COMPLETIONS OF QUADRANGLES IN PROJECTIVE PLANES

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1. Introduction. This paper discusses projective planes from the viewpoint of their classification into singly-generated and non-singly-generated planes. (Singly-generatedness, a property explicit in Hall **(6)** and Wagner **(19)**, and implicit in Hughes **(10)**, is defined in Section 5.) The elements (points and lines) of a singly-generated plane are expressible in four basic points called a quadrangle. A "completion procedure" enables us to obtain expressions for the elements of a plane.

In the case of a non-singly-generated plane, the application of the completion procedure yields expressions for a proper subplane instead of for the whole plane. Thus every non-singly-generated plane is the union of singly-generated proper subplanes. The main theorem (in Section 6) of this paper is that for planes of order less than twelve, a plane is non-singly-generated only if it is Desarguesian.

2. Free and restricted extensions of a quadrangle. The axioms of a *partial plane* given by Hall **(4)** state that we have a symmetric incidence relation between the set Σ of points and the set σ of lines such that:

I. There is at most one line of σ incident with two distinct points of Σ.

II. There is at most one point of Σ incident with two distinct lines of σ.

Suppose π is a partial plane whose elements belong to $\Sigma \cup \sigma$. We can obtain a new partial plane π' whose elements belong to $\Sigma' \cup \sigma'$, $\Sigma \subset \Sigma'$, $\sigma \subset \sigma'$, by a free extension of π (see Hall, **4**). There are two basic types of free extensions; one extension adds only points and the other adds only lines. A *free extension* of π is obtained by either (i) adding *for each pair of distinct points P, Q* a new line m' when and only when there is *no* line m on which both P and Q lie, or (ii) adding *for each pair of distinct lines p, q* a new point M' when and only when there is *no* point M lying on both p and q. The original incidence relation is extended to the new elements so that it remains symmetric and so that each of the new elements is defined to be incident to precisely the two dual elements which caused it to exist.

When we say *free extensions* of a partial plane π, we mean the partial planes

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$\{\pi_i\}$ where $\pi = \pi_0$ and π_{i+1} is a free extension of π_i; in fact, the extensions to odd-numbered partial planes will be here, by convention, additions of lines and to even-numbered planes, additions of points.

A *free plane* according to Hall **(4)** is the complete free extension of π_0, defined as

$$\bigcup_{i=0}^{\infty} \pi_i,$$

where the π_i are free extensions of π_0 and so that in some partial plane π_{i_0} there exists a set of four points such that no three of them are collinear. It is easy to verify that the free plane satisfies the axioms of projective geometry given by Hall **(7**, p. 346**)**.

Let us consider now the partial plane π_0 consisting of four distinct points: $\pi_0 : \{A, B, C, D\}$ and its free extension $\pi_1 : \{AB, AC, AD, BC, BD, CD\} \cup \pi_0$. The partial plane π_1 is also known as the *complete quadrangle* **(**see Coxeter **2)** whereas the partial plane π_0 is *here* called a *quadrangle.*

The free extension of π_1 is $\pi_2 : \{E = AB \cap CD, F = AC \cap BD, G = AD \cap BC\} \cup \pi_1$ and its extension is $\pi_3 : \{EF, FG, EG\} \cup \pi_2$. The new elements of π_2 and π_3 introduce respectively the *vertices* and *sides* of the *diagonal triangle* of the complete quadrangle **(2)**.

It is interesting to note that π_3 has seven points and nine lines while the smallest projective plane, the *Fano configuration* **(7**, p. 405**)**, consists of seven points and seven lines. Let us apply this free extension procedure to the Fano configuration and explain this difference. Let the Fano configuration consist of $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ with lines $P_1P_2P_3, P_1P_4P_5, P_1P_6P_7, P_2P_4P_6, P_2P_5P_7, P_3P_4P_7, P_3P_5P_6$. Now set $A = P_1$, $B = P_2$, $C = P_4$, $D = P_7$. Thus $E = P_3$, $F = P_5$, $G = P_6$. However, in this case EF, FG, EG are the same line $P_3P_5P_6$; hence the diagonal triangle is "degenerated." In this case, π_3 is replaced by a smaller set π_3'. In fact, we could obtain π_3' by just adding the restricting condition $EF = EG$. Thus π_3' is not a free extension but instead is a restricted extension due to identifications expressed by identities imposed upon the free extension elements. The identities can be imposed in order to develop new planes or, as in this case, they can be forced by the embedding of the extension procedure in an existing plane.

3. Expressions for the free extensions. Let us consider again the free extensions of a quadrangle. Each element of this free plane will be identified by the simplest possible expressions in terms of A, B, C, D, juxtaposition, and $\cap$. These unique canonical expressions will be called the free plane *polynomials.*

In order to define polynomials, let us find out which expressions reduce automatically to simpler expressions. If the expression is a single letter, then the expression is a polynomial. If the expression is not a single letter, then in order to be a valid expression of an element, it must either consist of two subexpressions representing points and in juxtaposition or two subexpressions representing lines and joined by $\cap$. The expression will be a polynomial provided that these two subexpressions are polynomials and if the expression describes a new element produced in an extension by the elements described by the subexpressions. Therefore, it is desirable to have a method to discover whether two polynomials placed in juxtaposition or joined by $\cap$ give a new element, or an element already present.

