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Inverse dynamics modeling of a (3-UPU)+(3-UPS+S) serial-parallel manipulator

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SUMMARY
The inverse dynamics model of a novel (3-UPU)+(3-UPS+S) serial–parallel manipulator (S-PM) formed by a 3-UPU PM and a 3-UPS+S PM connected in serial is studied in this paper. First, the inverse position, velocity, and acceleration of this S-PM are studied systematically. Second, the velocity mapping relations between each component and the terminal platform of (3-UPU)+(3-UPS+S) S-PM are derived. Third, the dynamics model of the whole (3-UPU)+(3-UPS+S) S-PM is established by means of the principle of virtual work. The process for establishing the dynamics model of this S-PM is fit for other S-PMs.

KEYWORDS: Dynamics; Kinematics; Serial-parallel manipulators.

1. Introduction

In recent years, some serial–parallel manipulators (S-PMs) have attracted much attention in the field of robotics.[1–13] This class of manipulators is composed of several parallel manipulators (PMs) connected in serial. Thus, the S-PMs have higher stiffness than serial manipulators (SMs) and a larger workspace than PMs. In this aspect, Tanio[2] presented an S-PM composed of two serially connected PMs and gave its closed-form solution for the position problem using a vector approach. Romdhane[3] designed and analyzed a hybrid S-PM formed by a pure translational PM which has a PPP-type passive leg and a pure rotational PM which has an S-type passive leg, and used Euler angles to analyze the direct position problem. Zheng *et al.*[4] studied the kinematics of a hybrid S-PM formed by a pure translational 3-UPU PM and a pure rotational 3-UPU PM by using quaternions. Lu and Hu[5–7] studied the kinematics, statics, and stiffness for several S-PMs formed by two PMs and extended their research to the S-PMs formed by an optional number of PMs connected in serial. Gallardo-Alvarado *et al.*[8,9] studied the kinematics and dynamics of this class of manipulators via screw theory and the principle of virtual work. Liang and Ceccarelli[10,11] designed and analyzed this class of S-PMs used as a waist–trunk system for a humanoid robot.

Dynamics is an important topic in mechanism theory. For the dynamics modeling, the most commonly used approaches are the Newton–Euler method,[12,13] the Lagrange formulation,[14,15] and the principle of virtual work.[16,17] The Newton–Euler method uses the free body diagrams of the rigid bodies. This method is comprehensive in that a complete solution for all the forces and kinematic variables can be obtained. However, since this method needs to solve all the internal reactions of the manipulators, it leads to large computations. In this aspect, Dasgupta[12] derived a closed-form dynamics model for the Stewart platform manipulator considering all dynamic and gravity effects.

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Ji[13] studied the influence of the leg inertia on the dynamics model of the Stewart platform. The Lagrangian formulation is a very useful method for solving the inverse dynamics of manipulators. This method describes the dynamics of a mechanical system from the concept of work and energy with the adoption of a generalized coordinate framework. The dynamics formula has well analytical and orderly structure and thus has advantages for control. In this aspect, Lee *et al.*[14] derived the equations of dynamics formulated in joint-space using the Lagrangian formulation. Abdellatif *et al.*[15] presented a computational highly efficient method to derive explicit equations of dynamics of a 6-degree of freedom (DOF) PM using the Lagrangian formulation. The principle of virtual work is considered as a simple approach for mechanism dynamics. Based on the principle of virtual work, the constraint wrenches of the manipulators have been eliminated from the formulation. This allows us to reduce the complexity of the inverse dynamics. In this aspect, Gosselin,[16] Tsai[17] studied the inverse dynamics of the Stewart–Gough PM using the principle of virtual work, respectively. In addition, some other methods have also been suggested in dynamics analysis of manipulators including the screw theory,[18] the recursive matrix method,[19] Kane equation,[20] and the influence coefficient method.[21]

Although some efforts have been spent on S-PMs,[2–11] the architectures of this class of manipulators are very limited. For this reason, this paper presents a novel (3-UPU)+(3-UPS+S) S-PM. The lower PM of this S-PM adopts a 3-DOF 3-UPU PM with three translations, which has some potential advantages in pure translational PM family.[22] The upper PM of this S-PM adopts a 3-DOF 3-UPS+S PM with three rotations, which has a passive leg and thus can greatly enhance the stiffness. As the pure translational and pure rotational motions are completely determined by the lower and upper PMs respectively, the S-PM can be controlled easily to achieve its translations and rotations compared with the general 6-DOF S-PMs. This manipulator has some potential applications for the robot arms, the machine tools, the surgical manipulators, the tunnel borers, and the satellite surveillance platform.

Due to the complicated couplings and various constraints in structure for S-PMs, solving the inverse kinematics and dynamics is a challenging work. The previous research for such manipulators mainly focused on the forward kinematics based on the principle of motional superposition of the lower and upper PMs. However, there are few efforts made towards the inverse kinematics and dynamics of S-PMs.[2–11] Because of their highly nonlinear relations between joint variables and position/orientation of the end effectors for the S-PMs, solving the inverse dynamics based on Newton–Euler method or Lagrange formulation involves large computations. Inversely, the principle of virtual work can eliminate the constraint wrenches and reduce the complexity effectively. For this reason, this paper focuses on establishing the dynamics of the novel (3-UPU)+(3-UPS+S) S-PM using the principle of virtual work. The method used in this paper can be used to guide the dynamics modeling for other S-PMs.

2. Inverse Kinematics of the (3-UPU)+(3-UPS+S) S-PM

2.1. *Description of the (3-UPU)+(3-UPS+S) S-PM*

The architecture of (3-UPU)+(3-UPS+S) S-PM is shown in Fig. 1, which is composed of a lower 3-UPU PM and an upper 3-UPS+S PM. Let the PMs from the bottom to top be the ith ($i = 1, 2$) PM. Let r_{ij} be the jth leg of ith PM. Let R_{ijk} ($i = 1, 2; j = 1, 2, 3$) be the kth R joint of the jth leg for the ith PM. Let $\perp$ be a perpendicular constraint and $\|$ be a parallel constraint, respectively.

The lower 3-UPU PM is an unsymmetrical PM with 3 translational DOFs. It includes a lower platform A, an upper platform B and three UPU active leg r_{1j} ($j = 1, 2, 3$) with linear actuators. Each UPU-type leg connects A with B by a U joint at A_i, a prismatic joint P along r_{1j}, and a U joint at B_i. The U joint is comprised of two intercrossed revolute joints R_{1j1} and R_{1j2} ($j = 1, 3$).

For the 3-UPU PM, the first and the third legs are two symmetrical legs. The geometrical constraints for the two legs can be expressed as following:

$$R_{1j1}\perp A,\ \ R_{1j1}\perp R_{1j2},\ \ R_{1j2}||R_{1j3},\ \ R_{1j3}\perp R_{1j4},\ \ R_{1j4}\perp B\ (j = 1, 3) \tag{1a}$$

The geometrical constraints for the unsymmetrical UPU-type leg can be expressed as following:

$$R_{121}\perp A_1A_3,\ \ R_{122}||R_{123},\ \ R_{121}\perp r_{12},\ \ R_{123}\perp R_{124},\ \ R_{124}\perp B_1B_3 \tag{1b}$$

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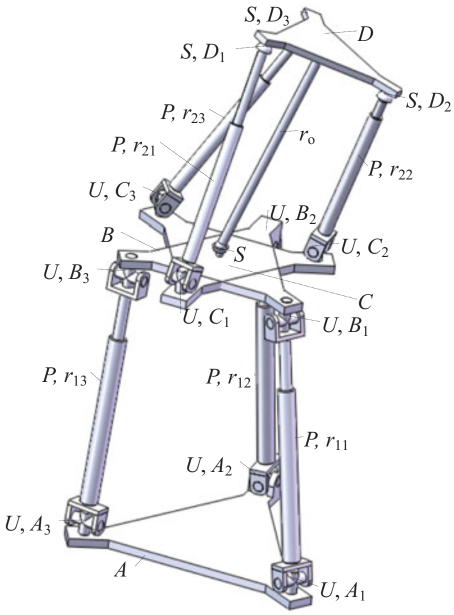

Fig. 1. CAD model of (3-UPU)+(3-UPS+S) PM.

The upper 3-UPS+S PM is composed of a lower platform C, an upper platform D, three UPS-type active legs r_{2j} (j = 1, 2, 3) with linear actuators, and one S-type constrained leg r_o. Each UPS-type leg connects C with D by a U joint at C_i, a prismatic joint P along r_{2j}, and an S joint at D_i. The S-type constrained leg r_o connects C with D by an S joint on C at o_c, a rigid rod perpendicularly fixed to D at o_d. For this PM, the geometrical constraints can be expressed as following:

$$r_o \perp D \tag{1c}$$

The four platforms A, B, C, and D are regular triangles with three vertices A_i, B_i, C_i, D_i ($i = 1$, 2, 3) and one center point o_a, o_b, o_c, o_d, respectively. B and C are fixed connected and have an angle of 60 degrees between them.

2.2. Inverse position kinematics of (3-UPU)+(3-UPS+S) S-PM

The inverse position kinematics of (3-UPU)+(3-UPS+S) S-PM solves the actuated variables from a given position and orientation in a given pose.

Let $\{A\}$, $\{B\}$, $\{C\}$, and $\{D\}$ be the coordinate systems at the center o_a, o_b, o_c, and o_d with x_N, y_N, and z_N (N = A, B, C, D) are three orthogonal coordinate axes and some constraints ($x_N || N_1N_3$, $y_N \perp N_1N_3$, $z_N \perp N$) are satisfied. Then $\{A\}$ and $\{D\}$ are the base coordinate system and terminal coordinate system for the whole (3-UPU)+(3-UPS+S) S-PM, respectively.

Let ${}^M\boldsymbol{Q}, {}^M\boldsymbol{v}_Q, {}^M\boldsymbol{a}_Q$ be the position, velocity, and acceleration vectors of the point Q expressed in the coordinate systems $\{M\}(M = A, B, C, D)$, respectively. Let ${}^M_N R$, ${}^M_N \omega$, and ${}^M_N \boldsymbol{\varepsilon}$ be the rotational matrix, angular velocity, and angular acceleration of $\{N\}$ ($N = A, B, C, D$) relative to $\{M\}$.

The position vector of A_j ($j = 1, 2, 3$) in $\{A\}$ can be expressed as follows:

$${}^A\boldsymbol{A}_1 = \frac{E_1}{2}\begin{pmatrix} q \\ -1 \\ 0 \end{pmatrix}, \quad {}^A\boldsymbol{A}_2 = \begin{pmatrix} 0 \\ E_1 \\ 0 \end{pmatrix}, \quad {}^A\boldsymbol{A}_3 = -\frac{E_1}{2}\begin{pmatrix} q \\ 1 \\ 0 \end{pmatrix}, \quad q = \sqrt{3}. \tag{2a}$$

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The position vector of C_j ($j = 1, 2, 3$) in $\{C\}$ can be expressed as follows:

$$ {}^{C}\boldsymbol{C}_1 = \frac{E_2}{2}\begin{pmatrix} q \\ -1 \\ 0 \end{pmatrix}, \quad {}^{C}\boldsymbol{C}_2 = \begin{pmatrix} 0 \\ E_2 \\ 0 \end{pmatrix}, \quad {}^{C}\boldsymbol{C}_3 = -\frac{E_2}{2}\begin{pmatrix} q \\ 1 \\ 0 \end{pmatrix}, \quad q = \sqrt{3}, \tag{2b} $$

where E_1 denotes the distance from o_a to A_j and E_2 denotes the distance from o_c to C_j.

The position vector of B_j ($j = 1, 2, 3$) in $\{B\}$ can be expressed as follows:

$$ {}^{B}\boldsymbol{B} = \frac{e_1}{2}\begin{pmatrix} q \\ -1 \\ 0 \end{pmatrix}, \quad {}^{B}\boldsymbol{B}_2 = \begin{pmatrix} 0 \\ e_1 \\ 0 \end{pmatrix}, \quad {}^{B}\boldsymbol{B}_3 = -\frac{e_1}{2}\begin{pmatrix} q \\ 1 \\ 0 \end{pmatrix}, \quad q = \sqrt{3}. \tag{2c} $$

The position vector of D_j ($j = 1, 2, 3$) in $\{D\}$ can be expressed as follows:

$$ {}^{D}\boldsymbol{D}_1 = \frac{e_2}{2}\begin{pmatrix} q \\ -1 \\ 0 \end{pmatrix}, \quad {}^{D}D_2 = \begin{pmatrix} 0 \\ e_2 \\ 0 \end{pmatrix}, \quad {}^{D}D_3 = -\frac{e_2}{2}\begin{pmatrix} q \\ 1 \\ 0 \end{pmatrix}, \quad q = \sqrt{3}, \tag{2d} $$

where e_1 denotes the distance from o_b to B_i and e_2 denotes the distance from o_d to D_i.

