2s(1 - \lambda) - \frac{\lambda}{2}.$$

Evaluating it on $z = (4\delta_0 + 3/4)s - \delta_0$ yields

$$\left(\left(4\delta_0 - \frac{1}{4}\right)s - \delta_0 - s^2\right)\frac{\partial g}{\partial s}\bigg|_{z=(4\delta_0+3/4)s-\delta_0} = 2s(1 - \lambda) - \frac{\lambda}{2},$$

which is not possible because g is a polynomial. This concludes the proof in this case.

Case 3. $\sigma_0 \notin \{-1, 4\delta_0 - 1/4\}$. The unique possible exponential factors must be of the form $E = \exp(g)$ with $g \in \mathbb{C}[x, y]$. Moreover, by theorems 2.1 and 2.2, if there is a Liouvillian first integral then the cofactor must be of the form $2s$. So,

$$(z - s^2 - s)\frac{\partial g}{\partial s} + (z - (1 + \sigma_0)s - \delta_0)\frac{\partial g}{\partial z} = 2s.$$

Evaluating the above equation on $z = (1 + \sigma)s + \delta_0$, we get

$$(\delta - s^2 + \sigma_0 s)\frac{\partial g}{\partial s}\bigg|_{z=(1+\sigma_0)s+\delta_0} = 2s,$$

which is not possible because g is a polynomial. This concludes the proof. □

7. Proof of theorem 1.2(b)

The proof of statement (b.1) is clear. To the complete proof of statement (b) in theorem 1.2 will be a direct consequence of the following theorem.

THEOREM 7.1. *The unique dynamical systems* (4.1) *which admit a Liouvillian first integral are the ones with the conditions* $\sigma_1 = \pm 6\gamma^2/25$.

Proof. In view of theorem 1.1(b) if $\sigma_1 = 6\gamma^2/5$ there exists an irreducible invariant algebraic curve $G_1(s, z)$ with G_1 given in statement (b.2) of theorem 1.1. The cofactor is of the form $K = 6\gamma/5$. Clearly,

$$V = G_1^{-5\gamma/6}$$

is an inverse integrating function and by theorem 2.2 system (3.1) admits a Liouvillian first integral. The first integral H can be obtained solving (2.2). Doing so, we get that a Liouvillian first integral is given in (1.7).

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Analogously, in view of theorem 1.1(b) if $\sigma_1 = -6\gamma^2/5$ there exists an irreducible invariant algebraic curve $G_2(s, z)$ with G_2 given in statement (b.3) of theorem 1.1. The cofactor is of the form $K = 6\gamma/5$. Clearly,

$$V = G_2^{-5\gamma/6}$$

is an inverse integrating function and by theorem 2.2 system (3.1) admits a Liouvillian first integral. The first integral H can be obtained solving (2.2). Doing so, we get that a Liouvillian first integral is given in (1.8).

Now we shall show that in any other case there are no Liouvillian first integrals. In these other cases, the unique possible exponential factors must be of the form $E = \exp(g)$ with $g \in \mathbb{C}[X, Y]$. Moreover, by theorems 2.1 and 2.2, if there is a Liouvillian first integral then the cofactor must be of the form $-\gamma$. So,

$$Y\frac{\partial g}{\partial X} + (\gamma Y + 6X^2 - \sigma_1 X)\frac{\partial g}{\partial Y} = -\gamma.$$

Evaluating the above equation on $Y = X = 0$ we get $0 = \gamma$, which is not possible. This concludes the proof. □

8. Proof of theorem 1.2(c)

In view of theorem 1.1(c) if $\alpha = \beta = 0$, the polynomial $y - mx^2/2 - \gamma x - c_0$ is a first integral. This proves statement (c.2).

If $\alpha = \beta\gamma/m$ with $\beta \neq 0$ then there exists the irreducible invariant algebraic curve $y + \beta/m = 0$. It turns out that $V = y + \beta/m$ is an inverse integrating factor and by theorem 2.2 system (3.1) admits a Liouvillian first integral. Solving (2.2) we obtain the first integral in (1.9) proving statement (c.2).

Finally, to conclude the proof of theorem 1.2(c) we shall see that there are no Liouvillian first integrals in any other case. Note that these other cases, the unique possible exponential factors must be of the form $E = \exp(g)$ with $g \in \mathbb{C}[x, y]$. Moreover, by Theorems 2.1 and 2.2, if there is a Liouvillian first integral then the cofactor must be of the form $mx + \gamma$. So,

$$y\frac{\partial g}{\partial x} + (\alpha + \beta x + \gamma y + mxy)\frac{\partial g}{\partial Y} = -mx - \gamma.$$

Evaluating the above equation on $y = 0$, $x = -\alpha/\beta$ we get $0 = m\alpha/\beta - \gamma$, which is not possible because $\alpha \neq \beta\gamma/m$. This concludes the proof.

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