Every element, except the original four points A, B, C, D, came from two dual elements which we shall call the *parents* (this name was suggested by T. S. Motzkin). Let us recall that the parents have only one *offspring*, the element they produce. If two elements have a common parent, then they will not together have offspring, since this common parent is the dual element incident with these two elements. Also an element and one of its "grandparents" cannot together have offspring, since a certain parent of the given element is already incident with it and the grandparent.

We have shown two situations in which two elements of the same type cannot produce offspring. Now we set out to show that these are the only such situations. Suppose we have two elements which already have a dual element incident with both of them. In the first place an element is incident with a dual element in this free plane if and only if one of these is the parent and the other is the offspring. This follows immediately from the way incidence is defined in the free plane and from our choice of four points for the original partial plane. Thus now we consider how this dual element is related to the original two elements. Clearly this dual element is not an offspring to both, since this is the situation where a *new* element is produced. If this dual element is an offspring of one of the elements and a parent of the other, then we have the second situation mentioned above. Finally, if this dual element is a parent of both elements, then we have the first situation.

Using the rules, we now generate the first six partial planes of the free plane. The number in parenthesis indicates the number of new elements in a partial plane:

$\pi_0 : \{A, B, C, D\} \quad (4);$

$\pi_1 : \{AB = a,\ AC = b,\ AD = c,\ BC = d,\ BD = e,\ CD = f\} \cup \pi_0 \quad (6);$

$\pi_2 : \{AB \cap CD = E,\ AC \cap BD = F,\ AD \cap BC = G\} \cup \pi_1 \quad (3);$

$\pi_3 : \{(AB \cap CD)(AC \cap BD) = EF = g,\ EG = h,\ FG = i\} \cup \pi_2 \quad (3);$

$\pi_4 : \{AB \cap FG = H,\ b \cap h = I,\ c \cap g = J,\ d \cap g = K,\ e \cap h = L,$
$f \cap i = M\} \cup \pi_3 \quad (6);$

$\pi_5 : \{j = CH, k = DH, l = HI, m = HJ, n = HK, o = HL, p = BI, q = DI,$
$r = IJ,\ s = IK,\ t = IM,\ u = BJ,\ v = CJ,\ w = JL,\ x = JM,\ y = AK,$
$z = DK,\ \alpha = KL,\ \beta = KM,\ \gamma = AL,\ \delta = CL,\ \epsilon = LM,\ \zeta = AM,$
$\eta = BM\} \cup \pi_4 \quad (24).$

These results are well known and appear in many places; for example, this is found in Hall **(5)** and in the works of Lombardo-Radice **(12, 13)**. Maisano **(14)** generalizes these results in two ways: first, in counting the elements in

π_n for any n, and second, in considering other initial planes for π_0 than the quadrangle.

Let us consider a plane of order 3 with the usual affine co-ordinates and the notation of Hall for ideal points (**7**, p. 353). Let us develop the points and lines as follows: $A:(\infty)$, $B:(0)$, $C:(0,0)$, $D:(1,1)$. Then $AB:l_\infty$, $AC:x=0$, $AD:x=1$, $BC:y=0$, $BD:y=1$, $CD:y=x$, $E:(1)$, $F:(0,1)$, $G:(1,0)$. Then we have $EF:y=x+1$, $EG:y=x+2$, $FG:y=2x+1$, $H:(2)$, $I:(0,2)$, $J:(1,2)$, $K:(2,0)$, $L:(2,1)$, $M:(2,2)$. This shows that the first thirteen points of the free plane correspond to distinct points of the plane of order 3. Now one can discover that the remaining lines of the geometry are $AKLM:x=2$, $BIJM:y=2$, $CHJL:y=2x$, $DHIK:y=2x+2$. These results can also be found in the works of Lombardo-Radice. We shall use these facts in Section 8 of this paper.

4. Identifications in the free plane. Now we consider the problem of making identifications in the free plane. The procedure is as follows: (1) we start with the partial plane $\pi_0 = \pi_0^{**}$; (2) from a partial plane π_i^{**} we obtain a partial plane π_i^* by making some identifications (possibly none) between elements of π_i^{**}; (3) from a partial plane π_i^*, we obtain the partial plane π_{i+1}^{**} as a free extension of π_i^*. To simplify notation, however, we shall signify π_i^{**} by π_i; in fact, most of the time $\pi_i^{**} = \pi_i$ (the free plane) anyway, and when this identity does not occur, the context will serve to clarify which meaning of π_i is intended. Then we write $\pi_i' = \pi_i - \pi_{i-1}$, i.e. π_i' is the set of new elements of π_i.

In order to describe *acceptable identifications* at any stage in which identifications are made, the following convention is enforced. In making an identification ρ between some element of π_i' and some element of π_i, there may result forced identifications among elements of π_j, $j < i$. If any of these forced identifications are not used in going from π_j $(=\pi_j^{**})$ to π_j^*, then ρ is not an acceptable identification and will not be used in going from π_i to π_i^*. In particular, any forced identifications between elements of π_j which force ρ to occur cannot be used in going from π_j to π_j^*.