The position vector of B_j ($j = 1, 2, 3$) in $\{A\}$ can be expressed as following:

$$ {}^{A}\boldsymbol{B}_j = {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{B}_j + {}^{A}\boldsymbol{o}_b. \tag{3a} $$

The position vector of $\boldsymbol{D}_j$ ($j = 1, 2, 3$) in $\{C\}$ can be expressed as following:

$$ {}^{C}\boldsymbol{D}_i = {}^{C}_{D}\boldsymbol{R}\,{}^{D}\boldsymbol{D}_j + {}^{C}\boldsymbol{o}_d. \tag{3b} $$

The position vector of $\boldsymbol{C}_j$ ($j = 1, 2, 3$) in $\{B\}$ can be expressed as following:

$$ {}^{B}\boldsymbol{C}_j = {}^{B}_{C}\boldsymbol{R}\,{}^{B}\boldsymbol{B}_j. \tag{4a} $$

The rotational transformation matrix ${}^{A}_{C}R$ can be expressed as following:

$$ {}^{A}_{C}\boldsymbol{R} = {}^{A}_{B}\boldsymbol{R}\,{}^{B}_{C}\boldsymbol{R}. \tag{4b} $$

The rotational transformation matrix ${}^{A}_{D}R$ can be expressed as following:

$$ {}^{A}_{D}\boldsymbol{R} = {}^{A}_{B}\boldsymbol{R}\,{}^{B}_{C}\boldsymbol{R}\,{}^{C}_{D}\boldsymbol{R}. \tag{4c} $$

B and C have an angle of 60 degrees between them, which leads to

$$ {}^{B}_{C}R = \frac{1}{2}\begin{bmatrix} 1 & -q & 0 \\ q & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4d} $$

As $r_o \perp D$, the position vector of the center of the terminal platform o_d can be expressed as follows:

$$ {}^{A}\boldsymbol{o}_d = {}^{A}\boldsymbol{o}_c + \boldsymbol{r}_o, \quad \boldsymbol{r}_o = \boldsymbol{r}_o\,{}^{A}z_D, \tag{5a} $$

$$ {}^{A}\boldsymbol{o}_c = {}^{A}\boldsymbol{o}_d - \boldsymbol{r}_o\,{}^{A}z_D. \tag{5b} $$

Here, $\boldsymbol{r}_o$ and $\boldsymbol{r}_o$ denote the length and the vector of constrained leg respectively for the 3-UPS+S PM, ${}^{A}z_D$ is the third column vector of ${}^{A}_{D}\boldsymbol{R}$. When ${}^{A}\boldsymbol{o}_d$ and ${}^{A}_{D}\boldsymbol{R}$ are given, ${}^{A}\boldsymbol{o}_c$ can be solved from Eq. (5b) directly.

As B and C are fixed connected with their centers kept coincidence, it leads to

$$ {}^{A}\boldsymbol{o}_b = {}^{A}\boldsymbol{o}_c. \tag{6a} $$

As the lower PM has no rotations, it leads to

$$ {}^{A}_{B}R = {}^{A}_{C}R = E_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6b} $$

The position vectors of the vertices for each platform in $\{A\}$ can be expressed as follows:

$$ {}^{A}\boldsymbol{B}_j = {}^{A}\boldsymbol{o}_b + {}^{A}_{B}\boldsymbol{R}^{B}\boldsymbol{B}_j, \quad {}^{A}\boldsymbol{C}_j = {}^{A}\boldsymbol{o}_b + {}^{A}_{C}\boldsymbol{R}^{C}C_j, \quad {}^{A}\boldsymbol{D}_j = {}^{A}\boldsymbol{o}_d + {}^{A}_{D}\boldsymbol{R}^{D}D_j. \tag{7} $$

The length of r_{ij} $(i = 1, 2; j = 1, 2, 3)$ can be solved as follows:

$$ r_{1j} = \left| {}^{A}\boldsymbol{B}_j - {}^{A}\boldsymbol{A}_j \right|. \tag{8a} $$

$$ r_{2j} = \left| {}^{A}\boldsymbol{D}_j - {}^{A}\boldsymbol{C}_j \right|. \tag{8b} $$

When the position and orientation of the terminal platform are given, the position vector of the vertices for each platform can be solved from Eqs. (5)–(7), and the inverse kinematics of (3-UPU)+(3-UPS+S) S-PM can be solved from Eqs. (8a) and (8b), subsequently.

2.3. *Inverse velocity of (3-UPU)+(3-UPS+S) S-PM*

The inverse velocity analysis of the (3-UPU)+(3-UPS+S) S-PM is to determine the required velocities of actuators from a given velocity of the terminal platform in a given pose.

Let $\boldsymbol{t} = [t_x t_y t_z]^T$, $\boldsymbol{s} = [s_x s_y s_z]^T$ be two arbitrary vectors, $\hat{\boldsymbol{t}}$ be a skew-symmetric matrix. There must be

$$ \hat{\boldsymbol{t}} = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}, \quad \hat{\boldsymbol{t}} = -\hat{\boldsymbol{t}}^T, \quad \boldsymbol{t} \times \boldsymbol{s} = \hat{\boldsymbol{t}}\boldsymbol{s}. \tag{9} $$

Let $v_{r_{ij}}$, $a_{r_{ij}}$ be the velocity and acceleration along r_{ij}, $\boldsymbol{\omega}_{r_{ij}}$ be the angular vector of r_{ij}.
By differentiating both sides of Eq. (5a) with respect to time, it leads to

$$ {}^{A}\boldsymbol{v}_{o_d} = {}^{A}\boldsymbol{v}_{o_c} + {}^{A}_{D}\boldsymbol{\omega} \times {}^{A}\boldsymbol{r}_o. \tag{10a} $$

The velocity of o_c can be expressed as following:

$$ {}^{A}\boldsymbol{v}_{o_c} = {}^{A}\boldsymbol{v}_{o_d} - {}^{A}_{D}\boldsymbol{\omega} \times {}^{A}\boldsymbol{r}_o. \tag{10b} $$

As B and C are fixed connected with their centers kept coincidence, it leads to

$$ {}^{A}\boldsymbol{v}_{o_b} = {}^{A}\boldsymbol{v}_{o_c}. \tag{11a} $$

As the lower PM has no rotations, it leads to

$$ {}^{A}_{B}\boldsymbol{\omega} = \boldsymbol{0}_{3\times 1}, \tag{11b} $$

$$ {}^{A}\boldsymbol{v}_{B_j} = {}^{A}\boldsymbol{v}_{Cj} = {}^{A}\boldsymbol{v}_{o_b} = {}^{A}\boldsymbol{v}_{o_c}. \tag{11c} $$

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From Eqs. (10b) and (11c), the velocity of r_{1j} ($j = 1, 2, 3$) for the 3-UPU PM can be derived as following:

$$v_{r_{1j}} = {}^A\boldsymbol{v}_{B_j} \cdot {}^A\boldsymbol{\delta}_{1j} = ({}^A\boldsymbol{v}_{o_d} - {}^A_D\boldsymbol{\omega} \times {}^A\boldsymbol{r}_o) \cdot {}^A\boldsymbol{\delta}_{1j} = \left[\, {}^A\boldsymbol{\delta}_{1j}^T \quad ({}^A\boldsymbol{\delta}_{1j} \times {}^A\boldsymbol{r}_o)^T \,\right] \begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix},$$
$${}^A\boldsymbol{\delta}_{1j} = \frac{{}^A\boldsymbol{B}_j - {}^A\boldsymbol{A}_j}{\left|{}^A\boldsymbol{B}_j - {}^A\boldsymbol{A}_j\right|}. \tag{12}$$

The velocity of vertices D_j can be expressed as following:

$${}^A\boldsymbol{v}_{D_j} = {}^A\boldsymbol{v}_{o_d} + {}^A_D\boldsymbol{\omega} \times {}^A\boldsymbol{e}_{2j}, \quad {}^A\boldsymbol{e}_{2j} = {}^A\boldsymbol{D}_j - {}^A\boldsymbol{o}_d. \tag{13}$$

From Eqs. (10b), (11c), and (13), the velocity of r_{2j} ($j = 1, 2, 3$) can be derived as following:

$$v_{r_{2j}} = ({}^A\boldsymbol{v}_{D_j} - {}^A\boldsymbol{v}_{Cj}) \cdot {}^A\boldsymbol{\delta}_{2j} = \left[\, \boldsymbol{O}_{3\times1}^T \quad ({}^A\boldsymbol{e}_{2j} + {}^A\boldsymbol{r}_o)^T \times {}^A\boldsymbol{\delta}_{2j}^T \,\right] \begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix}, \quad {}^A\boldsymbol{\delta}_{2j} = \frac{{}^A\boldsymbol{D}_j - {}^A\boldsymbol{C}_j}{\left|{}^A\boldsymbol{D}_j - {}^A\boldsymbol{C}_j\right|}. \tag{14}$$

From Eqs. (12) and (14), it leads to

$$\boldsymbol{V}_r = \boldsymbol{J} \begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\omega \end{bmatrix}, \quad \boldsymbol{J} = \begin{bmatrix} {}^A\boldsymbol{\delta}_{11}^T & ({}^A\boldsymbol{\delta}_{11} \times {}^A\boldsymbol{r}_o)^T \\ {}^A\boldsymbol{\delta}_{12}^T & ({}^A\boldsymbol{\delta}_{12} \times {}^A\boldsymbol{r}_o)^T \\ {}^A\boldsymbol{\delta}_{13}^T & ({}^A\boldsymbol{\delta}_{13} \times {}^A\boldsymbol{r}_o)^T \\ \boldsymbol{O}_{3\times1}^T & ({}^A\boldsymbol{e}_{21} + {}^A\boldsymbol{r}_o)^T \times {}^A\boldsymbol{\delta}_{21}^T \\ \boldsymbol{O}_{3\times1}^T & ({}^A\boldsymbol{e}_{22} + {}^A\boldsymbol{r}_o)^T \times {}^A\boldsymbol{\delta}_{22}^T \\ \boldsymbol{O}_{3\times1}^T & ({}^A\boldsymbol{e}_{23} + {}^A\boldsymbol{r}_o)^T \times {}^A\boldsymbol{\delta}_{23}^T \end{bmatrix}, \quad \boldsymbol{V}_r = \begin{bmatrix} v_{r_{11}} \\ v_{r_{12}} \\ v_{r_{13}} \\ v_{r_{21}} \\ v_{r_{22}} \\ v_{r_{23}} \end{bmatrix}, \tag{15}$$

where $\boldsymbol{J}$ is the Jacobian for (3-UPU)+(3-UPS+S) S-PM.

2.4. *Inverse acceleration of (3-UPU)+(3-UPS+S) S-PM*

The inverse acceleration analysis of the (3-UPU)+(3-UPS+S) S-PM is to determine the required accelerations of actuators from a given velocity/acceleration of the terminal platform in a given pose.

By differentiating both sides of Eq. (12) with respect to time, it leads to

$$\boldsymbol{a}_{r_{1j}} = {}^A\boldsymbol{a}_{B_j} \cdot {}^A\boldsymbol{\delta}_{1j} + {}^A\boldsymbol{v}_{B_j} \cdot {}^A\dot{\boldsymbol{\delta}}_{1j} = {}^A\boldsymbol{a}_{B_j} \cdot {}^A\boldsymbol{\delta}_{1j} + {}^A\boldsymbol{v}_{B_j} \cdot ({}^A\boldsymbol{\omega}_{r_{1j}} \times {}^A\boldsymbol{\delta}_{1j}), \tag{16a}$$

where ${}^A\boldsymbol{a}_{B_j} = {}^A\boldsymbol{a}_{o_d} - {}^A_D\boldsymbol{\varepsilon} \times {}^A\boldsymbol{r}_o - {}^A_D\boldsymbol{\omega} \times ({}^A_D\boldsymbol{\omega} \times {}^A\boldsymbol{r}_o)$.

