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Travelling wave solutions of the degenerate Kolmogorov–Petrovski–Piskunov equation

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We prove the existence of a family of Travelling Wave (TW) solutions for a large class of scalar reaction-diffusion equations with degenerate, nonlinear diffusion coefficients and monostable nonlinear reaction terms. We also investigate stability. Specifically, we show that, as in the linear diffusion case [6], the slowest TW in the family yields the asymptotic rate of the propagation of disturbances from the unstable rest state in these systems. In addition, we give conditions on the reaction term and diffusion coefficient ensuring the existence of interfaces.

1 Introduction

In the present paper, we study the asymptotic behaviour of solutions of the Initial Value Problem (IVP) for a degenerate parabolic equation

$$u_t(x,t) = (\phi(u))_{xx} + \psi(u) \tag{1.1}$$

in $D = \{(x,t) : x \in \mathbb{R}, 0 \leqslant t < +\infty\}$ with the initial condition

$$u(x,0) = u_0(x). \tag{1.2}$$

We assume that $\phi(u)$ is a non-negative continuous function on $[0,1]$ with $\phi(u) > 0$ and $\phi'(u) > 0$ on $(0,1]$, but $\phi(0) = \phi'(0) = 0$. The function $\psi(u)$ is non-negative on $[0,1]$ and $\psi(0) = \psi(1) = 0$. Therefore, (1.1) is a quasilinear parabolic equation when $u(x,t) > 0$, and it degenerates at points where $u(x,t) = 0$. More detailed assumptions on $\phi(u)$ and $\psi(u)$ are given in the next section. In particular, the class of problems that we consider includes equations of the form

$$u_t = (u^{\alpha+1})_{xx} + u^{\beta}(1-u), \quad \alpha > 0, \beta \geqslant 1. \tag{1.3}$$

Besides the existence and uniqueness of a solution of (1.1), (1.2), our assumptions

(Hypotheses 2.1) guarantee the continuity of the flux $-(\phi(u))_x$ across the interface (Theorem 2.1) and imply comparison theorems (Theorems 2.2 and 2.3): the only properties of solutions of (1.1), (1.2) that are essential for our analysis. However, the assumptions we use are probably not optimal, and our conclusions should hold for a larger class of equations of type (1.1), whose solutions have the general properties mentioned above.

With $\psi(u) \equiv 0$, (1.1) can be interpreted as a nonlinear heat equation with temperature-dependent heat conductivity. In this case, $u(x, t)$ stands for temperature. Applications of the equations of this type to modelling filtration processes, gas diffusion, phenomena in the atmosphere and in the ocean, and phase transitions are now classical; more recent applications include problems in mathematical biology (e.g. see Gurtin & MacCamy [15], Medvedev *et al.* [21] and Murray [22]) and medicine (see Chen & Friedman [9] and Sherratt [28], and the references therein). Because of their numerous applications in mathematical physics and biology, nonlinear degenerate equations of the type (1.1) with and without $\psi(u)$ have been studied extensively. We refer the reader elsewhere [3, 17, 25] for surveys on the results of the theory of degenerate parabolic equations and a bibliography. On the other hand, if $\phi(u) \equiv u$, (1.1) becomes a famous Kolmogorov–Petrovski–Piskunov (KPP) equation. It was introduced by Fisher [10] and by Kolmogorov, Petrovski and Piskunov [19] as a model for the propagation of advantageous genes in biological populations. The asymptotic behaviour of solutions of the KPP equation with an initial condition given by a non-negative function bounded by 1 whose support is restricted to the negative half of the real line was studied in an exhaustive manner in Kolmogorov *et al.* [19]. In particular, if $\phi(u) = u$ and $\psi(u) = u(1-u)$, it was shown that there exists a family of TW solutions $u(x, t) = U_c(x - ct)$ with all velocities $c \geqslant 2$. Moreover, the solutions of the IVP approach the TW solution $U_2(x)$ with minimal speed $c = 2$ as $t \to \infty$ in the sense

$$\sup_{x \in \mathbb{R}} |u(x + m(t), t) - U_2(x)| \to 0 \text{ and } m'(t) \to 2 \quad \text{as } t \to \infty$$

for some continuously differentiable function $m(t)$.

Many methods have been developed for studying stability of the TW solutions of the KPP equation and other nonlinear parabolic equations and system of equations. We note techniques based on maximum principle, linear stability analysis, the use of Evans functions, renormalization group analysis, probabilistic methods, and variational methods. For a discussion of the main results and references, see Volpert *et al.* [30]. In particular, Aronson & Weinberger [6] proposed a method for studying the asymptotic behaviour of solutions of the IVPs for (1.1) with linear diffusion and similar equations, using maximum principle and phase-plane analysis of certain second order Ordinary Differential Equations (ODEs). This paper shows how to construct sub- and super-solutions for IVPs, which in the case of the KPP equation imply that

$$\lim_{t \to \infty} u(x + ct, t) = \begin{cases} 1, & 0 \leqslant c < 2, \\ 0, & c > 2, \end{cases}$$

provided the support of the initial data $u_0(x)$ is restricted to the negative half of $\mathbb{R}$. Physically, this means that bounded perturbations of the unstable rest state propagate with an asymptotic speed 2, that is the minimal speed of the TWs of the KPP equation.

In particular, this means that no TW solution U_c with speed $c > 2$ can attract solutions of the IVP with compactly-supported initial data.

The attractiveness of the method of Aronson & Weinberger [6] is that it yields conclusions about the asymptotic behaviour of solutions of the IVPs from a simple phase-plane analysis of an ODE. This method was shown to work for problems in $\mathbb{R}^n$ in Aronson & Weinberger [7]. The purpose of the present paper is to extend the techniques of Aronson & Weinberger [6] to additionally cover degenerate nonlinear equations of the type (1.1). For this we need to do two things: first, perform a phase-plane analysis showing the existence of a family of TWs of (1.1); and secondly, construct suitable sub- and super-solutions of (1.1). We do both steps by adapting the corresponding techniques from Aronson & Weinberger [6]. In fact, after the change of independent variables at the beginning of § 3 the phase-plane analysis is almost the same as that used for finding TWs for equations with linear diffusion, though we think that our presentation is more geometric than that given in Aronson & Weinberger [6]. As to constructing sub- and super-solutions from the trajectories of the corresponding ODEs, the only difference from Aronson & Weinberger [6] is that we need to take into account the behaviour of solutions near the interface. We deal with this issue by using the continuity of the flux across the interface.

As a class of special solutions of nonlinear parabolic equations, TWs are important for a number of reasons. In many cases, they determine the asymptotic states of solutions of IVPs. It is also known that TWs can be used for studying different properties of solutions of the IVPs. For instance, in Gilding & Kersner [13], it was shown that the question of existence of interfaces in IVPs for some scalar degenerate reaction-convection-diffusion equations, including (1.1), is equivalent to that of the existence of *finite* TWs (for the definition of finite TWs, see (3.2)). Also, in Galactionov *et al.* [12] TWs were used to study the regularity of solutions of IVPs near the interface, as well as for the derivation of equations for interface motion. Therefore, the knowledge of TW solutions of reaction-diffusion equations may be quite helpful for understanding the general properties of solutions of the corresponding IVPs. For some subclasses of (1.1), TW solutions have been found by several authors. In Aronson [2], a family of TW solutions was shown to exist for (1.3) with $\alpha > 0$ and $\beta = 1$; and for $\alpha > 0$ and $\beta \in \mathbb{R}$ the phase-plane analysis for TWs was performed in de Pablo & Vazquez [29]. Also, finite TWs of (1.1) with $\phi''(0) \neq 0$ were found in Sànchez-Garduno & Maini [26]. However, our phase-plane analysis differs from these [2, 26, 29], and we are not aware of any analysis of TWs of (1.1) with assumptions on coefficients as general as those in the present paper. Likewise, to our knowledge the statements on the asymptotic rate of propagation of disturbances for (1.1) (Theorems 4.1 and 4.2) have not been proved before, though some ideas on extending results of Aronson & Weinberger [6] to (1.3) were sketched in Aronson [2].

The paper is organized as follows. In § 2 we give detailed hypotheses on the functions $\phi(u)$ and $\psi(u)$, which guarantee the existence of generalized solutions, the continuity of flux $-\frac{\partial\phi(u)}{\partial x}$, and monotonicity results. Details of the proof of flux continuity, which closely follows the method of Aronson [1], are relegated to the Appendix. § 3 contains the phase-plane analysis, which is carried out using centre manifold methods following a transformation in the waveframe ('time') coordinate that simplifies the ODE systems. This leads to the main theorem of this section on the existence of TWs with all speeds

$c \geqslant c^* > 0$. In §4, we prove that in the travelling frame of coordinates $\xi = x - ct, c > 0$, solutions of the IVPs (1.1), (1.2) converge to 0 if $c > c^*$, or to 1 if $0 < c < c^*$, for initial data given by the step function (4.1) (Theorem 4.1). With the additional restrictions on the reaction term in (1.1) $1 \leqslant \beta < 1 + \alpha$, $\alpha > 0$, we extend this result to cover IVPs with any continuous compactly supported initial data between 0 and 1 (Theorem 4.2). These theorems are extensions to the degenerate equation (1.1) of analogous results for scalar reaction-diffusion equations with linear diffusion.

2 Problem formulation and basic properties of solutions

First we specify the assumptions on the smoothness of the functions in (1.1). Let $\Phi(v)$ and $\Psi(v)$ denote the inverse functions to $\phi(u)$ and $k(u) \equiv \phi'(u)$, respectively:

$$\Phi(\phi(u)) = u, \quad \text{and} \quad \Psi(k(u)) = u, \quad u \geqslant 0.$$

Throughout the paper we shall assume that the following conditions hold:

Hypotheses 2.1

$$(a) \qquad \Phi(v) \in C^{\alpha_1}([0,1]), \text{ for some } \alpha_1 \in (0,1); \tag{2.1}$$

$$(b) \qquad \phi(u) \in C^{1+\alpha_2}([0,1]) \bigcap C^{4+\alpha_2}([m,1]), \text{ for some } \alpha_2 \in (0,1), \forall 0 < m < 1;$$

$$(c) \qquad \phi^{(i)} \geqslant 0, i = 0,1,2; \quad k(u) \equiv \phi'(u) > 0, \phi''(u) > 0, u > 0; \quad \phi(0) = 0;$$

$$(d) \qquad \lim_{u\to 0} \tfrac{k(u)}{u^\alpha} = \mathscr{C}_1, \quad \text{for some} \quad \mathscr{C}_1 \quad \text{and} \quad \alpha > 0;$$

$$(e) \qquad \psi(u) \in C([0,1]) \bigcap C^{1+\alpha_2}([m,1]), \text{ for some } \alpha_2 \in (0,1), \forall 0 < m < 1;$$

$$(f) \qquad \psi(u) > 0 \text{ for } 0 < u < 1, \psi(0) = \psi(1) = 0, \text{ and } \psi'(1) < 0;$$

$$(g) \qquad \lim_{u\to 0} \tfrac{\psi(u)}{u^\beta} = \mathscr{C}_2, \quad \text{for some} \quad \mathscr{C}_2 \quad \text{and} \quad \beta \geqslant 1;$$

$$(h) \qquad \tfrac{(v\Psi'(v))'}{\Psi'(v)} \text{ and } \tfrac{\psi\circ\Psi}{\Psi'}(v) \text{ are continuously differentiable functions for } v \geqslant 0;$$

(i) $0 \leqslant u_0(x) \leqslant 1$, $u_0(x)$ is continuous on $\mathbb{R}$; $\phi(u_0(x))$ is Lipschitz continuous $\mathbb{R}$.

