The stability of a strut under thrust, when buckling is resisted by a force proportional to the displacement. By S. GOLDSTEIN, B.A., St John's College.

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1. The problem discussed in this paper is that of the stability of a strut of length l, either clamped at both ends or pinned at both ends, under a thrust R, with buckling resisted by a force equal to K times the displacement. The problem was suggested to me by Mr H. Jeffreys, of St John's College, as being of some geophysical interest (see 6). It later appeared that it might have some connection with recent work on the collapse of thin films on water (see 9).

2. *Fundamental Differential Equation.*

The usual elementary theory is used*. The strut being of fixed length, we know by the principle of the exchange of stabilities that when R is gradually increased from zero till a position of equilibrium of the strut other than the straight form just becomes possible, then the straight form itself becomes unstable. We therefore seek for the relation between R and l which holds when such a position of equilibrium exists. The axis of x is taken along the central line of the strut in its unbuckled state, which is also taken to be the line of thrust. The axis of y is in the plane of buckling.

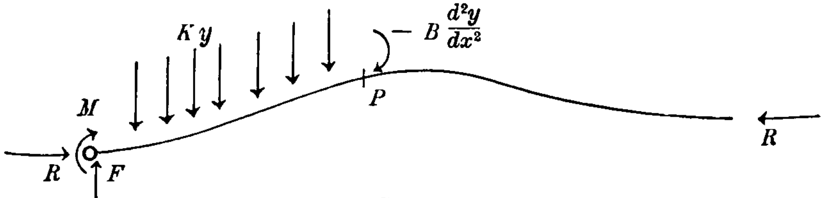

Fig. 1

Consider first a clamped strut. Take the origin O at one end, and let P be a point whose abscissa is x, the strut being supposed just buckled. Take moments about P for the portion OP of the strut. This gives the equation

$$B\frac{d^2y}{dx^2} + Ry + \int_0^x Ky(\xi)(x-\xi)\,d\xi - M - Fx = 0 \quad \ldots(1),$$

where M is the bending moment and F the shearing force at the clamp, as shown (Fig. 1): B is the flexural rigidity of the strut, and $y(\xi)$ is the value of y at a distance ξ from the end.

* Prescott, *Applied Elasticity*, pp. 82 *et seq.* (1924).

Differentiating this equation twice with respect to x gives

$$B\frac{d^4y}{dx^4}+R\frac{d^2y}{dx^2}+Ky=0 \qquad \text{......(2).}$$

This equation is unaltered if the origin be taken at any other point along the central line; and the same equation clearly holds for a pinned rod.

3. *Clamped Strut:* $R^2 < 4KB$.

If $R^2 < 4KB$ the equation

$$Bm^4+Rm^2+K=0 \qquad \text{......(3)}$$

has roots of the form $\pm\alpha\pm i\beta$, where α and β are real and positive.

Let the ends be $x=0$ and $x=l$. At the ends y and dy/dx vanish. The solution of (2) is

$$y=A\cosh\alpha x\cos\beta x+B\cosh\alpha x\sin\beta x+C\sinh\alpha x\cos\beta x + D\sinh\alpha x\sin\beta x \text{......(4).}$$

By the conditions at $x=0$, $A=0$ and $\beta B+\alpha C=0$, and we can write

$$y=D\sinh\alpha x\sin\beta x-\alpha H\cosh\alpha x\sin\beta x+\beta H\sinh\alpha x\cos\beta x \qquad \text{......(5).}$$

The conditions at $x=l$ give, on eliminating H/D and simplifying,

$$\beta^2\sinh^2\alpha l-\alpha^2\sin^2\beta l=0,$$

or

$$\beta\sinh\alpha l=\pm\alpha\sin\beta l \qquad \text{......(6).}$$

This cannot be satisfied with $l>0$, since $\sinh\alpha l>\alpha l$ and $\sin\beta l<\beta l$.

