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Asymptotics toward the diffusion wave for a one-dimensional compressible flow through porous media

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Consider the Cauchy problem for a one-dimensional compressible flow through porous media,

$$\begin{aligned} v_t - u_x &= 0, \quad x \in \boldsymbol{R}, \quad t > 0, \\ u_t + p(v)_x &= -\alpha u, \\ (v, u)|_{t=0} &= (v_0, u_0)(x). \end{aligned}$$

Hsiao and Liu showed that the solution (v, u) behaves as the diffusion wave $(\bar{v}, \bar{u})$, i.e. the solution of the porous-media equation due to the Darcy law. The optimal convergence rates have been obtained by Nishihara and co-workers. When $v_0(x)$ has the same constant state at $x = \pm\infty$, the convergence rate $\|(v - \bar{v})(\cdot, t)\|_{L^\infty} = O(t^{-1})$ obtained is 'optimal', since $\|\bar{v}(\cdot, t)\|_\infty = O(t^{-1/2})$. However, this 'optimal' convergence rate is less sufficient to determine the location of the diffusion wave. Our aim in this paper is to obtain the 'truly optimal' convergence rate by choosing suitably located diffusion waves.

1. Introduction

Subsequent to [13, 17], we investigate the asymptotic behaviour of solutions to the Cauchy problem

$$\left.\begin{aligned} v_t - u_x &= 0, \quad x \in \boldsymbol{R}, \quad t > 0, \\ u_t + p(v)_x &= -\alpha u, \\ (v, u)|_{t=0} &= (v_0, u_0)(x), \end{aligned}\right\} \tag{1.1}$$

which models a one-dimensional compressible flow through porous media. Here, v (> 0) is the specific volume, u is the velocity, $p = p(v) = v^{-\gamma}$ is the pressure with the adiabatic constant $\gamma \geqslant 1$ and α is a positive constant. The initial data are assumed to have constant states $(v_\pm, u_\pm)$, $v_\pm > 0$, at $x = \pm\infty$,

$$\lim_{x \to \pm\infty} (v_0, u_0)(x) = (v_\pm, u_\pm), \quad v_\pm > 0. \tag{1.2}$$

The solution to (1.1), (1.2) is expected to behave as that to

$$\left.\begin{aligned} \bar{v}_t - \bar{u}_x &= 0, \\ p(\bar{v})_x &= -\alpha \bar{u}, \end{aligned}\right\} \tag{1.3}$$

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due to the Darcy law. By (1.3), $(\bar{v},\bar{u})$, called the diffusion wave, is determined by the porous-media equation

$$\left.\begin{aligned} \bar{v}_t+\frac{1}{\alpha}p(\bar{v})_{xx}=0, \quad \text{with } \bar{v}|_{x=\pm\infty}=v_\pm,\\ p(\bar{v})_x=-\alpha\bar{u}. \end{aligned}\right\} \tag{1.4}$$

When $v_+\neq v_-$, the diffusion wave is a similarity solution of the form

$$\bar{v}=\bar{v}(x+x_0,t)=\psi\left(\frac{x+x_0}{\sqrt{t+1}}\right), \quad \psi(\pm\infty)=v_\pm, \tag{1.5}$$

where a shift x_0 is uniquely determined by the initial data (v_0,u_0). Convergence of the solution (v,u) to the diffusion wave $(\bar{v},\bar{u})$ was first shown by Hsiao and Liu [2,3] under suitable smallness conditions. The convergence rates were investigated in [13] and the optimal convergence rates

$$\|(v-\bar{v},u-\bar{u})(\cdot,t)\|_{L^p}=O(t^{-1/2-(1-1/p)/2},t^{-1-(1-1/p)/2}), \tag{1.6}$$

as $t\to\infty$ for $p\geqslant 2$ have been recently obtained in [17].

On the other hand, when

$$v_+=v_-=:\underline{v} \quad \text{and, for simplicity,} \quad u_+=u_-=0, \tag{1.7}$$

$\bar{v}$ is taken asymptotically as the solution to (1.4), with its initial data $\bar{v}_0(x)$, or

$$\bar{v}(x,t)\sim\underline{v}+\frac{\delta_{0v}}{\sqrt{4\pi a(t+1)}}\exp\left(-\frac{(x+x_0)^2}{4a(t+1)}\right), \quad a=\frac{-p'(\underline{v})}{\alpha}, \tag{1.8}$$

where

$$\delta_{0v}=\int_{-\infty}^{\infty}\bar{v}_0(x)\,\mathrm{d}x.$$

In this case, the convergence rate

$$\|(v-\bar{v},u-\bar{u})(\cdot,t)\|_{L^\infty}=O(t^{-1},t^{-3/2}) \tag{1.9}$$

(treat the L^∞-norm for simplicity) was obtained in [13]. As easily seen,

$$\|\bar{v}(\cdot,t)-\underline{v},\bar{u}(\cdot,t)\|_{L^\infty}=O(t^{-1/2},t^{-1}), \tag{1.10}$$

the convergence rate (1.9) is 'optimal'. However, the 'optimal' convergence rate (1.9) is not satisfactory in the sense that the location of the diffusion wave $(\bar{v},\bar{u})(x,t)$ is not determined, since

$$\|\bar{v}(\cdot+x_0,t)-\bar{v}(\cdot,t)\|_{L^\infty}=O(t^{-1}),$$

which is derived by

$$\bar{v}(x+x_0,t)-\bar{v}(x,t)\sim\frac{\delta_{0v}}{\sqrt{4\pi a(t+1)}}\int_0^1\exp\left(-\frac{(x+\theta x_0)^2}{4a(t+1)}\right)\left\{-\frac{(x+\theta x_0)x_0}{2a(t+1)}\right\}\mathrm{d}\theta.$$

Hence it is necessary to obtain

$$\|(v,u)(\cdot,t)-(\bar{v},\bar{u})(\cdot,t)\|_{L^\infty}=o(t^{-1},t^{-3/2}) \tag{1.11}$$

for a suitably located wave $(\bar{v},\bar{u})$.

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Our aim in this paper is to reconsider (1.1), (1.2), in case of (1.7), and to obtain the 'truly optimal' convergence rate satisfying (1.11) by deciding the location of the diffusion wave $(\bar{v}, \bar{u})$. As far as we know, there have been few works in which the location of the diffusion wave around the constant state is considered.

We now explain the idea of how to decide the location of the diffusion wave by the following simple problem:

$$\left.\begin{aligned} v_t - v_{xx} = 0, \quad x \in \boldsymbol{R}, \quad t > 0, \\ v|_{t=0} = v_0(x), \\ \int_{-\infty}^{\infty} v_0(x) = \delta_0 \neq 0. \end{aligned}\right\} \tag{1.12}$$

The solution v to (1.12) is expected to behave as

$$\bar{v}(x + x_0, t) = \frac{\delta_0}{\sqrt{4\pi(t+1)}} \exp\left(-\frac{(x+x_0)^2}{4(t+1)}\right) \tag{1.13}$$

or

$$\bar{v}_t - \bar{v}_{xx} = 0, \qquad \int_{-\infty}^{\infty} \bar{v}(x + x_0, t)\,\mathrm{d}x = \delta_0. \tag{1.13$'$}$$

Since $\|\bar{v}(\cdot, t)\|_{L^\infty} = O(t^{-1/2})$ and $\|\bar{v}(\cdot + x_0, t) - \bar{v}(\cdot, t)\|_{L^\infty} = O(t^{-1})$, we need to obtain

$$\|v(\cdot, t) - \bar{v}(\cdot + x_0, t)\|_{L^\infty} = o(t^{-1}) \tag{1.14}$$

for a suitably chosen x_0, in the same manner as (1.11). To derive (1.14), define

$$V(x, t) = \int_{-\infty}^{x} v(y, t)\,\mathrm{d}y, \qquad \bar{V}(x + x_0, t) = \int_{-\infty}^{x+x_0} \bar{v}(y, t)\,\mathrm{d}y. \tag{1.15}$$