By differentiating both sides of Eq. (14) with respect to time, it leads to

$$\begin{aligned} a_{r_{2j}} &= ({}^A\boldsymbol{v}_{D_j} - {}^A\boldsymbol{v}_{Cj}) \cdot {}^A\dot{\boldsymbol{\delta}}_{2j} + ({}^A\boldsymbol{a}_{D_j} - {}^A\boldsymbol{a}_{Cj}) \cdot {}^A\boldsymbol{\delta}_{2j} \\ &= ({}^A\boldsymbol{v}_{D_j} - {}^A\boldsymbol{v}_{Cj}) \cdot ({}^A\boldsymbol{\omega}_{r_{2j}} \times {}^A\boldsymbol{\delta}_{2j}) + ({}^A\boldsymbol{a}_{D_j} - {}^A\boldsymbol{a}_{Cj}) \cdot {}^A\boldsymbol{\delta}_{2j}, \end{aligned} \tag{16b}$$

where ${}^A\boldsymbol{a}_{D_j} = {}^A\boldsymbol{a}_{o_d} + {}^A_D\boldsymbol{\varepsilon} \times e_{2j} + {}^A_D\boldsymbol{\omega} \times ({}^A_D\boldsymbol{\omega} \times e_{2j}), {}^A\boldsymbol{a}_{Cj} = {}^A\boldsymbol{a}_{B_j}$.

In Eqs. (16a) and (16b), ${}^A\boldsymbol{\omega}_{r_{1j}}$ and ${}^A\boldsymbol{\omega}_{r_{2j}}$ are two unknowns, which are frequently used in the dynamics model. In what follows, ${}^A\boldsymbol{\omega}_{r_{1j}}$ and ${}^A\boldsymbol{\omega}_{r_{2j}}$ will be solved in the compact form.

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The angular velocity $\boldsymbol{\omega}_{r_{1j}}$ of r_{1j} for the lower 3-UPU PM can be expressed as following:

$$^{A}\boldsymbol{\omega}_{r_{1j}} = \dot{\theta}_{1j1}^{A}\boldsymbol{R}_{1j1} + \dot{\theta}_{1j2}^{A}\boldsymbol{R}_{1j2}. \tag{17a}$$

Here, $\dot{\theta}_{1j1}$ and $\dot{\theta}_{1j2}$ denote the angular velocity of R_{j1} and R_{j2}, respectively.
Cross-multiplying both sides of Eq. (17a) by $\boldsymbol{r}_{1j}$ yields

$$\begin{aligned}&\dot{\theta}_{1j1}^{A}\boldsymbol{R}_{1j1} \times^{A} \boldsymbol{r}_{1j} + \dot{\theta}_{1j2}^{A}\boldsymbol{R}_{1j2} \times^{A} \boldsymbol{r}_{1j} =^{A} \boldsymbol{\omega}_{r1j} \times^{A} \boldsymbol{r}_{1j}\\ &\quad =^{A} \boldsymbol{v}_{Bj} - v_{r1j}^{A}\boldsymbol{\delta}_{1j} = -^{A}\hat{\boldsymbol{\delta}}_{1j}^{2A}\boldsymbol{v}_{ob} +^{A} \hat{\boldsymbol{\delta}}_{i}^{2A}\hat{\boldsymbol{e}}_{1jB}^{A}\boldsymbol{\omega}, \quad ^{A}\boldsymbol{r}_{1j} =^{A} \boldsymbol{B}_{j} -^{A} \boldsymbol{A}_{j}.\end{aligned} \tag{17b}$$

Dot-multiplying both sides of Eq. (17b) by $\boldsymbol{R}_{2i}$, it leads to

$$\dot{\theta}_{1j1}(^{A}\boldsymbol{R}_{1j1} \times^{A} \boldsymbol{r}_{1j}) \cdot^{A} \boldsymbol{R}_{1j2} =^{A} \boldsymbol{R}_{1j2}^{T}(^{A}\boldsymbol{v}_{Bj} - v_{r1j}^{A}\boldsymbol{\delta}_{1j}) =^{A} \boldsymbol{R}_{1j2}^{T}(-^{A}\hat{\boldsymbol{\delta}}_{1j}^{2A}\boldsymbol{v}_{ob} +^{A} \hat{\boldsymbol{\delta}}_{1j}^{2A}\hat{\boldsymbol{e}}_{1jB}^{A}\boldsymbol{\omega}). \tag{18a}$$

Dot-multiplying both sides of Eq. (17b) by $\boldsymbol{R}_{1i}$, it leads to

$$\dot{\theta}_{1j2}(\boldsymbol{R}_{1j2} \times^{A} \boldsymbol{r}_{1j}) \cdot \boldsymbol{R}_{1j1} =^{A} \boldsymbol{R}_{1j1}^{T}(^{A}\boldsymbol{v}_{Bj} - v_{r1j}^{A}\boldsymbol{\delta}_{1j}) = \boldsymbol{R}_{1j1}^{T}(-^{A}\hat{\boldsymbol{\delta}}_{1j}^{2A}\boldsymbol{v}_{ob} +^{A} \hat{\boldsymbol{\delta}}_{1j}^{2A}\hat{\boldsymbol{e}}_{1jB}^{A}\boldsymbol{\omega}). \tag{18b}$$

From Eqs. (18a) and (18b), it leads to

$$\begin{aligned}\dot{\theta}_{1j1} &= \frac{(^{A}\boldsymbol{v}_{Bj} - v_{r1j}^{A}\boldsymbol{\delta}_{1j}) \cdot^{A} \boldsymbol{R}_{1j2}}{(^{A}\boldsymbol{R}_{1j1} \times^{A} \boldsymbol{R}_{1j2}) \cdot^{A} \boldsymbol{r}_{1j}} = \frac{(^{A}\hat{\boldsymbol{\delta}}_{1j}^{2A}\boldsymbol{v}_{ob} -^{A} \hat{\boldsymbol{\delta}}_{1j}^{2A}\hat{\boldsymbol{e}}_{1jB}^{A}\boldsymbol{\omega}) \cdot^{A} \boldsymbol{R}_{1j2}}{(^{A}\boldsymbol{R}_{1j1} \times^{A} \boldsymbol{R}_{1j2}) \cdot^{A} \boldsymbol{r}_{1j}},\\ \dot{\theta}_{1j2} &= -\frac{(^{A}\boldsymbol{v}_{Bj} - v_{r1j}^{A}\boldsymbol{\delta}_{1j}) \cdot^{A} \boldsymbol{R}_{1j1}}{(^{A}\boldsymbol{R}_{1j1} \times^{A} \boldsymbol{R}_{1j2}) \cdot^{A} \boldsymbol{r}_{1j}} = -\frac{(^{A}\hat{\boldsymbol{\delta}}_{1j}^{2A}\boldsymbol{v}_{ob} -^{A} \hat{\boldsymbol{\delta}}_{1j}^{2A}\hat{\boldsymbol{e}}_{1jB}^{A}\boldsymbol{\omega}) \cdot^{A} \boldsymbol{R}_{1j1}}{(^{A}\boldsymbol{R}_{1j1} \times^{A} \boldsymbol{R}_{1j2}) \cdot^{A} \boldsymbol{r}_{1j}}.\end{aligned} \tag{19}$$

From Eqs. (17a) and (19), it leads to

$$\begin{aligned}^{A}\boldsymbol{\omega}_{r1j} &= \dot{\theta}_{1j1}^{A}\boldsymbol{R}_{1j1} + \dot{\theta}_{1j2}^{A}\boldsymbol{R}_{1j2} = \frac{(^{A}\boldsymbol{R}_{1j1}^{A}\boldsymbol{R}_{1j2}^{T} -^{A} \boldsymbol{R}_{1j2}^{A}\boldsymbol{R}_{1j1}^{T})(^{A}\boldsymbol{v}_{Bj} - v_{r1j}^{A}\boldsymbol{\delta}_{1j})}{(^{A}\boldsymbol{R}_{1j1} \times^{A} \boldsymbol{R}_{1j2}) \cdot^{A} \boldsymbol{r}_{1j}}\\ &= \frac{(^{A}\boldsymbol{R}_{1j1} \times^{A} \boldsymbol{R}_{1j2}^{T}) \times (^{A}\boldsymbol{v}_{Bj} - v_{r1j}^{A}\boldsymbol{\delta}_{1j})}{(^{A}\boldsymbol{R}_{1j1} \times^{A} \boldsymbol{R}_{1j2}) \cdot^{A} \boldsymbol{r}_{1j}} = \frac{(^{A}\boldsymbol{R}_{1j1}^{A}\boldsymbol{R}_{1j2}^{T} -^{A} \boldsymbol{R}_{1j2}^{A}\boldsymbol{R}_{1j1}^{T})^{A}\hat{\boldsymbol{\delta}}_{1j}^{2}(^{A}\boldsymbol{v}_{b_o} -^{A} \hat{\boldsymbol{e}}_{1jB}^{A}\boldsymbol{\omega})}{(^{A}\boldsymbol{R}_{1j1} \times^{A} \boldsymbol{R}_{1j2}) \cdot^{A} \boldsymbol{r}_{1j}}\\ &= \begin{bmatrix} \frac{(^{A}\boldsymbol{R}_{1j1}^{A}\boldsymbol{R}_{1j2}^{T} -^{A} \boldsymbol{R}_{1j2}^{A}\boldsymbol{R}_{1j1}^{T})^{A}\hat{\boldsymbol{\delta}}_{1j}^{2}}{(^{A}\boldsymbol{R}_{1j1} \times^{A} \boldsymbol{R}_{1j2}) \cdot^{A} \boldsymbol{r}_{1j}} & \frac{-(^{A}\boldsymbol{R}_{1j1}^{A}\boldsymbol{R}_{1j2}^{T} -^{A} \boldsymbol{R}_{1j2}^{A}\boldsymbol{R}_{1j1}^{T})^{A}\hat{\boldsymbol{\delta}}_{1j}^{2A}\hat{\boldsymbol{e}}_{1j}}{(^{A}\boldsymbol{R}_{1i} \times^{A} \boldsymbol{R}_{2i}) \cdot^{A} \boldsymbol{r}_{1j}} \end{bmatrix} \begin{bmatrix} ^{A}\boldsymbol{v}_{ob}\\ ^{A}_{B}\boldsymbol{\omega} \end{bmatrix}.\end{aligned} \tag{20}$$

Using the same method, the angular velocity of r_{2j} for the upper PM in $\{C\}$ can be derived as

$$\begin{aligned}^{C}\boldsymbol{\omega}_{r2j} &= \dot{\theta}_{2j1}^{C}\boldsymbol{R}_{2j1} + \dot{\theta}_{2j2}^{C}\boldsymbol{R}_{2j2} = \frac{(^{C}\boldsymbol{R}_{2j1} \times^{C} \boldsymbol{R}_{2j2}^{T}) \times (^{C}\boldsymbol{v}_{Dj} - v_{r2j}^{C}\boldsymbol{\delta}_{2j})}{(^{C}\boldsymbol{R}_{2j1} \times^{C} \boldsymbol{R}_{2j2}) \cdot^{C} \boldsymbol{r}_{2j}}\\ &= \frac{(^{C}\boldsymbol{R}_{2j1}^{C}\boldsymbol{R}_{2j2}^{T} -^{C} \boldsymbol{R}_{2j2}^{C}\boldsymbol{R}_{2j1}^{T})^{C}\hat{\boldsymbol{\delta}}_{2j}^{2}(^{C}\boldsymbol{v}_{Dj} -^{C} \hat{\boldsymbol{e}}_{2jD}^{C}\boldsymbol{\omega})}{(^{C}\boldsymbol{R}_{2j1} \times^{A} \boldsymbol{R}_{2j2}) \cdot^{C} \boldsymbol{r}_{2j}}\\ &= \frac{(^{C}\boldsymbol{R}_{2j1}^{C}\boldsymbol{R}_{2j2}^{T} -^{C} \boldsymbol{R}_{2j2}^{C}\boldsymbol{R}_{2j1}^{T})^{C}\hat{\boldsymbol{\delta}}_{2j}^{2}(^{C}\hat{e}_{2j} +^{C} \hat{\boldsymbol{r}}_{o})_{D}^{C}\boldsymbol{\omega}}{(^{C}\boldsymbol{R}_{2j1} \times^{C} \boldsymbol{R}_{2j2}) \cdot^{C} \boldsymbol{r}_{2j}}.\end{aligned} \tag{21}$$