It is well-known that among the solutions of (1.1) of interest in applications, there are functions that lack the regularity required for classical solutions. In particular, some of the TW solutions, which we present in the next section, belong to this class. Therefore, solutions of (1.1) need to be understood in a generalized sense. We define a generalized solution of (1.1) following Kalashnikov [16]:

Definition 2.1 A bounded Hölder continuous function $u(x,t) \geqslant 0$ is called a generalized solution of (1.1) in the closed domain $G \subset D$, if $u(x,t)$ satisfies the integral identity

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} \{\phi(u)f_{xx} + uf_t + \psi(u)f\}\,dx\,dt - \int_{x_0}^{x_1} uf\,dx\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \phi(u)f_x\,dt\Big|_{x_0}^{x_1} = 0 \tag{2.2}$$

for any $t_0 < t_1$, $x_0 < x_1$ such that $\Pi \equiv [t_0,t_1] \times [x_0,x_1] \subset G$ and for any trial function $f(x,t)$ defined on Π and continuous together with its derivatives f_t, f_x, f_{xx} and equal to zero on the lateral boundaries of $\Pi: f(x,t) = 0, x = x_0, x = x_1$.

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If, in addition, the trial functions are non-negative and the equality sign in (2.2) is relaxed to $\leqslant$ ($\geqslant$), then $u(x,t)$ is called a generalized subsolution (supersolution) of (1.1).

Remark 2.1 Let $u(x,t)$ be a Hölder continuous function on G. Moreover, assume that $u(x,t)$ is bounded for finite values of t and is a sufficiently smooth function in G away from a finite number of nonintersecting continuous curves $J_k = \{(j_k(t),t),\ t \geqslant 0\}$, and satisfies on $D - \bigcup_k J_k$ the equation

$$u_t - (\phi(u))_{xx} - \psi(u) = 0$$

in classical sense. If $\frac{\partial \phi(u)}{\partial x}$ is continuous on J_k, then by integration by parts, one can show that $u(x,t)$ is a generalized solution of (1.1) in G.

Definition 2.2 A function $u(x,t)$ is called a generalized solution of the IVP (1.1), (1.2) if it is a generalized solution of (1.1) and satisfies (1.2).

The well-posedness of the IVPs and initial boundary value problems for quasilinear degenerate parabolic equations has been studied extensively [24, 4, 16, 17, 18, 29] (this list is far from complete). For a comprehensive review of results of the theory of degenerate parabolic equations, we refer the reader to Kalashnikov [17] and Samarski *et al.* [25]. In many cases, the analysis follows the scheme developed in Oleinik *et al.* [24] for the porous medium equation. The existence and uniqueness of a bounded generalized solution of (1.1), (1.2), (2.1) follow by direct extension of the results in Oleinik *et al.* [24] (for a related proof of the existence and uniqueness of solution of the IVP for the nonlinear heat equation with absorption, see also Kalashnikov [16]). Moreover, at the points where $t > 0$ and $u(x,t) > 0$, the generalized solution $u(x,t)$ is a classical solution of (1.1).

As we shall see below, depending on the Hölder exponents α and β (Hypotheses 2.1d,g), the support of the generalized solutions of (1.1), (1.2) may remain bounded, provided the support of the initial data is bounded. At the points where the density $u(x,t)$ vanishes (i.e. *the interface*), the spatial derivative $\frac{\partial u(x,t)}{\partial x}$ may have a discontinuity. However, the density flux $-\frac{\partial \phi(u(x,t))}{\partial x}$ remains continuous across the interface, as we show in the next theorem.

Theorem 2.1 *There exists a continuous derivative* $\frac{\partial \phi(u(x,t))}{\partial x}$ *everywhere in* $D_T \equiv \mathbb{R} \times (0,T]$. *In particular,*

$$\frac{\partial \phi(u(x,t))}{\partial x} = 0 \quad \text{if} \quad u(x,t) = 0. \tag{2.3}$$

Proof of Theorem 2.1 The proof of the theorem closely follows the lines of the proof of the analogous statement for the porous medium equation in Aronson [1]. For completeness, we present the proof in the Appendix. □

We now formulate two theorems on monotonic dependence of generalized solutions of (1.1) on the initial and boundary conditions. We then use these results to prove Theorems 2.4 and 2.5 on the asymptotic behaviour of the generalized solutions of the IVPs for (1.1) with specifically chosen initial conditions. These theorems are generalizations to (1.1) with a nonlinear diffusion coefficient of a very useful method for constructing sub- and super-solutions for scalar reaction diffusion-equations with linear diffusion [6]. We use them in §4 to determine the asymptotic rate of propagation of disturbances from the rest state.

Theorem 2.2 *Let $u(x,t)$ be a generalized subsolution of* (1.1), (1.2) *and $w(x,t)$ be a generalized supersolution of* (1.1) *in D, such that*

$$u(x,0) \leqslant w(x,0), \quad x \in \mathbb{R}.$$

Then $u(x,t) \leqslant w(x,t)$ everywhere in D.

Proof of Theorem 2.2 is completely analogous to that of Theorem 4 of Kalashnikov [16]. □

We shall also need the following comparison principle for generalized solutions of (1.1) in the half-strip:

$$S_{[a,b]} = \{(x,t) : \ a \leqslant x \leqslant b, t \geqslant 0\}, \quad -\infty < a < b < \infty.$$

Theorem 2.3 *Let $u(x,t)$ be a generalized solution of* (1.1), (1.2) *in $S_{[a,b]}$ and let the functions $u^{\pm} \in C^{2,1}_{x,t}(S_{[a,b]})$ satisfy* (1.1) *in the interior of $S_{[a,b]}$,*

$$u^-(x,0) \leqslant u(x,0) \leqslant u^+(x,0) \quad \textit{for} \quad a \leqslant x \leqslant b,$$

and

$$u^-(a,t) \leqslant u(a,t) \leqslant u^+(a,t), \quad u^-(b,t) \leqslant u(b,t) \leqslant u^+(b,t), \quad t \geqslant 0.$$

Then $u^- \leqslant u \leqslant u^+$ in $S_{[a,b]}$.

Proof of Theorem 2.3 follows directly from the proof of Theorem 12 of Aronson *et al.* [4]. □

The main tool in our study of stability of TW solutions of the degenerate equation (1.1) is the method of constructing sub- and super-solutions for the solutions $v(\xi,t)$ of (1.1) viewed in the uniformly travelling frame of reference $\xi = x - ct$, $c \in \mathbb{R}$:

$$v_t = (\phi(v))_{\xi\xi} + cv_\xi + \psi(v), \tag{2.4}$$

using certain trajectories of the ODE

$$(\phi(q))'' + cq' + \psi(q) = 0. \tag{2.5}$$

For scalar equations with linear diffusion, this method was developed in Aronson *et al.* [6]. Below we adapt Lemma of Aronson *et al.* [6] to construct subsolutions to (1.1).

Remark 2.2 All definitions and theorems previously introduced in this section extend directly to solutions of (2.4).

Theorem 2.4 *Let $q(\xi) \in [0,1]$ be a solution of the ODE* (2.5) *in* (a,b) *such that* $q(a) = 0$ *and* $q(b) = 0$, $-\infty < a < b < \infty$. *By* $v(\xi,t)$, *denote the solution of the IVP for* (2.4) *with initial data*

$$v(\xi,0) = \begin{cases} q(\xi), & \xi \in (a,b), \\ 0, & \xi \in \mathbb{R} - (a,b). \end{cases} \tag{2.6}$$

Then $v(\xi,t)$ *is a nondecreasing function of* t *for each* $\xi \in \mathbb{R}$. *Moreover,*

$$\lim_{t\to\infty} v(\xi,t) = \tau(\xi) \quad \textit{uniformly in each bounded interval,}$$

where $\tau(\xi) \in [0,1]$ *is the smallest nonnegative function on* $\mathbb{R}$ *such that*

$$\begin{aligned} &(a)\ \tau(\xi) \quad \textit{solves} \quad (2.5) \quad \textit{if} \quad \tau(\xi) > 0, \\ &(b) \quad k(\tau(\xi))\tau'(\xi) = 0 \quad \textit{if} \quad \tau(\xi) = 0, \\ &(c) \qquad \tau(\xi) \geqslant q(\xi), \quad \xi \in (a,b). \end{aligned} \tag{2.7}$$

(*For* (*b*), *recall that* $k(u) \equiv \phi'(u)$ – *see* (2.1c) *above.*)

Proof of Theorem 2.4 Let $v(\xi,t)$ denote the generalized solution of the IVP (2.4), (2.6). We extend $q(\xi)$ by setting $q(\xi) = 0$ on $\mathbb{R} - (a,b)$ and consider it as a function on the whole real line. Note that the constant function 0 is a generalized solution of (1.1) and, therefore, $v(\xi,t) \geqslant 0$ in D by Theorem 2.2. By Theorem 2.3, $v(\xi,t) \geqslant q(\xi)$ in $S_{[a,b]}$, because $q(a) = 0 \leqslant v(a,t)$ and $q(b) = 0 \leqslant v(b,t)$. Hence,

$$v(\xi,t) \geqslant q(\xi) \equiv v(\xi,0), \ \ \xi \in \mathbb{R}, t \geqslant 0. \tag{2.8}$$

The functions $v(\xi,t)$ and $v(\xi,t+h), h > 0$ are the generalized solutions of (1.1) with initial data $v(\xi,0)$ and $v(\xi,h), h > 0$, respectively. Thus, from Theorem 2.2 and (2.8), it follows that

$$v(\xi,t+h) \geqslant v(\xi,t) \text{ in } D \ \forall\ h > 0,$$

i.e. $v(\xi,t)$ is a nondecreasing function of t for every $\xi \in \mathbb{R}$. Since $0 \leqslant v(\xi,t) \leqslant 1$ (by Theorem 2.2), the function

$$0 \leqslant \tau(\xi) = \lim_{t\to\infty} v(\xi,t) \leqslant 1, \ \ \xi \in \mathbb{R}, \tag{2.9}$$

is well-defined. By Dini's Theorem the convergence is uniform on bounded intervals, and therefore $\tau(\xi)$ is continuous on $\mathbb{R}$.