Then, by (1.12)–(1.13$'$),

$$\left.\begin{aligned} (V - \bar{V})_t - (V - \bar{V})_{xx} = 0, \\ (V - \bar{V})|_{t=0} = V_0(x) - \bar{V}(x + x_0, 0), \quad V_0(x) = \int_{-\infty}^{x} v_0(y)\,\mathrm{d}y. \end{aligned}\right\} \tag{1.16}$$

Since

$$V_0(x),\ \bar{V}(x + x_0, 0) \to \begin{cases} 0 & \text{as } x \to -\infty, \\ \delta_0(\neq 0) & \text{as } x \to \infty, \end{cases}$$

if $v_0(x) \to 0$ fast as $x \to \pm\infty$, the shift x_0 is uniquely determined by

$$\int_{-\infty}^{\infty} (V_0(x) - \bar{V}(x + x_0, 0))\,\mathrm{d}x = 0. \tag{1.17}$$

Thus we arrive at the definition

$$w(x, t) = \int_{-\infty}^{x} \int_{-\infty}^{y} (v(z, t) - \bar{v}(z + x_0, t))\,\mathrm{d}z\mathrm{d}y, \tag{1.18}$$

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which satisfies

$$\left.\begin{aligned} w_t - w_{xx} &= 0, \\ w|_{t=0} = w_0(x) &:= \int_{-\infty}^{x}\int_{-\infty}^{y} (v_0(z) - \bar{v}(z + x_0, 0))\,\mathrm{d}z\mathrm{d}y. \end{aligned}\right\} \tag{1.19}$$

If $w_0 \in L^1$, which is reasonably assumed, then we easily obtain

$$\|(w, w_x, w_{xx})(\cdot, t)\|_{L^\infty} = O(t^{-1/2}, t^{-1}, t^{-3/2}). \tag{1.20}$$

Therefore, the desired estimate

$$\|v(\cdot, t) - \bar{v}(\cdot + x_0, t)\|_{L^\infty} = O(t^{-3/2}) \tag{1.21}$$

is obtained, provided that the initial data $v_0(x)$ tend to zero suitably fast as $x \to \pm\infty$.

The simple idea on the linear heat equation can be applicable to our problem (1.1), (1.2) with (1.7). We reformulate our problem. Suppose first that

$$\int_{-\infty}^{\infty} (v_0(x) - \underline{v}, u_0(x))\,\mathrm{d}x =: (\delta_{0v}, \delta_{0u}), \quad \delta_{0v} \neq 0, \tag{1.22}$$

and that $(\bar{v}, \bar{u})$ is defined as the solution to

$$\left.\begin{aligned} \bar{v}_t - \frac{1}{\alpha}(-p(\bar{v}))_{xx} &= 0, \\ \bar{v}|_{t=0} &= \bar{v}_0(x; x_0), \\ \bar{u} &= -\frac{1}{\alpha}p(\bar{v})_x \end{aligned}\right\} \tag{1.23}$$

or

$$\left.\begin{aligned} \bar{v}_t - \bar{u}_x &= 0, \\ p(\bar{v})_x &= -\alpha\bar{u}, \\ (\bar{v}, \bar{u})|_{t=0} &= \left(\bar{v}_0(x), -\frac{1}{\alpha}p(\bar{v}_0(x))'\right) \end{aligned}\right\} \tag{1.24}$$

for x_0 uniquely determined later. Similar to [2], we introduce auxiliary functions

$$(\hat{v}, \hat{u})(x, t) = \left(-\frac{1}{\alpha}\mathrm{e}^{-\alpha t}m_0'(x), \mathrm{e}^{-\alpha t}m_0(x)\right) \tag{1.25}$$

for a function m_0 satisfying

$$\int_{-\infty}^{\infty} m_0(x)\,\mathrm{d}x = \delta_{0u}, \quad m_0 \in C_0^\infty(\boldsymbol{R}), \tag{1.26}$$

so that $(\hat{v}, \hat{u})$ satisfies

$$\left.\begin{aligned} \hat{v}_t - \hat{u}_x &= 0, \\ \hat{u}_t &= -\alpha\hat{u}, \\ (\hat{v}, \hat{u})|_{t=0} &= \left(-\frac{1}{\alpha}m_0'(x), m_0(x)\right). \end{aligned}\right\} \tag{1.27}$$

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By (1.1), (1.24) and (1.27),

$$(v-\bar{v}-\hat{v})_t-(u-\bar{u}-\hat{u})_x=0, \tag{1.28 a}$$

$$(u-\bar{u}-\hat{u})_t+(p(v)-p(\bar{v}))_x=-\alpha(u-\bar{u}-\hat{u})+\frac{1}{\alpha}p(\bar{v})_{xt}, \tag{1.28 b}$$

$$(v-\bar{v}-\hat{v},u-\bar{u}-\hat{u})|_{t=0}=\left(v_0-\bar{v}_0+\frac{1}{\alpha}m_0',u_0+\frac{1}{\alpha}p(\bar{v}_0)'-m_0\right)(x). \tag{1.28 c}$$

From (1.28 a),

$$\int_{-\infty}^{\infty}(v-\bar{v}-\hat{v})\,\mathrm{d}x=0,$$

since

$$\int_{-\infty}^{\infty}\left(v_0-\bar{v}_0+\frac{1}{\alpha}m_0'\right)(x)\,\mathrm{d}x=0.$$

Also, since

$$\int_{-\infty}^{\infty}\left(u_0+\frac{1}{\alpha}p(\bar{v}_0)'-m_0\right)(x)\,\mathrm{d}x=0,$$

we have

$$\int_{-\infty}^{\infty}(u-\bar{u}-\hat{u})\,\mathrm{d}x=0$$

by (1.28 b). Hence integrating (1.28 a) twice with respect to x yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\int_{-\infty}^{x}(v-\bar{v}-\hat{v})(y,t)\,\mathrm{d}y\mathrm{d}x=\int_{-\infty}^{\infty}(u-\bar{u}-\hat{u})\,\mathrm{d}x=0,$$

and hence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{x}(v-\bar{v}-\hat{v})(z,t)\,\mathrm{d}z\mathrm{d}x=\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{x}(v_0-\bar{v}_0)(z)\,\mathrm{d}z+\frac{1}{\alpha}m_0(x)\right\}\mathrm{d}x.$$

Hence we choose $\bar{v}_0(x;x_0)$ such that

$$\bar{v}_0(x;x_0)=\underline{v}+\frac{\delta_{0v}}{\sqrt{4\pi a}}\exp\left(-\frac{(x+x_0)^2}{4a}\right), \tag{1.29}$$

with a shift x_0 uniquely determined by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{x}(v_0(z)-\bar{v}_0(z;x_0))\,\mathrm{d}z=-\frac{1}{\alpha}\delta_{0u}. \tag{1.30}$$

Thus we arrive at the definition of the perturbation

$$w(x,t)=\int_{-\infty}^{x}\int_{-\infty}^{y}(v-\bar{v}-\hat{v})(z,t)\,\mathrm{d}z\mathrm{d}y, \tag{1.31}$$

and

$$(v,u)=(\bar{v}+\hat{v}+w_{xx},\bar{u}+\hat{u}+w_{xt}). \tag{1.31'}$$

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By (1.28), w satisfies the second-order wave equation

$$w_{tt}+p(\bar{v}+\hat{v}+w_{xx})-p(\bar{v})+\alpha w_t=\frac{1}{\alpha}(p(\bar{v})-p(\underline{v}))_t, \tag{1.32}$$

by integrating (1.28 a) twice and (1.28 b) once, with initial data

$$\begin{aligned}(w,w_t)|_{t=0}&=(w_0,w_1)(x)\\ &:=\bigg(\int_{-\infty}^{x}\bigg\{\int_{-\infty}^{y}(v_0(z)-\bar{v}_0(z;x_0))\,\mathrm{d}z+\frac{m_0(y)}{\alpha}\bigg\}\,\mathrm{d}y,\\ &\qquad\int_{-\infty}^{x}(u_0-m_0)(z)\,\mathrm{d}z+\frac{p(\bar{v}_0(x;x_0))-p(\underline{v})}{\alpha}\bigg).\end{aligned} \tag{1.33}$$

Our present goal is to obtain a global solution w to (1.32), (1.33) and its behaviour as $t\to\infty$, provided that (w_0,w_1) is suitably small. To do this, it is necessary to know the behaviours of $\bar{v}$, the solution of the porous-media equation.