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As the lower PM has no rotations, the motion of C relative to A is pure translation, which leads to

$$\begin{aligned} {}^{A}\boldsymbol{\omega}_{r2j} &= \dot{\theta}^{A}_{2j1}\boldsymbol{R}_{2j1} + \dot{\theta}^{A}_{2j2}\boldsymbol{R}_{2j2} = \frac{({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}^{T}_{2j2}) \times ({}^{A}\boldsymbol{v}_{Dj} - {}^{A}\boldsymbol{v}_{Cj} - v^{A}_{r2j}\boldsymbol{\delta}_{2j})}{({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}_{2j2}) \cdot {}^{A}\boldsymbol{r}_{2j}} \\ &= -\frac{({}^{A}\boldsymbol{R}^{A}_{2j1}\boldsymbol{R}^{T}_{2j2} - {}^{A}\boldsymbol{R}^{A}_{2j2}\boldsymbol{R}^{T}_{2j1})^{A}\hat{\boldsymbol{\delta}}^{2}_{2j}({}^{A}\hat{e}_{2j} + {}^{A}\hat{\boldsymbol{r}}_{o})^{A}_{D}\boldsymbol{\omega}}{({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}_{2j2}) \cdot {}^{A}\boldsymbol{r}_{2j}} \\ &= \begin{bmatrix} \mathbf{0}_{3\times 3} & -\dfrac{({}^{A}\boldsymbol{R}^{A}_{2j1}\boldsymbol{R}^{T}_{2j2} - {}^{A}\boldsymbol{R}^{A}_{2j2}\boldsymbol{R}^{T}_{2j1})^{A}\hat{\boldsymbol{\delta}}^{2}_{2j}({}^{A}\hat{e}_{2j} + {}^{A}\hat{\boldsymbol{r}}_{o})^{A}_{D}\boldsymbol{\omega}}{({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}_{2j2}) \cdot {}^{A}\boldsymbol{r}_{2j}} \end{bmatrix} \begin{bmatrix} {}^{A}\boldsymbol{v}_{o_d} \\ {}^{A}_{D}\boldsymbol{\omega} \end{bmatrix}, \end{aligned} \tag{22}$$

where ${}^{C}\boldsymbol{v}_{Dj} = {}^{A}\boldsymbol{v}_{Dj} - {}^{A}\boldsymbol{v}_{Cj}$.

Then, substituting the results to Eqs. (16a) and (16b), the inverse acceleration of the (3-UPU)+(3-UPS+S) S-PM can be determined.

3. Inverse Dynamics of the (3-UPU)+(3-UPS+S) S-PM

3.1. Velocity mapping between each component and the terminal platform

3.1.1. Velocity mapping between the legs and the terminal platform for 3-UPU PM.. From Eq. (10b), it leads to

$${}^{A}\boldsymbol{v}_{ob} = {}^{A}\boldsymbol{v}_{o_c} = \begin{bmatrix} \boldsymbol{E}_{3\times 3} & {}^{A}\hat{\boldsymbol{r}}_{o} \end{bmatrix} \begin{bmatrix} {}^{A}\boldsymbol{v}_{o_d} \\ {}^{A}_{D}\boldsymbol{\omega} \end{bmatrix}. \tag{23a}$$

From Eqs. (11b) and (23a), the velocity mapping relation between B and D can be derived as following:

$$\begin{bmatrix} {}^{A}\boldsymbol{v}_{ob} \\ {}^{A}_{B}\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \boldsymbol{E} & {}^{A}\hat{\boldsymbol{r}}_{o} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \end{bmatrix} \begin{bmatrix} {}^{A}\boldsymbol{v}_{o_d} \\ {}^{A}_{D}\boldsymbol{\omega} \end{bmatrix} = \boldsymbol{J}_{cd} \begin{bmatrix} {}^{A}\boldsymbol{v}_{o_d} \\ {}^{A}_{D}\boldsymbol{\omega} \end{bmatrix}. \tag{23b}$$

Let $r_{f_{1j}}$ be the distance from the mass center of the jth cylinder to point A_j. By means of Eq. (20), the velocity of the mass center of jth cylinder in $\{A\}$ for the 3-UPU PM can be expressed as following:

$$\begin{aligned} {}^{A}\boldsymbol{v}_{f_{1j}} &= {}^{A}\boldsymbol{\omega}_{r_{1j}} \times {}^{A}\boldsymbol{\delta}_{1j} r_{f_{1j}} = -r^{A}_{f_{1j}}\hat{\boldsymbol{\delta}}^{A}_{1j}\boldsymbol{\omega}_{r1j} \\ &= \begin{bmatrix} \dfrac{-r^{A}_{f_{1j}}\hat{\boldsymbol{\delta}}_{1j}({}^{A}\boldsymbol{R}^{A}_{1j1}\boldsymbol{R}^{T}_{1j2} - {}^{A}\boldsymbol{R}^{A}_{1j2}\boldsymbol{R}^{T}_{1j1})^{A}\hat{\boldsymbol{\delta}}^{2}_{1j}}{({}^{A}\boldsymbol{R}_{1j1} \times {}^{A}\boldsymbol{R}_{1j2}) \cdot {}^{A}\boldsymbol{r}_{1j}} & \dfrac{r^{A}_{f_{1j}}\hat{\boldsymbol{\delta}}_{1j}({}^{A}\boldsymbol{R}^{A}_{1j1}\boldsymbol{R}^{T}_{1j2} - {}^{A}\boldsymbol{R}^{A}_{1j2}\boldsymbol{R}^{T}_{1j1})^{A}\hat{\boldsymbol{\delta}}^{2A}_{1j}\hat{\boldsymbol{e}}_{1j}}{({}^{A}\boldsymbol{R}_{1i} \times {}^{A}\boldsymbol{R}_{2i}) \cdot {}^{A}\boldsymbol{r}_{1j}} \end{bmatrix} \begin{bmatrix} {}^{A}\boldsymbol{v}_{ob} \\ {}^{A}_{B}\boldsymbol{\omega} \end{bmatrix}. \end{aligned} \tag{24a}$$

From Eqs. (20) and (24a), it leads to

$$\begin{aligned} &\begin{bmatrix} {}^{A}\boldsymbol{v}_{f_{1j}} \\ {}^{A}\boldsymbol{\omega}_{r_{1j}} \end{bmatrix} = \boldsymbol{J}_{f_{1j}} \begin{bmatrix} {}^{A}\boldsymbol{v}_{o_d} \\ {}^{A}_{D}\boldsymbol{\omega} \end{bmatrix}, \\ &\boldsymbol{J}_{f_{1j}} = \begin{bmatrix} \dfrac{-r^{A}_{f_{1j}}\hat{\boldsymbol{\delta}}_{1j}({}^{A}\boldsymbol{R}^{A}_{1j1}\boldsymbol{R}^{T}_{1j2} - {}^{A}\boldsymbol{R}^{A}_{1j2}\boldsymbol{R}^{T}_{1j1})^{A}\hat{\boldsymbol{\delta}}^{2}_{1j}}{({}^{A}\boldsymbol{R}_{1j1} \times {}^{A}\boldsymbol{R}_{1j2}) \cdot {}^{A}\boldsymbol{r}_{1j}} & \dfrac{r^{A}_{f_{1j}}\hat{\boldsymbol{\delta}}_{1j}({}^{A}\boldsymbol{R}^{A}_{1j1}\boldsymbol{R}^{T}_{1j2} - {}^{A}\boldsymbol{R}^{A}_{1j2}\boldsymbol{R}^{T}_{1j1})^{A}\hat{\boldsymbol{\delta}}^{2A}_{1j}\hat{\boldsymbol{e}}_{1j}}{({}^{A}\boldsymbol{R}_{1i} \times {}^{A}\boldsymbol{R}_{2i}) \cdot {}^{A}\boldsymbol{r}_{1j}} \\ \dfrac{({}^{A}\boldsymbol{R}^{A}_{1j1}\boldsymbol{R}^{T}_{1j2} - {}^{A}\boldsymbol{R}^{A}_{1j2}\boldsymbol{R}^{T}_{1j1})^{A}\hat{\boldsymbol{\delta}}^{2}_{1j}}{({}^{A}\boldsymbol{R}_{1j1} \times {}^{A}\boldsymbol{R}_{1j2}) \cdot {}^{A}\boldsymbol{r}_{1j}} & -\dfrac{({}^{A}\boldsymbol{R}^{A}_{1j1}\boldsymbol{R}^{T}_{1j2} - {}^{A}\boldsymbol{R}^{A}_{1j2}\boldsymbol{R}^{T}_{1j1})^{A}\hat{\boldsymbol{\delta}}^{2A}_{1j}\hat{\boldsymbol{e}}_{1j}}{({}^{A}\boldsymbol{R}_{1j1} \times {}^{A}\boldsymbol{R}_{1j2}) \cdot {}^{A}\boldsymbol{r}_{1j}} \end{bmatrix} \boldsymbol{J}_{cd}, \end{aligned} \tag{24b}$$

where $\boldsymbol{J}_{f_{1j}}$ is a 6×6 Jacobian matrix of the jth cylinder of 3-UPU PM, which relates the velocity mapping relation between the jth cylinder of the 3-UPU PM and the terminal platform D.

Let r_{m1j} be the distance from the mass center of the jth piston to point B_j. By means of Eq. (20), the velocity of the mass center of the jth piston of lower PM can be

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expressed as

$$\begin{aligned} {}^A\boldsymbol{v}_{m1j} &= {}^A\boldsymbol{\omega}_{r_{1j}} \times {}^A\boldsymbol{\delta}_{1j}(r_{1j} - r_{m1j}) + {}^A\boldsymbol{\delta}_{1j}\dot{r}_{1j} \\ &= -(r_{1j} - r_{m1j})^A\boldsymbol{\delta}^A_{1j}\boldsymbol{\omega}_{r1j} + {}^A\boldsymbol{\delta}_{1j}\left(\begin{bmatrix} {}^A\boldsymbol{\delta}^T_{1j} & {}^A\hat{\boldsymbol{e}}_{1j} \times {}^A\boldsymbol{\delta}^T_{1j} \end{bmatrix}\begin{bmatrix} {}^A\boldsymbol{v}_{ob} \\ {}^A_B\boldsymbol{\omega} \end{bmatrix}\right) \\ &= \left[\frac{(r_{m1j}-r_{1j})^A\hat{\boldsymbol{\delta}}_{1j}({}^A\boldsymbol{R}^A_{1j1}\boldsymbol{R}^T_{1j2} - {}^A\boldsymbol{R}^A_{1j2}\boldsymbol{R}^T_{1j1})\hat{\boldsymbol{\delta}}^2_{1j}}{({}^A\boldsymbol{R}_{1j1}\times{}^A\boldsymbol{R}_{1j2})\cdot\ {}^A\boldsymbol{r}_{1j}} + {}^A\boldsymbol{\delta}^A_{1j}\boldsymbol{\delta}^T_{1j} \quad \frac{(r_{m1j}-r_{1j})^A\hat{\boldsymbol{\delta}}_{1j}({}^A\boldsymbol{R}^A_{1j1}\boldsymbol{R}^T_{1j2} - {}^A\boldsymbol{R}^A_{1j2}\boldsymbol{R}^T_{1j1})^A\hat{\boldsymbol{\delta}}^{2A}_{1j}\hat{\boldsymbol{e}}_{1j}}{({}^A\boldsymbol{R}_{1j1}\times{}^A\boldsymbol{R}_{1j2})\cdot\ {}^A\boldsymbol{r}_{1j}} - {}^A\boldsymbol{\delta}^A_{1j}\boldsymbol{\delta}^{TA}_{1j}\hat{\boldsymbol{e}}_{1j} \right] \\ &\quad \times \begin{bmatrix} {}^A\boldsymbol{v}_{ob} \\ {}^A_B\boldsymbol{\omega} \end{bmatrix}. \end{aligned} \tag{25}$$

