Proceedings of the Edinburgh Mathematical Society (2019) **62**, 655–665
doi:10.1017/S001309151800024X

COMPACTNESS OF COMMUTATORS OF ONE-SIDED SINGULAR INTEGRALS IN WEIGHTED LEBESGUE SPACES

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(Received 1 August 2017; first published online 10 January 2019)

Abstract We prove that if $p > 1$, $w \in A_p^+$, $b \in CMO$ and C_b^+ is the commutator with symbol b of a Calderón–Zygmund convolution singular integral with kernel supported on $(-\infty, 0)$, then C_b^+ is compact from $L^p(w)$ into itself.

Keywords: commutators; compact operators; compactness; weights

2010 *Mathematics subject classification:* Primary 42B25; 26D10

1. Introduction and the result

In this paper, we shall work on compactness of commutators of Calderón–Zygmund convolution operators on the real line. By a Calderón–Zygmund kernel on the real line we mean a function $K \in L^1_{\mathrm{loc}}(\mathbb{R} - \{0\})$ verifying the following properties:

(i) there exists a positive constant C such

$$\left| \int_{\varepsilon<|x|<N} K(x)\,\mathrm{d}x \right| \leq C \tag{1.1}$$

for all $\varepsilon > 0$ and $N > \varepsilon$, and $\lim_{\varepsilon\to 0^+} \int_{\varepsilon<|x|<1} K(x)\,\mathrm{d}x$ exists;

(ii) there exists a positive constant C such that

$$|K(x)| \leq \frac{C}{|x|} \tag{1.2}$$

for all $x \neq 0$;

(iii) there exists a positive constant C such that

$$|K(x-y)-K(x)| \leq \frac{C|y|}{|x|^2} \tag{1.3}$$

for all x, y with $|x| > 2|y| > 0$.

Given a Calderón–Zygmund kernel K, we can define the singular integral operator T_K by

$$T_K f(x) = \text{p. v.} \int_{\mathbb{R}} K(x-y) f(y)\,\mathrm{d}y$$

and the maximal singular integral operator T_K^* by

$$T_K^* f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} K(x-y) f(y)\,\mathrm{d}y \right|.$$

Coifman and Fefferman proved in [5] that if $p > 1$ and $w \in A_p$, where A_p designates the class of Muckenhoupt's weights, then T_K and T_K^* are bounded in $L^p(w)$, where

$$L^p(w) = \left\{ f : \mathbb{R} \to \mathbb{R} : \|f\|_{p,w} = \left(\int_{\mathbb{R}} |f|^p w \right)^{1/p} < \infty \right\}.$$

These results are essentially due to the fact that the operators T_K and T_K^* can be controlled by the Hardy–Littlewood maximal operator, whose 'good weights' are precisely the A_p weights (see [10]).

Aimar *et al.*, observed in [1] that there are Calderón–Zygmund kernels supported on $(-\infty, 0)$ and that the corresponding singular integrals and maximal singular integrals (called one-sided singular integrals) can be controlled by the one-sided Hardy–Littlewood maximal operator M^+ defined by

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|.$$

This allowed them to prove that the one-sided operators T_K and T_K^* are bounded in $L^p(w)$ for w in Sawyer's class A_p^+, which are the weights for which M^+ is bounded in $L^p(w)$ (see [13] for the result and [9] for an extension). The interest of the result rests on the fact that A_p^+ is wider than A_p.

The commutator with symbol b of the singular integral T_K was introduced by Calderón in his seminal work [3]. It is defined, in the principal value sense, by

$$C_b f(x) = \int_{\mathbb{R}} (b(x) - b(y)) K(x-y) f(y)\,\mathrm{d}y.$$

Coifman *et al.* proved in [6] that if $b \in BMO$ and $p > 1$, then C_b is bounded in $L^p(\mathbb{R})$. In 1986, Uchiyama went further and proved that C_b is compact from $L^p(\mathbb{R})$ into itself if and only if $b \in CMO$, where CMO is the closure of $C_c^\infty(\mathbb{R})$ in BMO [14]. In the weighted case, Pérez [12] proved that if $p > 1$, $b \in BMO$ and $w \in A_p$, then the commutator C_b is bounded in $L^p(w)$. Finally, as a weighted counterpart of Uchiyama's theorem, Clop and

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Cruz proved recently in [4] that if $p > 1$, $b \in CMO$ and $w \in A_p$, then C_b is compact from $L^p(w)$ into itself.