Consider the Cauchy problem with, general initial data $\tilde{v}_0(x)$ instead of $\bar{v}(x;x_0)$,

$$\left.\begin{aligned}\bar{v}_t-(-p(\bar{v}))_{xx}&=0,\\ \bar{v}|_{t=0}&=\tilde{v}_0(x),\\ \int_{-\infty}^{\infty}(\tilde{v}_0(x)-\underline{v})\,\mathrm{d}x&=\delta_{0v}.\end{aligned}\right\} \tag{1.34}$$

As an asymptotic profile, we take

$$\phi(x,t;x_1)=\underline{v}+\phi_1(x,t;x_1)+\phi_2(x,t;x_1), \tag{1.35}$$

with

$$\left.\begin{aligned}\phi_1(x,t;x_1)&=\frac{\delta_{0v}}{\sqrt{4\pi a(t+1)}}\exp\bigg(-\frac{(x+x_1)^2}{4a(t+1)}\bigg)=:\delta_{0v}G(x+x_1,t+1),\\ \phi_2(x,t;x_1)&=\int_0^t\int_{-\infty}^{\infty}G(x-y,t-\tau)\cdot b\{\phi(y,\tau;x_1)^2\}_{yy}\,\mathrm{d}y\mathrm{d}\tau,\end{aligned}\right\} \tag{1.36}$$

where a shift x_1 is uniquely determined by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{x}\{\tilde{v}_0(z)-(\underline{v}+\delta_{0v}G(z+x_1,1))\}\,\mathrm{d}z\mathrm{d}x=0 \tag{1.37}$$

and

$$a=\frac{-p'(\underline{v})}{\alpha},\qquad b=\frac{-p''(\underline{v})}{2\alpha}. \tag{1.38}$$

Denoting the usual Sobolev spaces by H^m, $W^{m,p}$ ($m=0,1,2,\ldots,1\leqslant p\leqslant\infty$), our first theorem is as follows.

THEOREM 1.1. *Let*

$$V_0:=\int_{-\infty}^{x}\int_{-\infty}^{y}\{\tilde{v}(z)-(\underline{v}+\delta_{0v}G(z+x_1,1))\}\,\mathrm{d}z\mathrm{d}y$$

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be in $H^s \cap L^1$ $(s \geqslant 3)$*. If both* $\|V_0\|_{H^s\cap L^1}$ *and* $|\delta_{0v}|$ *are suitably small, then there exists a unique (weak) solution* $\bar{v}$ *to (1.34) satisfying* $\bar{v} - \phi \in C([0,\infty); H^{s-2})$ *and decay properties*

$$\|\partial_x^j(\bar{v}(\cdot,t) - \phi(\cdot,t;x_1))\|_{L^p} \leqslant C(1+t)^{-(1-1/p)/2-j/2-1}\log{(2+t)} \tag{1.39}$$

for $0 \leqslant j \leqslant s-3$, $2 \leqslant p \leqslant \infty$,

$$\|\partial_x^j(\bar{v}(\cdot,t) - \phi(\cdot,t;x_1))\|_{L^p} \leqslant Ct^{-(1-1/p)/2-j/2-1}\log t, \quad t \geqslant 2, \tag{1.40}$$

for $0 \leqslant j \leqslant s-3$, $1 \leqslant p < 2$*, and*

$$\|\partial_x^{s-2}(\bar{v}(\cdot,t) - \phi(\cdot,t;x_1))\|_{L^2} \leqslant C\|V_0\|_{H^s\cap L^1}(1+t)^{-s/2}, \tag{1.41}$$

where $C = C_0(\|V_0\|_{H^s\cap L^1} + |\delta_{0v}|)$*. Moreover, if* $V_0 \in H^s \cap W^{s-1,1}$*, then (1.39) holds for* $0 \leqslant j \leqslant s-3$, $1 \leqslant p \leqslant \infty$ *for* $C = C_0(\|V_0\|_{H^s\cap W^{s-1,1}} + |\delta_{0v}|)$*.*

If the initial data $\tilde{v}_0(x)$ is given by $\bar{v}_0(x;x_0)$ in (1.29) with (1.30), then $x_1 = x_0$ and

$$\int_{-\infty}^{x}\int_{-\infty}^{y}\{\bar{v}(z) - (\underline{v} + \delta_{0v}G(z+x_0,1))\}\,\mathrm{d}z\mathrm{d}y \in H^\infty \cap W^{\infty,1},$$

and hence (1.39) holds for $j = 0,1,2,\dots,$ in which x_1 is changed to x_0. As is easily seen, since

$$\|\partial_x^j(\phi(\cdot,t;x_0) - \underline{v})\|_{L^p} \leqslant C\delta_{0v}(1+t)^{-(1-1/p)/2-j/2}$$

(see lemma 2.1 in § 2), and $\|V_0\|_{H^s\cap W^{s-1,1}}$ depends on δ_{0v}, we have the following corollary.

Corollary 1.2. *The solution* $(\bar{v},\bar{u})$ *to (1.23) or (1.24) with (1.29), (1.30) satisfies the decay properties*

$$\|\partial_x^j(\bar{v}(\cdot,t) - \underline{v})\|_{L^p} \leqslant C\delta_{0v}(1+t)^{-(1-1/p)/2-j/2}, \tag{1.42}$$

$$\|\partial_x^j\bar{u}(\cdot,t)\|_{L^p} \leqslant C\delta_{0v}(1+t)^{-(1-1/p)/2-(j+1)/2} \tag{1.43}$$

for $j = 0,1,2,\dots,$ $1 \leqslant p \leqslant \infty$*.*

Using this corollary and employing the L^2-energy method, we arrive at the following result.

Theorem 1.3. *Let* (w_0,w_1) *be defined as in (1.33). If both* $(w_0,w_1) \in H^3 \times H^2$ *and* $\delta_0 := |\delta_{0v}| + |\delta_{0u}|$ *are sufficiently small, then there exists a solution* (v,u) *to (1.1), (1.2) with (1.7), which satisfies the behaviours*

$$\|v-\bar{v}, u-\bar{u}\|_{L^2} \leqslant O((1+t)^{-1}, (1+t)^{-3/2}), \tag{1.44}$$

$$\|v-\bar{v}, u-\bar{u}\|_{L^\infty} \leqslant O((1+t)^{-5/4}, (1+t)^{-7/4}). \tag{1.45}$$

The estimates (1.44), (1.45) are enough to determine the location of the diffusion wave $(\bar{v},\bar{u})$. However, by the Green-function method, we will be further able to get the 'truly optimal' convergence rates, though unfortunately we could not delete 'log' from the rates.

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Theorem 1.4. *Suppose that*

$$(w_0, w_1) \in (H^4 \times H^3) \cap (L^1 \times L^1).$$

If both $\|w_0, w_1\|_{H^4 \times H^3}$ *and* $\delta_0 = |\delta_{0v}| + |\delta_{0u}|$ *are suitably small, then the solution* (v, u) *to (1.1), (1.2) with (1.7) satisfies, for* $t \geqslant 2$,

$$\|(v - \bar{v}, u - \bar{u})(\cdot, t)\|_{L^\infty} = O(t^{-3/2} \log t, t^{-2} \log t) \tag{1.46}$$

and

$$\|(v - \bar{v}, u - \bar{u})(\cdot, t)\|_{L^2} = O(t^{-5/4} \log t, t^{-7/4} \log t). \tag{1.47}$$

As an asymptotic profile of (v, u), we adopt $(\phi(x, t; x_0), -(1/\alpha)p(\phi(x, t; x_0))_x)$. Then, combining theorems 1.1 and 1.4, we get our main result.