Hence, *for values of R less than* $\sqrt{4KB}$, *there is stability for any length.*

4. *Clamped Strut:* $R^2 > 4KB$.

For $R^2>4KB$, take the origin in the middle. Then y and dy/dx vanish at $x=\pm\frac{1}{2}l$.

The roots of (3) are $\pm i\gamma$, $\pm i\delta$, where

$$B\gamma^2=\tfrac{1}{2}R+\sqrt{(\tfrac{1}{4}R^2-KB)} \qquad \text{......(7),}$$

$$B\delta^2=\tfrac{1}{2}R-\sqrt{(\tfrac{1}{4}R^2-KB)} \qquad \text{......(8),}$$

γ and δ being real and positive.

The solution of (2) is then

$$y=A\cos\gamma x+B\sin\gamma x+C\cos\delta x+D\sin\delta x \qquad \text{...(9),}$$

and the end conditions give

$$\begin{aligned}A\cos\tfrac{1}{2}\gamma l+C\cos\tfrac{1}{2}\delta l&=0,\\ B\sin\tfrac{1}{2}\gamma l+D\sin\tfrac{1}{2}\delta l&=0,\\ \gamma A\sin\tfrac{1}{2}\gamma l+\delta C\sin\tfrac{1}{2}\delta l&=0,\\ \gamma B\cos\tfrac{1}{2}\gamma l+\delta D\cos\tfrac{1}{2}\delta l&=0.\end{aligned}$$

Hence either $B = D = 0, \quad \beta \sin \alpha l + \alpha \sin \beta l = 0$ ......(10),

or $A = C = 0, \quad \beta \sin \alpha l - \alpha \sin \beta l = 0$ ......(11),

where $\alpha = \frac{1}{2}(\gamma + \delta), \quad \beta = \frac{1}{2}(\gamma - \delta)$ ......(12).

The expressions for y in the two cases are of the forms

$$\left.\begin{array}{l} y = E(\cos \tfrac{1}{2}\delta l \cos \gamma x - \cos \tfrac{1}{2}\gamma l \cos \delta x) \text{ corresponding to (10)} \\ y = F(\sin \tfrac{1}{2}\delta l \sin \gamma x - \sin \tfrac{1}{2}\gamma l \sin \delta x) \text{ corresponding to (11)} \end{array}\right\} \quad (13).$$

To consider the equation $\beta \sin \alpha l = \pm \alpha \sin \beta l \; (\alpha > \beta)$ let

$$\beta l = x, \qquad \beta/\alpha = p \qquad (p < 1) \quad \text{...........(14).}$$

The equation is $\pm p \sin (x/p) = \sin x$ ......(15).

We are concerned only with positive values of x.

Let $f(x) = \sin x - p \sin (x/p)$. Then, if r is an odd integer, we have

$$f(r\pi - \sin^{-1} p) > 0, \qquad f(r\pi + \sin^{-1} p) < 0,$$

and, if r is an even integer,

$$f(r\pi - \sin^{-1} p) < 0, \qquad f(r\pi - \sin^{-1} p) > 0.$$

Hence there is a root of $f(x) = 0$ in each of the interva..

$$(r\pi - \sin^{-1} p, \qquad r\pi + \sin^{-1} p).$$

Also $f(0) = 0, f'(x) > 0$ for $x < \frac{1}{2}\pi$. Therefore $f(x) > 0$ for $x < \frac{1}{2}\pi$. For $\frac{1}{2}\pi < x < \pi - \sin^{-1} p$, $\sin x > p$, and therefore $f(x) > 0$. Hence $f(x) > 0$ for $0 < x < \pi - \sin^{-1} p$.