The weighted inequalities for commutators of one-sided singular integrals with BMO symbols have also been studied. Lorente and Riveros proved in [8] that if $w \in A_p^+$, then the one-sided commutator

$$C_b^+ f(x) = \int_x^\infty (b(x) - b(y))K(x-y)f(y)\,\mathrm{d}y$$

is bounded in $L^p(w)$.

In this paper, we deal with the compactness of C_b^+ and prove the following theorem.

Theorem 1. *If $p > 1$, $b \in CMO$ and $w \in A_p^+$, then C_b^+ is compact from $L^p(w)$ into itself.*

Our result improves the theorem in [4], in the sense that, for one-sided singular integrals on the real line, we get compactness for a wider class of weights. In the proof, we follow the pattern of [4], but with several differences. The main one is that we apply the Frechet–Kolmogorov–Riesz theorem itself, instead of the weighted version used in [4], which is not applicable in our setting. It is worth noting that the technique of using the Frechet–Kolmogorov–Riesz theorem directly, instead of the weighted version, could be useful in order to improve the main result in [2] about the compactness of bilinear commutators in weighted spaces.

In the next section, we state several known results we will apply later. Finally, § 3 contains the proof of Theorem 1.

Throughout the paper, the letter C denotes a positive constant, not necessarily the same at each occurrence. We also use the following notation: if f is a positive measurable function and E is a measurable set, $f(E) = \int_E f$, and if f is a compactly supported function we will denote by $\operatorname{supp} f$ the support of f.

2. Preliminaries

We will say that a locally integrable positive function w, defined on the real line, verifies the condition A_p^+, $p > 1$, and write $w \in A_p^+$, if there is a positive constant C such that for all real numbers a, b, c with $a < b < c$ the inequality

$$\left(\int_a^b w\right)^{1/p}\left(\int_b^c \sigma\right)^{1/p'} \leq C(c-a)$$

holds, where p' is the conjugate exponent of p and $\sigma = w^{1-p'}$.

The condition A_p^- can be defined similarly, by simply changing the orientation in the real line. It is clear that, for $p > 1$, $w \in A_p^+$ if and only if $\sigma \in A_{p'}^-$.

We will use the following properties of one-sided weights.

Theorem 2. *If $p > 1$ and $w \in A_p^+$, then*

(i) *there is $q < p$ such that $w \in A_q^+$;*

(ii) *there is $C > 0$ such that for all real numbers a, b, c with $a < b < c$ and all measurable sets E with $E \subset (b, c)$ and $|E| \neq 0$, the inequality*

$$\frac{w(a,b)}{w(E)} \leq C\left(\frac{c-a}{|E|}\right)^p$$

holds;

(iii)

$$\int_{A+\delta}^{\infty} \frac{\sigma(y)}{(y-A)^{p'}}\,\mathrm{d}y < \infty,$$

where $\sigma = w^{1-p'}$, $A \in \mathbb{R}$ and $\delta > 0$.

Proof. The proofs of (i) and (ii) can be found in [11, 13], respectively. Let us prove (iii). Since $w \in A_p^+$, then $\sigma \in A_{p'}^-$. By (i), there is $q < p'$ such that $\sigma \in A_q^-$ and by the version of (ii) for A_q^- weights, we have

$$\begin{aligned}\int_{A+\delta}^{\infty} \frac{\sigma(y)}{(y-A)^{p'}}\,\mathrm{d}y &= \sum_{j=0}^{\infty} \int_{A+2^j\delta}^{A+2^{j+1}\delta} \frac{\sigma(y)}{(y-A)^{p'}}\,\mathrm{d}y \leq \sum_{j=0}^{\infty} \frac{1}{(2^j\delta)^{p'}} \int_{A+2^j\delta}^{A+2^{j+1}\delta} \sigma \\ &\leq C\sigma(A, A+\delta) \sum_{j=0}^{\infty} \frac{1}{(2^j\delta)^{p'}}(2^{j+1})^q \\ &= C\frac{\sigma(A, A+\delta)}{\delta^{p'}} \sum_{j=0}^{\infty} \frac{1}{2^{j(p'-q)}} < \infty. \qquad \square\end{aligned}$$

We will also apply the next results concerning A_p^+ weights.