From (2.8), (2.9) and monotonicity of $v(\xi,t)$ in t for each $\xi \in \mathbb{R}$ it follows that

$$\tau(\xi) = \lim_{t\to\infty} v(\xi,t) \geqslant v(\xi,0) = q(\xi), \ \ \xi \in [a,b].$$

Thus, (2.7c) holds. We prove now that the limiting function $\tau(\xi)$ satisfies the other two properties listed in (2.7). First, we show that in its positivity set, $\tau(\xi)$ is a smooth function and it satisfies (2.5) there. For this, suppose $\tau(\xi_0) > 0$ for some $\xi_0 \in \mathbb{R}$. Choose $\delta > 0$ and $\gamma > 0$ such that

$$\tau(\xi) \geqslant 2\gamma > 0 \text{ for } \xi \in [\xi_0 - 2\delta, \xi_0 + 2\delta].$$

Since $v(\xi,t)$ tends to $\tau(\xi)$ as $t \to \infty$ uniformly on bounded intervals, there exists $T > 0$ such that

$$v(\xi,t) > \gamma \ \text{ in } \ \Pi_{\xi_0,2\delta,T} \equiv [\xi_0 - 2\delta, \xi_0 + 2\delta] \times [T,\infty].$$

In $\Pi_{\xi_0,2\delta,T}$, $v(\xi,t)$ satisfies a uniformly parabolic equation:

$$v_t = (k(v)v_\xi)_\xi + cv_\xi + \psi(v),$$

with $0 < k_1 \leqslant k(v(\xi,t)) \leqslant k_2 < \infty$. In addition, $v(\xi,t)$ is bounded in $\Pi_{\xi_0,2\delta,T}$. Thus, from the interior Schauder estimates [11, 20], we have that $v_\xi, v_{\xi\xi}$, and v_t are uniformly bounded and are uniformly continuous functions in $\Pi_{\xi_0,\delta,T+1} \subset \Pi_{\xi_0,2\delta,T}$. Therefore, $v, v_\xi, v_{\xi\xi}$, and v_t are equicontinuous on $[\xi_0-\delta, \xi_0+\delta]$ considered as families of functions parametrized by t with $t \geqslant T$. By Arzela–Ascoli's Theorem, given any sequence of times $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ there exists a subsequence $\{t'_n\}$ such that $v_\xi, v_{\xi\xi}$, and $v_t(\xi, t'_n)$ converge uniformly on $[\xi_0-\delta, \xi_0+\delta]$ as $n \to \infty$. Therefore, $v_\xi(\xi, t'_n) \to \tau'(\xi)$ and $v_{\xi\xi}(\xi, t'_n) \to \tau''(\xi)$ uniformly on $[\xi_0-\delta, \xi_0+\delta]$ as $n \to \infty$. Moreover, $v_t(\xi, t'_n) \to 0$ on $[\xi_0-\delta, \xi_0+\delta]$ as $n \to \infty$, because $v_t(\xi,t)$ is uniformly continuous in $\Pi_{\xi_0,\delta,T+1}$ and $v(\xi,t)$ tends to $\tau(\xi)$ monotonically. Taking the limit along $\{t'_n\}$ in (2.4), we conclude that $\tau(\xi)$ satisfies (2.5) at ξ_0. Hence (2.7a) holds.

The statement of (2.7b) can be shown as follows. Fix $\xi^* \in \mathbb{R}$. If for some $t^* \geqslant 0$ $v(\xi^*, t^*) > 0$, then by monotonicity of $v(\xi,t)$ in t, it will remain bounded away from zero for all later times, and

$$\tau(\xi^*) = \lim_{t\to\infty} v(\xi^*, t) > 0.$$

Therefore, it remains to consider the case when

$$v(\xi^*, t) = 0, \quad t \geqslant t^*,$$

for some $t^* \geqslant 0$. As in the proof of Theorem 2.1 (see (A 17)) we can show that for an arbitrary sufficiently small $\delta > 0$

$$\frac{|\phi(v(\xi'_2,t)) - \phi(v(\xi'_1,t))|}{|\xi'_2 - \xi'_1|} \leqslant C_1\delta^\nu,\ |\xi^* - \xi'_i| < \delta, i = 1,2,\ t \geqslant t^*,\ 0 < \nu \leqslant 1, \tag{2.10}$$

where positive constant C_1 does not depend upon δ and t. By taking (2.10) to the limit as $t \to \infty$, we see that the flux $\frac{\partial\phi(\tau(\xi))}{\partial\xi} \equiv k(\tau(\xi))\tau'(\xi)$ is equal to zero at the interface. Therefore, (2.7b) holds.

Finally, we show that $\tau(\xi)$ is the smallest non-negative function on $\mathbb{R}$ satisfying (2.7a–c). For this, suppose that $\sigma(\xi)$ is another such function. Then $\sigma(\xi)$ is a generalized solution of (1.1) on $\mathbb{R}$, because it satisfies (2.7a,b) (see Remark 2.1). Moreover, from (2.7c) we have

$$\sigma(\xi) \geqslant q(\xi) = v(\xi, 0).$$

Therefore, by Theorem 2.2, $v(\xi,t) \leqslant \sigma(\xi)$ and

$$\lim_{t\to\infty} v(\xi,t) = \tau(\xi) \leqslant \sigma(\xi). \qquad \square$$

The following theorem is a counterpart of Theorem 2.4. It provides a method for constructing supersolutions for (2.4).

Theorem 2.5 *Let $q(\xi) \in [0,1]$ be a solution of the ODE (2.5) in (a,∞), for some $a \in \mathbb{R}$ such that*

$$\begin{aligned} &(a)\quad q(a) = 1,\\ &(b)\quad q'(\xi) \leqslant 0, \xi > a,\\ &(c)\quad k(q(\xi))q'(\xi) \text{ is continuous for } \xi > a. \end{aligned} \tag{2.11}$$

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By $v(\xi,t)$ denote the solution of the IVP for (2.4) *with initial data*

$$v(\xi,0) = p(\xi) \equiv \begin{cases} 1, & \xi \leqslant a, \\ q(\xi), & \xi > a \end{cases} \tag{2.12}$$

Then $v(\xi,t)$ is a non-increasing function of t for each ξ. Moreover,

$$\lim_{t\to\infty} v(\xi,t) = \tau(\xi) \quad \textit{uniformly in each bounded interval,}$$

where $\tau(\xi) \in [0,1]$ is the largest non-negative function such that

$$\begin{aligned} &(a)\ \tau(\xi) \quad \textit{solves} \quad (2.5), \quad \textit{if} \quad \tau(\xi) > 0, \\ &(b) \quad k(\tau(\xi))\tau'(\xi) = 0, \quad \textit{if} \quad \tau(\xi) = 0, \\ &(c) \qquad \tau(\xi) \leqslant q(\xi), \quad \xi \geqslant a. \end{aligned} \tag{2.13}$$

Proof of Theorem 2.5 Note that the constant function 1 is a supersolution of (2.4) and $v(\xi,0) \leqslant 1$. By Theorem 2.2, $v(\xi,t) \leqslant 1$ for all $\xi \in \mathbb{R}$ and $t \geqslant 0$. Next we show that $p(\xi)$ defined in (2.12) is a generalized supersolution of (2.4). For this we need to verify that

$$\begin{aligned} L(p,f;\xi_0,\xi_1,t_0,t_1) \quad \equiv \quad & \int_{t_0}^{t_1}\int_{\xi_0}^{\xi_1} \{\phi(p)f_{\xi\xi} - cpf_\xi + pf_t + \psi(p)f\}\, d\xi\, dt \\ & - \int_{\xi_0}^{\xi_1} pf\, d\xi\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \phi(p)f_\xi\, dt\Big|_{\xi_0}^{\xi_1} \geqslant 0 \end{aligned} \tag{2.14}$$

for any $t_0 < t_1$, $\xi_0 < \xi_1$ such that $\Pi \equiv [\xi_0,\xi_1] \times [t_0,t_1] \subset \mathbb{R}^2_+ = \{(\xi,t) : \xi \in \mathbb{R},\ t \geqslant 0\}$ and for any trial function $f(\xi,t) \in C^{2,1}_{x,t}(\Pi)$ equal to zero on the lateral boundaries of Π: $f(\xi_{1,2},t) = 0$ (see Definition 2.1). Clearly, $p_1(\xi) = 1$ is a generalized solution of (2.4) in $H_a = \{(\xi,t) :\ \xi \leqslant a, t \geqslant 0\}$. Remark 2.1 and (2.11c)) imply that $q(\xi)$ is a generalized solution of (2.4) in $G_a = \{(\xi,t) :\ \xi \geqslant a, t \geqslant 0\}$. Therefore, if $\xi_0 < \xi_1 \leqslant a$ or $a \leqslant \xi_0 < \xi_1$ then

$$L(p,f;\xi_0,\xi_1,t_0,t_1) = 0,$$

for any trial function f, $t_1 > t_0 \geqslant 0$. It remains to prove (2.14) for $\xi_0 < a < \xi$. By splitting the double integral in (2.14) into two integrals over $[\xi_0,a] \times [t_0,t_1]$ and $[a,\xi_1] \times [t_0,t_1]$, integrating by parts and straightforward calculations, we obtain

$$\begin{aligned} L(p,f;\xi_0,\xi_1,t_0,t_1) = & \int_{t_0}^{t_1}\int_{\xi_0}^{a} ((\phi(p))_{\xi\xi} + cp_\xi + \psi(p))f\, d\xi\, dt \\ & + \int_{t_0}^{t_1}\int_{a}^{\xi_1} ((\phi(q))_{\xi\xi} + cq_\xi + \psi(q))f\, d\xi\, dt \\ & - \int_{t_0}^{t_1} (\phi(p))_\xi f\, dt\,\Big|_{\xi=\xi_0}^{\xi=\xi_1}. \end{aligned} \tag{2.15}$$

By (2.12), the two double integrals on the right-hand side of (2.15) vanish. Furthermore, recalling that $\xi_0 < a < \xi_1$, the definition of $p(\xi)$ (2.12) and (2.11 b)) we have

$$(\phi(p))_\xi|_{\xi=\xi_0} = 0 \text{ and } (\phi(p))_\xi|_{\xi=\xi_1} = k(q(\xi_1))q'(\xi_1) \leqslant 0.$$

These relations and (2.15) imply that $L(p,f;\xi_0,\xi_1,t_0,t_1) \geqslant 0$ for any non-negative trial function f. Thus, $p(\xi)$ is a supersolution and $v(\xi,t) \leqslant p(\xi)$ for $t \geqslant 0$, because $v(\xi,0) = p(\xi)$. Finally, we have $v(\xi,t) \leqslant v(\xi,0)$ for any $t \geqslant 0$.

The rest of the proof is completely analogous to that of Theorem 2.4. □

3 Phase-plane analysis

In the present section, we study TW solutions of (1.1). Specifically, we seek nonnegative bounded solutions of (1.1) of the form

$$u(x,t) = U(\xi), \quad \xi = x - ct, \quad c \in \mathbb{R}. \tag{3.1}$$

The purpose of this section is two-fold: first, we show existence of a family of TWs with velocities $c \geqslant c^* > 0$; secondly, we set the ground for studying stability of these waves by specifying the trajectories, which will be used for constructing sub- and super-solutions in the next section.

Among the TW solutions of (1.1) we distinguish those $U(\xi)$ such that

$$U(\xi) = 0 \text{ for } \xi \geqslant \xi_0, \quad \text{for some } \xi_0 \in \mathbb{R}. \tag{3.2}$$

We refer to such TWs as *finite*.