Theorem 1.5 (main theorem). *Under the assumptions of theorem 1.4, it holds that*

$$\left\| \left(v - \phi(x, t; x_0), u + \frac{1}{\alpha} p(\phi(x, t; x_0))_x \right) \right\|_{L^\infty} = O(t^{-3/2} \log t, t^{-2} \log t) \tag{1.48}$$

and

$$\left\| \left(v - \phi(x, t; x_0), u + \frac{1}{\alpha} p(\phi(x, t; x_0))_x \right) \right\|_{L^2} = O(t^{-5/4} \log t, t^{-7/4} \log t). \tag{1.49}$$

We finally state the related works. Following the pioneering work of Hsiao and Liu [2, 3], there are developments in several directions. One is to obtain the convergence rates just treated in this paper (see [12, 13, 17]). Second, weak solutions should be treated when no smallness condition is assumed (see [4, 7–9, 18, 20]). Moreover, the initial–boundary-value problems, the problems for the full system including the equation from the conservation of energy, etc., have been considered in [5, 6, 10, 11, 15, 16] and the references therein (see also the book [1] by Hsiao).

The plan of this paper is as follows. In § 2, we first investigate the behaviour of solutions to the porous-media equation, so that the decay rates of $\bar{v} - \underline{v}$ are obtained. In § 3, applying the L^2-energy method, we prove theorem 1.3. Finally, we prove the optimal convergence rate (theorem 1.4) by using the Green function of the heat equation.

2. The porous-media equation

We consider the Cauchy problem for the porous-media equation

$$\left. \begin{aligned} \bar{v}_t - (-p(\bar{v}))_{xx} = 0, \\ \bar{v}|_{t=0} = \tilde{v}_0(x), \\ \int_{-\infty}^{\infty} (\tilde{v}_0(x) - \underline{v})\, \mathrm{d}x = \delta_{0v} \neq 0. \end{aligned} \right\} \tag{2.1}$$

Here, let $\tilde{v}_0(x)$ be given.

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First, the asymptotic profile $\phi(x,t;x_1)$ of $\bar{v}$ is defined by (1.35) with (1.36)–(1.38). Of course, ϕ_1, ϕ_2 satisfy

$$\phi_{1t} - a\phi_{1xx} = 0, \qquad \phi_1|_{t=0} = \frac{\delta_{0v}}{\sqrt{4\pi a}} \exp\biggl(-\frac{(x+x_1)^2}{4a}\biggr) =: \phi_0(x;x_1) - \underline{v}, \tag{2.2}$$

$$\phi_{2t} - a\phi_{2xx} = b(\phi_1^2)_{xx}, \qquad \phi_2|_{t=0} \equiv 0, \tag{2.3}$$

respectively, and so

$$(\phi_1+\phi_2)_t - a(\phi_1+\phi_2)_{xx} - b(\phi_1+\phi_2)^2_{xx} = -b(2\phi_1\phi_2 + \phi_2^2)_{xx},$$

i.e.

$$\phi_t - a\phi_{xx} - b(\phi - \underline{v})^2_{xx} = -b(2\phi_1\phi_2+\phi_2^2)_{xx} \qquad \phi|_{t=0} = \phi_0(x) := \phi_0(x;x_1). \tag{2.4}$$

Denote the perturbation by

$$V(x,t) = \int_{-\infty}^{x}\int_{-\infty}^{y} (\bar{v}-\phi)(z,t)\,\mathrm{d}z\mathrm{d}y \quad \text{or} \quad \bar{v} = \phi + V_{xx}. \tag{2.5}$$

Then V satisfies

$$\left.\begin{aligned} V_t - (-p(\phi+V_{xx}) + p(\underline{v}) + a(\phi-\underline{v}) + b(\phi-\underline{v})^2) = b(2\phi_1\phi_2+\phi_2^2),\\ V|_{t=0} = V_0(x) = \int_{-\infty}^{x}\int_{-\infty}^{y} (\tilde{v}_0(z) - \phi_0(z;x_1))\,\mathrm{d}z\mathrm{d}y. \end{aligned}\right\} \tag{2.6}$$

Linearizing this around $\underline{v}$, we have the reformulated problem

$$\left.\begin{aligned} V_t - aV_{xx} = 2b\phi_1\phi_2 + 2b\phi_1 V_{xx} + O(\phi_1^3 + \phi_2^2 + V_{xx}^2) =: 2b\phi_1\phi_2 + \tilde{F}(V_{xx}),\\ V|_{t=0} = V_0(x). \end{aligned}\right\} \tag{2.7}$$

To estimate V, it is necessary to have the properties of ϕ_i $(i=1,2)$.

LEMMA 2.1. *Let ϕ_1, ϕ_2 be given by (1.36). Then they satisfy the following estimates:*

$$\|\partial_x^j \phi_1(\cdot,t;x_1)\|_{L^p} \leqslant C\delta_{0v}(1+t)^{-(1-1/p)/2-j/2}, \tag{2.8}$$

$$\|\partial_x^j \phi_2(\cdot,t;x_1)\|_{L^p} \leqslant C\delta_{0v}^2(1+t)^{-(1-1/p)/2-(1+j)/2} \tag{2.9}$$

for $j=0,1,2,\dots$ and $1 \leqslant p \leqslant \infty$.

Proof. The estimate (2.8) is well known. We only show (2.9) for $j=0$. For $t \geqslant 2$,

$$\begin{aligned} \|\phi_2(t)\|_{L^\infty} &\leqslant C\biggl(\int_0^{t/2} \|\partial_x^2 G(\cdot,t-\tau)\|_{L^\infty}\|\phi_1^2(\cdot,\tau)\|_{L^1}\,\mathrm{d}\tau \\ &\qquad + \int_{t/2}^{t} \|\partial_x G(\cdot,t-\tau)\|_{L^1}\|\phi_1\phi_{1y}(\cdot,\tau)\|_{L^\infty}\,\mathrm{d}\tau\biggr) \\ &\leqslant C\delta_{0v}^2 t^{-3/2}\int_0^t (1+\tau)^{-1/2}\,\mathrm{d}\tau \\ &\leqslant C\delta_{0v}^2 t^{-1} \end{aligned}$$

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and

$$\begin{aligned}\|\phi_2(t)\|_{L^1} &\leqslant C\biggl(\int_0^{t/2} \|\partial_x^2 G(\cdot,t-\tau)\|_{L^1}\|\phi_1^2(\cdot,\tau)\|_{L^1}\,\mathrm{d}\tau \\ &\qquad + \int_{t/2}^{t} \|\partial_x G(\cdot,t-\tau)\|_{L^1}\|\phi_1\phi_{1y}(\cdot,\tau)\|_{L^1}\,\mathrm{d}\tau\biggr) \\ &\leqslant C\delta_{0v}^2 t^{-1}\int_0^t (1+\tau)^{-1/2}\,\mathrm{d}\tau \\ &\leqslant C\delta_{0v}^2 t^{-1/2}.\end{aligned}$$

By combining these with $\|\phi_2(t)\|_{L^1\cap L^\infty} \leqslant C$ $(t \leqslant 2)$, the desired estimate (2.9) is obtained. □

Using lemma 2.1, we have the following theorem.