Theorem 3 (Sawyer [13]). *Let $p > 1$. Then the one-sided maximal operator M^+ is bounded in $L^p(w)$ if and only if $w \in A_p^+$.*

Theorem 4 (Aimar *et al.* [1]). *Let $p > 1$, let w be a positive measurable function on the real line, and let K be a Calderón–Zygmund kernel supported on $(-\infty, 0)$. If $w \in A_p^+$, then the maximal singular integral T_K^* is bounded in $L^p(w)$.*

Theorem 5 (Lorente and Riveros [8]). *If $p > 1$, $b \in BMO$ and $w \in A_p^+$, then there exists $C > 0$ such that the inequality*

$$\|C_b^+ f\|_{p,w} \leq C\|b\|_{BMO}\|f\|_{p,w}$$

holds for all $f \in L^p(w)$.

If X and Y are Banach spaces, a linear operator $T : X \to Y$ is compact if $T(B_1^*)$ is precompact, i.e. it has compact closure in Y, where B_1^* designates the closed unit ball of X. It is clear that compactness implies boundedness and that if T_n is a sequence of

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compact operators which converges to the operator T in the operator norm, then T is compact.

The next result, which will be crucial in the proof of our theorem, is known as the Frechet–Kolmogorov–Riesz theorem, and characterizes the precompact subsets of Lebesgue spaces.

Theorem 6 (Yosida [15]). *Let $1 \leq p < \infty$. A subset $H \subset L^p(\mathbb{R})$ is precompact if and only if the following conditions hold:*

(a) *H is bounded in $L^p(\mathbb{R})$;*

(b) *$\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h) - f(x)|^p \,\mathrm{d}x = 0$ uniformly in $f \in H$;*

(c) *$\lim_{M\to -\infty} \int_{-\infty}^{M} |f|^p = 0$ uniformly in $f \in H$;*

(d) *$\lim_{M\to \infty} \int_{M}^{\infty} |f|^p = 0$ uniformly in $f \in H$.*

3. Proof of Theorem 1

Proof. We will work with $b \in C_c^\infty(\mathbb{R})$, since for $b \in CMO$ we can find a sequence $\{b_j\} \subset C_c^\infty(\mathbb{R})$ such that $\|b_j - b\|_{BMO} \to 0$ and therefore, by Theorem 5 [8],

$$\|C_{b_j}^+ - C_b^+\|_{L^p(w)\to L^p(w)} = \|C_{b_j-b}^+\|_{L^p(w)\to L^p(w)} \leq C\|b_j - b\|_{BMO} \to 0.$$

For each $\nu > 0$, let φ^ν be a differentiable function of one real variable such that $\varphi^\nu(x) = 1$ if $x \leq -\nu$, $\varphi^\nu(x) = 0$ if $x \geq -\nu/2$, $\|\varphi^\nu\|_\infty = 1$ and $\|(\varphi^\nu)'\|_\infty = 2/\nu$. Let $K^\nu = K \cdot \varphi^\nu$. It is easy to see that K^ν is a Calderón–Zygmund kernel supported on $(-\infty, 0)$. This truncation technique goes back to [7].

Let $C_b^{+,\nu}$ be the commutator associated with the kernel K^ν, i.e.

$$C_b^{+,\nu} f(x) = \int_x^\infty (b(x) - b(y))K^\nu(x-y)f(y)\,\mathrm{d}y.$$

For $x \in \mathbb{R}$, by the properties of the kernels K and K^ν, we have

$$\begin{aligned} |C_b^{+,\nu} f(x) - C_b^+ f(x)| &= \left| \int_x^{x+\nu} (b(x) - b(y))K(x-y)f(y)\,\mathrm{d}y \right. \\ &\quad \left. - \int_{x+(\nu/2)}^{x+\nu} (b(x) - b(y))K^\nu(x-y)f(y)\,\mathrm{d}y \right| \\ &\leq \int_x^{x+\nu} |b(x) - b(y)||K(x-y)||f(y)|\,\mathrm{d}y \end{aligned}$$