Substituting (3.1) into (1.1), we obtain the equation for $U(\xi)$:

$$(k(U)U')' + cU' + \psi(U) = 0, \tag{3.3}$$

where $k(U) \equiv \phi'(U)$. We rewrite (3.3) as a system of first order ODEs:

$$\begin{cases} k(U)U' = W, \\ k(U)W' = -cW - k(U)\psi(U). \end{cases} \tag{3.4}$$

By changing the independent variable to η via the transformation

$$d\eta = k^{-1}(U(\xi))\,d\xi, \tag{3.5}$$

we rewrite (3.4) as

$$\begin{pmatrix} \dot{U} \\ \dot{W} \end{pmatrix} = \begin{pmatrix} W \\ -cW - f(U) \end{pmatrix} \equiv \mathscr{F}(U,W), \tag{3.6}$$

where $\dot{()} = \frac{d}{d\eta}()$ and $f(U) = k(U)\psi(U)$. In the following we refer to the independent (waveframe) coordinates ξ and η as 'time'.

By (2.1d,g), in the neighbourhood of zero, we have

$$f(U) = a_\gamma U^\gamma + o(U^\gamma); \ \gamma \equiv \alpha + \beta, \ a_\gamma = \mathscr{C}_1\mathscr{C}_2, \tag{3.7}$$

where the constants $\alpha, \beta, \mathscr{C}_1$, and $\mathscr{C}_2$ are the same as in (2.1 d,g). In addition, $f \in C^1([0,1])$ and $f(0) = f(1) = 0, f(U) > 0, U \in (0,1)$. In the presentation that follows, we shall assume that $\gamma > 1$, since the case of $\gamma = 1$ was analysed in Aronson & Weinberger [6]. Theorem 3.1 holds for $\gamma > 1$, regardless of the individual values of α and β. However, the

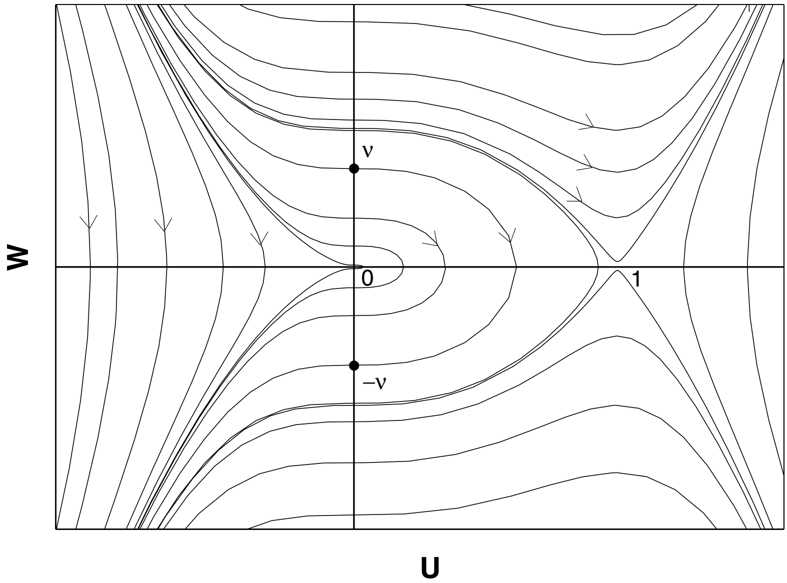

FIGURE 1. The phase-plane of (3.6) for $c = 0$. The phase-planes depicted on Figures 1–4 are schematic. As noted in the proof of Lemma 3.2, the fixed point $(1, 0)$ need not be hyperbolic (as depicted here).

conditions on α and β in (2.1) are necessary for the well-posedness of the IVPs for (1.1), and for the finiteness results at the end of this section.

Now we are in a position to state the main result of this section.

Theorem 3.1 *If $\gamma > 1$ there exists $c^*(\gamma) > 0$ such that*

1) *there are no TW solutions of* (1.1) *if* $0 \leqslant c < c^*(\gamma)$;
2) *there is a family of TW solutions of* (1.1) *for all speeds* $c \geqslant c^*(\gamma)$.

First, we prove several auxiliary lemmas, showing the existence of certain trajectories of (3.6) for small and large $c > 0$. Then we use these trajectories and continuous dependence of (3.6) on c to show the existence of the TW solutions. Finally, we specify which of these TWs are finite.

We start with the case of small $c > 0$. By $F(U)$ we denote a primitive of f: $F(U) = \int_0^U f(s)ds$.

Lemma 3.2 *For $c = 0$ and any $v \in (0, \sqrt{2F(1)})$ there is a nonnegative solution $U(\eta)$ of* (3.6) *connecting two points* $(0, v)$ *and* $(0, -v)$ *with $U < 1$ throughout* (*Figure* 1).

Proof of Lemma 3.2 For $c = 0$, examination of the phase-plane of (3.6) about the fixed point $(1, 0)$ reveals that the branches of its stable and unstable manifolds lying in $u < 1$

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belong to the level set of the following first integral:

$$\frac{W^2}{2} + F(U) = F(1).$$

The level sets of

$$\frac{W^2}{2} + F(U) = \frac{v^2}{2} \in (0, F(1)) \tag{3.8}$$

contain the solutions claimed in the lemma. By integrating (3.8), we obtain the passage time from $(0, v)$ to $(0, -v)$:

$$T_v = 2\int_0^{U_v} \frac{dU}{\sqrt{v^2 - 2F(U)}}, \quad U_v = F^{-1}\left(\frac{v^2}{2}\right). \tag{3.9}$$

T_v is finite, because $F \in C^2([0, 1])$ and the singularity in (3.9) is integrable. □

The behaviour of the trajectories near the origin, described in Lemma 3.2, persists for small positive $c > 0$. In particular, for sufficiently small $c > 0$, there exist trajectories encircling the origin in the positive U semiplane with finite transit time between the points on the $U = 0$ axis. This is summarized in the following lemma.

Lemma 3.3 *For sufficiently small $c > 0$, there exists $\bar{v}(c)$ with $\bar{v}(c) \to \sqrt{2F(1)}$ as $c \to 0$, such that every trajectory $\mathcal{T}_v$ passing through $(0, -v)$ with $v \in (0, \bar{v}(c))$ also passes through a point $(0, v^+), v^+ > 0$ with finite transit time between these points and $U < 1$ throughout.*

Proof of Lemma 3.3 follows from Lemma 3.2 and continuous dependence on c. □

We now specify the local behaviour of trajectories near the origin. Linearization of (3.6) near the origin shows that $D\mathcal{F}(0, 0)$ has an eigenvalue $\lambda_1 = -c < 0$ with eigenvector $(1, -c)^T$; the other eigenvalue $\lambda_2 = 0$ with eigenvector $(1, 0)^T$. For $c > 0$, there is, therefore, a strong stable manifold $W^{ss}(0, 0)$ tangent at $(0, 0)$ to $(1, -c)^T$ and a centre manifold $W^c(0, 0)$ tangent at $(0, 0)$ to $(1, 0)^T$. The latter can be approximated near the origin by a standard centre-manifold calculation [8]:

$$W = -\frac{a_\gamma}{c} U^\gamma + o(U^\gamma). \tag{3.10}$$

The location of $W^{ss}(0, 0)$ in the positive W semiplane can be estimated as follows. Consider the curve

$$\Gamma = \left\{(U, W) : W = -cU + \frac{p(U)}{c}\right\}, \quad p(U) = \int_0^U \frac{f(s)}{s} ds.$$

Note that $p(0) = 0$; $p(U) > 0, U \in (0, 1)$, and $p(U) \sim U^\gamma$ for small $U > 0$. Let

$$L(U, W) = W + cU - \frac{p(U)}{c}.$$

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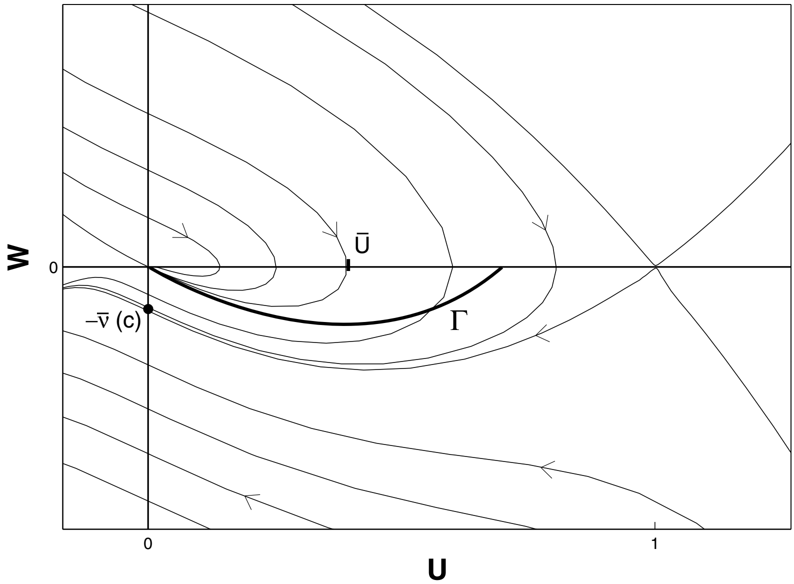

FIGURE 2. The phase-plane of (3.6) for $c > 0$, small: bounding $W^{ss}(0,0)$ via Γ.

Note that Γ is the zero-level set of L and compute

$$\left.\frac{dL}{dt}\right|_{\Gamma} = (\nabla L, \mathscr{F})|_{\Gamma} = -\frac{1}{c^2}p(U)p'(U) < 0, \quad U \in (0,1). \tag{3.11}$$

From (3.11) we conclude that the trajectories of (3.6) cross Γ from above. Therefore, $W^{ss}(0,0)$ lies strictly above Γ for $U > 0$. For sufficiently small $c > 0$, Γ intersects the U axis at $U_0 : 0 < U_0 < 1$. Hence, W^{ss} intersects the U axis at some point $\bar{U}$ with $0 < \bar{U} < U_0 < 1$. In the first quadrant under the direction of the vector field, $W^{ss}(0,0)$ must cross in negative time the W axis at some point $(0, v^-)$ (Figure 2).

We turn now to the case of large $c > 0$. The following lemma shows the existence of the heteroclinic orbits connecting $(1,0)$ and $(0,0)$, corresponding to the TW solutions of (1.1).

Lemma 3.4 *For* $c > c^+ \equiv 2\sqrt{M}$, $M \equiv \max_{u\in(0,1)} \frac{f(u)}{u}$, $W^u(1,0) \bigcap W^c(0,0) \neq \emptyset$.