Let $\mathbf{A} \in \pi_i'$, $\mathbf{B} \in \pi_j'$, $j < i$, and j and i have the same parity. Let the parents of $\mathbf{A}$ be α and β, and let the parents of $\mathbf{B}$ be γ and δ. Then we want to show that the identification $\mathbf{A} = \mathbf{B}$ is not an acceptable identification. First, let us assume $\beta = \gamma$. Then the identification can be written as $\alpha\gamma = \gamma\delta$. Let $\alpha = \lambda\chi \cap \rho\mu$ (or $(\lambda \cap \chi)(\rho \cap \mu)$). Also let ω be incident with elements $\lambda\chi$ and $\gamma\delta$; let Ω be incident with elements $\rho\mu$ and $\gamma\delta$. Now we notice that $\alpha\gamma = \gamma\delta$ implies $\alpha = \omega = \Omega$. On the other hand $\alpha = \omega = \Omega$ implies $\alpha\gamma = \gamma\delta$. Since α, ω, Ω are in π_k, $k < i$, $\alpha\gamma = \gamma\delta$ $(\mathbf{A} = \mathbf{B})$ is not an acceptable identification.

Now suppose α, β, γ, δ are all distinct. Then $\alpha\beta = \gamma\delta$ $(\mathbf{A} = \mathbf{B})$ implies $\alpha\gamma = \gamma\delta$ and $\beta\gamma = \gamma\delta$. By the above arguments for $\alpha\gamma = \gamma\delta$, it follows that this implies identifications ρ_1 and ρ_2, respectively, in lower-numbered partial planes. Also, by the same arguments, ρ_1 and ρ_2 imply $\alpha\gamma = \gamma\delta$ and $\beta\gamma = \gamma\delta$ respectively. Finally $\alpha\gamma = \gamma\delta$ and $\beta\gamma = \gamma\delta$ imply $\alpha\beta = \gamma\delta$. Therefore $\alpha\beta = \gamma\delta$ ($\mathbf{A} = \mathbf{B}$) is not an acceptable identification.

Now let us consider the degenerate cases for the above argument. We assumed that $\mathbf{B}$, α, and β had parents. (The grandparents of α and β were used to clarify which elements were of the same type and which elements were dual to each other; the existence of grandparents is not essential to the argument.) First, suppose that $\mathbf{B}$ is an element of π_0. In this case $\mathbf{A}$ would be defined as $PQ \cap RS$ for some points P, Q, R, S. Then $\mathbf{A} = \mathbf{B}$ implies that $\mathbf{B} \in PQ$ and $\mathbf{B} \in RS$. If $\mathbf{B} \in PQ$ and $\mathbf{B} \in RS$ in the free plane, then $\mathbf{A}$ would not have been produced. Otherwise the identifications $BP = PQ$ and $\mathbf{B}R = RS$ are implied by $\mathbf{A} = \mathbf{B}$. On the other hand these forced identifications imply $\mathbf{A} = \mathbf{B}$, thus $\mathbf{A} = \mathbf{B}$ is not an acceptable identification.

If $\mathbf{B}$ has parents, then $j \geqslant 1$, and $i \geqslant 3$. At least one of α and β lies in π'_{i-1}; let α be that element. Then α has parents. Now suppose β does not have parents, then $\beta \in \pi_0 \neq \pi'_{i-1}$. Now, $\mathbf{A} = \mathbf{B}$ is $\alpha\beta = \gamma\delta$, which implies $\alpha\gamma = \gamma\delta$ and $\beta\gamma = \gamma\delta$. Now $\alpha\gamma = \gamma\delta$ implies an identification ρ_1 in some partial plane π_k, $k < i$; while $\beta\gamma = \gamma\delta$ is an identification in some partial plane $\pi_{k'}$, $k' < i$. Now ρ_1 implies $\alpha\gamma = \gamma\delta$, and $\alpha\gamma = \gamma\delta$ and $\beta\gamma = \gamma\delta$ imply $\alpha\beta = \gamma\delta$. Therefore $\mathbf{A} = \mathbf{B}$ is not an acceptable identification in this case. Therefore, we have completed the proof of the following theorem.

THEOREM 4.1. *If* $\mathbf{A} \in \pi_i'$ *and* $\mathbf{B} \in \pi_j'$ *and* $\mathbf{A} = \mathbf{B}$ *is an acceptable identification, then* $i = j$.

Any identifications between elements of π_1' or π_2' produce degenerate partial planes in which there is no set of four points with all subsets of three noncollinear. Any identification between elements of π_3' forces the Fano configuration. Assuming no identifications in π_3', it is easy to see there are no allowable identifications in π_4'. It turns out that π_5' is the first place where identifications are made. In particular, Desargues' theorem forces identifications between elements of π_5'. At this time, the author knows of no plane of finite order in which identifications do not occur between elements of π_5'.

5. Some theorems old and new about singly-generatedness. Whenever a plane has the property that there exist four points such that *all* the elements can be described in expressions involving only the four points, then we say the plane is *singly-generated.* In Section 2 we demonstrated that the plane of order 2 is singly-generated. On the other hand, if one takes any four points (no subset of three collinear) of the geometry of order four, the set of elements obtained by the restrictive extension scheme forms a proper subplane of order two. Thus we have an example of a *non-singly-generated* plane.

THEOREM 5.1. *A Desarguesian plane of order* p^α *is singly-generated if and only if* $\alpha = 1$.