For $\pi + \sin^{-1} p < x < 2\pi - \sin^{-1} p$, $\sin x < -p$ and $f(x) < 0$; for $2\pi + \sin^{-1} p < x < 3\pi - \sin^{-1} p$, $\sin x > p$ and $f(x) > 0$; and so on. Hence there is no root of $f(x) = 0$ outside the intervals

$$(r\pi - \sin^{-1} p, \qquad r\pi + \sin^{-1} p).$$

Similarly there is a root of $\sin x + p \sin x/p = 0$ in each of the intervals $(r\pi - \sin^{-1} p,\ r\pi + \sin^{-1} p)$, and no root outside these intervals. There is only one root of each equation in each interval, since otherwise we should have $\frac{1}{2}p\pi < \sin^{-1} p$. The root of one equation is in the first half of each interval, the other equation having its root in the second half. The first equation

$$(p \sin x/p = \sin x)$$

has its first root the smaller if the greatest integer contained in p^{-1} is even: it has its second root the smaller if the greatest integer contained in $2p^{-1}$ is odd: its third root the smaller if the greatest integer in $3p^{-1}$ is even, and so on.

All these results are easily obtained by drawing the graphs of $\sin x$ and $\pm p \sin x/p$ for a few values of p.

The first root for x is then certainly between $\frac{1}{2}\pi$ and π, so that βl lies between $\frac{1}{2}\pi$ and π. For the second root βl is between π and $\frac{3}{2}\pi$: for the third root between $\frac{3}{2}\pi$ and 2π, and so on. Now β

vanishes when $R^2 = 4KB$ and increases with R, so that if we suppose l to remain fixed and R to increase from $2\sqrt{KB}$, instability will first manifest itself when the first root is reached, for which βl lies between $\frac{1}{2}\pi$ and π. Further, if l is comparable with continental dimensions the first root will be reached by βl for a value of R very near to $2\sqrt{KB}$ (see 7).

5. *Clamped Strut: wave-length and number of nodes after buckling.*

The expressions for y in (13) can also be written in the forms

$$\left.\begin{aligned} y/E &= \sin\beta\left(\tfrac{1}{2}l + x\right)\sin\alpha\left(\tfrac{1}{2}l - x\right) + \sin\beta\left(\tfrac{1}{2}l - x\right)\sin\alpha\left(\tfrac{1}{2}l + x\right) \\ y/F &= \sin\beta\left(\tfrac{1}{2}l + x\right)\sin\alpha\left(\tfrac{1}{2}l - x\right) - \sin\beta\left(\tfrac{1}{2}l - x\right)\sin\alpha\left(\tfrac{1}{2}l + x\right) \end{aligned}\right\} \quad (16),$$

and we may get some notion of the number of nodes corresponding to the various roots of (15) by taking $\beta l = m\pi$, where m is a positive integer. This reduces the forms (13) or (16) to

$$\left.\begin{aligned} y &= G\cos m\pi x/l \cos m\pi x/pl \ (m \text{ odd}) \\ &= -G\sin m\pi x/l \sin m\pi x/pl \ (m \text{ even}) \end{aligned}\right\} \text{ for the symmetrical modes} \quad \ldots\ldots(17),$$

$$\left.\begin{aligned} y &= H\cos m\pi x/l \sin m\pi x/pl \ (m \text{ odd}) \\ &= -H\sin m\pi x/l \cos m\pi x/pl \ (m \text{ even}) \end{aligned}\right\} \text{ for the asymmetrical modes} \quad \ldots\ldots(18).$$

For $m = 1$ the number of internal nodes is of the order of p^{-1} and the wave-length is of the order of $2pl$. For higher values of m the number of nodes is usually somewhat greater than mp^{-1}. We may compare this with the results for $K = 0$, in which case there are no nodes when buckling first occurs, one node for the first symmetrical mode, two for the second asymmetrical mode, and so on. (Of course, the higher modes have practical application only if the lower ones are prevented by constraints.)

