Proceedings of the Royal Society of Edinburgh, **131A**, 621–646, *2001*

Best constants in the L^2-Nash inequality

Emmanuel Humbert
University of Paris VI, Institut de Mathématiques,
175 rue du Chevaleret, 75013 Paris, France
(humbert@math.jussieu.fr)

(MS received 17 December 1999; accepted 18 May 2000)

We prove that, contrary to the L^1-Nash inequality, there exists a second best constant for the L^2-Nash inequality on any smooth compact Riemannian manifold.

1. Introduction

The Nash inequality in $\mathbb{R}^n$ asserts that there exists $A > 0$ such that, for all $u \in \mathcal{D}(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} u^2 \, \mathrm{d}x\right)^{1+2/n} \leqslant A \int_{\mathbb{R}^n} |\nabla u|^2 \, \mathrm{d}x \left(\int_{\mathbb{R}^n} |u| \, \mathrm{d}x\right)^{4/n}.$$

For $n \geqslant 3$, this inequality is obtained by combining the Sobolev inequality,

$$\left(\int_{\mathbb{R}^n} |u|^{2n/(n-2)} \, \mathrm{d}x\right)^{(n-2)/n} \leqslant A \int_{\mathbb{R}^n} |\nabla u|^2 \, \mathrm{d}x,$$

and Hölder's inequality,

$$\left(\int_{\mathbb{R}^n} u^2 \, \mathrm{d}x\right)^{1+2/n} \leqslant \left(\int_{\mathbb{R}^n} |u| \, \mathrm{d}x\right)^{4/n} \left(\int_{\mathbb{R}^n} |u|^{2n/(n-2)} \, \mathrm{d}x\right)^{(n-2)/n}$$

By [3], the best constant in the Nash inequality is

$$A_0(n) = \frac{(n+2)^{(n+2)/n}}{2^{2/n} n \lambda_1(\mathcal{B}) |\mathcal{B}|^{2/n}},$$

where $|\mathcal{B}|$ denotes the Euclidean volume of the unit ball $\mathcal{B}$ in $\mathbb{R}^n$ and $\lambda_1(\mathcal{B})$ is the first Neumann eigenvalue of the Laplacian for radial functions on $\mathcal{B}$.

In this paper we let (M, g) be a smooth compact Riemannian n-manifold. Without loss of generality, we may assume that $\mathrm{Vol}(M) = 1$. The previous Nash inequality is clearly not true on M. It has to be modified by adding another term. One may add an L^1-term to get what we will refer to as the L^1-Nash inequality, $\forall u \in C^\infty(M)$,

$$\left(\int_M u^2 \, \mathrm{d}v_g\right)^{1+2/n} \leqslant A \int_M |\nabla u|_g^2 \, \mathrm{d}v_g \left(\int_M |u| \, \mathrm{d}v_g\right)^{4/n} + B \left(\int_M |u| \, \mathrm{d}v_g\right)^{2+4/n}. \tag{1.1}$$

This inequality was studied in [4] by Druet, Hebey and Vaugon. They showed that, for every $\epsilon > 0$, there exists $B_\epsilon > 0$ such that it is true with $A = A_0(n) + \epsilon$ and

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

$B = B_\epsilon$. More importantly, they also exhibited the rather unexpected fact that the existence of B, such that it is true with $A = A_0(n)$, depends on the geometry of M.

We consider in this paper the following inequality, $\forall u \in C^\infty(M)$:

$$\left(\int_M u^2 \,\mathrm{d}v_g\right)^{1+2/n} \leqslant \left(A\int_M |\nabla u|_g^2 \,\mathrm{d}v_g + B\int_M u^2 \,\mathrm{d}v_g\right)\left(\int_M |u| \,\mathrm{d}v_g\right)^{4/n} \qquad N(A,B)(u)$$

We refer to this inequality as the L^2-Nash inequality. As in the case of the L^1-Nash inequality, for $n \geqslant 3$, it is obtained by combining the Sobolev inequality and Hölder's inequality. Concerning terminology, we say that $N(A,B)$ *is valid* if $N(A,B)(u)$ is true for all $u \in C^\infty(M)$.

In this paper we study the sharp L^2-Nash inequality $N(A_0(n),B)$. In this case, contrary to the sharp L^1-Nash inequality, we prove that B always exists. Among other references, a similar study was done in [2,5,6] on what concerns classical sharp Sobolev inequalities. We especially point out the beautiful and inspiring reference [5] by Druet, which we somehow follow here. However, note that our partial differential equation and the concentration phenomenon we face are of different nature. In particular, the difficulties we have to deal with are distinct. In order to be easily understood, the proof is done in the case of compact Riemannian manifolds. This contains all the technical difficulties. Concerning manifolds with boundary and complete manifolds, we only give a sketch of the proof. A general reference on sharp Sobolev-type inequalities is given in [7].