THEOREM 2.2. *Let $V_0 \in H^s \cap L^1 (s \geqslant 3)$ be sufficiently small and $|\delta_{0v}|$ also small. Then problem (2.7) has a unique solution $V \in C([0,\infty); H^s \cap L^1) \cap C((0,\infty); W^{2,1})$ with $V_x \in L^2(0,\infty; H^s)$, which satisfies the following for $C = C_0(\|V_0\|_{H^s\cap L^1} + |\delta_{0v}|)$,*

$$\|\partial_x^k V(t)\|_{L^1} \leqslant Ct^{-k/2}\log t, \qquad t \geqslant 2, \quad k = 0,1,2, \tag{2.10}$$

$$\|\partial_x^k V(t)\|_{L^p} \leqslant C(1+t)^{-(1-1/p)-k/2}\log(2+t), \quad t \geqslant 0, \quad k = 0,1,2, \quad 2 \leqslant p \leqslant \infty, \tag{2.11}$$

$$\|\partial_x^k V(t)\|_{L^2} \leqslant C(1+t)^{-k/2}, \qquad t \geqslant 0, \quad k = 3,4,\dots,s, \tag{2.12}$$

and

$$\int_0^t \sum_{k=0}^{s} (1+\tau)^k \|\partial_x^k V(t)\|_{L^2}^2\,\mathrm{d}\tau \leqslant C, \tag{2.13}$$

and, moreover,

$$\|\partial_x^k V(t)\|_{L^p} \leqslant C(1+t)^{-(1-1/p)/2-k/2}\log(2+t), \qquad t \geqslant 0, \quad k = 0,1,\dots,s-1, \quad 2 \leqslant p \leqslant \infty, \tag{2.14}$$

$$\|\partial_x^k V(t)\|_{L^p} \leqslant Ct^{-(1-1/p)/2-k/2}\log t, \quad t \geqslant 2, \quad k = 0,1,\dots,s-1, \quad 1 \leqslant p < 2. \tag{2.15}$$

In addition, if $V_0 \in W^{s-1,1}$, then it holds that, for $t \geqslant 0$,

$$\|\partial_x^k V(t)\|_{L^p} \leqslant C(1+t)^{-(1-1/p)/2-k/2}\log(2+t), \qquad k = 0,1,\dots,s-1, \quad 1 \leqslant p \leqslant \infty, \tag{2.16}$$

for $C = C_0(\|V_0\|_{H^s\cap W^{s-1,1}} + |\delta_{0v}|)$.

Theorem 1.1 is a direct consequence of theorem 2.2, since

$$\bar{v}(x,t) - \phi(x,t;x_1) = V_{xx}.$$

The case $\tilde{v}(x) = \bar{v}_0(x)$ is a special case with $s = \infty$. So we easily have corollary 1.2.

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Proof of theorem 2.2. The method is similar to that in [19], so we only sketch the proof.

Define the solution space $X_M(T)(T \leqslant \infty)$ by

$$X_M(T) = \{v \mid v \in C([0,T]; H^s \cap L^1) \cap C((0,T]; W^{2,1}), \\ v_x \in L^2(0,T; H^s) \text{ with } \|v\|_{X(T)} \leqslant M\},$$

where

$$\begin{aligned} \|v\|_{X(T)} = \sup_{0<t\leqslant T} \sum_{k=0}^{2} t^{k/2} (\log(1+t))^{-1} \|\partial_x^k v(t)\|_{L^1} \\ + \max_{0\leqslant t\leqslant T} \biggl(\sum_{k=0}^{2} (\log(2+t))^{-1} \\ \times \{(1+t)^{1/4+k/2} \|\partial_x^k v(t)\| + (1+t)^{1/2+k/2} \|\partial_x^k v(t)\|_{L^\infty}\} \biggr) \\ + \sum_{k=3}^{s} (1+t)^{k/2} \|\partial_x^k v(t)\| + \biggl(\int_0^t \sum_{k=0}^{s} (1+\tau)^k \|\partial_x^k v_x(\tau)\|^2 \, d\tau \biggr)^{1/2}. \end{aligned} \tag{2.17}$$

From now on, we denote the L^2-norm simply by $\|\cdot\|$.

We define the iteration $\{V^{(n)}(x,t)\}$ by

$$V^{(0)}(x,t) = \int G(x-y,t) V_0(y) \, dy + \int_0^t \int G(x-y,t-\tau) 2b\phi_1\phi_2(y,\tau) \, dy d\tau,$$

$$V^{(n+1)}(x,t) = V^{(0)}(x,t) + \int_0^t \int G(x-y,t-\tau) F(V_{xx}^{(n)}(y,\tau)) \, dy d\tau,$$

and show that $\{V^{(n)}(x,t)\}$ is a Cauchy sequence in $X_{M_0\varepsilon}(T)$ for some positive constant M_0 independent of T, provided that $\|V_0\|_{H^s\cap L^1} + \delta_{0v}$ is sufficiently small. Here and hereafter, the integrand $(-\infty,\infty)$ will be abbreviated. In fact, for $t \geqslant 2$,

$$\begin{aligned} \|V^{(0)}(t)\|_{L^\infty} &\leqslant (1+t)^{-1/2} \|V_0\|_{L^1} \\ &\quad + C\biggl(\int_0^{t/2} \|G\|_{L^\infty} \|\phi_1\phi_2\|_{L^1} \, d\tau + \int_{t/2}^t \|G\|_{L^1} \|\phi_1\phi_2\|_{L^\infty} \, d\tau \biggr) \\ &\leqslant C(\|V_0\|_{L^1} + \delta_{0v}^3) t^{-1/2} \log t \end{aligned} \tag{2.18}$$

and

$$\|V^{(0)}(t)\|_{L^1} \leqslant C(\|V_0\|_{L^1} + \delta_{0v}^3) \log t. \tag{2.19}$$

The estimates of first- and second-order derivatives are similar. Moreover, $V^{(0)}$ satisfies

$$V_t^{(0)} - aV_{xx}^{(0)} = 2b\phi_1\phi_2, \tag{2.20 a}$$

$$V^{(0)}|_{t=0} = V_0(x). \tag{2.20 b}$$

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Hence, differentiating (2.20 a) k times with respect to x and multiplying the resultant equation by $(1+t)^k\partial_x^k V$ $(k=0,1,\dots,s)$, we have

$$(1+t)^k\|\partial_x^k V^{(0)}(t)\|^2+\int_0^t(1+\tau)^k\|\partial_x^k V_x^{(0)}(\tau)\|^2\,\mathrm{d}\tau\leqslant C(\|V_0\|_{H^s}^2+\delta_{0v}^3). \quad (2.21)$$

Noting that $\|\partial_x^k V^{(0)}(t)\|_{L^\infty}\leqslant C\|V^{(0)}(t)\|_{H^s}$ for $k=0,1,2$, and combining (2.18)–(2.21), we have

$$\|V^{(0)}\|_{X_T}\leqslant C_1(\|V_0\|_{H^s\cap L^1}+\delta_{0v}^{3/2})\quad(\leqslant\varepsilon).$$

In a similar fashion to that above, we have that, if $\|V^{(n)}\|_{X_T}\leqslant M_0\varepsilon$, then

$$\|V^{(n+1)}\|_{X_T}\leqslant C_1\varepsilon+C_2(\delta_{0v}\cdot\varepsilon+\varepsilon^2)\leqslant M_0\varepsilon$$

if $C_1+C_2(\delta_{0v}+\varepsilon)\leqslant M_0$. Moreover, taking ε smaller if necessary, we have that $\{V^{(n)}\}$ is a Cauchy sequence in $X_{<_0\varepsilon}(T)$ for any $T\leqslant\infty$.

Thus we obtain the solution V to (2.7) that satisfies (2.10)–(2.13).

Again, estimating the expression

$$V(x,t)=\int G(x-y,t)V_0(y)\,\mathrm{d}y+\int_0^t\int G(x-y,t-\tau)(2b\phi_1\phi_2+\tilde F(V_{xx}))(y,\tau)\,\mathrm{d}y\mathrm{d}\tau,$$

we have (2.14), (2.15) for $t\geqslant 2$. In particular, note that $\partial_x^{s-1}V$ can be estimated by

$$\partial_x^{s-1}V=\int\partial_x^{s-1}GV_0\,\mathrm{d}y+\int_0^t\int\partial_x G(x-y,t-\tau)\partial_y^{s-2}(2b\phi_1\phi_2+\tilde F(V_{xx}))\,\mathrm{d}y\mathrm{d}\tau.$$

Since $V\in H^s$. For $t\leqslant 2$, $\|\partial_x^k V(t)\|_{L^p}\leqslant C$ $(p\geqslant 2,\ k\leqslant s)$ completes (2.14). Also, if $V_0\in W^{s-1,1}$, then (2.16) holds.