Proof of Lemma 3.4 We consider a positively invariant region T bounded by $U = 1, W = 0$ and $W = -\sqrt{M}U$ (Figure 3). Clearly the vector field of (3.6) points into this triangle across the first two sides. On the third side, we have

$$\frac{dW}{dU} = -c - \frac{f(U)}{W} = \frac{f(U)}{\sqrt{M}U} - c \leqslant \sqrt{M} - c.$$

Thus, for $c > 2\sqrt{M}$ the flow is into T. From linearization at $(1,0)$, we see that the left-hand branch of $W^u(1,0)$ enters T. There are no fixed points in the interior of T, and hence, via index theory, no other limit sets [14]. The ω−limit set of the points on $W^u(1,0)$

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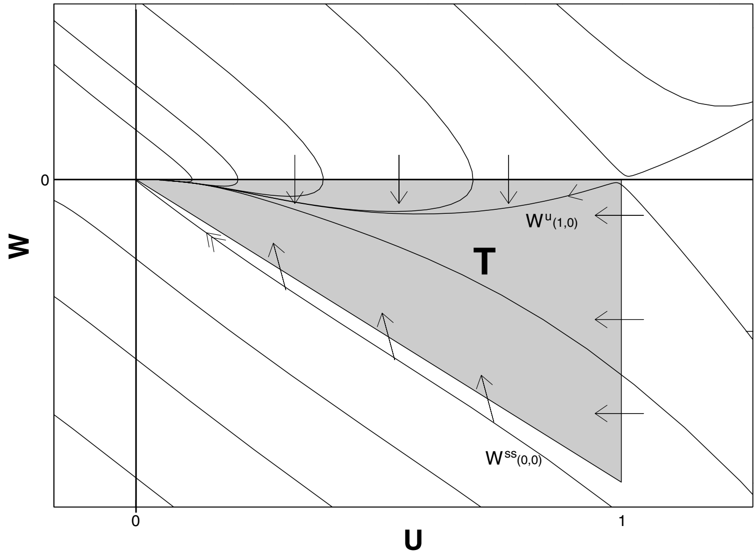

FIGURE 3. The phase-plane of (3.6) for $c > 0$, large: the trapping region, T.

must be the origin, the only point in the closure of T where the vector field vanishes (Figure 3). The statement of the lemma follows, since $W^{ss}(0,0) \bigcap T = \emptyset$. □

Corollary 3.5 *There exists $c = c^* > 0$, for which $W^u(1,0) \bigcap W^{ss}(0,0) \neq \emptyset$.*

Proof of Corollary 3.5 As we have shown above, for small $c > 0$ $W^u(1,0)$ lies strictly below $W^{ss}(0,0)$ (Fig. 4a); while for large c $W^u(1,0)$ lies strictly above $W^{ss}(0,0)$ (Fig. 4b). The statement of the lemma follows from the continuous dependence on c and from the intermediate value theorem. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1 Note that for $W < 0$ the vertical component of of the vector field of (3.6) at every point increases monotonically as c increases. Therefore, $W^u(1,0)$ monotonically moves up while $W^{ss}(0,0)$ moves down as c increases (Figure 5). By Corollary 3.5 the two manifolds meet at $c = c^*(\gamma)$. Therefore, there are no TW solutions for $c < c^*(\gamma)$. For all $c \geqslant c^*(\gamma)$ $W^u(1,0)$ limits on $(0,0)$. Moreover, by (3.10) for $c > c^*(\gamma)$ $W^u(1,0)$ is tangent to $W^c(0,0)$ at the origin. □

Remark 3.1 The assumption $\psi'(1) < 0$ (Hypotheses 2.1 f)) guarantees that the fixed point $(1,0)$ of (3.6) is hyperbolic. This condition can be relaxed to include functions whose asymptotic behaviour as $u \to 1-0$ is characterized by $\psi(u) \sim \mathscr{C}(1-u)^q$, $q \geqslant 2$, $\mathscr{C} > 0$. In this case, the fixed point $(1,0)$ is no longer hyperbolic. Examination of the phase-plane of (3.6) near $(1,0)$ reveals that there is a one-dimensional stable manifold $W^s(1,0)$ and a

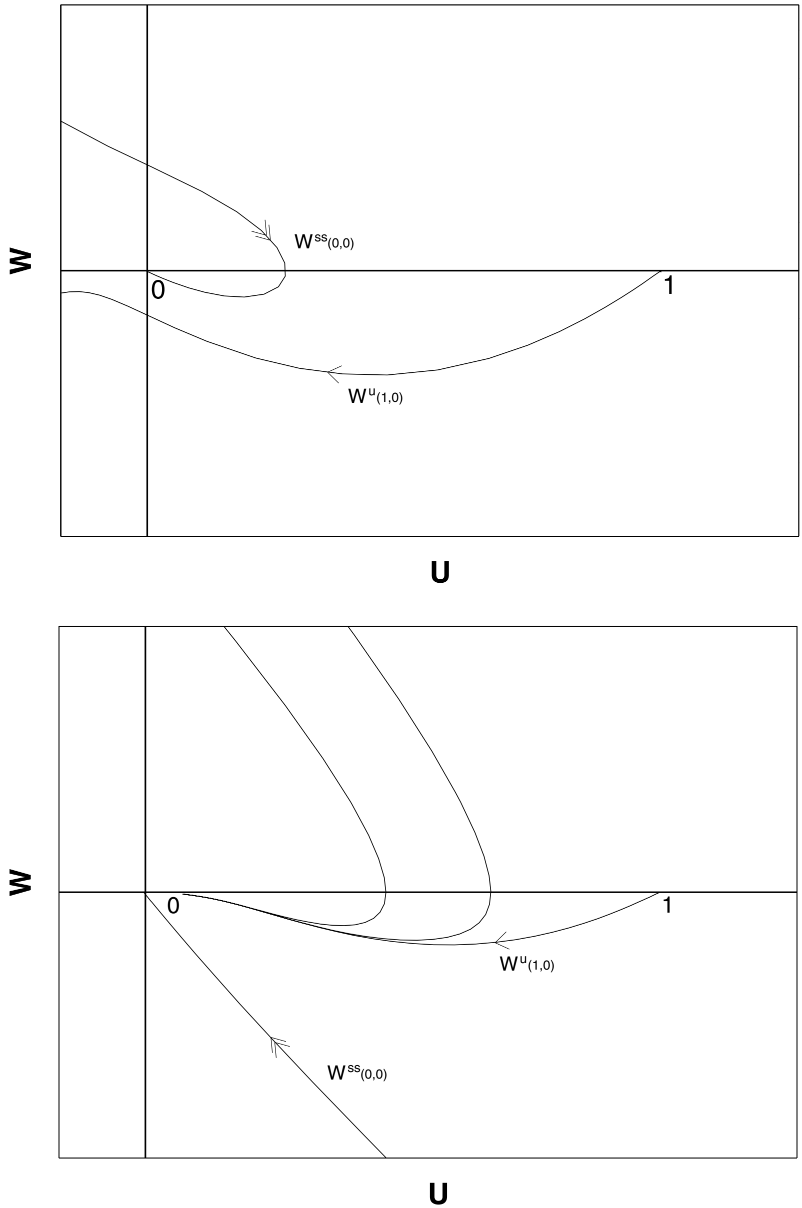

FIGURE 4. The creation of a heteroclinic connection: (a) $c < c^*$, (b) $c > c^*$.

centre manifold $W^c(1,0)$. A standard centre-manifold analysis [8, 14] yields that $W^c(1,0)$ enters a positively invariant region T of the phase-plane of (3.6) (see Figure 3). Moreover, the flow on $W^c(1,0)$ is directed inside T. With this in mind, by considering $W^c(1,0)$ instead of $W^u(1,0)$ in the proofs of Lemmas 3.2 and 3.4, one easily proves that there exists $c = c^* > 0$, for which $W^c(1,0) \bigcap W^{ss}(0,0) \neq \emptyset$. Consequently, the heteroclinic orbits connecting $(1,0)$ and $(0,0)$ exist for $c \geqslant c^*$.

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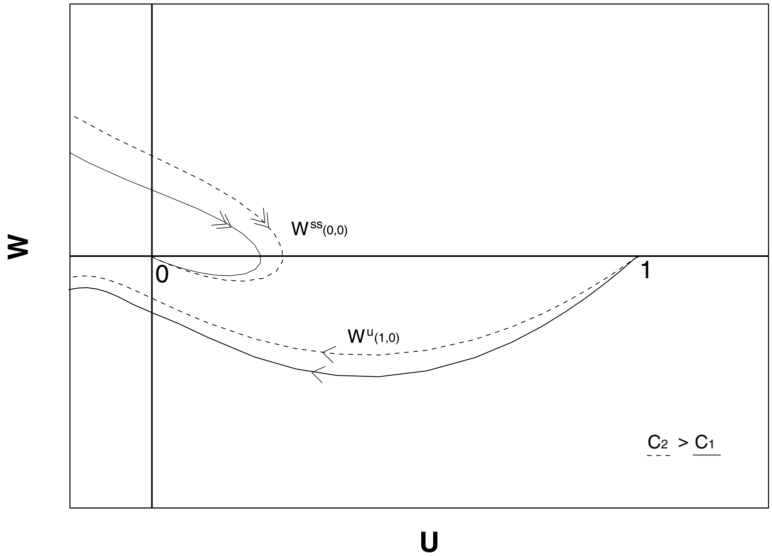

FIGURE 5. Movements of the manifolds as c increases.

We next determine which of the TWs are finite. Let $W = h_c(U)$, $0 \leqslant U \leqslant 1$ denote the heteroclinic orbit connecting $(1,0)$ and $(0,0)$. Linearization of (3.6) about the origin and (3.10) yields

$$h_c(U) = \begin{cases} -cU + o(U), & c = c^*(\gamma), \\ -\frac{a_\gamma}{c} U^\gamma + o(U^\gamma), & c > c^*(\gamma). \end{cases} \tag{3.12}$$

Heteroclinic orbits that reach the origin in finite time in ξ correspond to finite TWs. Thus, we need to estimate the passage time from $(\epsilon, h_c(\epsilon))$ to the origin, where ϵ is small and positive. By ξ_0 and ξ_ϵ we denote the instants when the trajectory passes through $(0,0)$ and $(\epsilon, h_c(\epsilon))$, respectively. Then, from (3.4) we have

$$\int_{\xi_\epsilon}^{\xi_0} d\xi = \int_\epsilon^0 \frac{k(U)}{h_c(U)} dU. \tag{3.13}$$

Taking into account that $k(U) \sim U^\alpha$ and (3.12), we conclude that the integral in (3.13) converges only if $c = c^*(\gamma)$ and $\alpha > 0$ and hence that the corresponding TW is finite. If $c > c^*(\gamma), \beta \geqslant 1$ the passage time is infinite.

The existence of finite TW solutions of (1.1) has an immediate consequence to the behaviour of solutions of the IVP for (1.1): namely, it implies the existence of interfaces. As was shown in the previous paragraph, $\alpha > 0$ and $\beta \geqslant 1$ the equation (1.1) has finite travelling wave solutions. By the results of Gilding & Kersner [13], this implies that the supports of solutions of the IVP for (1.1), (1.2) remain bounded from the left (right) for all finite times provided the initial data $u_0(x)$ have bounded support from the left (right).