Define $\mathcal{A}$ by

$$\mathcal{A} = \{A > 0 \mid \exists\ B > 0 \text{ such that } N(A,B) \text{ is valid}\}.$$

Mimicking what was done in [4], one easily gets that

$$\inf(\mathcal{A}) = A_0(n).$$

Conversely, we prove the following.

THEOREM 1.1. *Let (M,g) be a smooth compact Riemannian n-manifold. Then there exists $B > 0$ such that, for all $u \in C^\infty(M)$,*

$$\left(\int_M u^2 \,\mathrm{d}v_g\right)^{1+2/n} \leqslant \left(A_0(n)\int_M |\nabla u|_g^2 \,\mathrm{d}v_g + B\int_M u^2 \,\mathrm{d}v_g\right)\left(\int_M |u| \,\mathrm{d}v_g\right)^{4/n}.$$

In other words, there always exists B for which $N(A_0(n),B)$ is valid.

For $n = 1$ (i.e. for $M = S^1$), a simple proof of theorem 1.1 goes through a partition of unity argument. Such a proof extends to flat manifolds. Independently, an extension of this result to manifolds with boundary and complete manifolds will be sketched at the end of § 2.

We now define

$$B_0 = \inf\{B \in \mathbb{R} \text{ such that } N(A_0(n),B) \text{ is valid}\}.$$

In last section, we compute B_0 for the circle S^1 using a non-trivial argument. Giving the explicit value for B_0 in the other cases is a difficult problem. At the moment, we only obtain the following result in the general case.

THEOREM 1.2. *Let (M, g) be a smooth compact Riemannian n-manifold. Then*

$$B_0 \geqslant \max\biggl(\mathrm{Vol}(M)^{-2/n}, \frac{|\mathcal{B}|^{-2/n}}{6n}\biggl(\frac{2}{n+2} + \frac{n-2}{\lambda_1}\biggr)\biggl(\frac{n+2}{2}\biggr)^{2/n} \max_{x\in M} S_g(x)\biggr),$$

where $|\mathcal{B}|$ is the volume of the unit ball $\mathcal{B}$ in $\mathbb{R}^n$, λ_1 is the first non-zero Neumann eigenvalue of the Laplacian on radial functions on $\mathcal{B}$, $\mathrm{Vol}(M)$ *is the volume of (M, g) and $S_g(x)$ is the scalar curvature of g at x.*

We now say that $u \in H_1^2(M)$, $u \not\equiv 0$, is an extremal function for the sharp L^2-inequality $N(A_0(n), B_0)$ if

$$\biggl(\int_M u^2 \,\mathrm{d}v_g\biggr)^{1+2/n} = \biggl(A_0(n)\int_M |\nabla u|_g^2 \,\mathrm{d}v_g + B_0 \int_M u^2 \,\mathrm{d}v_g\biggr)\biggl(\int_M |u| \,\mathrm{d}v_g\biggr)^{4/n}.$$

We then prove, with a very simple argument, the following result.

THEOREM 1.3. *Let (M, g) be a smooth compact Riemannian n-manifold with $n \geqslant 1$. Suppose that the L^1-Nash inequality (1.1) is true, with $A = A_0(n)$ and some B. Then there exists $u \in H_1^2(M)$, $u \not\equiv 0$, an extremal function for the sharp L^2-Nash inequality $N(A_0(n), B_0)$.*

Together with the results of [4], this leads to the following result. A discussion on the n-dimensional Cartan–Hadamard conjecture can be found in [7].

COROLLARY 1.4. *There exists extremal functions for the sharp L^2-Nash inequality $N(A_0(n), B_0)$ on any smooth compact Riemannian n-manifold of non-positive sectional curvature when the n-dimensional Cartan–Hadamard conjecture is true. In particular, there exist extremal functions for the sharp L^2-Nash inequality $N(A_0(n), B_0)$ on any smooth compact Riemannian manifold of non-positive sectional curvature and dimension 2, 3 or 4.*

A more in-depth study of the existence of extremal functions for the sharp L^2-Nash inequality will be done in [8]. In § 5 below, we point out the example of S^1, where constant functions are extremal functions for the sharp L^2-Nash inequality. We also exhibit a counterexample where this is false.