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$$+ \int_{x+(\nu/2)}^{x+\nu} |b(x) - b(y)||K^\nu(x-y)||f(y)|\,dy$$
$$\leq C\|b'\|_\infty \int_x^{x+\nu} |f| + C\|b'\|_\infty \int_{x+(\nu/2)}^{x+\nu} |f|$$
$$\leq C\|b'\|_\infty \nu M^+ f(x). \tag{3.1}$$

Then, since M^+ is bounded in $L^p(w)$, we obtain

$$\|C_b^{+,\nu} f - C_b^+ f\|_{p,w} \leq C\nu \|f\|_{p,w},$$

which implies that C_b^+ is the limit, as ν tends to 0, of the operators $C_b^{+,\nu}$ in the operator norm. Therefore, we will prove that if $w \in A_p^+$ and $b \in C_c^\infty(\mathbb{R})$, then $C_b^{+,\nu}$ is compact in $L^p(w)$.

Assume that $w \in A_p^+$ and $b \in C_c^\infty(\mathbb{R})$. We have to prove that

$$S = \{C_b^{+,\nu} f : \|f\|_{p,w} \leq 1\}$$

is precompact in $L^p(w)$, which is equivalent to proving that $H = Sw^{1/p} = \{uw^{1/p} : u \in S\}$ is precompact in $L^p(\mathbb{R})$ (without the weight). This is due to the fact that the function $T(f) = fw^{1/p}$ is a bijective isometry from $L^p(w)$ to $L^p(\mathbb{R})$. We will show that H verifies the conditions of Theorem 6. Condition (a) follows immediately by Theorem 5.

In order to prove (c) and (d), we have to see that

$$\lim_{M\to\infty} \int_M^\infty |C_b^{+,\nu} f|^p w = 0 \quad \text{uniformly in } f \text{ with } \|f\|_{p,w} \leq 1 \tag{3.2}$$

and

$$\lim_{M\to-\infty} \int_{-\infty}^M |C_b^{+,\nu} f|^p w = 0 \quad \text{uniformly in } f \text{ with } \|f\|_{p,w} \leq 1. \tag{3.3}$$

It is clear that (3.2) holds: if $x > \sup(\operatorname{supp} b)$, then $C_b^{+,\nu} f(x) = 0$. Let us see that (3.3) holds. Let M_0 be a negative number such that $(M_0/2) < \inf(\operatorname{supp} b)$ and let $M < M_0$. If $x < M$, then $y - x > (|x|/2)$ for all $y \in \operatorname{supp} b$. Applying (1.2) and the Hölder inequality, we have

$$|C_b^{+,\nu} f(x)| \leq \int_x^\infty |b(y)||K^\nu(x-y)||f(y)|\,dy \leq \|b\|_\infty \int_{\operatorname{supp} b} |K^\nu(x-y)||f(y)|\,dy$$
$$\leq C\|b\|_\infty \int_{\operatorname{supp} b} \frac{|f(y)|}{y-x}\,dy \leq C\|b\|_\infty \frac{1}{|x|} \int_{\operatorname{supp} b} |f(y)|\,dy$$
$$\leq C\|b\|_\infty \frac{1}{|x|} \|f\|_{p,w} \left(\int_{\operatorname{supp} b} \sigma\right)^{1/p'} \leq C\|b\|_\infty \frac{1}{|x|} \left(\int_{\operatorname{supp} b} \sigma\right)^{1/p'}.$$

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Then

$$\begin{aligned}\int_{-\infty}^{M} |C_b^{+,\nu} f|^p w &\le C\|b\|_\infty^p \left(\int_{-\infty}^{M} \frac{w(x)}{|x|^p}\,\mathrm{d}x\right)\left(\int_{\mathrm{supp}\ b} \sigma\right)^{p/p'} \\ &= C\|b\|_\infty^p \left(\int_{\mathrm{supp}\ b} \sigma\right)^{p/p'} \sum_{j=1}^{\infty} \int_{2^j M}^{2^{j-1}M} \frac{w(x)}{|x|^p}\,\mathrm{d}x \\ &\le C\|b\|_\infty^p \left(\int_{\mathrm{supp}\ b} \sigma\right)^{p/p'} \sum_{j=1}^{\infty} \frac{1}{(2^{j-1}|M|)^p} \int_{2^j M}^{2^{j-1}M} w(x)\,\mathrm{d}x. \end{aligned} \tag{3.4}$$