Thus we have completed theorem 2.2. □

3. L^2-energy method

Consider the linearized problem for the reformulated one (1.32) with (1.33),

$$w_{tt}-aw_{xx}+w_t=F(w_{xx}), \quad (3.1\,a)$$

$$(w,w_t)|_{t=0}=(w_0,w_1)(x), \quad (3.1\,b)$$

where $\alpha=1$ without loss of generality, $a=-p'(\underline{v})>0$, $b=-\frac{1}{2}p''(\underline{v})$ and

$$\begin{aligned}F&=-p(\underline{v}+(\bar v-\underline{v}+\hat v+w_{xx}))+p(\underline{v}+(\bar v-\underline{v}))+p'(\underline{v})w_{xx}+(p(\bar v)-p(\underline{v}))_t\\&=b(\bar v-\underline{v})w_{xx}+O(|\hat v|+|\bar v-\underline{v}|^3+w_{xx}^2+|\bar v_t|).\end{aligned} \quad (3.2)$$

To obtain the time-global solution to (3.1), we combine the local existence theorem with the *a priori* estimates.

PROPOSITION 3.1 (local existence). *Suppose that* $(w_0,w_1)\in H^3\times H^2$, *with*

$$\|w_0,w_1\|_{H^3\times H^2}\leqslant M.$$

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Then there is a positive number T_0 such that problem (3.1) has a unique solution $w \in C^i([0,T_0];H^{3-i})$, $i=0,1,2,3$, with

$$\sup_{0\leqslant t\leqslant T_0} \|w, w_t\|_{H^3\times H^2} \leqslant C_0 M$$

for a constant $C_0>0$.

Proposition 3.2 (*a priori* estimates). *Let $w \in C^i([0,T];H^{3-i})$, $i=0,1,2,3$, be a solution to (3.1) for any $T>0$. Then there is some constant $\varepsilon_0>0$ such that, if $\|w_0,w_1\|_{H^3\times H^2}+\delta_0 \leqslant \varepsilon_0$, $\delta_0 = |\delta_{0v}|+|\delta_{0u}|$, then it holds that*

$$\begin{aligned}\sum_{k=0}^{3}(1+t)^k\|\partial_x^k w(t)\|^2 + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k w_t(t)\|^2 \\ + \sum_{k=0}^{1}(1+t)^{k+3}\|\partial_x^k w_{tt}(t)\|^2 + (1+t)^4\|w_{ttt}(t)\|^2 \\ + \int_0^t \biggl(\sum_{k=0}^{2}(1+\tau)^k\|\partial_x^k w_x(\tau)\|^2 + \sum_{k=0}^{2}(1+\tau)^{k+1}\|\partial_x^k w_t(\tau)\|^2 \\ + \sum_{k=0}^{1}(1+\tau)^{k+3}\|\partial_x^k w_{tt}(\tau)\|^2\biggr)\,\mathrm{d}\tau \\ \leqslant C(\|w_0,w_1\|_{H^3\times H^2}+\delta_0).\end{aligned} \tag{3.3}$$

Propositions 3.1 and 3.2 yield the global existence theorem.

Theorem 3.3. *Suppose that $(w_0,w_1)\in H^3\times H^2$ and that $\|w_0,w_1\|_{H^3\times H^2}+\delta_0$ is small. Then there exists a unique solution $w\in C^i([0,\infty);H^{3-i})$ to (3.1), which satisfies*

$$\begin{aligned}\sum_{k=0}^{3}(1+t)^{k/2}\|\partial_x^k w(t)\| + \sum_{k=0}^{2}(1+t)^{1+k/2}\|\partial_x^k w_t(t)\| \\ + \sum_{k=0}^{1}(1+t)^{3/2+k/2}\|\partial_x^k w_{tt}(t)\| + (1+t)^2\|w_{ttt}(t)\| \\ \leqslant C(\|w_0,w_1\|_{H^3\times H^2}+\delta_0).\end{aligned}$$

We only show the *a priori* estimates. Suppose that

$$\begin{aligned}N(T) := \sup_{0\leqslant t\leqslant T}\biggl(\sum_{k=0}^{3}(1+t)^k\|\partial_x^k w(t)\|^2 + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k w_t(t)\|^2 \\ + \int_0^t\biggl(\sum_{k=0}^{2}(1+\tau)^k\|\partial_x^k w_x(\tau)\|^2 + \sum_{k=0}^{2}(1+\tau)^{k+1}\|\partial_x^k w_t(\tau)\|^2\biggr)\,\mathrm{d}\tau\biggr) \\ < \varepsilon \quad (\leqslant 1).\end{aligned} \tag{3.4}$$

Similarly to [13], we show (3.3) by series of estimates.

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Estimate 1. Multiplying (3.1 a) by $w_t+\lambda w$ for $0<\lambda\ll 1$ and integrating it over $\boldsymbol{R}\times[0,t]$, we have

$$\begin{aligned}
&\int(\tfrac{1}{2}w_t^2+\lambda w_t w+\tfrac{1}{2}\lambda w^2+\tfrac{1}{2}aw_x^2)(x,t)\,\mathrm{d}x\\
&\quad+\int_0^t\int((1-\lambda)w_t^2+\tfrac{1}{2}\lambda a w_x^2)(x,\tau)\,\mathrm{d}x\mathrm{d}\tau\\
&\qquad\leqslant C\|w_0,w_1\|^2_{H^3\times H^2}+\int_0^t\int F(w_{xx})(w_t+\lambda w)(x,\tau)\,\mathrm{d}x\mathrm{d}\tau\\
&\qquad\leqslant C\|w_0,w_1\|^2_{H^3\times H^2}\\
&\qquad\quad+\int_0^t(\tfrac{1}{2}w_t^2+C\{\|\bar v-\underline{v}\|_{L^\infty}\|w_{xx}(\tau)\|^2+\|\hat v\|^2\\
&\qquad\qquad\qquad+\|\bar v-\underline{v}\|^6_{L^6}+\|w_{xxx}(\tau)\|\|w_{xx}(\tau)\|^3\})\,\mathrm{d}\tau\\
&\qquad\quad+\int_0^t C(\|\bar v-\underline{v}\|_{L^\infty}+\|w_{xx}(\tau)\|^2+\|\hat v\|\\
&\qquad\qquad\qquad+\|\bar v-\underline{v}\|^3_{L^6}+\|w_{xx}(\tau)\|^2_{L^4})\|w(\tau)\|\,\mathrm{d}\tau\\
&\qquad\leqslant C(\|w_0,w_1\|^2_{H^3\times H^2}+\delta_0+N(T)^2)\\
&\qquad\quad+C(\delta_0+N(T))\int_0^t(1+\tau)^{-5/4}\|w(\tau)\|\,\mathrm{d}\tau.
\end{aligned}$$

Hence

$$\begin{aligned}
\|(w_t,w_x,w)(t)\|^2+\int_0^t\|(w_t,w_x)(\tau)\|^2\,\mathrm{d}\tau&\leqslant C(\|w_0,w_1\|_{H^3\times H^2}+\delta_0+N(T)^2)\\
&=:\bar C_0.
\end{aligned}\tag{3.5}$$

Here we have used

$$|\hat v(x,t)|\leqslant C\delta_0\mathrm{e}^{-\alpha t},\tag{3.6}$$

$$\||\bar v-\underline{v}|^3\|\leqslant C\|\bar v-\underline{v}\|^3_{L^6}\leqslant C\delta_0^3(1+t)^{-5/4},\tag{3.7}$$

etc., and Gronwall's inequality. Using (3.5) and multiplying (3.1 a) by $(1+t)w_t$ yields

$$(1+t)\|(w_t,w_x)(t)\|^2+\int_0^t(1+\tau)\|w_t(\tau)\|^2\,\mathrm{d}\tau\leqslant\bar C_0.\tag{3.8}$$