Concluding our phase-plane analysis, we note some of the properties of $W^{ss}(0,0)$. These properties easily follow from the phase-plane analysis presented above. They will be used in the next section to construct sub- and super-solutions which will enable us to determine the asymptotic rate of propagation of disturbances in the system described by (1.1). First, we consider the case $0 < c < c^*(\gamma)$. As we have shown above, we can choose arbitrarily small $c_0 > 0$ such that $c_0 < c$ and $W^{ss}_{c_0}(0,0)$ intersects the U−axis at some point $\bar{U}$: $0 < \bar{U} < 1$ (Figure 4a). By Γ_{c_0}, we denote the part of $W^{ss}_{c_0}(0,0)$ lying in the negative W semiplane. On the other hand, Γ_{c^*} coincides with the heteroclinic orbit connecting $(1,0)$ and $(0,0)$. By monotone continuous dependence of the vector field in the negative W semiplane on c, we conclude that in this semiplane $W^{ss}_c(0,0)$ lies between Γ_{c_0} and Γ_{c^*} for all $c_0 < c < c^*$. Moreover, the transit time from the point of intersection of $W^{ss}_c(0,0)$ and the U-axis to the origin is finite in ξ. Since $c_0 > 0$ can be chosen arbitrarily small these conclusions hold for all $0 < c < c^*(\gamma)$. In the first quadrant, $W^{ss}(0,0)$ bends up and back under the direction of the vector field and crosses (in negative time) the W-axis at some point $(0, v_c^+)$. Therefore, for each $0 < c < c^*(\gamma)$ $W^{ss}_c(0,0)$ passes through $(0, v_c^+ > 0)$, lies in the strip $0 \leqslant U < 1$, and hits the origin in finite time in ξ. For $c > c^*(\gamma)$, $W^{ss}_c(0,0)$ lies strictly below Γ_{c^*} in the negative W-semiplane; passes through the point $(1, v_c < 0)$ and reaches the origin in finite ξ-time.

4 Propagation of disturbances

In this section, we prove two results concerning stability of the rest states and the rate of propagation of disturbances in the systems, which are described by (1.1). The theorems of this sections are extensions of the corresponding results of Aronson & Weinberger [6] to the equations (1.1), (2.1). Our first result establishes the asymptotic rate of propagation for solutions of the IVP for (1.1), (2.1) with $\alpha > 0$, $\beta \geqslant 1$ and initial conditions

$$u(x,0) = \begin{cases} 1, & x \leqslant 0, \\ 0, & x > 0. \end{cases} \tag{4.1}$$

Specifically, we prove

Theorem 4.1 *Let $u(x,t)$ be the solution of the IVP* (1.1), (2.1) *with $\beta \geqslant 1, \alpha > 0$ and initial data* (4.1). *Then for any $x \in \mathbb{R}$*

$$\lim_{t\to\infty} u(x+ct,t) = \begin{cases} 1, & 0 < c < c^*(\gamma), \\ 0, & c > c^*(\gamma). \end{cases} \tag{4.2}$$

The convergence is uniform on bounded intervals of $\mathbb{R}$.

If, in addition to the conditions of Theorem 4.1, $1 \leqslant \beta < 1+\alpha$, $\alpha > 0$ then we claim

Theorem 4.2 *Let $u(x,t)$ be the solution of the IVP* (1.1), (2.1) *with continuous compactly supported initial data $u_0(x)$. If $1 \leqslant \beta < 1+\alpha, \alpha > 0$ then either $u_0(x) \equiv 0$ or $\lim_{t\to\infty} u(x,t) = 1$ for any $x \in \mathbb{R}$. The convergence is uniform on the bounded intervals of $\mathbb{R}$. Moreover,* (4.2) *holds.*

Corollary 4.3 *If $\beta = 1+\alpha$, $\alpha > 0$ then the statement of the theorem holds for continuous initial data whose support is large enough to include*

$$[a,b],\ b-a \geqslant \int_0^1 \frac{dw}{\sqrt{1-w^2}} = \frac{\pi}{2}.$$

Remark 4.1. Theorem 4.2 implies that $u \equiv 0$ is an unstable rest state, and $u \equiv 1$ is a stable one for (1.1), (2.1) with $1 \leqslant \beta < 1 + \alpha$. We believe that the statement of the Theorem 4.2 holds for a larger ranges of parameters α and β, including those of Theorem 4.1, but proving this would require additional techniques.

The proofs of the Theorems 4.1 and 4.2 use the phase-plane analysis of the previous section. First we prove an auxiliary lemma about the trajectories of (3.4) in the strip $P \equiv \{(U, W) : 0 \leqslant U \leqslant 1, W \in \mathbb{R}\}$.

Lemma 4.4 *Except for the stationary solutions corresponding to the two fixed points at* $(0, 0)$ *and* $(1, 0)$*, and the heteroclinic orbit connecting them (when it exists), all other trajectories leave* P *in finite forward or backwards time* ξ*.*

Proof of Lemma 4.4 The proof follows from a simple phase-plane analysis of (3.4) (see Figures 2 and 4). We prove the statement of the lemma for trajectories of (3.6). This yields the same conclusion for trajectories of (3.4), because finite intervals of time in η correspond to finite intervals in ξ.

Suppose $\mathcal{T}$ is the trajectory passing through the point $(U, W) \in P^+ \equiv P \bigcap \{(U, W) : W > 0\}$. The second equation of (3.6) yields that in P^+ the W-component of $\mathcal{T}$ increases in backwards time, and, therefore, from the first equation of (3.6) we conclude that $\mathcal{T}$ crosses the U-axis away from the origin in finite backwards time.

On the other hand, as long as $\mathcal{T}$ stays in $P^- \equiv P \bigcap \{(U, W) : W < 0\}$ we have

$$\dot{U} = W < 0.$$

Also, note that the vector field on $P^0 \equiv P \bigcap \{(U, W) : W = 0\}$ points downwards. Therefore, if $\mathcal{T}$ stays bounded away from the U—axis (in forward or backwards time), it will leave P^- in finite time. Clearly, if $\mathcal{T}$ crosses P^0 in backwards time, it leaves P in backwards time as we discussed above. Thus, it remains to consider only the trajectories that asymptote onto one of the fixed points: $(0, 0)$ and $(1, 0)$ in both forward and backwards time. All such trajectories are listed as exceptions in Lemma 4.4. □

Proof of Theorem 4.1 We rewrite (1.1) in the uniformly moving frame of coordinates $\xi = x - ct, c > 0$:

$$v_t = (\phi(v))_{\xi\xi} + cv_\xi + \psi(v), \tag{4.3}$$

where $v(\xi, t) = u(x, t) = u(\xi + ct, t)$.

We consider first the case $0 < c < c^*(\gamma)$. Let $q_c(\xi), \xi \in [a, b], -\infty < a < b < 0$, denote the U-component of the part of $W_c^{ss}(0, 0)$ such that $q_c(a) = q_c(b) = 0$ and $0 \leqslant q_c(\xi) < 1$, $\xi \in [a, b]$. As was shown at the end of §3, q_c is well-defined. By $\underline{v}(\xi, t)$ we denote the solution of (4.3) satisfying the initial condition

$$\underline{v}(\xi, 0) = \begin{cases} q_c(\xi), & \xi \in [a, b], \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 2.4,

$$\lim_{t \to \infty} \underline{v}(\xi, t) = \underline{\tau}(\xi) \quad \text{uniformly on each bounded interval of } \mathbb{R},$$

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where $0 \leqslant \underline{\tau}(\xi) \leqslant 1$ is the smallest nonnegative function on $\mathbb{R}$, which satisfies all the conditions in (2.7). By Lemma 4.4, the only candidates for $\underline{\tau}$ are: $\tau_0 \equiv 0$ and $\tau_1 \equiv 1$, because for $0 < c < c^*(\gamma)$ there is no heteroclinic orbit connecting the two equilibria, and all other trajectories of (3.6) either become greater than 1 or violate (2.7b). However, $\tau_0 \equiv 0$ is excluded on the basis of (2.7c). Thus, $\underline{\tau} \equiv 1$. Since $v(\xi, 0) \geqslant \underline{v}(\xi, 0)$ and $v(\xi, t) \leqslant 1$, from Theorem 2.2 we have

$$1 \geqslant \lim_{t\to\infty} v(\xi, t) \geqslant \lim_{t\to\infty} \underline{v}(\xi, t) = 1.$$

The statement of the theorem for $0 < c < c^*(\gamma)$ follows.

Similarly, for $c > c^*(\gamma)$, let $q_c(\xi), \xi \in [0, d]$ be the U-component of the part of $W^{ss}(0,0)$ lying in P^-. By the remarks at the end of the previous section, q_c is well-defined, $q(0) = 1$, $q(d) = 0$. Moreover, $d < \infty$. Let $\bar{v}(\xi, t)$ be the solution of the IVP for (4.2) with the initial condition

$$\bar{v}(\xi, t) = \begin{cases} 1, & \xi \leqslant 0, \\ q_c(\xi), & 0 < \xi \leqslant d, \\ 0, & \xi > d. \end{cases}$$

By Theorem 2.5,

$$\lim_{t\to\infty} \bar{v}(\xi, t) = \bar{\tau}(\xi) \quad \text{uniformly on each bounded interval of } \mathbb{R},$$

where $0 \leqslant \bar{\tau}(\xi) \leqslant 1$ is the largest non-negative function satisfying (2.13a–c). By Lemma 4.4, the only candidates for $\bar{\tau}(\xi)$ are: $\tau_0 \equiv 0$, $\tau_1 \equiv 1$, and the heteroclinic orbit. The latter, however, has infinite support as a function of ξ (see (3.12)), and takes positive values everywhere on $\mathbb{R}$, whereas $q_c(\xi) = 0$ for $\xi \geqslant d$. It follows that the heteroclinic orbit must be excluded because it does not satisfy (2.13c). The stationary solution $\tau_1 \equiv 1$ does not satisfy (2.13c) too. Thus, $\bar{\tau}(\xi) \equiv 0$. The statement of the theorem then follows from Theorem 2.2:

$$0 \leqslant \lim_{t\to\infty} v(\xi, t) \leqslant \lim_{t\to\infty} \bar{v}(\xi, t) = 0,$$

since $0 \leqslant v(\xi, 0) \leqslant \bar{v}(\xi, 0)$. □

Proof of Theorem 4.2 By $q_\epsilon(\xi)$, $0 < \epsilon < 1$ we denote the trajectory of (3.3) with $c = 0$ that connects the positive and negative W-semiaxes and passes through $(\epsilon, 0)$ in the U-W phase-plane (see Figure 1). For definiteness, we assume that $q_\epsilon(0) = 0, q'_\epsilon(0) > 0$. Such a trajectory exists and belongs to the level set of:

$$\frac{\dot{U}^2}{2} + F(U) = F(\epsilon), \; F(U) = \int_0^U f(s)\, ds.$$

The 'time of flight' from the positive to negative W-semiaxes is then given by

$$2\int_0^\epsilon \frac{du}{\sqrt{2(F(\epsilon) - F(u))}}.$$

After returning to the original time variable ξ (3.5) the time of flight is

$$T(\epsilon) = 2\int_0^\epsilon \frac{k(u)du}{\sqrt{2(F(\epsilon) - F(u))}}. \tag{4.4}$$

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By taking into account the asymptotic behaviours of $k(u)$ and $\psi(u)$ (see Hypotheses 2.1 d),g)) and (3.7), from (4.4) we have

$$\lim_{\epsilon\to 0} T(\epsilon) = \lim_{\epsilon\to 0} \mathscr{C}_1 \sqrt{\frac{2(\gamma+1)}{a_\gamma}} \int_0^\epsilon \frac{u^\alpha du}{\sqrt{\epsilon^{\alpha+\beta+1} - u^{\alpha+\beta+1}}}. \tag{4.5}$$