Since $w \in A_p^+$, by Theorem 2(i) there exists q with $1 < q < p$ such that $w \in A_q^+$. Applying Theorem 2(ii) with $a = 2^j M$, $b = 2^{j-1}M$, $c = 0$ and $E = (-1, 0) \subset (2^{j-1}M, 0)$, we obtain

$$w(2^j M, 2^{j-1}M) \le C w(-1, 0) 2^{jq} |M|^q.$$

Then the right-hand side of (3.4) is less than

$$C\|b\|_\infty^p \left(\int_{\mathrm{supp}\ b} \sigma\right)^{p/p'} w(-1,0) \sum_{j=1}^{\infty} \frac{1}{(2^j|M|)^{p-q}} = C\frac{1}{|M|^{p-q}}.$$

Therefore,

$$\int_{-\infty}^{M} |C_b^{+,\nu} f|^p w \le \frac{C}{|M|^{p-q}},$$

which tends to 0 as M tends to $-\infty$ uniformly in f with $\|f\|_{p,w} \le 1$.

Let us prove (b). Let $\varepsilon > 0$. By (c) there is $M_0 < 0$ such that for all $M \le M_0$ and all $f \in L^p(w)$ with $\|f\|_{p,w} \le 1$ the inequality

$$\int_{-\infty}^{M} |C_b^{+,\nu} f|^p w < \varepsilon \tag{3.5}$$

holds. Let $M_1 < M_0 - \nu$ and $M_2 > 0$ with $M_2 > \sup(\mathrm{supp}\ b) + \nu$. By L^p continuity of w, there exists $\delta_1 > 0$ such that for all h with $0 < |h| < \delta_1$ the inequalities

$$\|b'\|_\infty^p \left(\int_{M_1-\nu}^{M_2+\nu} \sigma\right)^{p/p'} \int_{M_1-\nu}^{M_2+\nu} |w^{1/p}(x) - w^{1/p}(x-h)|^p\,\mathrm{d}x < \varepsilon \tag{3.6}$$

and

$$\|b\|_\infty^p \left(\int_{M_2+\nu}^{\infty} \frac{\sigma(y)}{(y-M_2)^{p'}}\,\mathrm{d}y\right)^{p/p'} \int_{M_1-\nu}^{M_2+\nu} |w^{1/p}(x) - w^{1/p}(x-h)|^p\,\mathrm{d}x < \varepsilon \tag{3.7}$$

hold. Notice that it is possible to find such a δ_1 because σ is locally integrable and

$$\int_{M_2+\nu}^{\infty} \frac{\sigma(y)}{(y-M_2)^{p'}}\,\mathrm{d}y < \infty$$

by Theorem 2(iii).

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Let $\delta_2 > 0$ such that for all h with $0 < |h| < \delta_2$ the inequality

$$\|b'\|_\infty^p \int_{M_1}^{M_2} \left(\int_{x+(\nu/2)}^{x+h+(\nu/2)} \sigma \right)^{p/p'} w(x)\,\mathrm{d}x < \varepsilon \tag{3.8}$$

holds.

Let

$$\delta = \min\left\{\delta_1, \delta_2, \frac{\nu}{4}, \frac{\varepsilon^{1/p}}{\|b'\|_\infty}, \frac{\nu\varepsilon^{1/p}}{\|b\|_\infty}\right\}.$$

Then if $\|f\|_{p,w} \le 1$ and $0 < |h| < \delta$, we have

$$\int_{\mathbb{R}} |C_b^{+,\nu} f(x+h) w^{1/p}(x+h) - C_b^{+,\nu} f(x) w^{1/p}(x)|^p \,\mathrm{d}x = \int_{-\infty}^{M_1} + \int_{M_1}^{M_2} = J_1 + J_2,$$

since $C_b^{+,\nu} f(x) = 0$ and $C_b^{+,\nu} f(x+h) = 0$ for all $x > M_2$.