6. *Note on the Formation of Mountains by Horizontal Compression.*

According, for example, to the thermal contraction theory of mountain formation, the process of the cooling of the earth was such as to leave the crust in a state of horizontal compression, under which it gave way by crumpling. The objection has been raised* that horizontal compression would give deformations only on a very small scale, a thin crust being unable to transmit considerable stresses across regions of continental extent without buckling. We shall see that for buckling to occur the depth would have to be a good deal less than is generally considered admissible. The depth with which we are concerned is the depth of the crust down to the level of no strain. The criticism applies not only to

* E.g. Bowie, *J. Franklin Inst.* vol. 198, p. 188 (Aug. 1924).

the thermal contraction theory, but to any theory of horizontal compression.

In buckling the crust would have to take with it the matter underneath, so that the problem is substantially the same as the one that we have been considering. Further, for regions of continental extent, the breadth is sufficiently large compared with the depth to prevent buckling horizontally, so that our two-dimensional theory applies.

To obtain an estimate of the depth required, we take a strut of rectangular section of depth d and breadth b. Then $B = EI$, where E is Young's modulus and I is the moment of inertia of a cross-section about the neutral line in its plane; $I = \frac{1}{12}bd^3$ and $E = 6 \times 10^{11}$ dynes/cm.^{2} roughly, so that $B = 5 \times 10^{10} bd^3$ dynes $\times$ cm.^{2} $K = bg\rho$, where ρ is the density of the material which the crust must move in buckling. Then $K = 3b \times 10^3$ dynes/cm.^{2} roughly, and $4BK = 6 \times 10^{14} b^2 d^3$ dynes^2. In order that buckling may be produced before crumpling, we must have both R/bd less than the breaking stress, which is nearly 19 dynes/cm.^{2}, and R greater than $2\sqrt{BK}$. Hence we must have

$$6 \times 10^{14} b^2 d^3 < R^2 < 10^{18} b^2 d^2, \text{ or } d < 1{\cdot}6 \times 10^3 \text{ cm.};$$

i.e. *the depth of the level of no strain must not exceed* 16 *metres*! The present depth of the level of no strain is between 75 and 80 kilometres, and its depth at the last epoch of mountain formation was not much different*.

It follows that the stabilizing effect of gravity is such as to ensure that any possible horizontal stress can be transmitted without buckling, and yield will therefore occur first by fracture at the weakest point; this is what is actually needed for mountain formation.

7. *Numerical Example of the Buckling of a Clamped Strut.*

For America l is of the order of 4×10^8 cm., and for the Alpine fold of order 3×10^7 cm. Hence, if we take the depth to be less than the critical depth, the lowest value of R that would produce buckling would correspond to a value of β of order, say 10^{-7} cm.$^{-1}$, and would be very near to $2\sqrt{KB}$. If we take $d = 10^3$ cm. we have $B = 5b \times 10^{19}$ dynes $\times$ cm.^{2}, $4BK = 6b^2 \times 10^{23}$ dynes^2 and the critical value of R is $7{\cdot}75b \times 10^{11}$ dynes. If we take $R = 7{\cdot}8b \times 10^{11}$ dynes we get

$$\gamma = 9{\cdot}3 \times 10^{-3} \text{ cm.}^{-1}, \quad \delta = 8{\cdot}3 \times 10^{-3} \text{ cm.}^{-1},$$

$$\alpha = 8{\cdot}8 \times 10^{-3} \text{ cm.}^{-1}, \quad \beta = 5 \times 10^{-4} \text{ cm.}^{-1}, \quad p = 0{\cdot}06$$

$\sin^{-1} p = 0{\cdot}06$. For l of the order of 10^7 cm. the value of β, and therefore also of R, is already too large. This shows how near

* Jeffreys, *Phil. Mag.* vol. 32, pp. 583 and 585 (Dec. 1916).

R must be to its critical value $2\sqrt{BK}$ for l of anything like this order of magnitude, and how near βl must be to π for the first root in such cases. (β and p both continually increase from zero as R increases from $2\sqrt{BK}$.)

The value of R chosen will be just large enough to cause buckling for a length of the order of $2\pi \times 10^5$ cm. The wave-length would then be of the order of $7{\cdot}54 \times 10^3$ cm., and the number of internal nodes would be 16 or 17.