Remark. Singly-generated Desarguesian planes are Moebius nets. Pickert (**16**, p. 139) states that they exist, up to isomorphisms, only as the classical *rational net* (**2**) and the planes of prime order.

CONJECTURE. *Every finite non-Desarguesian plane is singly-generated.*

THEOREM 5.2. *There exists a non-singly-generated non-Desarguesian geometry.*

Proof. First we wish to establish that the number of elements of a singly-generated geometry does not exceed the cardinality of the integers. To do this, we note that every expression of the restricted extension scheme uses only seven distinct symbols. Thus if we let these seven symbols be the marks for writing integers in base seven, the number of expressions does not exceed the number of integers.

We choose to use an affine plane instead of the corresponding projective plane. An affine Desarguesian plane may be characterized by the fact that every co-ordinatization of it can be done by pairs of elements from a skew field (Segre, **17**; Bruck, **1**). So we shall now choose a non-associative ring which will co-ordinatize an affine plane and have more than a countable number of elements.

Consider the Cayley algebra over the real numbers. The points for the geometry will be pairs (x, y) of elements of the algebra, and lines will be equations of the form $x = k$, and $y = xm + b$. In the system addition defines an Abelian group. Furthermore, both distributive laws hold. Also there are inverses for non-zero elements so that multiplication is a loop. From this we have enough to guarantee the existence of a plane (**7**, p. 362).

THEOREM 5.3. *The dual plane of a singly-generated plane is also singly-generated.*

Proof. Suppose π is a singly-generated plane such that every one of its elements can be written as a polynomial involving A, B, C, D. Suppose the dual plane π' is not singly-generated and, therefore, any four of its points A', B', C', D' generate a proper subplane of π'. In particular, consider the A' of π' to be the $a = AB$ of π, B' of π' to be $d = BC$ of π, C' of π' to be $f = CD$ of π, and D' of π' to be $c = AD$ of π. Then $A'B'$ of π' is B of π, $B'C'$ of π' is C of π, $C'D'$ of π' is D of π, $D'A'$ of π' is A of π. Now consider an element P' of π' which is not a member of the proper subplane generated by A', B', C', D'. Then there is a line p of π associated with the point P' of π'. Now p can be expressed as a polynomial of A, B, C, D. This means that P' can be expressed in terms of $A'B'$, $B'C'$, $C'D'$, $D'A'$ or equivalently as an expression involving A', B', C', D'. Since P' is not in the proper subplane generated by A', B', C', D' no such expression should exist. This contradiction shows that A', B', C', D' singly-generate π'.

Finally we have a corollary to the following theorem of A. Wagner (**20**):

WAGNER'S THEOREM. *A projective plane whose collineation group is transitive on quadrangles is Desarguesian.*

Now, if two ordered quadrangles A, B, C, D and A', B', C', D' singly-generate the plane and if all the identifications of the plane in terms of the first quadrangle agree with the identifications made between elements generated by the second quadrangle, then there is an element α of the collineation group such that $\alpha A = A'$, $\alpha B = B'$, $\alpha C = C'$, $\alpha D = D'$. Now if every quadrangle completes the plane in the same way, i.e., every identification scheme is isomorphic to every other scheme, then the collineation group is transitive on quadrangles and the plane is Desarguesian.

6. Planes whose order is less than twelve. It has been shown that for orders $n = 2, 3, 4, 5, 7$, and 8 the Desarguesian planes are the only type of plane **(5, 6, 9)**. There is no geometry of order 6 **(18)**. For $n = 9$, we have four known geometries: the Desarguesian plane, the Hughes plane **(7)**, the Hall plane **(4)**, and the dual of the Hall plane. No geometries are known for order 10, and no geometry besides the Desarguesian one is known for order 11.

GLEASON'S THEOREM **(3)**. *Every finite plane in which every quadrangle can be completed to a Fano configuration is a Desarguesian plane of order* 2^α.

BRUCK'S THEOREM **(7**, p. 398**)**. *If a plane of order n has a subplane of order m, then either $n = m^2$ or $n \geqslant m^2 + m$.*

LEMMA 6.1. *The only plane of order 9 which is not singly-generated is the Desarguesian one.*

The proof of this lemma will be given in the next section.

THEOREM 6.1. *Every non-Desarguesian plane of order less than 12 is singly-generated.*

Proof. The introductory remarks of this section indicate that we need only consider the orders 9, 10, and 11. Lemma 6.1 takes care of order 9. Suppose there is a non-singly-generated plane of order 10 or 11. Then by Bruck's theorem, there are no subplanes of order three or any higher order. Therefore, all subplanes are Fano configurations, but because the plane is non-singly-generated, every quadrangle completes to a Fano configuration. Now by Gleason's theorem, these planes would be Desarguesian planes (and actually do not exist).

7. Proof of Lemma 6.1. By Bruck's theorem, all subplanes are of order 2 or of order 3. We begin by characterizing these subplanes as they appear in the digraph complete set of Latin squares associated with a finite plane **(15)**. Also we use the notation of Hall for ideal elements **(7**, p. 353**)**.