2. Proof of theorem 1.1

The proof of theorem 1.1 proceeds in four steps, which are quite similar to the ones of [5]. The specific problems of the L^2-Nash inequality appear in step 2, where the method used in [5] does not work, and especially in step 4, where the special form of our inequality leads to several difficulties. We assume here that $n \geqslant 2$. The proof of the theorem then proceeds by contradiction. We assume that, for all $B > 0$, there exists $u \in C^\infty(M)$ such that

$$\biggl(\int_M u^2 \,\mathrm{d}v_g\biggr)^{1+2/n} > \biggl(A_0(n)\int_M |\nabla u|_g^2 \,\mathrm{d}v_g + B \int_M u^2 \,\mathrm{d}v_g\biggr)\biggl(\int_M |u| \,\mathrm{d}v_g\biggr)^{4/n}.$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

This is clearly equivalent to $\tilde{\mu}_\alpha = \inf_{u\in\Lambda} \tilde{I}_\alpha(u) < A_0(n)^{-1}$ for all $\alpha \geqslant 0$, where

$$\tilde{I}_\alpha(u) = \left(\int_M |\nabla u|_g^2 \, \mathrm{d}v_g + \alpha\right)\left(\int_M |u| \, \mathrm{d}v_g\right)^{4/n}$$

and

$$\Lambda = \left\{ u \in C^\infty(M) \mid \int_M u^2 \, \mathrm{d}v_g = 1 \right\}.$$

In order to carry over the non-differentiability of $\tilde{I}_\alpha$, we define for $\epsilon, \alpha > 0$ and $u \in \Lambda$,

$$\tilde{I}_{\epsilon,\alpha}(u) = \left(\int_M |\nabla u|_g^2 \, \mathrm{d}v_g + \alpha\right)\left(\int_M |u|^{1+\epsilon} \, \mathrm{d}v_g\right)^{4/(n(1+\epsilon))}$$

and

$$\mu_{\epsilon,\alpha} = \inf_{u\in\Lambda} \tilde{I}_{\epsilon,\alpha}(u).$$

One may check that $\lim_{\epsilon\to 0} \mu_{\epsilon,\alpha} = \tilde{\mu}_\alpha$. Hence, for $\alpha < 0$, we can choose $(\epsilon_\alpha)_\alpha$ such that $\lim_{\alpha\to\infty} \epsilon_\alpha = 0$ and $\mu_{\epsilon_\alpha,\alpha} < A_0(n)^{-1}$. We let

$$\mu_\alpha = \mu_{\epsilon_\alpha,\alpha} \quad \text{and} \quad I_\alpha = \tilde{I}_{\epsilon_\alpha,\alpha}.$$

Mimicking what was done in [4], one may prove that there exists $u_\alpha \in C^2(M)$ such that $I_\alpha(u_\alpha) = \mu_\alpha$ with $u_\alpha \geqslant 0$. Moreover, writing the Euler equation of u_α, we get that, in the sense of distributions,

$$2A_\alpha \Delta_g u_\alpha + \frac{4}{n} B_\alpha u_\alpha^{\epsilon_\alpha} = k_\alpha u_\alpha, \tag{E_α}$$

where Δ_g stands for the Laplacian with the minus sign convention, and where

$$A_\alpha = \left(\int_M u_\alpha^{1+\epsilon_\alpha} \, \mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))}$$

$$B_\alpha = \left(\int_M |\nabla u_\alpha|_g^2 \, \mathrm{d}v_g + \alpha\right)\left(\int_M u_\alpha^{1+\epsilon_\alpha} \, \mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))-1}$$

$$k_\alpha = \frac{4}{n}\mu_\alpha + 2\int_M |\nabla u_\alpha|_g^2 \, \mathrm{d}v_g \left(\int_M u_\alpha^{1+\epsilon_\alpha} \, \mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))}.$$

By the Sobolev embedding theorem, we have $u_\alpha \in L^{2n/(n-2)}(M)$ and then, by classical methods, we prove that $u_\alpha \in C^2(M)$.

Remark 2.1. In the following, all the limits are taken as $\alpha \to \infty$. Moreover, we assume that all the sequences have a limit (not necessarily finite) by passing to a subsequence.

REMARK 2.2. Since $\mu_\alpha < A_0(n)^{-1}$, we have

$$\int_M u_\alpha^{1+\epsilon_\alpha}\,dv_g \to 0.$$

From $N(A_0(n)+\epsilon, B_\epsilon)(u_\alpha)$, where $\epsilon > 0$ is small, we get

$$\liminf \int_M |\nabla u_\alpha|_g^2\,dv_g \biggl(\int_M u_\alpha^{1+\epsilon_\alpha}\,dv_g\biggr)^{4/(n(1+\epsilon_\alpha))} \geqslant (A_0(n)+\epsilon)^{-1}.$$

In addition, since $\mu_\alpha < A_0(n)^{-1}$, it is clear that

$$\limsup \int_M |\nabla u_\alpha|_g^2\,dv_g \biggl(\int_M u_\alpha^{1+\epsilon_\alpha}\,dv_g\biggr)^{4/(n(1+\epsilon_\alpha))} \leqslant A_0(n)^{-1}.$$

As one easily checks,

$$\lim A_\alpha \int_M |\nabla u_\alpha|_g^2\,dv_g = A_0(n)^{-1}, \tag{2.1}$$

$$\lim B_\alpha \int_M u_\alpha^{1+\epsilon_\alpha}\,dv_g = A_0(n)^{-1}, \tag{2.2}$$

$$\lim k_\alpha = \biggl(2+\frac{4}{n}\biggr)A_0(n)^{-1}, \tag{2.3}$$

$$\lim A_\alpha \alpha = 0. \tag{2.4}$$

Now let $a_\alpha = A_\alpha^{1/2}$. Also let x_α be a point of M such that $u_\alpha(x_\alpha) = \|u_\alpha\|_\infty$.