From Eqs. (20) and (25), it leads to

$$\begin{bmatrix} {}^A\boldsymbol{v}_{m1j} \\ {}^A\boldsymbol{\omega}_{r1j} \end{bmatrix} = \boldsymbol{J}_{m1j}\begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix},$$

$$\boldsymbol{J}_{m1j} = \begin{bmatrix} \frac{(r_{m1j}-r_{1j})^A\hat{\boldsymbol{\delta}}_{1j}({}^A\boldsymbol{R}^A_{1j1}\boldsymbol{R}^T_{1j2} - {}^A\boldsymbol{R}^A_{1j2}\boldsymbol{R}^T_{1j1})\hat{\boldsymbol{\delta}}^2_{1j}}{({}^A\boldsymbol{R}_{1j1}\times{}^A\boldsymbol{R}_{1j2})\cdot\ {}^A\boldsymbol{r}_{1j}} + {}^A\boldsymbol{\delta}^A_{1j}\boldsymbol{\delta}^T_{1j} & \frac{(r_{m1j}-r_{1j})^A\hat{\boldsymbol{\delta}}_{1j}({}^A\boldsymbol{R}^A_{1j1}\boldsymbol{R}^T_{1j2} - {}^A\boldsymbol{R}^A_{1j2}\boldsymbol{R}^T_{1j1})^A\hat{\boldsymbol{\delta}}^{2A}_{1j}\hat{\boldsymbol{e}}_{1j}}{({}^A\boldsymbol{R}_{1j1}\times{}^A\boldsymbol{R}_{1j2})\cdot\ {}^A\boldsymbol{r}_{1j}} - {}^A\boldsymbol{\delta}^A_{1j}\boldsymbol{\delta}^{TA}_{1j}\hat{\boldsymbol{e}}_{1j} \\ \frac{({}^A\boldsymbol{R}^A_{1j1}\boldsymbol{R}^T_{1j2} - {}^A\boldsymbol{R}^A_{1j2}\boldsymbol{R}^T_{1j1})^A\hat{\boldsymbol{\delta}}^2_{1j}}{({}^A\boldsymbol{R}_{1j1}\times{}^A\boldsymbol{R}_{1j2})\cdot\ {}^A\boldsymbol{r}_{1j}} & -\frac{({}^A\boldsymbol{R}^A_{1j1}\boldsymbol{R}^T_{1j2} - {}^A\boldsymbol{R}^A_{1j2}\boldsymbol{R}^T_{1j1})^A\hat{\boldsymbol{\delta}}^{2A}_{1j}\hat{\boldsymbol{e}}_{1j}}{({}^A\boldsymbol{R}_{1j1}\times{}^A\boldsymbol{R}_{1j2})\cdot\ {}^A\boldsymbol{r}_{1j}} \end{bmatrix}\boldsymbol{J}_{cd}, \tag{26}$$

where $\boldsymbol{J}_{m1j}$ is a 6 × 6 Jacobian matrix of the *j*th piston of 3-UPU PM, which relates the velocity mapping relations between *j*th piston of the 3-UPU PM and the terminal platform *D*.

3.1.2. Velocity mapping between the legs of 3-UPS+S PM and the terminal platform.. Let $o_{f_{2j}}$ be the mass center of the *j*th cylinder in *i*th PM, $r_{f_{2j}}$ be the distance from $o_{f_{2j}}$ to the point C_j. The position and velocity vectors of $o_{f_{2j}}$ in $\{A\}$ can be expressed as follows:

$${}^A\boldsymbol{o}_{f_{2j}} = {}^A\boldsymbol{C}_i + r^A_{f_{2j}}\boldsymbol{\delta}_{2j}, \tag{27a}$$

$$\begin{aligned} {}^A\boldsymbol{v}_{f_{2j}} &= {}^A\boldsymbol{v}_{o_a} + r^A_{f_{2j}}\boldsymbol{\omega}_{r_{2j}} \times {}^A\boldsymbol{\delta}_{2j} = {}^A\boldsymbol{v}_{o_d} - {}^A_D\boldsymbol{\omega} \times {}^A\boldsymbol{r}_o + r^A_{f_{2j}}\boldsymbol{\omega}_{r_{2j}} \times {}^A\boldsymbol{\delta}_{2j} \\ &= \left[\boldsymbol{E}_{3\times3} \quad \frac{r^A_{f_{2j}}\hat{\boldsymbol{\delta}}_{2j}({}^A\boldsymbol{R}^A_{2j1}\boldsymbol{R}^T_{2j2} - {}^A\boldsymbol{R}^A_{2j2}\boldsymbol{R}^T_{2j1})^A\hat{\boldsymbol{\delta}}^2_{2j}({}^A\hat{e}_{2j} + {}^A\hat{\boldsymbol{r}}_o)}{({}^A\boldsymbol{R}_{2j1}\times{}^A\boldsymbol{R}_{2j2})\cdot\ {}^A\boldsymbol{r}_{2j}} + {}^A\hat{\boldsymbol{r}}_o \right]\begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix}. \end{aligned} \tag{27b}$$

From Eqs. (22) and (27b), it leads to

$$\begin{bmatrix} {}^A\boldsymbol{v}_{f_{2j}} \\ {}^A\boldsymbol{\omega}_{r_{2j}} \end{bmatrix} = \boldsymbol{J}_{f_{2j}}\begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix}, \quad \boldsymbol{J}_{f_{2j}} = \begin{bmatrix} \boldsymbol{E}_{3\times3} & \frac{r^A_{f_{2j}}\hat{\boldsymbol{\delta}}_{2j}({}^A\boldsymbol{R}^A_{2j1}\boldsymbol{R}^T_{2j2} - {}^A\boldsymbol{R}^A_{2j2}\boldsymbol{R}^T_{2j1})^A\hat{\boldsymbol{\delta}}^2_{2j}({}^A\hat{e}_{2j} + {}^A\hat{\boldsymbol{r}}_o)}{({}^A\boldsymbol{R}_{2j1}\times{}^A\boldsymbol{R}_{2j2})\cdot\ {}^A\boldsymbol{r}_{2j}} + {}^A\hat{\boldsymbol{r}}_o \\ \boldsymbol{0}_{3\times3} & -\frac{({}^A\boldsymbol{R}^A_{2j1}\boldsymbol{R}^T_{2j2} - {}^A\boldsymbol{R}^A_{2j2}\boldsymbol{R}^T_{2j1})^A\hat{\boldsymbol{\delta}}^2_{2j}({}^A\hat{e}_{2j} + {}^A\hat{\boldsymbol{r}}_o){}^A_D\boldsymbol{\omega}}{({}^A\boldsymbol{R}_{2j1}\times{}^A\boldsymbol{R}_{2j2})\cdot\ {}^A\boldsymbol{r}_{2j}} \end{bmatrix}, \tag{27c}$$

where $\boldsymbol{J}_{f_{2j}}$ is a 6 × 6 Jacobian matrix of the *j*th cylinder for 3-UPS+S PM, which relates the velocity mapping relations between the *j*th cylinder of the 3-UPS+S PM and the terminal platform *D*.

Let $o_{m_{2j}}$ be the mass center of the *j*th piston for the *i*th PM, $r_{m_{2j}}$ be the distance from $o_{m_{2j}}$ to the point D_j. The position and velocity vectors of $o_{m_{2j}}$ in $\{A\}$ can be expressed as follows:

$${}^A\boldsymbol{o}_{m_{2j}} = {}^A\boldsymbol{D}_i - r^A_{m_{2j}}\boldsymbol{\delta}_{2j}, \tag{28a}$$

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$$
\begin{aligned}
{}^{A}\boldsymbol{v}_{m_{2j}} &= {}^{A}\boldsymbol{v}_{D_i} - r^{A}_{m_{2j}}\boldsymbol{\omega}_{r_{2j}} \times {}^{A}\boldsymbol{\delta}_{2j} = {}^{A}\boldsymbol{v}_{o_d} + {}^{A}_{D}\boldsymbol{\omega} \times {}^{A}\boldsymbol{e}_{2j} - r^{A}_{m_{2j}}\boldsymbol{\omega}_{r_{2j}} \times {}^{A}\boldsymbol{\delta}_{2j} \\
&= \begin{bmatrix} \boldsymbol{E}_{3\times 3} & -\frac{r^{A}_{m_{2j}}\hat{\boldsymbol{\delta}}_{2j}({}^{A}\boldsymbol{R}^{A}_{2j1}\boldsymbol{R}^{T}_{2j2} - {}^{A}\boldsymbol{R}^{A}_{2j2}\boldsymbol{R}^{T}_{2j1}){}^{A}\hat{\boldsymbol{\delta}}^{2}_{2j}({}^{A}\hat{e}_{2j} + {}^{A}\hat{\boldsymbol{r}}_{o})}{({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}_{2j2})\cdot {}^{A}\boldsymbol{r}_{2j}} - {}^{A}\hat{\boldsymbol{e}}_{i} \end{bmatrix} \begin{bmatrix} {}^{A}\boldsymbol{v}_{o_d} \\ {}^{A}_{D}\boldsymbol{\omega} \end{bmatrix}.
\end{aligned}
\tag{28b}
$$

From Eqs. (22) and (28b), it leads to

$$
\begin{bmatrix} {}^{A}\boldsymbol{v}_{m_{2j}} \\ {}^{A}\boldsymbol{\omega}_{r_{2j}} \end{bmatrix} = \boldsymbol{J}_{m_{2j}} \begin{bmatrix} {}^{A}\boldsymbol{v}_{o_d} \\ {}^{A}_{D}\boldsymbol{\omega} \end{bmatrix}, \quad \boldsymbol{J}_{m_{2j}} = \begin{bmatrix} \boldsymbol{E}_{3\times 3} & -\frac{r^{A}_{m_{2j}}\hat{\boldsymbol{\delta}}_{2j}({}^{A}\boldsymbol{R}^{A}_{2j1}\boldsymbol{R}^{T}_{2j2} - {}^{A}\boldsymbol{R}^{A}_{2j2}\boldsymbol{R}^{T}_{2j1}){}^{A}\hat{\boldsymbol{\delta}}^{2}_{2j}({}^{A}\hat{e}_{2j} + {}^{A}\hat{\boldsymbol{r}}_{o})}{({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}_{2j2})\cdot {}^{A}\boldsymbol{r}_{2j}} - {}^{A}\hat{\boldsymbol{e}}_{i} \\ \boldsymbol{0}_{3\times 3} & -\frac{({}^{A}\boldsymbol{R}^{A}_{2j1}\boldsymbol{R}^{T}_{2j2} - {}^{A}\boldsymbol{R}^{A}_{2j2}\boldsymbol{R}^{T}_{2j1}){}^{A}\hat{\boldsymbol{\delta}}^{2}_{2j}({}^{A}\hat{e}_{2j} + {}^{A}\hat{\boldsymbol{r}}_{o}){}^{A}_{D}\boldsymbol{\omega}}{({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}_{2j2})\cdot {}^{A}\boldsymbol{r}_{2j}} \end{bmatrix},
\tag{28c}
$$

where $\boldsymbol{J}_{m_{2j}}$ is a 6 × 6 Jacobian matrix of the *j*th piston for 3-UPS+S PM, which relates the velocity mapping relation between the *j*th piston of the 3-UPS+S PM and the terminal platform *D*.

Let $\boldsymbol{v}_{r_o}$ and $\boldsymbol{a}_{r_o}$ be the velocity and acceleration vectors of the mass center of r_o. Let $\boldsymbol{\omega}_{r_o}$ and ${}^{A}\boldsymbol{\varepsilon}_{r_o}$ be the angular velocity and angular acceleration vectors of r_o. It leads to

$$
{}^{A}\boldsymbol{\omega}_{r_o} = {}^{A}_{D}\boldsymbol{\omega}, \quad {}^{A}\boldsymbol{\varepsilon}_{r_o} = {}^{A}_{D}\boldsymbol{\varepsilon}.
\tag{29a}
$$

$\boldsymbol{v}_{r_o}$ can be solved by the following formula:

$$
{}^{A}\boldsymbol{v}_{r_o} = {}^{A}\boldsymbol{v}_{ob} + {}^{A}\boldsymbol{\omega}_{r_o} \times {}^{A}\boldsymbol{r}_{o}/2 = {}^{A}\boldsymbol{v}_{o_d} - {}^{A}\boldsymbol{\omega}_{r_o} \times {}^{A}\boldsymbol{r}_{o}/2 = \begin{bmatrix} \boldsymbol{E}_{3\times 3} & {}^{A}\hat{\boldsymbol{r}}_{o}/2 \end{bmatrix} \begin{bmatrix} {}^{A}\boldsymbol{v}_{o_d} \\ {}^{A}_{D}\boldsymbol{\omega} \end{bmatrix}.
\tag{29b}
$$

From (29a) and (29b), it leads to

$$
\begin{bmatrix} {}^{A}\boldsymbol{v}_{r_o} \\ {}^{A}\boldsymbol{\omega}_{r_o} \end{bmatrix} = \boldsymbol{J}_{r_o} \begin{bmatrix} {}^{A}\boldsymbol{v}_{o_d} \\ {}^{A}_{D}\boldsymbol{\omega} \end{bmatrix}, \quad \boldsymbol{J}_{r_o} = \begin{bmatrix} \boldsymbol{E}_{3\times 3} & -{}^{A}\hat{\boldsymbol{r}}_{o}/2 \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{E}_{3\times 3} \end{bmatrix},
\tag{30}
$$

where $\boldsymbol{J}_{r_o}$ is a 6 × 6 Jacobian matrix of the r_o, which relates the velocity mapping relations between r_o and the terminal platform *D*.