Estimate 2. Differentiating (3.1 a) in x yields

$$w_{xtt}-aw_{xxx}+w_{xt}=F(w_{xx})_x.\tag{3.9}$$

By (3.5), we can multiply (3.9) by $(1+t)(w_{xt}+\lambda w_x)$ for $0<\lambda\ll1$, which results in

$$(1+t)\|(w_{xt},w_{xx},w_x)(t)\|^2+\int_0^t(1+\tau)\|(w_{xt},w_{xx})(\tau)\|^2\,\mathrm{d}\tau\leqslant\bar C_0.\tag{3.10}$$

Again, multiplying (3.9) by $(1+t)^k w_{xt}$, $k=0,1,2$, and using (3.10), we have

$$(1+t)^2\|(w_{xt},w_{xx})(t)\|^2+\int_0^t(1+\tau)^2\|w_{xt}(\tau)\|^2\,\mathrm{d}\tau\leqslant\bar C_0.\tag{3.11}$$

ESTIMATE 3. Multiplying (3.9) by $-(1+t)^k(w_{xxxt}+\lambda w_{xxx})$, $k=0,1,2$, yields

$$(1+t)^2\|(w_{xxt},w_{xxx},w_{xx})(t)\|^2+\int_0^t\|(w_{xxt},w_{xxx})(\tau)\|^2\,\mathrm{d}\tau\leqslant\bar{C}_0. \tag{3.12}$$

Hence we can multiply (3.9) by $-(1+t)^k w_{xxxt}$, $k=0,1,2,3$, and obtain

$$(1+t)^3\|(w_{xxt},w_{xxx})(t)\|^2+\int_0^t(1+\tau)^3\|w_{xxt}(\tau)\|^2\,\mathrm{d}\tau\leqslant\bar{C}_0. \tag{3.13}$$

Combining estimates 1–3 and taking $N(T)$ ($\leqslant\varepsilon$) small, we get

$$N(T)\leqslant C(\|w_0,w_1\|^2_{H^3\times H^2}+\delta_0).$$

Moreover, since $F(w_{xx})\in C^i([0,T);H^{1-i})$, $i=0,1$, $u_{tt}\in C^i([0,T);H^{1-i})$. Using these facts, we continue the estimates.

ESTIMATE 4. Differentiating (3.1 a) in t yields

$$w_{ttt}-aw_{xxt}+w_{tt}=F(w_{xx})_t. \tag{3.14}$$

By (3.8), we can multiply (3.14) by $(1+t)^k(w_{tt}+\lambda w_t)$, $k=0,1,2$, so that

$$(1+t)^2\|(w_{tt},w_{xt},w_t)(t)\|^2+\int_0^t(1+\tau)^2\|(w_{tt},w_{xt})(\tau)\|^2\,\mathrm{d}\tau \leqslant C(\|w_0,w_1\|^2_{H^3\times H^2}+\delta_0)=:C_0. \tag{3.15}$$

Hence we multiply (3.14) by $(1+t)^k w_{tt}$, $k=0,1,2,3$, so that

$$(1+t)^3\|(w_{tt},w_{xt})(t)\|^2+\int_0^t(1+\tau)^3\|w_{tt}(\tau)\|^2\,\mathrm{d}\tau\leqslant C_0. \tag{3.16}$$

ESTIMATE 5. Finally, multiplying (3.14) by $-(1+t)^k(w_{xxtt}+\lambda w_{xxt})$, $k=0,1,2,3$, and $-(1+t)^k w_{xxtt}$, $k=0,1,\dots,4$, we obtain

$$(1+t)^3\|(w_{xtt},w_{xxt},w_{xt})(t)\|^2+\int_0^t(1+\tau)^3\|(w_{xtt},w_{xxt})(\tau)\|^2\,\mathrm{d}\tau\leqslant C_0 \tag{3.17}$$

and

$$(1+t)^4\|(w_{xtt},w_{xxt})(t)\|^2+\int_0^t(1+\tau)^4\|w_{xtt}(\tau)\|^2\,\mathrm{d}\tau\leqslant C_0, \tag{3.18}$$

respectively. Using (3.14), we have

$$(1+t)^4\|w_{ttt}(t)\|^2\leqslant C_0. \tag{3.19}$$

Combining (3.15)–(3.19) shows (3.3).

Continuing this process, we have the regularity theorem.

THEOREM 3.4. *Suppose that*

$$(w_0,w_1)\in H^s\times H^{s-1}\quad \textit{for } s\geqslant 3.$$

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If $\|w_0, w_1\|_{H^s\times H^{s-1}} + \delta_0$ *is small, then there exists a unique solution to (3.1),* $w \in C^i([0,\infty); H^{s-i})$, $i = 0, 1, \ldots, s$, *which satisfies*

$$(1+t)^{k/2+j}\|\partial_x^k \partial_t^j w(t)\| \leqslant C(\|w_0, w_1\|_{H^s\times H^{s-1}} + \delta_0) \tag{3.20}$$

for $j = 0, 1, \ldots, s-2$ *and* $k = 0, 1, \ldots, s-j$,

$$(1+t)^{(k+1)/2+s-2}\|\partial_x^k \partial_t^{s-1} w(t)\| \leqslant C(\|w_0, w_1\|_{H^s\times H^{s-1}} + \delta_0) \tag{3.21}$$

and

$$(1+t)^{s-1}\|\partial_t^s w(t)\| \leqslant C(\|w_0, w_1\|_{H^s\times H^{s-1}} + \delta_0). \tag{3.22}$$

Theorem 1.3 is a direct consequence of theorem 3.3. If we apply theorem 3.4 with $s \geqslant 4$ to our original problem (1.1), then the solution (v, u) is classical. However, this regularity theorem will be used in the next section, where we will try to obtain the 'truly optimal' convergence rates.

For the proof of theorem 3.4, we only note that we can differentiate (3.1 a) $s-2$ times with respect to t, from which we obtain the estimates

$$(1+t)^{2(s-2)}\|\partial_t^{s-2} w(t)\|^2, \qquad (1+t)^{2(s-2)+1}\|\partial_t^{s-2} w_x(t)\|^2,$$
$$(1+t)^{2(s-2)+2}\|\partial_t^{s-2} w_{xx}(t)\|^2 \leqslant C(\|w_0, w_1\|_{H^s\times H^{s-1}} + \delta_0)$$

in a similar fashion to estimates 4, 5. These derive (3.20). We cannot differentiate (3.1 a) more than $s-2$ times, and hence we only have (3.21), (3.22).

4. The Green-function method

We rewrite (3.1), using the Green function G of the heat equation, as

$$w(x,t) = \int G(x-y,t)w_0(y)\,\mathrm{d}y + \int_0^t \int G(x-y,t-\tau)(-w_{\tau\tau} + F(w_{yy}))(y,\tau)\,\mathrm{d}y\mathrm{d}\tau. \tag{4.1}$$

Remember that

$$F(w_{xx}) = b(\bar{v} - \underline{v})w_{xx} + O(|\hat{v}| + |\bar{v} - \underline{v}|^3 + w_{xx}^2 + |\bar{v}_t|). \tag{4.2}$$