Suppose we wish to generate a subplane from (0), (∞), (x_1, y_1), (x_2, y_2). First, it is imperative that no three of these are incident with the same line. Clearly the points (x_1, y_1), (x_2, y_2) of the affine plane are not incident with $l_\infty = (0)(\infty)$. Also, we do not allow $x_1 = x_2$ or $y_1 = y_2$, so that neither (0) nor (∞) is incident with the line $(x_1, y_1)(x_2, y_2)$. The points incident with the lines $(0)\ (x_i, y_i)$, $i = 1, 2$, are (a, y_i), and those incident with the lines $(\infty)(x_i, y_i)$ are (x_i, b). In addition, (x_1, y_1) and (x_2, y_2) are incident with a line which is represented by a row in one of the Latin squares. In this square there is also a line incident with (x_1, y_2) and (x_2, z). Thus we have:

$$
\begin{array}{ccccccccc}
- & - & - & - & - & - & - & - & - \\
- & - & y_1 & - & - & y_2 & - & - & - \\
- & - & - & - & - & - & - & - & - \\
- & - & y_2 & - & - & z & - & - & - \\
- & - & - & - & - & - & - & - & -
\end{array}
$$

This Latin square is associated with a slope m and the point (m) is, of course, incident with $(0)(\infty)$ and $(x_1, y_1)(x_2, y_2)$.

The subplane generated by (0), (∞), (x_1, y_1), (x_2, y_2) is a Fano configuration if and only if $z = y_1$. If we have a Fano configuration, the three points (i) $(0)(\infty) \cap (x_1, y_1)(x_2, y_2) = (m)$, (ii) $(0)(x_1, y_1) \cap (\infty)(x_2, y_2) = (x_2, y_1)$, (iii) $(\infty)(x_1, y_1) \cap (0)(x_2, y_2) = (x_1, y_2)$ must be incident with the same line. But (x_1, y_2), (x_2, z), and (m) are incident with the same line; hence $z = y_1$. On the other hand, $z = y_1$ allows us to build the Fano configuration from (x_1, y_1), (x_2, y_2), (x_1, y_2), (x_2, y_1) by forming the lines which must occur: $(x_1, y_1)(x_2, y_2)(m)$, $(x_1, y_2)(x_2, y_1)(m)$, $(x_1, y_1)(x_1, y_2)(\infty)$, $(x_2, y_1)(x_2, y_2)(\infty)$, $(x_1, y_1)(x_2, y_1)(0)$, $(x_1, y_2)(x_2, y_2)(0)$, and $(m)(0)(\infty)$.

If $z \neq y_1$ and if we stipulate that the plane generated by (0), (∞), (x_1, y_1), (x_2, y_2) is of order 3, then an immediate characterization is available. Now $(0)(x_2, z) \cap (x_1, y_1)(x_2, y_2)$ is a point (x_3, z) by the nature of our co-ordinates. Also, $(x_1, y_2)(x_2, z) \cap (\infty)(x_3, z)$ is (x_3, w). Furthermore, $(0)(x_1, y_1) \cap (x_1, y_2)(x_2, z)$ is (x_4, y_1). Now we have five points incident with a line: (m), (x_1, y_2), (x_2, z), (x_3, w), and (x_4, y_1); hence, two of these must be the same. Since (m) does not pair up with any of the others and x_1, x_2, and x_3 are all distinct, the only possible alike pair is (x_3, w) and (x_4, y_1), which should then be called (x_3, y_1). The lines $x = x_1$ and $y = z$ are incident with the point (x_1, z). We may now obtain (x_2, y_1) and (x_3, y_2). Thus we have the nine distinct points of the affine plane. Furthermore, the last two points mentioned are the only ones available for the lines determined by (m) and (x_1, z). Thus we have:

$$
\begin{array}{ccccccccc}
- & - & - & - & - & - & - & - & - \\
- & - & y_1 & - & y_2 & - & - & z & - \\
- & - & - & - & - & - & - & - & - \\
- & - & y_2 & - & z & - & - & y_1 & - \\
- & - & - & - & - & - & - & - & - \\
- & - & z & - & y_1 & - & - & y_2 & - \\
- & - & - & - & - & - & - & - & -
\end{array}
$$

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LEMMA 7.1. *The planes of order* 9 *which have the elementary Abelian group table for the additive loop are either Desarguesian or they are singly-generated.*

Proof. According to a recent result by Hall, Swift, and Killgrove **(4)**, any plane that has the elementary Abelian group table for the additive loop must be one of the four known planes of order 9.

Now let us consider a representative for each plane other than the Desarguesian plane. First, the Hughes plane has the following Latin square **(8**; 1.44.1.1**)**:

0	6	3	1	5	8	2	4	7
1	7	4	2	3	6	0	5	8
2	8	5	0	4	7	1	3	6
<u>3</u>	0	6	<u>8</u>	1	5	7	2	4
4	1	7	6	2	3	8	0	5
5	2	8	7	0	4	6	1	3
6	3	0	5	8	1	4	7	2
<u>7</u>	4	1	<u>3</u>	6	2	5	8	0
<u>8</u>	5	2	<u>4</u>	7	0	3	6	1

Now (0, 3) and (3. 8) (the ones underlined in the fourth row above) fail to generate a subplane of order three or two. By Bruck's theorem all proper subplanes of a plane of order 9 must be of order 2 or 3. Therefore, this plane is singly-generated.