3.2. *Inverse dynamics modeling*

The inverse dynamics analysis of the (3-UPU)+(3-UPS+S) S-PM is to determine the required forces of actuators from the given kinematics of the terminal platform in a given pose.

3.2.1. The inertia force and torque of each component.. Let $m_{f_{ij}}, \boldsymbol{I}_{f_{ij}}, \boldsymbol{f}_{f_{ij}}, \boldsymbol{n}_{f_{ij}}$ and $\boldsymbol{G}_{f_{ij}}$ ($i = 1, 2; j = 1, 2, 3$) be the mass, inertia matrix, inertia force, inertia torque, and the gravity of the *j*th cylinder in the *i*th PM, respectively. Let $m_{mij}, \boldsymbol{I}_{mij}, \boldsymbol{f}_{mij}, \boldsymbol{n}_{mij}$, and $\boldsymbol{G}_{m_{ij}}$ be the mass, inertia matrix, inertia force, inertia torque, and the gravity of the *j*th piston in the *i*th PM, respectively. Let $m_{o_i}, \boldsymbol{I}_{o_i}, \boldsymbol{f}_{o_i}, \boldsymbol{n}_{o_i}$, and $\boldsymbol{G}_{o_i}$ be the mass, inertia matrix, inertia force, inertia torque, and the gravity of the upper platform for *i*th PM. Let $m_{r_o}, \boldsymbol{I}_{r_o}, \boldsymbol{f}_{r_o}, \boldsymbol{n}_{r_o}$, and $\boldsymbol{G}_{r_o}$ be the mass, inertia matrix, inertia force, inertia torque, and the gravity of r_o. Let $\boldsymbol{F}_{o2}$, $\boldsymbol{T}_{o2}$ be the workloads applied onto *D* at o_d.

Differentiating both sides of Eq. (20) with respect to time, the angular acceleration of r_{1j} for the lower PM can be derived as following:

$$
\begin{aligned}
{}^{A}\boldsymbol{\varepsilon}_{r1j} = {} & \frac{-\dot{\theta}^{A}_{1j1}\boldsymbol{R}_{1j2} \times ({}^{A}\boldsymbol{v}_{Bj} - v^{A}_{r1j}\boldsymbol{\delta}_{1j}) + \left({}^{A}\boldsymbol{R}_{1j1} \times {}^{A}\boldsymbol{R}^{T}_{1j2}\right) \times ({}^{A}\boldsymbol{a}_{Bj} - a^{A}_{r1j}\boldsymbol{\delta}_{1j} - v^{A}_{r1j}\boldsymbol{\omega}_{r1j} \times {}^{A}\boldsymbol{\delta}_{1j})}{({}^{A}\boldsymbol{R}_{1j1} \times {}^{A}\boldsymbol{R}_{1j2})\cdot {}^{A}\boldsymbol{r}_{1j}} \\
& - \frac{({}^{A}\boldsymbol{R}_{1j1} \times {}^{A}\boldsymbol{R}^{T}_{1j2}) \times ({}^{A}\boldsymbol{v}_{Bj} - v^{A}_{r1j}\boldsymbol{\delta}_{1j})}{\left[({}^{A}\boldsymbol{R}_{1j1} \times {}^{A}\boldsymbol{R}_{1j2})\cdot {}^{A}\boldsymbol{r}_{1j}\right]^{2}} \left[{}^{A}\boldsymbol{a}_{Bj} \cdot \left({}^{A}\boldsymbol{R}_{1j1} \times {}^{A}\boldsymbol{R}^{T}_{1j2}\right)\right].
\end{aligned}
\tag{31a}
$$

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Differentiating both sides of Eq. (22) with respect to time, the angular acceleration of r_{2j} can be derived as following:

$$
{}^{A}\boldsymbol{\varepsilon}_{r2j} = \frac{-\dot{\theta}^{A}_{2j1}\boldsymbol{R}_{2j2} \times ({}^{A}\boldsymbol{v}_{Dj} - {}^{A}\boldsymbol{v}_{Cj} - v^{A}_{r2j}\boldsymbol{\delta}_{2j}) + \left({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}_{2j2}\right) \times ({}^{A}\boldsymbol{a}_{D_i} - {}^{A}\boldsymbol{a}_{C_i} - a^{A}_{r2j}\boldsymbol{\delta}_{2j} - v^{A}_{r2j}\boldsymbol{\omega}_{r2j} \times {}^{A}\boldsymbol{\delta}_{2j})}{({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}_{2j2}) \cdot {}^{A}\boldsymbol{r}_{2j}}
- \frac{\left({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}_{2j2}\right) \times ({}^{A}\boldsymbol{v}_{Dj} - {}^{A}\boldsymbol{v}_{Cj} - v^{A}_{r2j}\boldsymbol{\delta}_{2j})}{\left[({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}_{2j2}) \cdot {}^{A}\boldsymbol{r}_{2j}\right]^2}\left[({}^{A}\boldsymbol{v}_{Dj} - {}^{A}\boldsymbol{v}_{Cj}) \cdot ({}^{A}\boldsymbol{R}_{2j1} \times {}^{A}\boldsymbol{R}_{2j2})\right]. \tag{31b}
$$

Differentiating both sides of Eq. (24a) with respect to time, the acceleration of the mass center of the jth cylinder of the lower PM can be derived as following:

$$
\boldsymbol{a}_{r1j} = {}^{A}\boldsymbol{\varepsilon}_{r1j} \times {}^{A}\boldsymbol{\delta}_{1j} r_{1j} + {}^{A}\boldsymbol{\omega}_{r1j} \times ({}^{A}\boldsymbol{\omega}_{r1j} \times {}^{A}\boldsymbol{\delta}_{1j}) r_{1j}. \tag{32a}
$$

Differentiating both sides of Eq. (25) with respect to time, the acceleration of the mass center of the jth piston of the lower PM can be derived as following:

$$
{}^{A}\boldsymbol{a}_{m1j} = {}^{A}\boldsymbol{\varepsilon}_{r1j} \times {}^{A}\boldsymbol{\delta}_{1j}(r_{1j} - r_{m1j}) + {}^{A}\boldsymbol{\omega}_{r1j} \times ({}^{A}\boldsymbol{\omega}_{r1j} \times {}^{A}\boldsymbol{\delta}_{1j})(r_{1j} - r_{m1j})
+ {}^{A}\boldsymbol{\delta}_{1j}\ddot{r}_{1j} + 2({}^{A}\boldsymbol{\omega}_{r1j} \times {}^{A}\boldsymbol{\delta}_{1j})\dot{r}_{1j} \tag{32b}
$$

Differentiating both sides of Eq. (27b) with respect to time, the acceleration of the mass center of the jth cylinder of the upper PM can be derived as following:

$$
{}^{A}\boldsymbol{a}_{f_{2j}} = {}^{A}\boldsymbol{a}_{o_a} + r^{A}_{f_{2j}}\boldsymbol{\varepsilon}_{f_{2j}} \times {}^{A}\boldsymbol{\delta}_{2j} + r^{A}_{f_{2j}}\boldsymbol{\omega}_{f_{2j}} \times ({}^{A}\boldsymbol{\omega}_{f_{2j}} \times {}^{A}\boldsymbol{\delta}_{2j}). \tag{33a}
$$

From Eq. (28b), it leads to

$$
{}^{A}\boldsymbol{a}_{m_{2j}} = {}^{A}\boldsymbol{a}_{D_i} - r^{A}_{m_{2j}}\boldsymbol{\varepsilon}_{r_{2j}} \times {}^{A}\boldsymbol{\delta}_{2j} - r^{A}_{m_{2j}}\boldsymbol{\omega}_{r_{2j}} \times ({}^{A}\boldsymbol{\omega}_{r_{2j}} \times {}^{A}\boldsymbol{\delta}_{2j}). \tag{33b}
$$

Differentiating both sides of Eq. (29b) with respect to time, $\boldsymbol{a}_{r_o}$ can be solved as following:

$$
{}^{A}\boldsymbol{a}_{r_o} = {}^{A}\boldsymbol{a}_{o_d} - {}^{A}\boldsymbol{\varepsilon}_{r_o} \times {}^{A}\boldsymbol{r}_o/2 - {}^{A}\boldsymbol{\omega}_{r_o} \times ({}^{A}\boldsymbol{\omega}_{r_o} \times {}^{A}\boldsymbol{r}_o)/2. \tag{34}
$$

From Eqs. (31)–(34), the corresponding inertia force, torque, and the gravity can be derived as follows:

$$
\begin{aligned}
{}^{A}\boldsymbol{f}_{f_{ij}} &= -m^{A}_{f_{ij}}\boldsymbol{a}_{f_{ij}}, {}^{A}\boldsymbol{G}_{f_{ij}} = m_{f_{ij}}\boldsymbol{g}, {}^{A}\boldsymbol{n}_{f_{ij}} = -{}^{A}\boldsymbol{I}^{A}_{f_{ij}}\boldsymbol{\varepsilon}_{r_{ij}} - {}^{A}\boldsymbol{\omega}_{r_{ij}} \times ({}^{A}\boldsymbol{I}^{A}_{fij}\boldsymbol{\omega}_{rij}),\\
{}^{A}\boldsymbol{f}_{mij} &= -m^{A}_{m_{ij}}\boldsymbol{a}_{m_{ij}}, {}^{A}\boldsymbol{G}_{m_{ij}} = m_{m_{ij}}\boldsymbol{g}, {}^{A}\boldsymbol{n}_{m_{ij}} = -{}^{A}\boldsymbol{I}^{A}_{mij}\boldsymbol{\varepsilon}_{m_{ij}} - {}^{A}\boldsymbol{\omega}_{r_{ij}} \times ({}^{A}\boldsymbol{I}^{A}_{m_{ij}}\boldsymbol{\omega}_{r_{ij}}),\\
{}^{A}\boldsymbol{f}_{o_i} &= -m^{A}_{o_i}\boldsymbol{a}_{o_i}, {}^{A}\boldsymbol{G}_{oi} = m_{o_i}\boldsymbol{g}, {}^{A}\boldsymbol{n}_{o_i} = -{}^{A}\boldsymbol{I}^{A}_{o_i}\boldsymbol{\varepsilon}_{o_i} - {}^{A}\boldsymbol{\omega}_{oi} \times ({}^{A}\boldsymbol{I}^{A}_{oi}\boldsymbol{\omega}_{oi}), {}^{A}\boldsymbol{I}_{o_1} = {}^{A}_{B}R^{B}\boldsymbol{I}_{o_1},\\
{}^{A}\boldsymbol{f}_{r_o} &= -m^{A}_{r_o}\boldsymbol{a}_{r_o}, {}^{A}\boldsymbol{G}_{r_o} = m_{r_o}\boldsymbol{g}, {}^{A}\boldsymbol{n}_{r_o} = -{}^{A}\boldsymbol{I}^{A}_{r_o}\boldsymbol{\varepsilon}_{r_o} - {}^{A}\boldsymbol{\omega}_{r_o} \times ({}^{A}\boldsymbol{I}^{A}_{r_o}\boldsymbol{\omega}_{r_o}),\\
{}^{A}\boldsymbol{I}_{f_{ij}} &= {}^{A}_{ij}R^{ij}\boldsymbol{I}_{f_{ij}}, {}^{A}\boldsymbol{I}_{m_{ij}} = {}^{A}_{ij}R^{ij}\boldsymbol{I}_{m_{ij}}, {}^{A}\boldsymbol{I}_{o_1} = {}^{A}_{B}R^{B}\boldsymbol{I}_{o_1}, {}^{A}\boldsymbol{I}_{o_2} = {}^{A}_{D}R^{D}\boldsymbol{I}_{o_1},\\
{}^{A}_{ij}R &= [\,\boldsymbol{R}_{ij2} \quad \boldsymbol{\delta}_{ij} \times \boldsymbol{R}_{ij2} \quad \boldsymbol{\delta}_{ij}\,],
\end{aligned} \tag{35}
$$

where ${}^{A}_{ij}R$ denotes the rotational matrix of $\{ij\}$ relative to $\{A\}$. $\{ij\}$ is a coordinate frame with $\boldsymbol{R}_{ij2}$, $\boldsymbol{\delta}_{ij} \times \boldsymbol{R}_{ij2}$, and $\boldsymbol{\delta}_{ij}$ are the diction vectors corresponding to their three orthogonal coordinate axes, which is used to express the inertia matrices.