STEP 1. For all $\delta > 0$,

$$\liminf \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} u_\alpha^{1+\epsilon_\alpha}\,dv_g}{\int_M u_\alpha^{1+\epsilon_\alpha}\,dv_g} > 0.$$

For $x \in B(0,\delta) \subset \mathbb{R}^n$, let

$$g_\alpha(x) = (\exp_{x_\alpha})^* g(a_\alpha x),$$
$$\varphi_\alpha(x) = \|u_\alpha\|_\infty^{-1} u_\alpha(\exp_{x_\alpha}(a_\alpha x)).$$

We easily get

$$\Delta_{g_\alpha}\varphi_\alpha + \frac{2}{n}\|u_\alpha\|_\infty^{-1+\epsilon_\alpha} B_\alpha \varphi_\alpha^{\epsilon_\alpha} = \tfrac{1}{2}k_\alpha\varphi_\alpha. \tag{$\tilde{E}_\alpha$}$$

Since $\Delta_g u_\alpha(x_\alpha) \geqslant 0$, we get from (E_α) and (2.3),

$$\|u_\alpha\|_\infty^{\epsilon_\alpha} B_\alpha \leqslant C\|u_\alpha\|_\infty. \tag{2.5}$$

Since $\|\varphi_\alpha\|_{L^\infty(B(0,\delta))} \leqslant 1$, one gets from $(\tilde{E}_\alpha)$ and standard methods that, for $a \in {]0,1[}$, $\|\varphi_\alpha\|_{C^{1,a}B(0,\delta)} \leqslant C$. Hence $(\varphi_\alpha)_\alpha$ is equicontinuous and, by Ascoli's theorem, there exists $\varphi \in C^0(B(0,\delta))$ such that $\varphi_\alpha \to \varphi$ in $C^0(B(0,\delta))$. We have

$$\varphi(0) = \lim \varphi_\alpha(0) = 1, \tag{2.6}$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

and also

$$\begin{aligned}\int_{B(0,\delta)} \varphi_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_{g_\alpha} &= \|u_\alpha\|_\infty^{-(1+\epsilon_\alpha)} A_\alpha^{-n/2} \int_{B_{x_\alpha}(\delta a_\alpha)} u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g \\ &= \|u_\alpha\|_\infty^{-(1+\epsilon_\alpha)} A_\alpha^{-(n/4)(1-\epsilon_\alpha)} \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g}{\int_M u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g} \\ &\leqslant \|u_\alpha\|_\infty^{-1} A_\alpha^{-n/4} \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g}{\int_M u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g}, \end{aligned} \tag{2.7}$$

since $\|u_\alpha\|_\infty^{\epsilon_\alpha} \geqslant 1$, equation (2.5) implies that $\|u_\alpha\|_\infty \geqslant C \cdot B_\alpha$ and, since $A_\alpha \to 0$, equation (2.2) implies $B_\alpha \geqslant C \cdot A_\alpha{}^{-(n/4)(1+\epsilon_\alpha)} \geqslant C \cdot A_\alpha^{-n/4}$. Inequality (2.7) then becomes

$$\int_{B(0,\delta)} \varphi_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_{g_\alpha} \leqslant C \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g}{\int_M u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g}.$$

Moreover,

$$\int_{B(0,\delta)} \varphi_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_{g_\alpha} \to C > 0 \tag{2.8}$$

by (2.6) and since $g_\alpha \to \xi$ in $C^2(K)$. Finally, we get

$$\frac{\int_{B_{x_\alpha}(\delta a_\alpha)} u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g}{\int_M u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g} \geqslant C > 0,$$

which ends the proof of the step.

Remark 2.3. Coming back to (2.7) and (2.8), one easily gets that

$$A_\alpha^{n/4} \|u_\alpha\|_\infty \to C > 0. \tag{2.9}$$

Step 2. Let $(c_\alpha)_\alpha$ be a sequence of positive numbers such that $a_\alpha / c_\alpha \to 0$. Then

$$\lim \frac{\int_{B_{x_\alpha}(c_\alpha)} u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g}{\int_M u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g} = 1.$$

Let $\eta \in C^\infty(\mathbb{R})$ be such that

(i) $\eta([0, \frac{1}{2}]) = \{1\}$,

(ii) $\eta([1, +\infty[) = \{0\}$,

(iii) $0 \leqslant \eta \leqslant 1$.