We shall see in 9 that the results for a pinned strut are practically the same.

(NOTE.—If the wave-length does not come out to be larger than, or at any rate comparable with, the depth, the theory certainly ceases to have any practical significance.)

8. *Strut pinned at both ends:* $R^2 < 4KB$.

Equation (2) still holds. Take the origin in the middle. The end conditions are $y = 0$, $d^2y/dx^2 = 0$ at $x = \pm \frac{1}{2}l$.

If $R^2 < 4KB$ we may write, as before,

$$y = A \cosh \alpha x \cos \beta x + B \cosh \alpha x \sin \beta x + C \sinh \alpha x \cos \beta x + D \sinh \alpha x \sin \beta x \quad (\alpha,\ \beta \text{ real and positive}).$$

The end conditions give

$$\left.\begin{array}{ll}\text{either } B = C = 0, & \cosh^2 \tfrac{1}{2}\alpha l \cos^2 \tfrac{1}{2}\beta l + \sinh^2 \tfrac{1}{2}\alpha l \sin^2 \tfrac{1}{2}\beta l = 0\\ \text{or } \quad A = D = 0, & \cosh^2 \tfrac{1}{2}\alpha l \sin^2 \tfrac{1}{2}\beta l + \sinh^2 \tfrac{1}{2}\alpha l \cos^2 \tfrac{1}{2}\beta l = 0\end{array}\right\} \quad (19),$$

and again there cannot be buckling if $R^2 < 4KB$.

9. *Pinned Strut:* $R^2 > 4KB$.

If $R^2 > 4KB$ we may write

$$y = A \cos \gamma x + B \sin \gamma x + C \cos \delta x + D \sin \delta x,$$

where γ and δ are given by (7) and (8).

The end conditions give

$$\begin{aligned} A \cos \tfrac{1}{2}\gamma l + C \cos \tfrac{1}{2}\delta l &= 0,\\ B \sin \tfrac{1}{2}\gamma l + D \sin \tfrac{1}{2}\delta l &= 0,\\ \gamma^2 A \cos \tfrac{1}{2}\gamma l + \delta^2 C \cos \tfrac{1}{2}\delta l &= 0,\\ \gamma^2 B \sin \tfrac{1}{2}\gamma l + \delta^2 D \sin \tfrac{1}{2}\delta l &= 0. \end{aligned}$$

Since, in general, $\gamma \neq \delta$, it appears that we must have

$$A \cos \tfrac{1}{2}\gamma l = B \sin \tfrac{1}{2}\gamma l = C \cos \tfrac{1}{2}\delta l = D \sin \tfrac{1}{2}\delta l = 0 \quad (20).$$

Hence three of A, B, C, D must be zero and either γl or δl must be equal to $n\pi$, where n is a positive integer.

As R increases from $2\sqrt{KB}$, γ increases and δ decreases, and buckling will first take place when γ or δ becomes equal to a multiple of π/l. We have then that

$$y = A \cos n\pi x/l \text{ or } B \sin n\pi x/l \quad \text{.........(21)},$$

where $n\pi/l$ is the multiple of π/l first reached by

$$\{\tfrac{1}{2}R/B \pm \sqrt{(\tfrac{1}{4}R^2/B^2 - K/B)}\}^{\frac{1}{2}},$$

as R increases from $2\sqrt{KB}$. The wave-length is $2l/n$ and the number of internal nodes is $n - 1$.