Using the substitution $w = (\frac{u}{\epsilon})^{\alpha+1}$ in (4.5), we find that

$$\lim_{\epsilon\to 0} T(\epsilon) = 2C_\gamma \epsilon^{\frac{1+\alpha-\beta}{2}} \int_0^1 \frac{dw}{\sqrt{1-w^{1+\frac{\beta}{\alpha+1}}}},\ C_\gamma = \frac{\mathscr{C}_1}{2(\alpha+1)} \sqrt{\frac{2(\gamma+1)}{a_\gamma}}. \tag{4.6}$$

It is easy to see that for $\alpha > 0$ and $\beta \geqslant 1$ the integral on the right-hand side of (4.6) is convergent:

$$\int_0^1 \frac{dw}{\sqrt{1-w^{1+\frac{\beta}{\alpha+1}}}} \leqslant \int_0^1 \frac{dw}{\sqrt{1-w}} = 2. \tag{4.7}$$

In the rest of the proof, we assume that $\alpha > 0$ and $\beta \geqslant 1$. Then (4.6) and (4.7) imply

$$\lim_{\epsilon\to 0} T(\epsilon) = 0.$$

We now summarize the properties of $q_\epsilon(\xi)$: by choosing $\epsilon > 0$ sufficiently small, the positive part of $q_\epsilon(\xi)$ and the components of its support can be made arbitrarily small. Next we use $q_\epsilon(\xi)$ to construct subsolutions for the IVP for (1.1) with initial data $u_0(x)$.

Assume that $u_0(x) \not\equiv 0$. Then by continuity of $u_0(x)$, there exist $\delta > 0$ and $[a,b]$ such that $u_0(x) \geqslant \delta$, $x \in [a,b]$. By the properties of $q_\epsilon(\xi)$ discussed above, for sufficiently small $0 < \epsilon < \delta$ we have

$$|q_\epsilon(\xi)| < \delta,\ 0 < T(\epsilon) \leqslant b - a.$$

By $v(x,t)$ denote the solution of (1.1) with initial data

$$v(x,0) = \begin{cases} q_\epsilon(x-a), & x \in [a, a+T(\epsilon)] \subseteq [a,b], \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 2.4,

$$\lim_{t\to 0} v(x,t) = \tau(x), \text{ uniformly on bounded intervals of } \mathbb{R},$$

where $\tau(x)$ is the smallest non-negative function on $\mathbb{R}$ satisfying (2.7). A simple inspection of the phase-plane of (2.5) with $c = 0$, as in Lemma 4.4, leads to the conclusion that $\tau(x) \equiv 1$. We omit the details. Therefore,

$$\lim_{t\to 0} u(x,t) = 1, \text{ uniformly on bounded intervals of } \mathbb{R}. \tag{4.8}$$

This concludes the proof of the first part of the theorem. It remains to show that (4.2) holds. For $c > c^*(\gamma)$, we refer to the proof of Theorem 4.1, as it applies after obvious changes. As to the case $c < c^*(\gamma)$, the proof of Theorem 4.1 applies after the following remarks. By construction of $\underline{v}(\xi,t)$ (see the proof of Theorem 4.1), $\underline{v}(\xi,0)$ has bounded

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support and $v(\xi, 0) < 1$. By (4.8), there exists $\underline{t} > 0$ such that $u(\xi, \underline{t}) \geqslant v(\xi, 0), \xi \in \mathbb{R}$. Therefore, $\underline{v}(\xi, t)$ can be used as a subsolution for $t > \underline{t}$. In particular, $u(x, t - \underline{t}) \geqslant \underline{v}(\xi, t), t > \underline{t}$. By arguing further as in the proof of Theorem 4.1, we conclude the present proof for $c > c^*(\gamma)$. □

Proof of Corollary 4.3 This follows from the proof of Theorem 4.2 after straightforward modifications. □

5 Conclusions

In the present paper, we have studied TW solutions for a large class of scalar reaction-diffusion equations with degenerate, nonlinear diffusion coefficients and monostable non-linear reaction terms. We have proved existence of a family of such solutions with velocities equal or exceeding some critical value $c^*(\gamma)$ (Theorem 3.1). Using techniques based on the Maximum Principle, we have shown that the critical speed $c^*(\gamma)$ is equal to the asymptotic rate of propagation of the solutions of the IVP for (1.1), (2.1) with compactly-supported initial data. The present paper therefore generalizes the results on asymptotic rate of propagation of disturbances known for scalar equations with linear diffusion [6] to a large class of equations with nonlinear diffusion coefficients.

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Appendix A Proof of Theorem 2.1

For any point $(x, t), t > 0$ in the positivity set of $u(x, t)$ the statement of the theorem follows from the theory for quasilinear parabolic equations [11, 20]. Therefore, we only need to show the continuity of the flux at (x_0, t_0) where $u(x_0, t_0) = 0$.

The proof of the theorem relies on an *a priori* estimate of the spatial derivative of $v(x, t) \equiv k(u(x, t))$ in the neighbourhood of (x_0, t_0). To obtain this estimate we use the method of Bernstein in the form in which it was used in Aronson [1] to obtain analogous estimates for the porous medium equation. To this end, recall that $\Psi(v)$ denotes the inverse function of $k(u)$, and rewrite (1.1) in terms of $v(x, t)$:

$$v_t = v v_{xx} + \theta(v)(v_x)^2 + g(v), \tag{A 1}$$

where

$$\theta(v) = \frac{(v\Psi'(v))'}{\Psi'(v)} \quad \text{and} \quad g(v) = \frac{\psi \circ \Psi}{\Psi'}(v).$$

Note that $\theta(v) \geqslant 0$ in R (see (A 4)). According to Hypotheses 2.1(h), $\theta(v)$ and $g(v)$ are continuously differentiable functions for non-negative v.

Let $R = (a, b) \times (\tau, T]$ be a rectangle containing (x_0, t_0), i.e. $a < x_0 < b$ and $0 < \tau < t_0 < T$. By choosing the dimensions of R sufficiently small we can make $u(x, t)$ and

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$v(x,t) \equiv k(u(x,t))$ arbitrarily small in R. In particular, we choose R to satisfy the following three technical conditions:

$$(1)\ \ |v(x,t)| \leqslant \frac{1}{9M}, \quad M \equiv \max_{[0,k(1)]} |\theta(v)|, \tag{A 2}$$

$$(2)\ \ C_1 u^\alpha \leqslant k(u(x,t)) \leqslant C_2 u^\alpha \text{ in } R, \tag{A 3}$$

$$(3)\ \ \theta_1(v) = (v\Psi'(v))' \geqslant 0 \text{ in } R, \tag{A 4}$$

where $\Psi(v)$ is the inverse function of $k(u)$. The first condition can be satisfied by the continuity of $v(x,t)$. The second one reflects the local behaviour of $k(u(x,t))$ near zero, and it is consistent with (2.1d). As to the third condition, consider the primitive of θ_1: $\theta_2(v) = v\Psi'(v)$. The differentiable function $\theta_2(v)$ is positive for small $v > 0$, because

$$\Psi'(v) = \frac{1}{k'(u)} > 0, \text{ for } v = k(u) > 0, \text{ by (2.1c), and}$$

$$\theta_2(0) = \lim_{v\to 0} v\frac{\Psi(v) - \Psi(0)}{v} = \lim_{v\to 0} \Psi(v) = \Psi(0) = 0.$$

For sufficiently small v (and, therefore, for sufficiently small u), $\theta_1(v) = \theta_2'(v) \geqslant 0$.

Lemma A.1 *Let u be a smooth positive classical solution of* (1.1) *in the rectangle* $R \equiv (a,b) \times (0,T]$ *(i.e. $u \in C^{2,1}_{x,t}(R) \bigcap C^0(\bar{R})$, $u_{xxx} \in C^0(R)$) and let $R_1 \equiv (a_1,b_1) \times (\tau, T]$ be any subrectangle of R with $a < a_1 < b_1 < b$ and $0 < \tau < T$. Then*

$$\left|\frac{\partial k(u(x,t))}{\partial x}\right| \leqslant C_3 \quad in \quad \bar{R}_1, \tag{A 5}$$

where the positive constant C_3 depends only upon α_1, $a_1 - a$, $b - b_1$, and $\max_{\bar{R}} u(x,t)$.

Proof of Lemma A 5 Let $\varrho(w)$ be a smooth function, which will be specified below. Denote $v = \varrho(w)$ and rewrite (A 1) in terms of w:

$$w_t = \varrho w_{xx} + \left(\varrho\frac{\varrho''}{\varrho'} + \tilde{\theta}\varrho'\right)(w_x)^2 + \tilde{g}(w), \tag{A 6}$$

where $\tilde{\theta}(w) = \theta(\varrho(w))$ and $\tilde{g}(w) = \frac{g(\varrho(w))}{\varrho'(w)}$. Differentiating (A 6) with respect to x and multiplying both sides by $p \equiv w_x$, we obtain

$$\frac{1}{2}(p^2)_t - \varrho p p_{xx} = \left\{\varrho'' + \varrho\left(\frac{\varrho''}{\varrho'}\right)' + (\tilde{\theta}\varrho')'\right\} p^4 + \left\{\varrho' + 2\left(\varrho\frac{\varrho''}{\varrho'} + \tilde{\theta}\varrho'\right)\right\} p^2 p_x + \tilde{g}' p^2. \tag{A 7}$$

Let $0 \leqslant \zeta = \zeta(x) \leqslant 1$ be a smooth cut-off function such that $\zeta = 1$ on $\bar{R}_1$ and $\zeta = 0$ in the neighbourhood of the lateral boundaries of R. Denote $z = \zeta^2 p^2$. At a point $(\bar{x}, \bar{t}) \in R$, where z attains a maximum we have $z_t \geqslant 0$, $z_{xx} \leqslant 0$, and $z_x = 0$, i.e.

$$\varrho z_{xx} - z_t \leqslant 0 \quad\text{and}\quad \zeta^2 p p_x = -\zeta\zeta_x p^2. \tag{A 8}$$

Combining (A 7) and (A 8) and following exactly the same steps as in the proof of Lemma

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of Aronson [1], we obtain

$$-\left(\varrho'' + \varrho\left(\frac{\varrho''}{\varrho'}\right)' + (\tilde{\theta}\varrho')'\right)p^4\zeta^2 \leqslant -\left(\varrho' + 2\left(\varrho\frac{\varrho''}{\varrho'} + \tilde{\theta}\varrho'\right)\right)\zeta\zeta_x p^3 + (3\varrho\zeta_x^2 - \varrho\zeta\zeta_{xx} + \tilde{g}')p^2. \qquad \text{(A 9)}$$