For $k \in \mathbb{N}$, we let $\eta_{\alpha,k} = (\eta(c_\alpha^{-1} d_g(x, x_\alpha)))^{2^k}$. Multiplying (E_α) by $\eta_{\alpha,k}^2 u_\alpha$ and integrating over M gives

$$\begin{aligned} 2A_\alpha \int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2\,\mathrm{d}v_g - 2A_\alpha \int_M |\nabla \eta_{\alpha,k}|_g^2 u_\alpha^2\,\mathrm{d}v_g \\ + \frac{4}{n} B_\alpha \int_M \eta_{\alpha,k}^2 u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g = k_\alpha \int_M (\eta_{\alpha,k} u_\alpha)^2\,\mathrm{d}v_g. \end{aligned} \tag{2.10}$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

Using $N(A_0(n)+\epsilon, B_\epsilon)(\eta_{\alpha,k}u_\alpha)$, one easily checks that

$$
\begin{aligned}
2A_\alpha \int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 \,\mathrm{d}v_g - 2A_\alpha \int_M |\nabla \eta_{\alpha,k}|_g^2 u_{\alpha_g}^2 \,\mathrm{d}v_g + \frac{4}{n} B_\alpha \int_M \eta_{\alpha,k}^2 u_\alpha^{1+\epsilon_\alpha} \,\mathrm{d}v_g \\
\leqslant k_\alpha \biggl((A_0(n)+\epsilon) \int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 \,\mathrm{d}v_g \biggl(\int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} \,\mathrm{d}v_g \biggr)^{4/(n(1+\epsilon_\alpha))} \\
+ B_\epsilon \int_M (\eta_{\alpha,k} u_\alpha)^2 \,\mathrm{d}v_g \biggl(\int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} \,\mathrm{d}v_g \biggr)^{4/(n(1+\epsilon_\alpha))} \biggr)^{n/(n+2)}.
\end{aligned}
\tag{2.11}
$$

Moreover, with the assumption on $(c_\alpha)_\alpha$,

$$
|\nabla \eta_{\alpha,k}|_g^2 \leqslant \frac{C}{c_\alpha^2} \quad \Rightarrow \quad A_\alpha \int_M |\nabla \eta_{\alpha,k}|_g^2 u_\alpha^2 \,\mathrm{d}v_g \to 0.
$$

Now let

$$
\lambda_k = \lim \frac{\int_M \eta_{\alpha,k}^2 u_\alpha^{1+\epsilon_\alpha} \,\mathrm{d}v_g}{\int_M u_\alpha^{1+\epsilon_\alpha} \,\mathrm{d}v_g}, \qquad \tilde{\lambda}_k = \lim \frac{\int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} \,\mathrm{d}v_g}{\int_M u_\alpha^{1+\epsilon_\alpha} \,\mathrm{d}v_g}.
$$

From the definition of $\eta_{\alpha,k}$, we get, for all $k \in \mathbb{N}$,

$$
\lambda_{k+1} \leqslant \tilde{\lambda}_{k+1} \leqslant \lambda_k \leqslant \tilde{\lambda}_k \leqslant \mu = \lim \frac{\int_{B_{x_\alpha}(c_\alpha)} u_\alpha^{1+\epsilon_\alpha} \,\mathrm{d}v_g}{\int_M u_\alpha^{1+\epsilon_\alpha} \,\mathrm{d}v_g} \tag{2.12}
$$

and, by step 1,

$$
\exists\, C > 0 \text{ such that } \forall k \in \mathbb{N},\ \lambda_k \geqslant C. \tag{2.13}
$$

Let us now prove that $\lambda_k \leqslant \tilde{\lambda}_k^2$. Let

$$
L_k = \lim A_\alpha \int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 \,\mathrm{d}v_g.
$$

Note that (2.2) and the definition of A_α imply

$$
\lim B_\alpha \int_M \eta_{\alpha,k}^2 u_\alpha^{1+\epsilon_\alpha} \,\mathrm{d}v_g = \lambda_k A_0(n)^{-1}
$$

and

$$
k_\alpha \int_M (\eta_{\alpha,k} u_\alpha)^2 \,\mathrm{d}v_g \leqslant C.
$$

In particular, equation (2.10) gives $L_k < +\infty$. We also clearly have, by (2.1) and (2.2),

$$
\lim \int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 \,\mathrm{d}v_g \biggl(\int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} \,\mathrm{d}v_g \biggr)^{4/(n(1+\epsilon_\alpha))} = L_k \tilde{\lambda}_k^{4/n}.
$$