Let us consider the Hall plane which has the following Latin square **(8**; 1.44.6**)**:

0	<u>4</u>	<u>8</u>	5	2	6	7	3	1
1	<u>7</u>	<u>4</u>	2	3	8	6	5	0
2	<u>8</u>	<u>5</u>	3	0	7	1	4	6
3	1	6	4	8	2	5	0	7
4	0	7	1	6	5	3	8	2
5	2	3	0	7	1	4	6	8
6	3	2	7	5	0	8	1	4
7	6	1	8	4	3	0	2	5
8	5	0	6	1	4	2	7	3

Now (1, 4) and (2, 8) (the ones underlined in the first row above) fail to generate a subplane of order 3 or of order 2. Therefore, this plane is singly-generated. By Theorem 5.3, the remaining plane which is dual to this one must also be singly-generated, which proves the lemma.

The method of proof for Lemma 6.1 is now clear. We must show that a non-singly-generated plane will have precisely the elementary Abelian group table for the additive loop. Therefore, our procedure is to obtain contradictions in attempting the construction of other types of Latin squares for the lines with slope 1. By permuting the rows and columns of such Latin squares, we can always make the first row and first column be the digits 0 through 8 in order from left to right and downward. Gleason's theorem shows that there must be at least one subplane of order 3. Without loss of generality, it can have the affine co-ordinates involving only 0, 1, 2. With this convention, with the idea of forcing the second row to be different from that of the elementary Abelian group, we obtain (Case 1):

$$\begin{array}{ccccccccc} 0 & 1 & 2 & \underline{3} & 4 & \underline{5} & 6 & 7 & 8 \\ 1 & 2 & 0 & 4 & \underline{\underline{3}} & \underline{\underline{6}} & 5 & 8 & 7 \\ 2 & 0 & 1 & \underline{5} & \underline{\underline{a}} & b & c & d & e \end{array}$$

By the Latin square property, $a = 6$ (Case 1.1) or $a = 7$ (Case 1.2.) If $a = 6$ and $b = 3$ (Case 1.1.1) then two Fano configurations $(3, 3)$, $(5, 5)$, $(3, 5)$, $(5, 3)$ (underlined once) and $(4, 3)$, $(5, 6)$, $(4, 6)$, $(5, 3)$ are produced. Then, by the Latin square property, $d \neq 7, 8$ and $e \neq 7, 8$, but $a \neq 7, 8$ and $b \neq 7, 8$. This gives us a contradiction since the remaining c cannot accommodate both 7 and 8. Therefore, in Case 1.1.2 we can assume $b = 7$ and $c = 8$. Then $e = 3$, since we want $(4, 3)$, $(4, 6)$, $(5, 6)$, $(5, 7)$, $(8, 7)$ to be points of a plane of order 3. In this case, $(3, 3)$, $(5, 5)$, $(3, 5)$, $(8, 8)$, $(8, 3)$ shows that b should be 8 instead of 7. This contradiction eliminates Case 1.1.2 and hence Case 1.1 also.

If $a = 7$, then $b = 8$ (Case 1.2.1) or $c = 8$ (Case 1.2.2). If $b = 8$, then $e = 3$ in order to obtain the subplane of order 3 involving $(3, 3)$, $(5, 5)$, $(8, 8)$, $(3, 5)$, $(5, 8)$. By the Latin square property, $d = 6$ as 6 has no other place to go. Finally, $c = 4$. If $c = 8$, then $d = 4$ in order to obtain the subplane of order 3 involving $(3, 4)$, $(3, 5)$, $(6, 5)$, $(6, 8)$, $(7, 8)$. By the Latin square property $e = 6$ and then $b = 3$. It can be shown, by properly switching rows and columns and by renaming the elements of either Case 1.2.1 or Case 1.2.2, that we obtain (Case 2):

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 0 & \underline{4} & 5 & \underline{3} & 7 & 8 & \underline{6} \\ 2 & 0 & 1 & \underline{f} & g & h & i & j & \underline{k} \end{array}$$

Thus we have proved the following lemma.

LEMMA 7.2. *Any non-singly-generated plane of order* 9 *has an additive loop whose table agrees with the elementary Abelian group in the first two rows.*

The following six equalities are equivalent in that if any one of them holds, they all hold: $f = 5$, $g = 3$, $h = 4$, $i = 8$, $j = 6$, $k = 7$. If we assume that $f \neq 5$, then by proper naming and permuting of the last three columns, we can say that $f = 6$ (Case 2.1). Now $g = 7$ or $g = 8$. If $g = 7$ (Case 2.1.1), $h = 8$, and $k = 3$ or 4 or 5. If (in Case 2.1.1) $k = 3$, then we obtain a contradiction since the subplane elements underlined indicate that $h = 4$. Similarly, if $k = 4$, then $(4, 4)$, $(8, 8)$, $(8, 4)$ fail to generate a plane of order 2 or of order 3. Finally, if $k = 5$, then $(3, 4)$, $(3, 6)$, $(8, 6)$, $(8, 5)$ fail to complete to a subplane of order 3. Now if $g = 8$ (Case 2.1.2), $h = 7$, and $k = 3$ or 4 or 5. If $k = 3$, then $(3, 3)$, $(8, 8)$, $(3, 6)$, $(8, 3)$ fail to complete to a subplane. If $k = 5$, then we obtain a contradiction similarly. If $k = 4$ *and* $j = 3$, then we have a contradiction. Therefore, we have as the only possibilities of Case 2.1:

Case 2.1.2.1

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	6	8	7	3	5	4
3	l	m	0	n	p	q	r	s

Case 2.1.2.2

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	6	8	7	3	5	4
3	t	u	7	v	w	x	0	y

Case 2.1.2.3

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	6	8	7	3	5	4
3	z	α	8	β	γ	δ	ϵ	0

Now $w = 1$ or $w = 6$. If $w = 6$, then $y = 1$ and $v = 2$ and $x = 8$, which produces a contradiction since $x = 8$ implies $y = 7$ and the 0 is a 6. If $w = 1$, then $u = 6$, and then $y = 8$, which produces a contradiction. Likewise $\gamma = 1$ is forced in Case 2.1.2.3; then $7 = \alpha$, since 7 has no other place. Also $6 = \beta$; it has no other place either. Then we obtain a contradiction by considering (2, 0), (4, 5), (8, 6), (8, 0), (4, 6). This leaves us with only Case 2.1.2.1, which can be divided into Case 2.1.2.1.1 where $p = 1$. Case 2.1.2.1.2 where $p = 6$ and $s = 1$, and Case 2.1.2.1.3 where $p = 8$ and $r = 1$. In Case 2.1.2.1.2, $l = 7$ is forced; then $r = 8$ is forced, which gives us a contradiction. In Case 2.1.2.1.3, $6 = l$ since it has no other place. Also $7 = n$. Then (1, 1), (4, 4), (7, 7), (7, 1), (4, 7) fail to complete to a plane. We have now, by placing the 8:

Case 2.1.2.1.1.1

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	6	8	7	3	5	4
3	8	7	0	6	1			5

Case 2.1.2.1.1.2

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	6	8	7	3	5	4
3	6	8	0	7	1		4	

Case 2.1.2.1.1.3

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	6	8	7	3	5	4
3	5	4	0	2	1	8	6	7
8			5	3	4	2		

Now 2 cannot be placed in the left square without creating a contradiction; likewise 5 cannot be placed in the middle square. In the right square we bring the last row to the fifth row in order to use more information. Inspection shows that this fifth row is either 876543210 or 867534201. In the former case, the subplane with (1, 0), (1, 7), (5, 7), (5, 4), (8, 4), (8, 4), (8, 7), (8, 0) fails to complete. In the latter case, the set (0, 0), (0, 8), (1, 1), (8, 8), (8,0) fails to complete to a subplane.

Now let us consider the other possibility, $f = 5$ (Case 2.2). Then we can state the following lemma.

LEMMA 7.3. *Any non-singly-generated plane of order* 9 *has an additive loop whose table agrees with the elementary Abelian group in the first three rows.*

Now without loss of generality we have:

Case 2.2.1

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	5	3	4	8	6	7
3	a	b	0	c	d	e	f	g

Case 2.2.2

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	5	3	4	8	6	7
3	h	i	6	j	k	0	l	m

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Case 2.2.1 breaks down into two subcases: $d = 1$ (Case 2.2.1.1) and $d = 6$, $f = 1$ (Case 2.2.1.2). Then Case 2.2.1.1 breaks down into two subcases: $c = 2$ (Case 2.2.1.1.1) and $c = 6$, $f = 2$ (Case 2.2.1.1.2). Case 2.2.1.1.1 cannot place 6, 7, 8. Case 2.2.1.1.2 cannot have $a = 7$ because the points $(1, 1)$, $(7, 7)$, $(7, 2)$ will not complete to a subplane. Therefore, $a = 8$ and $b = 7$. Points $(2, 0)$, $(2, 7)$, $(3, 4)$, $(3, 0)$, $(6, 7)$ indicate that $e = 4$, while points $(1, 0)$, $(1, 8)$, $(3, 5)$, $(3, 0)$, $(6, 8)$ indicate that $e = 5$. Therefore, we only need consider Case 2.2.1.2, which in turn breaks down into three subcases: $c = 2$ (Case 2.2.1.2.1), $c = 7$ and $g = 2$ (Case 2.2.1.2.2), and $c = 8$, $e = 2$ (Case 2.2.1.2.3). Case 2.2.1.2.1 cannot have $a = 8$; therefore $a = 7$, $b = 8$. However, one set of points indicates $d = 5$ while another set indicates $g = 5$. In Case 2.2.1.2.2 points $(1, 2)$, $(5, 3)$, $(5, 6)$, $(8, 6)$, $(8, 2)$ produce a contradiction immediately. Likewise in Case 2.2.1.2.3 there is an immediate contradiction.

Analysing Case 2.2.2, we have two subcases: $k = 1$ (Case 2.2.2.1) and $k = 8$, $l = 1$ (Case 2.2.2.2). Case 2.2.2.1 has three subcases: $j = 8$, $l = 2$ (Case 2.2.2.1.1), $j = 7$, $m = 2$ (Case 2.2.2.1.2), and $j = 2$ (Case 2.2.2.1.3). Case 2.2.2.1.1 has a set of points which produce a contradiction, while in Case 2.2.2.1.2 one first shows that $h = 4$ can be forced and then the contradiction, can easily be found. Inspection will show that in Case 2.2.2.1.3 $h = 8$ is forced, and then $m = 5$ is forced, which in turn produces a contradiction with other points.