When l is of the order $2\pi \times 10^5$ cm. and d is 10^3 cm., so that $4BK = 6b^2 \times 10^{23}$ dynes^2 and R increases from $7{\cdot}75b \times 10^{11}$ dynes as before (see 7), γ increases and δ decreases from $8{\cdot}8 \times 10^{-5}$ cm.$^{-1}$, while π/l is 5×10^{-6} cm.$^{-1}$ When $R = 7{\cdot}8b \times 10^{11}$ dynes, $\gamma = 9{\cdot}3 \times 10^{-4}$ cm.$^{-1}$ and $\delta = 8{\cdot}3 \times 10^{-4}$ cm.$^{-1}$, so that a multiple of π/l is reached for a value of R nearer the critical value. Thus the pinned strut buckles before a clamped strut of the same length would do so. Also, for the case considered, n is 17 or 18, so that the wave-length and the number of nodes are about the same as for a clamped strut.

10. *The Compression and Collapse of Surface Films on Water.*

In a series of papers* Mr N. K. Adam has discussed the behaviour, under compression, of surface films on water. These films are of one molecule thickness, and when the area per molecule is sufficiently reduced (to about $2{\cdot}2 \times 10^{-15}$ cm.^{2}) the film behaves like a solid and there is a linear relation between the compression and the thrust. The linear relation holds when the thrust is increased from 3 or 4 dynes/cm. until we reach the stage at which the film gives way†.

The atoms being in equilibrium under their mutual attractions and repulsions, we suppose that when the system is compressed, the restoring force on any atom is approximately proportional to its relative displacement, and the usual elementary elastic theory may be applied. First, we can calculate from the compressibility the value of E, Young's modulus for the film. Then if the film were to buckle, it would have to raise and lower the water with it, so that we can employ the theory of 2–5, 8 and 9 to find under what thrust instability should first manifest itself. The first effect would be to cause buckling, which in a film of one molecule thickness would necessarily be imperceptible; but very shortly after this the film should give way. We assume that the film does not collapse before the thrust reaches its theoretical value for buckling.

11. *The Value of Young's Modulus for a Solid Film.*

We deal with a breadth of 1 cm. Then‡, since any error in experiment will increase the observed compressibility, we shall use the lowest value obtained, 0·4 per cent. for a thrust of 10 dynes.

* *Proc. Roy. Soc.* A, vol. 99, p. 336 (1921); vol. 101, p. 452 (1922), etc. References to the two papers mentioned will be given as Part I and Part II.

† Part I. ‡ Part II, p. 458.

We may take the depth of a film as being of the order of $2{\cdot}5 \times 10^{-7}$ cm.* Then E is equal to the force per unit depth divided by the compressibility. This gives $E = O\,(10^{10})$ dynes/cm.^{2}

12. *The Collapse of a Solid Film.*

The value of the thrust that produces collapse cannot be determined with accuracy, but is anything from 20 to 45 dynes†. If we suppose buckling to precede collapse the value of R that gives buckling should not be very different from the value for collapse, preference being given to the smaller values.

Now $B = EI$, $I = \frac{1}{12}d^3$, $K = g$, and with our data and the above value for E, $B = O\,(10^{-11})$ dynes $\times$ cm.^{2}, $BK = O\,(10^{-8})$ dynes^2.

If we were to take R to be 16 dynes, we should, employing the results of 4, find that $\beta = O\,(4{\cdot}5 \times 10^5)$ cm.$^{-1}$ This value of β is much too large for a length anywhere between 2 or 3 cm. to 20 or 30 cm. We want a value of β not greater than about 1 cm.$^{-1}$, and this means a value of R of the order of 10^{-4} dynes, which is nonsense. With R 16 dynes, values of B between 6×10^{-2} and $6{\cdot}4 \times 10^{-2}$ dynes $\times$ cm.^{2} give values of β between 2 and 0, so that for this value of R we should want $B = O\,(6 \times 10^{-2})$ dynes cm.^{2}, $E = O\,(6 \times 10^{19})$ dynes/cm.^{2}

The results if we considered the film as pinned would be practically the same, and would show the same discrepancy.

Clearly then there are other factors to be considered that make for stability.