Equation (2.11) then leads to

$$
2L_k + \frac{4}{n} A_0(n)^{-1} \lambda_k \leqslant \biggl(2 + \frac{4}{n}\biggr) A_0(n)^{-1} ((A_0(n)+\epsilon) L_k \tilde{\lambda}_k^{4/n})^{n/(n+2)}.
$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

If $\tilde{L}_k = A_0(n)L_k$, we obtain, since ϵ was arbitrary,

$$2\tilde{L}_k + \frac{4}{n}\lambda_k \leqslant \left(2 + \frac{4}{n}\right)\tilde{L}_k^{n/(n+2)}\tilde{\lambda}_k^{4/(n+2)}.$$

Now, for x, y, z, let

$$f(x,y,z) = \left(2 + \frac{4}{n}\right)x^{n/(n+2)}y^{4/(n+2)} - \left(\frac{4}{n}z + 2x\right).$$

Differentiating in x, we see that $\forall x,y,z > 0$, $f(x,y,z) \leqslant f(y^2,y,z)$, and then

$$f(\tilde{L}_k, \tilde{\lambda}_k, \lambda_k) \leqslant f(\tilde{\lambda}_k^2, \tilde{\lambda}_k, \lambda_k) = \frac{4}{n}(\tilde{\lambda}_k^2 - \lambda_k).$$

We then get $\lambda_k \leqslant \tilde{\lambda}_k^2$. Now, from (2.12), (2.13), we get $\forall N \in \mathbb{N}$, $0 < C \leqslant \lambda_0^N \leqslant \mu$. Since $\mu \leqslant 1$, we have $\mu = 1$, which proves the step.

STEP 3. There exists $C > 0$ such that, for all $x \in M$,

$$u_\alpha(x)d(x,x_\alpha)^{n/2} \leqslant C,$$

where d denotes the distance for g.

We proceed by contradiction. Suppose that the following assumption is true:

$$\exists\ y_\alpha \in M \text{ such that } u_\alpha(y_\alpha)d(y_\alpha,x_\alpha)^{n/2} \to +\infty. \tag{H}$$

Let

$$v_\alpha = u_\alpha(y_\alpha)d(y_\alpha,x_\alpha)^{n/2}.$$

We can assume that

$$v_\alpha = \|u_\alpha(\cdot)d(\cdot,x_\alpha)^{n/2}\|_\infty.$$

First, we prove that, if ν is small enough,

$$B_{y_\alpha}(u_\alpha(y_\alpha)^{-2/n}) \cap B_{x_\alpha}(a_\alpha v_\alpha^\nu) = \emptyset. \tag{2.14}$$

It is here enough to show that

$$d(x_\alpha,y_\alpha) \geqslant u_\alpha(y_\alpha)^{-2/n} + a_\alpha v_\alpha^\nu,$$

or, equivalently, that

$$v_\alpha^{2/n-\nu} \geqslant v_\alpha^{-\nu} + a_\alpha u_\alpha(y_\alpha)^{2/n}.$$

If $\nu < 2/n$, from (H), we get that $v_\alpha^{2/n-\nu} \to +\infty$ and $v_\alpha^{-\nu} \to 0$. Hence it still has to be proved that $a_\alpha u_\alpha(y_\alpha)^{2/n} \leqslant C$. We have $a_\alpha u_\alpha(y_\alpha)^{2/n} \leqslant a_\alpha\|u_\alpha\|_\infty^{2/n}$. Since $a_\alpha = A_\alpha^{1/2}$, and by (2.9), this gives $a_\alpha\|u_\alpha\|_\infty^{2/n} \leqslant C$. Equation (2.14) then follows.

We let, for $x \in B(0,1)$,

$$h_\alpha(x) = (\exp_{y_\alpha})^* g(l_\alpha x),$$
$$\psi_\alpha(x) = u_\alpha(y_\alpha)^{-1}u_\alpha(\exp_{y_\alpha}(l_\alpha x)),$$

where

$$l_\alpha = \|u_\alpha\|_\infty^{-(n+4)/2n}u_\alpha(y_\alpha)^{1/2}.$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

On $B(0,1)$, we have

$$\Delta_{h_\alpha}\psi_\alpha = \frac{k_\alpha \|u_\alpha\|_\infty^{-(1+4/n)} u_\alpha(y_\alpha)}{2A_\alpha}\psi_\alpha - \frac{2B_\alpha \|u_\alpha\|_\infty^{-(1+4/n)} u_\alpha(y_\alpha)^{\epsilon_\alpha}}{nA_\alpha}\psi_\alpha^{\epsilon_\alpha}. \qquad (E'_\alpha)$$