The estimation of J_1 is simple: by (3.5) we have

$$J_1 \le 2C \int_{-\infty}^{M_0} |C_b^{+,\nu} f|^p w < 2C\varepsilon.$$

In order to estimate J_2, we split it as follows:

$$\begin{aligned} J_2 &\le C \int_{M_1}^{M_2} |C_b^{+,\nu} f(x+h)|^p |w^{1/p}(x+h) - w^{1/p}(x)|^p \,\mathrm{d}x \\ &\quad + C \int_{M_1}^{M_2} |C_b^{+,\nu} f(x+h) - C_b^{+,\nu} f(x)|^p w(x)\,\mathrm{d}x = J_{21} + J_{22}. \end{aligned}$$

It is clear that

$$J_{21} \le \int_{M_1-\nu}^{M_2+\nu} |C_b^{+,\nu} f(x)|^p |w^{1/p}(x) - w^{1/p}(x-h)|^p \,\mathrm{d}x.$$

If $x \in (M_1 - \nu, M_2 + \nu)$, then

$$\begin{aligned} |C_b^{+,\nu} f(x)| &\le \int_x^{M_2+\nu} |b(x) - b(y)||K^\nu(x-y)||f(y)|\,\mathrm{d}y \\ &\quad + \int_{M_2+\nu}^{\infty} |b(x)||K^\nu(x-y)||f(y)|\,\mathrm{d}y = I(x) + II(x). \end{aligned}$$

Applying the mean value theorem, condition (1.2) and the Hölder inequality, we obtain

$$I(x) \le C\|b'\|_\infty \int_x^{M_2+\nu} |f(y)|\,\mathrm{d}y \le C\|b'\|_\infty \left(\int_x^{M_2+\nu} \sigma \right)^{1/p'} \le C\|b'\|_\infty \left(\int_{M_1-\nu}^{M_2+\nu} \sigma \right)^{1/p'}.$$

On the other hand, $II(x) = 0$ if $x \notin \operatorname{supp} b$. Otherwise, $x < M_2$ and we have

$$II(x) \le C\|b\|_\infty \int_{M_2+\nu}^{\infty} \frac{|f(y)|}{y-x}\,\mathrm{d}y \le C\|b\|_\infty \left(\int_{M_2+\nu}^{\infty} \frac{\sigma(y)}{(y-M_2)^{p'}}\,\mathrm{d}y \right)^{1/p'}.$$

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Then, by (3.6) and (3.7),

$$J_{21} \leq C\|b'\|_\infty^p \left(\int_{M_1-\nu}^{M_2+\nu} \sigma\right)^{p/p'} \int_{M_1-\nu}^{M_2+\nu} |w^{1/p}(x) - w^{1/p}(x-h)|^p \,\mathrm{d}x$$

$$+C\|b\|_\infty^p \left(\int_{M_2+\nu}^{\infty} \frac{\sigma(y)}{(y-M_2)^{p'}}\right)^{p/p'} \int_{M_1-\nu}^{M_2+\nu} |w^{1/p}(x) - w^{1/p}(x-h)|^p \,\mathrm{d}x < C\varepsilon.$$

In order to estimate J_{22}, we first write suitably the difference $C_b^{+,\nu} f(x+h) - C_b^{+,\nu} f(x)$:

$$\begin{aligned} C_b^{+,\nu} f(x+h) - C_b^{+,\nu} f(x) &= \int_{x+h}^{\infty} (b(x+h) - b(y))(K^\nu(x+h-y) \\ &\quad - K^\nu(x-y)) f(y)\,\mathrm{d}y \\ &\quad + \int_{x+h}^{\infty} (b(x+h) - b(x)) K^\nu(x-y) f(y)\,\mathrm{d}y \\ &\quad + \int_{x+h}^{x} (b(x) - b(y)) K^\nu(x-y) f(y)\,\mathrm{d}y \\ &= \overline{I}(x) + \overline{II}(x) + \overline{III}(x). \end{aligned}$$

It is clear that $\overline{III}(x) = 0$ for all x. On the other hand, taking into account that if $x + h < y < x + (\nu/2)$, then $K^\nu(x-y) = 0$, and applying the mean value theorem and Theorem 4, we have

$$\int_{M_1}^{M_2} |\overline{II}|^p w \leq \|b'\|_\infty^p |h|^p \int_{M_1}^{M_2} \left| \int_{x+(\nu/2)}^{\infty} K^\nu(x-y) f(y)\,\mathrm{d}y \right|^p w(x)\,\mathrm{d}x$$

$$\leq C\|b'\|_\infty^p |h|^p \|T_{K^\nu}^* f\|_{p,w}^p \leq C\|b'\|_\infty^p |h|^p \|f\|_{p,w}^p \leq C\|b'\|_\infty^p |h|^p < C\varepsilon.$$