3.2.2. Dynamics formula derivation.. From the principle of virtual work, it leads to

$$\begin{aligned}
&\boldsymbol{F}_q^T \boldsymbol{V}_r + \sum_{i=1}^{2}\sum_{j=1}^{3}\left(\begin{bmatrix} {}^A\boldsymbol{f}_{f_{ij}}^T + {}^A\boldsymbol{G}_{f_{ij}}^T & {}^A\boldsymbol{n}_{fi_j}^T \end{bmatrix}\begin{bmatrix} {}^A\boldsymbol{v}_{fi_j} \\ {}^A\boldsymbol{\omega}_{rij} \end{bmatrix} + \begin{bmatrix} {}^A\boldsymbol{f}_{mij}^T + {}^A\boldsymbol{G}_{m_{ij}}^T & {}^A\boldsymbol{n}_{m_{ij}}^T \end{bmatrix}\begin{bmatrix} {}^A\boldsymbol{v}_{mij} \\ {}^A\boldsymbol{\omega}_{rij} \end{bmatrix}\right) \\
&+ \begin{bmatrix} {}^A\boldsymbol{f}_{o_1}^T + {}^A\boldsymbol{G}_{o_1}^T & {}^A\boldsymbol{n}_{o_1}^T \end{bmatrix}\begin{bmatrix} {}^A\boldsymbol{v}_{ob} \\ {}^A_B\boldsymbol{\omega} \end{bmatrix} + \begin{bmatrix} {}^A\boldsymbol{F}_{o_2}^T + {}^A\boldsymbol{f}_{o_2}^T + {}^A\boldsymbol{G}_{o_2}^T & {}^A\boldsymbol{T}_{o_2}^T + {}^A\boldsymbol{n}_{o_2}^T \end{bmatrix}\begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix} \\
&+ \begin{bmatrix} {}^A\boldsymbol{f}_{r_o}^T + {}^A\boldsymbol{G}_{r_o}^T & {}^A\boldsymbol{n}_{r_o}^T \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_{r_o} \\ \boldsymbol{\omega}_{r_o} \end{bmatrix} = 0. \qquad (36a)
\end{aligned}$$

Substituting Eqs. (15), (23b), (24b), (26), (27c), (28c), and (30) into Eq. (36a), it leads to

$$\begin{aligned}
&\boldsymbol{F}_q^T \boldsymbol{J}\begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix} + \sum_{i=1}^{2}\sum_{j=1}^{3}\left(\begin{bmatrix} {}^A\boldsymbol{f}_{f_{ij}}^T + {}^A\boldsymbol{G}_{f_{ij}}^T & {}^A\boldsymbol{n}_{fi_j}^T \end{bmatrix}\boldsymbol{J}_{f_{ij}}\begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix} + \begin{bmatrix} {}^A\boldsymbol{f}_{mij}^T + {}^A\boldsymbol{G}_{m_{ij}}^T & {}^A\boldsymbol{n}_{m_{ij}}^T \end{bmatrix}\boldsymbol{J}_{m_{ij}}\begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix}\right) \\
&+ \begin{bmatrix} {}^A\boldsymbol{f}_{o_1}^T + {}^A\boldsymbol{G}_{o_1}^T & {}^A\boldsymbol{n}_{o_1}^T \end{bmatrix}\boldsymbol{J}_{cd}\begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix} + \begin{bmatrix} {}^A\boldsymbol{F}_{o_2}^T + {}^A\boldsymbol{f}_{o_2}^T + {}^A\boldsymbol{G}_{o_2}^T & {}^A\boldsymbol{T}_{o_2}^T + {}^A\boldsymbol{n}_{o_2}^T \end{bmatrix}\begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix} \\
&+ \begin{bmatrix} {}^A\boldsymbol{f}_{r_o}^T + {}^A\boldsymbol{G}_{r_o}^T & {}^A\boldsymbol{n}_{r_o}^T \end{bmatrix}\boldsymbol{J}_{r_o}\begin{bmatrix} {}^A\boldsymbol{v}_{o_d} \\ {}^A_D\boldsymbol{\omega} \end{bmatrix} = 0. \qquad (36b)
\end{aligned}$$

From Eq. (36b), the formula for solving dynamics is derived as following:

$$\begin{aligned}
\boldsymbol{F}_q = -(\boldsymbol{J}^{-1})^T &\left(\boldsymbol{J}_{cd}^T\begin{bmatrix} \boldsymbol{f}_{o_1} + \boldsymbol{G}_{o_1} \\ \boldsymbol{n}_{o_1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{F}_{o_2} + \boldsymbol{f}_{o_2} + \boldsymbol{G}_{o_2} \\ \boldsymbol{T}_{o_2} + \boldsymbol{n}_{o_2} \end{bmatrix}\right. \\
&\left. + \sum_{i=1}^{2}\sum_{j=1}^{3}\left(\boldsymbol{J}_{fij}^T\begin{bmatrix} \boldsymbol{f}_{f_{ij}} + \boldsymbol{G}_{f_{ij}} \\ \boldsymbol{n}_{f_{ij}} \end{bmatrix} + \boldsymbol{J}_{mij}^T\begin{bmatrix} \boldsymbol{f}_{m_{ij}} + \boldsymbol{G}_{mi_j} \\ \boldsymbol{n}_{mij} \end{bmatrix}\right) + \boldsymbol{J}_{r_o}^T\begin{bmatrix} \boldsymbol{f}_{r_o} + \boldsymbol{G}_{r_o} \\ \boldsymbol{n}_{r_o} \end{bmatrix}\right). \qquad (36c)
\end{aligned}$$

4. Workspace

The workspace of (3-UPU)+(3-UPS+S) S-PM can be solved using CAD variation geometry approach.[5] In CAD software, the simulation mechanism of the (3-UPU)+(3-UPS+S) S-PM can be easily constructed. When given the maximum extension $r_{1max} = 1.6$ m and the minimum extension $r_{i\min} = 1.2$ m for r_{1j} ($j = 1, 2, 3$), the maximum extension $r_{2\max} = 1.2$ m and the minimum extension $r_{2\min} = 0.85$ m for r_{2j} ($j = 1, 2, 3$), and the increment $\delta r_i = 0.05$ m ($i = 1, 2$) for each active leg, by varying the driving dimensions of r_{ij} in the given extent, the simulation mechanism varies correspondingly and the position components of the center of the terminal platform are solved automatically. The workspace of the (3-UPU)+(3-UPS+S) S-PM is formed by some sub-workspaces. When four of r_{ij} ($i = 1, 2; j = 1, 2, 3$) reach their limited values, varying the remaining two of r_{ij} from $r_{i\min}$ to $r_{i\max}$, each sub-workspace can be constructed. The construction processes of one sub-workspace are described as follows:

Step 1: Set $r_{13} = r_{1\max}$, $r_{21} = r_{22} = r_{23} = r_{2\max}$, $r_{11} = r_{1\min} + (j-1)\delta r_i (j = 1, \ldots, n_1)$, where $n_1 = (r_{1\max} - r_{1\min})/\delta r_1$.

Step 2: Set $j = 1$, increasing r_{12} by δr_1 at each increment from $r_{1\min}$ to $r_{1\max}$, the position components ($X_o Y_o Z_o$) are solved. By transferring the position solutions into spatial spline curves in the 3D software, a spatial curve is formed from the solved points.

Step 3: Repeat step 2 – except set $j = 2, \ldots, n_1$, other spatial curves can be constructed. Constructing the n_1 spatial curves $c_j (j = 1, \ldots, n_1)$ by the loft command, one sub-workspace surface can be obtained.

Step 4: Repeat steps 1–3: except set r_{ij} verifying versus Table I, other sub-workspace can be obtained.

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Table I. The construction processes of sub-workspace.

r_{11}	r_{12}	r_{13}	r_{21}	r_{22}	r_{23}
$r_{1\min} - r_{1\max}$	$r_{1\min} - r_{1\max}$	$r_{1\max}$	$r_{2\max}$	$r_{2\max}$	$r_{2\max}$
$r_{1\min} - r_{1\max}$	$r_{1\max}$	$r_{1\min} - r_{1\max}$	$r_{2\max}$	$r_{2\max}$	$r_{2\max}$
$r_{1\max}$	$r_{1\min} - r_{1\max}$	$r_{1\min} - r_{1\max}$	$r_{2\max}$	$r_{2\max}$	$r_{2\max}$
$r_{1\min}$	$r_{1\min} - r_{1\max}$	$r_{1\max}$	$r_{2\min} - r_{2\max}$	$r_{2\max}$	$r_{2\max}$
$r_{1\min}$	$r_{1\min}$	$r_{1\max}$	$r_{2\min} - r_{2\max}$	$r_{2\min} - r_{2\max}$	$r_{2\max}$
$r_{1\min} - r_{1\max}$	$r_{1\min}$	$r_{1\max}$	$r_{2\max}$	$r_{2\min} - r_{2\max}$	$r_{2\max}$
$r_{1\max}$	$r_{1\min}$	$r_{1\min} - r_{1\max}$	$r_{2\max}$	$r_{2\min} - r_{2\max}$	$r_{2\max}$
$r_{1\max}$	$r_{1\min}$	$r_{1\min}$	$r_{2\max}$	$r_{2\min} - r_{2\max}$	$r_{2\min} - r_{2\max}$
$r_{1\max}$	$r_{1\min} - r_{1\max}$	$r_{1\min}$	$r_{2\max}$	$r_{2\max}$	$r_{2\min} - r_{2\max}$
$r_{1\min} - r_{1\max}$	$r_{1\max}$	$r_{1\min}$	$r_{2\max}$	$r_{2\max}$	$r_{2\min} - r_{2\max}$
$r_{1\min}$	$r_{1\max}$	$r_{1\min}$	$r_{2\min} - r_{2\max}$	$r_{2\max}$	$r_{2\min} - r_{2\max}$
$r_{1\min}$	$r_{1\max}$	$r_{1\min} - r_{1\max}$	$r_{2\min} - r_{2\max}$	$r_{2\max}$	$r_{2\max}$
$r_{1\min}$	$r_{1\min} - r_{1\max}$	$r_{1\min}$	$r_{2\min} - r_{2\max}$	$r_{2\max}$	$r_{2\min}$
$r_{1\min} - r_{1\max}$	$r_{1\min} - r_{1\max}$	$r_{1\min}$	$r_{2\max}$	$r_{2\max}$	$r_{2\min}$
$r_{1\min} - r_{1\max}$	$r_{1\min}$	$r_{1\min}$	$r_{2\max}$	$r_{2\min} - r_{2\max}$	$r_{2\min}$
$r_{1\min} - r_{1\max}$	$r_{1\min}$	$r_{1\min}$	$r_{2\max}$	$r_{2\min}$	$r_{2\min} - r_{2\max}$
$r_{1\min} - r_{1\max}$	$r_{1\min}$	$r_{1\min} - r_{1\max}$	$r_{2\max}$	$r_{2\min}$	$r_{2\max}$
$r_{1\min}$	$r_{1\min}$	$r_{1\min} - r_{1\max}$	$r_{2\min} - r_{2\max}$	$r_{2\min}$	$r_{2\max}$
$r_{1\min}$	$r_{1\min}$	$r_{1\min} - r_{1\max}$	$r_{2\min}$	$r_{2\min} - r_{2\max}$	$r_{2\max}$
$r_{1\min}$	$r_{1\min} - r_{1\max}$	$r_{1\min} - r_{1\max}$	$r_{2\min}$	$r_{2\max}$	$r_{2\max}$
$r_{1\min}$	$r_{1\min} - r_{1\max}$	$r_{1\min}$	$r_{2\min}$	$r_{2\max}$	$r_{2\min} - r_{2\max}$
$r_{1\min}$	$r_{1\min}$	$r_{1\min}$	$r_{2\min} - r_{2\max}$	$r_{2\min}$	$r_{2\min} - r_{2\max}$
$r_{1\min}$	$r_{1\min}$	$r_{1\min}$	$r_{2\min} - r_{2\max}$	$r_{2\min} - r_{2\max}$	$r_{2\min}$
$r_{1\min}$	$r_{1\min}$	$r_{1\min}$	$r_{2\min}$	$r_{2\min} - r_{2\max}$	$r_{2\min} - r_{2\max}$