Moreover,

$$h_\alpha \to \xi \quad \text{in } C^2(B(0,1)). \qquad (2.15)$$

We have

$$\|u_\alpha\|_{L^\infty(B_{y_\alpha}(u_\alpha(y_\alpha)^{-2/n}))} \leqslant C \cdot u_\alpha(y_\alpha).$$

To see this, note that, by the very first definition of y_α, for all $x \in B_{y_\alpha}(u_\alpha(y_\alpha)^{-2/n})$, we have

$$u_\alpha(y_\alpha) d(x_\alpha, y_\alpha)^{n/2} \geqslant u_\alpha(x) d(x_\alpha, x)^{n/2}. \qquad (2.16)$$

Moreover, since $x \in B_{y_\alpha}(u_\alpha(y_\alpha)^{-2/n})$,

$$d(y_\alpha, x) \leqslant u_\alpha(y_\alpha)^{-2/n}$$

and, by (H), $u_\alpha(y_\alpha)^{-2/n} \leqslant \frac{1}{2} d(x_\alpha, y_\alpha)$. So we have

$$d(x, x_\alpha) \geqslant d(x_\alpha, y_\alpha) - d(x, y_\alpha) \geqslant d(x_\alpha, y_\alpha) - u_\alpha(y_\alpha)^{-2/n} \geqslant \tfrac{1}{2} d(x_\alpha, y_\alpha).$$

Coming back to (2.16), the result comes immediately. Since $l_\alpha \leqslant u_\alpha(y_\alpha)^{-2/n}$, it follows that $\|\psi_\alpha\|_{L^\infty(B(0,1))} \leqslant C$. From (2.5), (2.9) and the fact that, by (2.2), $B_\alpha A_\alpha^{1/4} n(1+\epsilon_\alpha) \to C > 0$, we get

$$\|u_\alpha\|_\infty^{\epsilon_\alpha} \to C. \qquad (2.17)$$

Now, from (2.5), (2.9) and (2.17), we see that (E'_α) has bounded coefficients. Hence standard arguments imply that the sequence $(\psi_\alpha)_\alpha$ is bounded in $C^{1,a}(B(0,1))$ $(0 < a < 1)$. As in step 1, one may find $\psi \in C^0(B(0,1))$ such that, up to a subsequence,

$$\psi_\alpha \to \psi \quad \text{in } C^0(B(0,1)).$$

Here, ψ is such that $\psi(0) = 1$, and then

$$\int_{B(0,1)} \psi \, \mathrm{d}x > 0. \qquad (2.18)$$

However, by (2.15),

$$\int_{B(0,1)} \psi \, \mathrm{d}x = \lim \int_{B(0,1)} \psi_\alpha^{1+\epsilon_\alpha} \, \mathrm{d}v_{h_\alpha},$$

and, as one can check,

$$\int_{B(0,1)} \psi_\alpha^{1+\epsilon_\alpha} \, \mathrm{d}v_{h_\alpha} = \beta_\alpha,$$

where

$$\beta_\alpha = A_\alpha^{(n/4)(1+\epsilon_\alpha)} u_\alpha(y_\alpha)^{-(1+\epsilon_\alpha)} l_\alpha^{-n} \frac{\int_{B_{y_\alpha}(l_\alpha)} u_\alpha^{1+\epsilon_\alpha} \, \mathrm{d}v_g}{A_\alpha^{(n/4)(1+\epsilon_\alpha)}}.$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

If we prove that $\lim \beta_\alpha = 0$, we get a contradiction with (2.18), which ends the proof of the step. First, let

$$m_\alpha = \frac{u_\alpha(y_\alpha)}{\|u_\alpha\|_\infty}.$$

Clearly, by (2.9),

$$\beta_\alpha \leqslant C m_\alpha^{-(n/2+1)} \frac{\int_{B_{y_\alpha}(u_\alpha(l_\alpha))} u_\alpha^{1+\epsilon_\alpha} \, dv_g}{\int_M u_\alpha^{1+\epsilon_\alpha} \, dv_g}.$$

By the previous step and (2.14),

$$\lim \frac{\int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-2/n})} u_\alpha^{1+\epsilon_\alpha} \, dv_g}{\int_M u_\alpha^{1+\epsilon_\alpha} \, dv_g} = 0. \tag{2.19}$$

If $m_\alpha \geqslant C > 0$, we have $\beta_\alpha \to 0$. Hence we assume that $\lim m_\alpha = 0$. We now proceed by induction to prove that

$$m_\alpha^{-((n+3)/(n+2))^k} \int_{B_{y_\alpha}(2^{-k} u_\alpha(y_\alpha)^{-2/n})} u_\alpha^2 \, dv_g \to 0. \tag{H_k}$$

First, we prove that (H_0) is true. We proved before that

$$\|u_\alpha\|_{L^\infty(B_{y_\alpha}(u_\alpha(y_\alpha)^{-2/n}))} \leqslant C \cdot u_\alpha(y_\alpha).$$