13. *The Effect of Surface Tension.*

Since in both compression and buckling the area of the water-film surface is changed, we may expect surface tension to play a part. For compression, if T is the surface tension,

$$E = (R + T)/(d \times \text{compressibility}).$$

Since the surface tension of a solid film cannot vary appreciably with additional compression, we may‡ take T to be 70 dynes/cm. at temperatures between 10° and 20° C., the temperature of Mr Adam's experiments on palmitic acid§. Then

$$E = O\,(8 \times 10^{10})\ \text{dynes/cm.}^2$$

The equation (2) becomes

$$B\frac{d^4y}{dx^4} + (R - T)\frac{d^2y}{dx^2} + Ky = 0 \qquad \text{......(22).}$$

The value of R that produces buckling must exceed T. According to the experimental results, however, the film collapses for values of R (20–45 dynes/cm.) less than T, and therefore

* Part II, p. 463. † Part I, p. 340.
‡ Cary and Rideal, *Proc. Roy. Soc.* A, vol. 109, p. 320 (1925).
§ Part I, p. 340.

less than the theoretical value for buckling. Hence instability must first manifest itself by a weakness in the film causing collapse; this happens before the film has a chance to buckle.

14. *The Frequency of Vibration of an Atom in a Solid Film.*

From the value of E we can make a very rough calculation for the frequency of vibration of an atom in a solid film. The cross-section of the head of a molecule is of the order of 26×10^{-16} cm.^{2}*, i.e. $(5 \times 10^{-8}$ cm.$)^2$. We take 5×10^{-8} cm. as the original distance D, between two atoms, that is decreased to $D - \delta$ by compression. Then we have

$$\text{Force per unit area across a cross-section} = E\delta/D.$$

If n is the number of atoms in the molecule, d the depth of the unimolecular layer, then the restoring force on an atom

$$= (E\delta/D) \div (n/dD) = Ed\delta/n.$$

The frequency is $\sqrt{\mu}/2\pi$, where

$$\mu = \text{restoring force} \div (\delta \times \text{mass}).$$

The mass of an atom is $1{\cdot}65 \times 10^{-24} A$ gm., where A is its atomic weight. Take $A = 12$ (for a carbon atom), and $n = 60$, for example. With $E = O\,(8 \times 10^{10})$ dynes/cm.^{2}, $\mu = O\,(1{\cdot}7 \times 10^{25})$ and the frequency $= O\,(6{\cdot}5 \times 10^{11})$.

We may compare this with the corresponding frequency for crystals, as obtained by the method of residual rays†. The frequencies are c/λ, where c is the velocity of light ($= 3 \times 10^{10}$ cm./sec.) and λ, the wave-length of the residual rays, is between $1{\cdot}2 \times 10^{-3}$ and $1{\cdot}2 \times 10^{-2}$ cm. for most crystals. This gives frequencies varying between $2{\cdot}5 \times 10^{13}$ and $2{\cdot}5 \times 10^{12}$.

15. *Summary.*

For a strut under thrust when buckling is resisted by a force proportional to the displacement, it is shown that for both a clamped and a pinned strut there is stability for any length when the thrust is less than a certain critical value; and the relation is found between the length and the first value of the thrust above this that will give buckling. A notion of the wave-length and number of nodes after buckling is obtained.

The results are applied to the theory of the formation of mountains by horizontal compression, and it is shown that the crust of the earth would be able to transmit, without buckling, stresses right up to the breaking stress across regions of continental extent, unless the depth down to the level of no strain were less than 16 metres.

* Part II, p. 463.

† Born, *Dynamik der Kristallgitter*, p. 623 (1923).

An attempt is made to employ the results also to consider the stability of solid surface films under compression, and it is seen that other factors, making for stability, must enter besides rigidity and gravity. This is supplied by surface tension, and it then appears that collapse due to a weakness in the film must precede buckling.

A rough calculation is given of the frequency of vibration of an atom in a solid film.

16. I have to thank Mr H. Jeffreys for advice and assistance in the preparation of this paper, and Mr E. K. Rideal for showing me an experiment similar to those of Mr Adam.