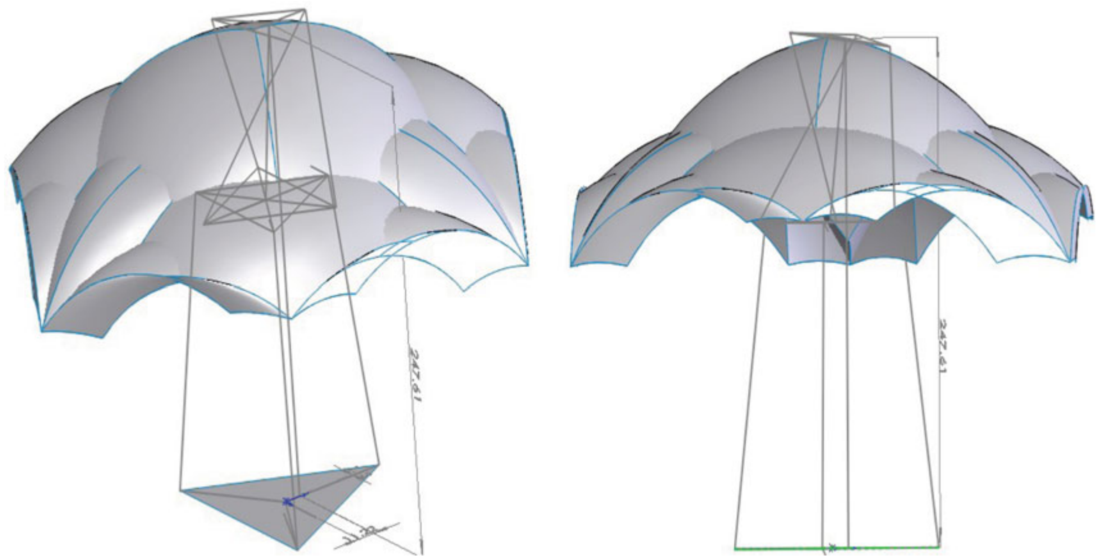

Fig. 2. A reachable workspace of the (3-UPU)+(3-UPS+S) S-PM: (a) the isometric view; (b) the front view.

The workspace of (3-UPU)+(3-UPS+S) S-PM is constructed as follows (see Fig. 2):

5. Analytic Solved Example

Set the dimension parameters of the (3-UPU)+(3-UPS+S) S-PM as: $E_1 = 120/\text{q}$ m, $e_1 = E_2 = 80/\text{q}$ m, $e_2 = 60/\text{q}$m, $r_o = 0.90$ m. Set the workloads applied onto D at o_d as: $\boldsymbol{F}_{o2} = [-20 - 30 - 60]^{\mathrm{T}}$, $\boldsymbol{T}_{o2} = [-30 - 30100]^{\mathrm{T}}$. Set the mass and inertial parameters as: $m_{o1} = 67.84$ Kg, $m_{o2} = 14.83$

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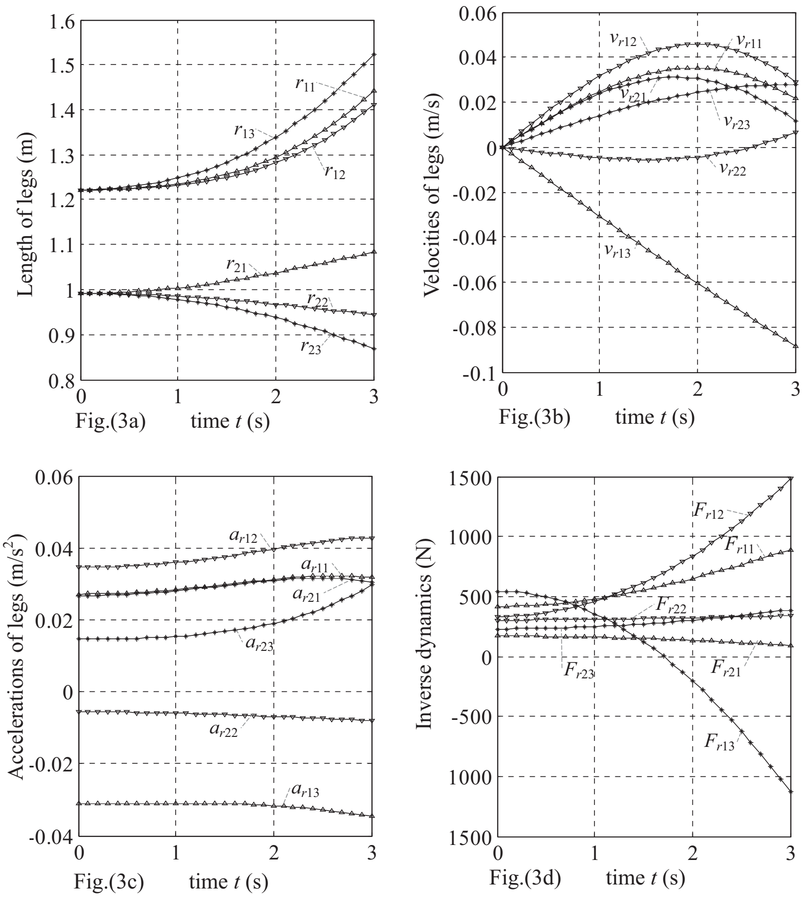

Fig. 3. Solved results of the (3-UPU)+(3-UPS+S) S-PM: (a) length of r_{ij}; (b) velocities of r_{ij}; (c) accelerations of r_{ij}; (d) active forces of r_{ij}.

Kg, $m_{f11} = m_{f12} = m_{f13} = 9.43$ Kg, $m_{f21} = m_{f22} = m_{f23} = 3.90$ Kg, $m_{m11} = m_{m12} = m_{m13} = 5.54$ Kg, $m_{m21} = m_{m22} = m_{m23} = 2.80$ Kg, $m_{ro} = 3.55$ Kg, ${}^{B}I_{o1} =$ diag[2.5725 2.5725 4.9206] Kg m^2, ${}^{D}I_{o2} =$ diag[0.3281 0.3281 0.6525] Kg m^2, ${}^{11}I_{f11} = {}^{12}I_{f12} = {}^{13}I_{f13} =$ diag[0.6596 0.6596 0.0133] Kg m^2, ${}^{21}I_{f21} = {}^{22}I_{f22} = {}^{23}I_{f23} =$ diag[0.2107 0.2092 0.004615] Kg m^2, ${}^{11}I_{m11} = {}^{12}I_{m12} = {}^{13}I_{m13} =$ diag[0.4249 0.4249 0.0065] Kg m^2, ${}^{21}I_{m21} = {}^{22}I_{m22} = {}^{23}I_{m23} =$ diag[0.1596 0.1596 0.0006] Kg m^2, ${}^{D}I_{ro} =$ diag[0.3237 0.3237 0.00739] Kg m^2.

Support the independent parameters (${}^{A}X_d, {}^{A}Y_d, {}^{A}Z_d$, α, β, λ) varying according constant accelerations with (0.03 m/s^2 0.02 m/s^2 0.03 m/s^2 -3° /s^2 2° /s^2 3° /s^2) begin at initial pose (0 m 0 m 2.10 m 0° 0° 0°) from immobile state. The extension, velocity, acceleration of active legs r_{ij} ($i = 1, 2$; $j = 1, 2, 3$) are solved as shown in Figs. 3(a)–3(c), the driving forces are solved as shown in Fig. 3(d).

Finally, in order to verify the numerical example, the results obtained using the analytical model are compared with the simulation results generated with a simulation model in Matlab/SimMechanics.[23]

Under Matlab/SimMechanics, the dimensional, mass, inertial, kinematics parameters and the workloads for the (3-UPU)+(3-UPS+S) S-PM are given according to the analytical model.

The simulation mechanism and the dynamics result generated in Matlab/SimMechanics for (3-UPU)+(3-UPS+S) S-PM are shown in Figs. 4(a) and 4(b), respectively.

From Figs. 3(d) and 4(b), it is shown that the numerical results of the case study using analytical model are in excellent agreement with the simulation result generated in Matlab/SimMechanics.

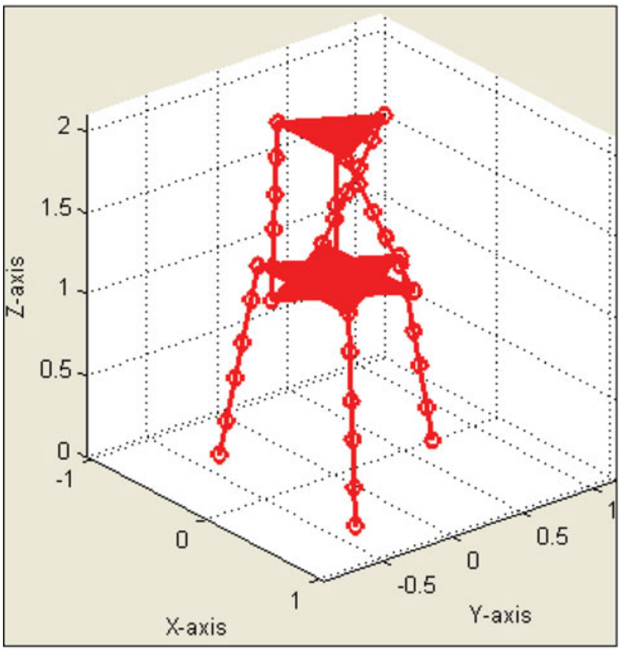

Fig.(4a) Simulation mechanism

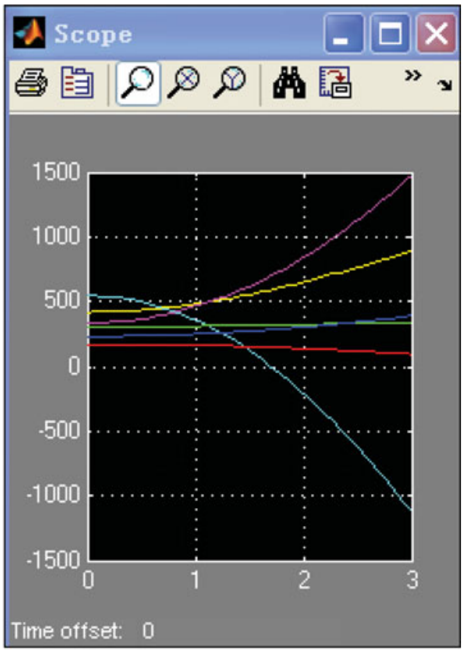

Fig.(4b) Dynamics simulation result

Fig. 4. Simulation results of the dynamics for (3-UPU)+(3-UPS+S) S-PM: (a) simulation mechanism; (b) dynamics simulation result.

6. Conclusion

The main contribution of this paper lies in the presentation and the derivation of the dynamics model of a novel (3-UPU)+(3-UPS+S) S-PM with 6 DOF. The translational and rotational motions of this S-PM can be easily controlled by the lower and the upper PMs respectively. The workspace analysis shows that this novel S-PM has large workspace. This S-PM has some potential applications for the robot arms, the machine tools, the surgical manipulators, the tunnel borers, and the satellite surveillance platform. The formulae for the inverse position, velocity, and acceleration of this S-PM are derived in explicit form. The dynamics model is established on the basis of the principle of virtual work and the kinematics model. The analytic results of the dynamics of this S-PM are verified by its simulation results. The established dynamics model presented in this paper will be valuable in the development of this S-PM. In addition, the analytic method for this S-PM is fit for other S-PMs.

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