As in [14], since

$$\begin{aligned}\int_0^{t/2} & G(x-y,t-\tau)(-w_{\tau\tau})(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\ &= \int_0^{t/2}\Big(-\frac{\partial}{\partial\tau}\int Gw_\tau + \int G_\tau w_{\tau\tau}\Big)\,\mathrm{d}\tau \\ &= \int G(x-y,t)w_1(y)\,\mathrm{d}y - \int G(x-y,\tfrac{1}{2}t)w_t(y,\tfrac{1}{2}t)\,\mathrm{d}y \\ &\qquad - \int_0^{t/2}\int G_t(x-y,t-\tau)w_\tau(y,\tau)\,\mathrm{d}y\mathrm{d}\tau,\end{aligned}$$

we have the following expression for w,

$$\begin{aligned} w(x,t) &= \psi(x,t) - \int G(x-y,\tfrac{1}{2}t)w_t(y,\tfrac{1}{2}t)\,\mathrm{d}y \\ &\quad - \int_0^{t/2}\int G_t(x-y,t-\tau)w_\tau(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\ &\quad - \int_{t/2}^{t}\int G(x-y,t-\tau)w_{\tau\tau}(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\ &\quad + \int_0^{t}\int G(x-y,t-\tau)F(w_{yy})(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\ &=: I_1+\cdots+I_5, \end{aligned} \tag{4.3}$$

with $I_5 =: I_{51}+\cdots+I_{54}$ from (4.2). Here,

$$\psi(x,t) = \int G(x-y,t)(w_0+w_1)(y)\,\mathrm{d}y \tag{4.4}$$

or

$$\left.\begin{aligned} \psi_t - a\psi_{xx} &= 0, \\ \psi|_{t=0} &= (w_0+w_1)(x). \end{aligned}\right\} \tag{4.4$'$}$$

Using the estimates obtained the preceding section, we estimate the right-hand side of (4.3).

Theorem 4.1. *Suppose that* $(w_0,w_1)\in H^s\times H^{s-1}$, *with* $s=4$ *as in theorem 3.4, and that* $(w_0,w_1)\in L^1$. *Then the solution* w *to (3.1) satisfies, for* $t\geqslant 2$,

$$\|\partial_x^k w(t)\| \leqslant Ct^{-1/4-k/2}\log t \quad \textit{and} \quad \|\partial_x^k w(t)\|_{L^\infty} \leqslant Ct^{-1/2-k/2}\log t \tag{4.5}$$

for $k=0,1,2,3$ *and*

$$\|\partial_x^k w_t(t)\| \leqslant Ct^{-5/4-k/2}\log t \quad \textit{and} \quad \|\partial_x^k w_t(t)\|_{L^\infty} \leqslant Ct^{-3/2-k/2}\log t \tag{4.6}$$

for $k=0,1,2$.

Theorem 1.4 is a direct consequence of theorem 4.1.

Proof of theorem 4.1. Since $s=4$, theorem 3.4 gives the estimate

$$\|\partial_x^k\partial_t^j w(t)\| \leqslant Ct^{-k/2-j}, \qquad j=0,1,2, \quad k=0,1,2,3, \tag{4.7}$$

$$\|\partial_x^k\partial_t^3 w(t)\| \leqslant Ct^{-(k+3)/2}, \quad k=0,1. \tag{4.8}$$

Using these, $I_1,\dots,I_4$ are easily estimated. For example,

$$\begin{aligned} \|\partial_x^3 I_3\|_{L^\infty} &\leqslant C\int_0^{t/2}\|G_{txxx}(t-\tau)\|\|w_\tau(\tau)\|\,\mathrm{d}\tau \\ &\leqslant Ct^{-1/4-1-3/2}\int_0^{t/2}(1+\tau)^{-1}\,\mathrm{d}\tau \\ &= Ct^{-11/4}\log t \\ &\leqslant Ct^{-2}, \end{aligned}$$

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$$\begin{aligned}\|\partial_x^3 I_4\|_{L^\infty} &\leqslant C\int_{t/2}^t \|G_x(t-\tau)\|\|w_{\tau\tau xx}(\tau)\|\,\mathrm{d}\tau\\ &\leqslant Ct^{-3}\int_{t/2}^t (t-\tau)^{-3/4}\,\mathrm{d}\tau\\ &= Ct^{-3+1/4}\\ &\leqslant Ct^{-2}.\end{aligned}$$

We now estimate I_5,

$$\begin{aligned}\|I_{51}\|_{L^\infty} &\leqslant \int_0^t \|G(t-\tau)\|_{L^\infty}\|(\bar v-\underline{v})w_{xx}(\tau)\|_{L^1}\,\mathrm{d}\tau\\ &\leqslant Ct^{-1/2}\int_0^{t/2}(1+\tau)^{-1/4-1}\,\mathrm{d}\tau + Ct^{-1/4-1}\int_{t/2}^t (t-\tau)^{-1/2}\,\mathrm{d}\tau\\ &\leqslant Ct^{-1/2},\\ \|I_{53}\|_{L^\infty} &\leqslant \int_0^t \|G(t-\tau)\|_{L^\infty}\|(\bar v-\underline{v})(\tau)\|_{L^3}^3\,\mathrm{d}\tau\\ &\leqslant Ct^{-1/2}\int_0^{t/2}(1+\tau)^{-1}\,\mathrm{d}\tau + Ct^{-1}\int_{t/2}^t (t-\tau)^{-1/2}\,\mathrm{d}\tau\\ &\leqslant Ct^{-1/2}\log t,\\ \|I_{54}\|_{L^\infty} &\leqslant \int_0^t \|G(t-\tau)\|_{L^\infty}\|w_{xx}(\tau)\|_{L^2}^2\,\mathrm{d}\tau\\ &\leqslant Ct^{-1/2}\int_0^{t/2}(1+\tau)^{-2}\,\mathrm{d}\tau + Ct^{-2}\int_{t/2}^t (t-\tau)^{-1/2}\,\mathrm{d}\tau\\ &\leqslant Ct^{-1/2}.\end{aligned}$$

Since $\hat v$ decays exponentially, I_2 is easily estimated. L^2-estimates and the higher order in x estimates are similar to the above. Thus we have (4.5).

To get (4.6), we differentiate (4.3) in t,

$$\begin{aligned}w_t(x,t) = \psi_t(x,t) &- \int G_t(x-y,\tfrac{1}{2}t)w_t(y,\tfrac{1}{2}t)\,\mathrm{d}y\\ &+ \int G(x-y,\tfrac{1}{2}t)(-w_{tt}(y,\tfrac{1}{2}t)+F(w_{yy})(y,\tfrac{1}{2}t))\,\mathrm{d}y\\ &- \int_0^{t/2}\int G_{tt}(x-y,t-\tau)w_\tau(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\\ &- \int_{t/2}^t\int G(x-y,t-\tau)w_{\tau\tau\tau}(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\\ &+ \int_0^{t/2}\int G_t(x-y,t-\tau)F(w_{yy})(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\\ &+ \int_{t/2}^t\int G(x-y,t-\tau)\partial_\tau F(w_{yy})(y,\tau)\,\mathrm{d}y\mathrm{d}\tau.\end{aligned} \tag{4.9}$$

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Here we have used integration by parts in τ. Similar estimates for (4.9) yield (4.6). We omit the details. □

REMARK 4.2. To apply theorem 4.1 to theorem 1.4, the estimates of w_{xx} and w_{tx} are needed. To estimate w_{tx}, we need the estimate of

$$\int_{t/2}^{t}\int G_x(x-y,t-\tau)w_{\tau\tau\tau}(y,\tau)\,\mathrm{d}y\mathrm{d}\tau =: II_{5x}$$

from the fifth term in the right-hand side of (4.9). If $(w_0, w_1) \in H^3 \times H^2$, then we have only $\|w_{\tau\tau\tau}(\tau)\| \leqslant C(1+\tau)^{-2}$, so that

$$\|II_{5x}\|_{L^\infty} \leqslant Ct^{-2}\int_{t/2}^{t}(t-\tau)^{-3/4}\,\mathrm{d}\tau \leqslant Ct^{-7/4}.$$

This decay order seems to be less sufficient. So we assume $s = 4$ in theorem 3.4. In (4.6), (4.7), we could not delete 'log'. These come from I_{53}, etc., which depend on the choice of the diffusion wave $\phi(x,t;x_0)$. We believe that 'log' will be deleted if the diffusion wave ϕ is selected more suitably. We also note that the L^1-convergence result should be obtained. But our present method will not be applicable.

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