If the coefficient of $\zeta^2 p^4$ in the last inequality is positive, then (A 9) will yield the desired bound on p, which in turn will imply a bound on $|v_x|$ in $\bar{R}_1$. The condition on this coefficient can be ensured by a suitable choice of ϱ. In particular, let $\epsilon = \max_{\bar{R}} v$ and consider

$$\varrho(w) = w(\kappa - w), \qquad \text{(A 10)}$$

where $\kappa = \frac{1}{\sqrt{2M}}$ and M is defined in (A 2). $\varrho(w)$ maps $[0, \frac{\kappa}{3}]$ onto $[0, \frac{1}{9M}]$ and, therefore, it is a twice continuously differentiable monotone function from $[0, \kappa_1]$ to $[0, \epsilon]$ for some $0 < \kappa_1 \leqslant \frac{\kappa}{3}$ (see (A 2)). With this choice of ϱ we will show that the coefficient of $\zeta^2 p^4$ is positive, i.e.

$$\mathscr{J} \equiv \varrho'' + \varrho\left(\frac{\varrho''}{\varrho'}\right)' + (\tilde{\theta}\varrho')' < 0 \quad \text{on} \quad \left[0, \frac{\kappa}{3}\right].$$

From (A 10) we have $(\frac{\varrho''}{\varrho'})' < 0$ and $\varrho''\tilde{\theta} \leqslant 0$ on $[0, \frac{\kappa}{3}]$ ($\tilde{\theta} \geqslant 0$ by (A 4)). Using these inequalities and (A 10), we obtain

$$\mathscr{J} \leqslant \varrho'' + \varrho'\tilde{\theta}' = \varrho'' + (\varrho')^2\theta' \leqslant -1 + \left(\frac{1}{\sqrt{2M}} - 2w\right)^2 M$$
$$\leqslant -1 + \left(\frac{1}{2M} + 4\left(\frac{\kappa}{3}\right)^2\right)M = \frac{-5}{18} < 0 \text{ on } \left[0, \frac{\kappa}{3}\right] \supset [0, \kappa_1]. \qquad \text{(A 11)}$$

Proceeding further as in Aronson [1], from (A 9) and (A 11) we obtain first the bound on $p = w_x$ and, therefore, on $v_x = \varrho'(w)w_x$. Since the estimates in our case are completely analogous to those in Aronson [1] for the porous medium equation, we omit the further details and refer the reader to Aronson [1]. □

Corollary A.2 *Let u be a smooth positive solution of* (1.1). *Then*

$$|k(u(x_2, t)) - k(u(x_1, t))| \leqslant C_4|x_2 - x_1|, \qquad \text{(A 12)}$$
$$|u(x_2, t) - u(x_1, t)| \leqslant C_5|x_2 - x_1|^{\nu}, x_1, x_2 \in \bar{R}_1, \qquad \text{(A 13)}$$

where $\nu = \min\{1, \alpha^{-1}\}$.

Proof of Corollary A.2 The first inequality (A 12) follows directly from Lemma A.1. To prove (A 13), note that if $\alpha > 1$

$$|u(x_2, t) - u(x_1, t)|^{\alpha} \leqslant |u^{\alpha}(x_2, t) - u^{\alpha}(x_1, t)| \leqslant C_1^{-1}|k(u(x_2, t)) - k(u(x_1, t))| \leqslant C_1^{-1}C_3|x_2 - x_1|, \qquad \text{(A 14)}$$

where the positive constants C_1 and C_3 are the same as in (A 3) and (A 5), respectively. If $\alpha > 1$ then (A 13) follows from (A 14). On the other hand, if $\alpha \leqslant 1$ then $\Psi'(v) = \frac{1}{k'(u)}$ is

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bounded in $\bar{R}_1$. Thus,

$$|u(x_2,t)-u(x_1,t)| \leqslant \max_{\bar{R}_1} |\Psi'(v)||k(u(x_2,t))-k(u(x_1,t))| \leqslant C_4|x_2-x_1|. \qquad \square$$

Remark A.1 The statement of the corollary remains valid for the generalized solution of (1.1), because it can be obtained as a limit of a sequence of positive classical solutions.

Now we are in a position to prove the theorem. Let $I_\delta(x_0) \subset (a_1,b_1)$ denote the interval such that $|x-x_0|<\delta$ for any $x \in I_\delta(x_0)$. The generalized solution of (1.1) $u(x,t)$ can be obtained as a limit of a decreasing sequence of positive classical solutions:

$$u(x,t) = \lim_{n\to\infty} w_n(x,t).$$

By Corollary A.2, we have

$$0 < w_n(x,t) \leqslant C_5\delta^\nu, \; x \in I_\delta(x_0), \; n = 1,2,\ldots. \tag{A 15}$$

Using integration by parts, we obtain

$$\phi(w_n(x_2,t_0)) - \phi(w_n(x_1,t_0)) = \int_{x_1}^{x_2} \frac{\partial\phi(w_n(x,t_0))}{\partial x}\,dx = \int_{x_1}^{x_2} k(w_n(x,t_0))\frac{\partial w_n(x,t_0)}{\partial x}$$
$$= k(w_n(x,t))w_n(x,t)|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\partial k(w_n(x,t_0))}{\partial x} w_n(x,t_0)\,dx. \tag{A 16}$$

Using (A 5) and (A 15), from (A 16) we have

$$|\phi(w_n(x_2,t_0)) - \phi(w_n(x_1,t_0))| \leqslant C_6\delta^\nu|x_2-x_1|, \; x_1,x_2 \in I_\delta(x_0).$$

By taking the limit as $n\to\infty$ in the last inequality, we derive

$$|\phi(u(x_2,t_0)) - \phi(u(x_1,t_0))| \leqslant C_6\delta^\nu|x_2-x_1|, \; x_1,x_2 \in I_\delta(x_0). \tag{A 17}$$

The continuity of the flux in the neighbourhood of the interface (x_0,t_0), as well as (2.3), follows from (A 17).

References

[1] Aronson, D. G. (1969) Regularity properties of flows through media. *SIAM J. Appl. Math.* **17**, 461–467.

[2] Aronson, D. G. (1980) Density-dependent interaction-diffusion systems. In: Stuart, W. E., Ray, W. H. and Conley, C. (eds.), *Dynamics and Modeling of Reactive Systems*. Academic Press.

[3] Aronson, D. G. (1985) The porous medium equation. In: Fasano. A. and Primicerio, M. (eds.), *Nonlinear Diffusion Problems: Lecture Notes in Mathematics 1224*. Springer-Verlag.

[4] Aronson, D. G., Crandall, M. G. & Peletier, L. A. (1982) Stabilization of solutions of a degenerate nonlinear diffusion problem. *Nonlin. Anal., Theory, Meth. and Appl.* **6**, 1001–1022.

[5] Aronson, D. G. & Vazquez, J. L. (1995) Anomalous exponents in nonlinear diffusion. *J. Nonlinear Sci.* **5**, 29–56.

[6] Aronson, D. G. & Weinberger, H. F. (1975) Nonlinear diffusion in population genetics, combustion, and nerve propagation. In: Goldstein, J. (ed.), *Lecture Notes in Mathematics 446*. Springer-Verlag.

[7] Aronson, D. G. & Weinberger, H. F. (1978) Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.* **30**, 33–76.

[8] Carr, J. (1981) *Applications of Centre Manifold Theory*. Springer-Verlag.

[9] Chen, X. & Friedman, A. (2000) A free boundary problem arising in a model of wound healing. *SIAM J. Math. Anal.* **32**, 778–800.

[10] Fisher, R. A. (1937) The wave of advance of advantageous genes. *Ann. Eugenics* **7**, 355–369.

[11] Friedman, A. (1964) *Partial Differential Equations of Parabolic Type*. Prentice-Hall.

[12] Galactionov, V. A., Shmarev, S. I. & Vazquez, J. L. (1999) Regularity of interfaces in diffusion processes under the influence of strong absorption. *Arch. Rational Mech. Anal.* **149**, 183–212.

[13] Gilding, B. H. & Kersner, R. (1996) The characterization of reaction-convection-diffusion processes by travelling waves. *J. Diff. Eqns.* **124**, 27–79.

[14] Guckenheimer, J. & Holmes, P. J. (1983) *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag.

[15] Gurtin, M. E. & MacCamy, R. C. (1977) On the diffusion of biological populations. *Math. Biosci.* **33**, 35–49.

[16] Kalashnikov, A. S. (1974) On the character of the propagation of disturbances in problems with nonlinear heat absorption. *Zhurnal Vychilitelnoi Matematiki i Matematicheskoi Physiki* **14**, 891–905.

[17] Kalashnikov, A. S. (1987) Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations. *Uspekhi Mat. Nauk* **42:2**, 135–176. (English translation in *Russian Math. Surveys* **42**(2), 169–222.)

[18] Kersner, R. (1978) Some properties of generalized solutions of quasilinear parabolic equations. *Acta Math. Acad. Sci. Hungar.* **32**, 301–330 (in Russian).

[19] Kolmogorov, A. N., Petrovskii, I. G. & Piscounoff (1937) A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem. *Bjul. Mosk. Gos. Univ.* **1**(7), 1–26. (English translation in Pelce, P. (ed.) (1988) *Dynamics of Curved Fronts*. Academic Press.)

[20] Ladyzhenskaia, O. A., Solonnikov, V. A. & Uraltseva, N. N. (1968) *Linear and Quasilinear Equations of Parabolic Type*. Translations of Mathematical Monographs, **23**. AMS.

[21] Medvedev, G. S., Kaper, T. J. & Kopell, N. (2000) Reaction-diffusion system with periodic front dynamics. *SIAM J. Appl. Math.* **60**, 1601–1638.

[22] Murray, J. D. (1993) *Mathematical Biology*. Springer-Verlag.

[23] Oleinik, O. A. (1957) On the equations of unsteady filtration type. *Dokl. Akad. Nauk SSSR*, **113**, 1210–1213.

[24] Oleinik, O. A., Kalashnikov, A. S. & Yui-Lin, Ch. (1958) The Cauchy problem and boundary value problems for equations of the type of unsteady filtration. *Izv. Akad. Nauk SSSR Ser. Mat.* **22**, 667–704.

[25] Samarski, A. A., Galaktionov, V. A., Kurdumov, S. P. & Mikhailov, A. P. (1995) *Blow-up in Quasilinear Parabolic Equations*. Walter de Gruyter.

[26] Sànchez-Garduno, F., Maini, P. K., & Kappos, M. E. (1996) A shooting argument approach to a sharp-type solution for nonlinear degenerate Fisher-KPP equations. *IMA J. on Appl. Math.* **57**, 211–221.

[27] Sànchez-Garduno, F. & Maini, P. K. (1997) Travelling wave phenomena in non-linear diffusion degenerate Nagumo equations. *J. Math. Biol.* **35**, 713–728.

[28] Sherratt, J. A. (1999) Travelling wave solutions of a mathematical model for tumor encapsulation. *SIAM J. Appl. Math.* **60**, 392–407.

[29] de Pablo, A. & Vazquez, J. L. (1991) Travelling waves and finite propagation in a reaction-diffusion equation. *J. Diff. Eqns.* **93**, 19–61.

[30] Volpert, A. I., Volpert, V. A. & Volpert, V. A. (1994) *Travelling Wave Solutions of Parabolic Systems*. Translations of Math. Monographs, **140**. AMS.