Finally, we will work with $\overline{I}$. First, we split $\overline{I}(x)$ in three terms:

$$\overline{I}(x) = \int_{x+h}^{\infty} (b(x+h) - b(y))(K^\nu(x+h-y) - K^\nu(x-y)) f(y)\,\mathrm{d}y = \int_B + \int_C + \int_D,$$

where

$$B = \left\{ y : y > x+h, x-y > -\frac{\nu}{2}, x+h-y < -\frac{\nu}{2} \right\},$$

$$C = \left\{ y : y > x+h, x-y < -\frac{\nu}{2}, x+h-y > -\frac{\nu}{2} \right\}$$

and

$$D = \left\{ y : y > x+h, x-y < -\frac{\nu}{2}, x+h-y < -\frac{\nu}{2} \right\}.$$

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Let us begin with the integral over D. By property (1.3) of the kernel K^ν, we have

$$\begin{aligned}\left|\int_D \cdots\right| &\leq \int_{x+(\nu/2)}^{\infty} |b(x+h)-b(y)||K^\nu(x+h-y)-K^\nu(x-y)||f(y)|\,\mathrm{d}y \\ &\leq C\|b\|_\infty |h| \int_{x+(\nu/2)}^{\infty} \frac{|f(y)|}{(y-x)^2}\,\mathrm{d}y = C\|b\|_\infty |h| \sum_{j=0}^{\infty} \int_{x+(2^j\nu/2)}^{x+((2^{j+1}\nu)/2)} \frac{|f(y)|}{(y-x)^2}\,\mathrm{d}y \\ &\leq C\|b\|_\infty |h| \sum_{j=0}^{\infty} \left(\frac{2}{2^j\nu}\right)^2 \int_{x+(2^j\nu/2)}^{x+((2^{j+1}\nu)/2)} |f(y)|\,\mathrm{d}y \leq C\|b\|_\infty |h| \frac{2}{\nu} M^+ f(x).\end{aligned}$$

Then, since $w \in A_p^+$ and $\|f\|_{p,w} \leq 1$, we obtain

$$\int_{M_1}^{M_2} \left|\int_D \cdots\right|^p w \leq C\|b\|_\infty^p |h|^p \left(\frac{2}{\nu}\right)^p \int_{M_1}^{M_2} |M^+ f(x)|^p w(x)\,\mathrm{d}x \leq C\|b\|_\infty^p |h|^p \frac{1}{\nu^p} < C\varepsilon.$$

The estimations for B and C are similar. Therefore, we only write the details for C. For the integral over C, we have to take into account that necessarily $h > 0$ (if $h < 0$, then $C = \emptyset$) and apply the mean value theorem, property (1.2) of K^ν and the Hölder inequality as follows:

$$\begin{aligned}\left|\int_C \cdots\right| &\leq \int_{x+(\nu/2)}^{x+h+(\nu/2)} |b(x+h)-b(y)||K^\nu(x-y)||f(y)|\,\mathrm{d}y \\ &\leq C\|b'\|_\infty \int_{x+(\nu/2)}^{x+h+(\nu/2)} \frac{|x+h-y|}{y-x}|f(y)|\,\mathrm{d}y \leq C\|b'\|_\infty \int_{x+(\nu/2)}^{x+h+(\nu/2)} |f(y)|\,\mathrm{d}y \\ &\leq C\|b'\|_\infty \|f\|_{p,w} \left(\int_{x+(\nu/2)}^{x+h+(\nu/2)} \sigma\right)^{1/p'} \leq C\|b'\|_\infty \left(\int_{x+(\nu/2)}^{x+h+(\nu/2)} \sigma\right)^{1/p'}.\end{aligned}$$

Then, by (3.8),

$$\int_{M_1}^{M_2} \left|\int_C \cdots\right|^p w \leq C\|b'\|_\infty^p \int_{M_1}^{M_2} \left(\int_{x+(\nu/2)}^{x+h+(\nu/2)} \sigma\right)^{p/p'} w(x)\,\mathrm{d}x < C\varepsilon. \qquad \square$$

Acknowledgements. This research has been supported in part by Ministerio de Economía y Competitividad, Spain (grant no. MTM2015-66157-C2-2-P) and Junta de Andalucía (grant no. FQM354).

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