In Case 2.2.2.2, it can be shown that $h = 4$, $i = 5$, by completing subplanes, and it can be shown that $j = 7$ and $m = 2$ by the Latin square property. We have proved the following lemma.

LEMMA 7.4. *Any non-singly-generated plane of order* 9 *has an additive loop whose table agrees with the elementary Abelian group in the first four rows.*

Case 2.2.2.2

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	5	3	4	8	6	7
3	4	5	6	7	8	0	1	2
4	a	b	c	d	e	f	g	h
5	j	k	l	m	n	p	q	r

Either $a = 3$ (Case 2.2.2.2.1) or $a = 5$ (Case 2.2.2.2.2). Case 2.2.2.2.1 can be broken down into three subcases: $b = 6$ (Case 2.2.2.2.1.1), $b = 7$ (Case 2.2.2.2.1.2), and $b = 8$ (Case 2.2.2.2.1.3). In Case 2.2.2.2.1.1, $c = 1$ and $g = 5$ are forced, then $d = 8$ and $e = 7$ are forced, then $f = 2$ and $h = 0$ are forced. Further inspection shows a contradiction. Similarly in Case 2.2.2.2.1.2, $d = 0$, $f = 5$, then $c = 8$, $e = 6$, then $g = 2$, $h = 1$, and then a contradiction. Finally in Case 2.2.2.2.1.3, $e = 2$, $h = 5$, then $c = 7$, $d = 6$, then $f = 1$, $g = 0$, and then a contradiction.

Now in Case 2.2.2.2.2, obviously $b = 3$, $j = 3$, $k = 4$. Now either $6 = d$

or $6 = e$. If $6 = d$, then the set $(0, 0)$, $(0, 4)$, $(4, 4)$, $(4, 6)$, $(6, 6)$ fails to complete to a subplane. Since $e = 6$, then $d = 8$, $c = 7$. Now the set $(1, 0)$, $(1, 5)$, $(3, 5)$, $(3, 7)$, $(8, 7)$ forces $h = 0$. Then $f = 1$ and $g = 2$. Now $l = 8$ since 8 has no other place to go in the sixth row. Then $m = 6$, $n = 7$. Then $p = 1$, $q = 2$, $r = 0$. Now we can consider:

0	$\underline{1}$	2	3	$\underline{4}$	5	6	$\underline{7}$	8
1	2	0	4	5	3	7	8	6
2	0	1	5	3	4	8	6	7
3	$\underline{4}$	5	6	$\underline{7}$	8	0	$\underline{1}$	2
4	5	3	7	8	6	1	2	0
5	3	4	8	6	7	2	0	1
6	s	t	u	v	w	x	y	z
7	α							
8								

Clearly, $u = 0$, $x = 3$. If $s = 8$, then $y = 3$ is forced, but this is a contradiction. Now $s = 7$ (the only other possibility) and by the underlined numbers, $v = 1$, $y = 4$. By the Latin square property, $t = 8$, $w = 2$, $z = 5$. By the Latin square property, $\alpha = 8$ since that is the only row and column available for 8. Then a subplane completion places $(3, 1)$ and $(8, 3)$. Then the Latin square property finishes the fourth and last column. Then a subplane completion places $(4, 2)$. Now $(5, 0)$, $(5, 1)$, $(4, 0)$ are placed automatically. Only one element can be placed in the eighth row and eighth column. Then the remaining elements are forced. This proves that any non-singly-generated plane of order 9 has the elementary Abelian group for the additive loop. Now by Lemma 7.1, the proof of Lemma 6.1 follows immediately.

8. Further justification for the conjecture. The work of A. Wagner **(19)** shows that the question, "Are all subplanes of finite planes Desarguesian?" has been settled. From this vantage point, disorder prevails. On the other hand, the results of Gleason **(3)** and Zappa **(21)** strengthen the conjecture: "If every quadrangle generates a subplane of order p, then it is Desarguesian." The conjecture in this paper is a natural generalization of the former conjecture. This justifies it as a proper open question. We further justify it by evidence obtained from some planes of order 16.

Consider the Veblen-Wedderburn plane of order 16 called $S(1)$ in Kleinfeld's paper **(11)**. Then choose $A : (0, 0)$, $B : (0, 1)$, $C : (1, 0)$, $D : (15, 15)$. Now $AB : x = 0$, $AC : y = 0$, $AD : y = x$, $BC : y = x + 1$, $BD : y = x8+1$, $CD : y = x14 + 14$. Then $E : (0, 14)$, $F : (11, 0)$, $G : (1)$. Finally, $CH : y = x11 + 11$ and $J : (3, 3)$ does not lie on CH (see the end of Section 3). Thus, this quadrangle generates the plane, since it does not generate a plane of order 2 or of order 3.

Consider the Veblen-Wedderburn plane $T(24)$. Then choose $A : (0, 0)$, $B : (0, 1)$, $C : (1, 0)$, $D : (2, 15)$. By a similar sequence of calculations one discovers that $J:(14,2)$ does not lie on $CH: y = x4 + 4$. Thus, this quadrangle generates the plane.

Consider the Veblen-Wedderburn plane $V(1)$ and use the quadrangle $A:(0,0)$, $B:(0,1)$, $C:(1,0)$, $D:(15,15)$ and one can calculate and discover that $J:(5,5)$ does not lie on $CH: y = x8 + 8$. Hence, this is also a singly-generated plane.

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