Hence we have, noting that $u_\alpha(y_\alpha) \to \infty$,

$$\begin{aligned}\int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-2/n})} u_\alpha^2 \, dv_g &\leqslant C u_\alpha(y_\alpha) \int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-2/n})} u_\alpha^{1+\epsilon_\alpha} \, dv_g \\ &\leqslant C m_\alpha \|u_\alpha\|_\infty \int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-2/n})} u_\alpha^{1+\epsilon_\alpha} \, dv_g.\end{aligned}$$

By (2.9) and (2.19),

$$\lim \|u_\alpha\|_\infty \int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-2/n})} u_\alpha^{1+\epsilon_\alpha} \, dv_g = 0;$$

(H_0) then follows. Now let $\epsilon_k = ((n+3)/(n+2))^k$ and suppose that (H_k) is true. Let us prove that (H_{k+1}) is true. Let $\eta_{\alpha,k}(x) = \eta(u_\alpha(y_\alpha)^{2/n} 2^k d_g(x, y_\alpha))$, where η is defined as in step 2. Multiplying (E_α) by

$$\frac{u_\alpha(\eta_{\alpha,k})^2}{m_\alpha^{\epsilon_k}}$$

and integrating over M, we obtain

$$\begin{aligned}2A_\alpha m_\alpha^{-\epsilon_k} &\int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 \, dv_g \\ &- 2A_\alpha m_\alpha^{-\epsilon_k} \int_M |\nabla \eta_{\alpha,k}|_g^2 u_\alpha^2 \, dv_g + \frac{4}{n} B_\alpha m_\alpha^{-\epsilon_k} \int_M \eta_{\alpha,k}^2 u_\alpha^{1+\epsilon_\alpha} \, dv_g \\ &= k_\alpha m_\alpha^{-\epsilon_k} \int_M (\eta_{\alpha,k} u_\alpha)^2 \, dv_g.\end{aligned} \tag{2.20}$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

By (H_k),

$$2A_\alpha m_\alpha^{-\epsilon_k}\int_M |\nabla\eta_{\alpha,k}|_g^2 u_\alpha^2\,\mathrm{d}v_g \leqslant CA_\alpha u_\alpha(y_\alpha)^{4/n} m_\alpha^{-\epsilon_k}\int_{B_{y_\alpha}(2^{-k}u_\alpha(y_\alpha)^{-2/n})} u_\alpha^2\,\mathrm{d}v_g \leqslant CA_\alpha u_\alpha(y_\alpha)^{4/n}.$$

Moreover, by (2.9), $A_\alpha u_\alpha(y_\alpha)^{4/n} = A_\alpha m_\alpha^{4/n}\|u_\alpha\|_\infty^{4/n} \leqslant C\cdot m_\alpha^{4/n} \to 0$. We also have, by (H_k) and (2.3),

$$k_\alpha m_\alpha^{-\epsilon_k}\int_M (\eta_{\alpha,k}u_\alpha)^2\,\mathrm{d}v_g \to 0.$$

Therefore, equation (2.20) gives

$$\left.\begin{aligned} 2A_\alpha\int_M |\nabla\eta_{\alpha,k}u_\alpha|_g^2\,\mathrm{d}v_g &\leqslant C\cdot m_\alpha^{\epsilon_k},\\ \frac{4}{n}B_\alpha\int_M \eta_{\alpha,k}^2 u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g &\leqslant C\cdot m_\alpha^{\epsilon_k}.\end{aligned}\right\}\tag{2.21}$$

Replacing $\eta_{\alpha,k}$ by $\sqrt{\eta_{\alpha,k}}$ and doing the same, we see that

$$\frac{4}{n}B_\alpha\int_M \eta_{\alpha,k}^{1+\epsilon_\alpha}u_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g \leqslant C\cdot m_\alpha^{\epsilon_k}.\tag{2.22}$$

Moreover, using $N(A,B)(\eta_{\alpha,k}u_\alpha)$, one easily checks that

$$\begin{aligned}\left(\int_M(\eta_{\alpha,k}u_\alpha)^2\,\mathrm{d}v_g\right)^{(n+2)/n} &\leqslant A\cdot\int_M|\nabla\eta_{\alpha,k}u_\alpha|_g^2\,\mathrm{d}v_g\left(\int_M(\eta_{\alpha,k}u_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))}\\ &\quad + B\cdot\int_M(\eta_{\alpha,k}u_\alpha)^2\,\mathrm{d}v_g\left(\int_M(\eta_{\alpha,k}u_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))}.\end{aligned}$$

Clearly, we have, in fact,

$$\begin{aligned}\left(\int_M(\eta_{\alpha,k}u_\alpha)^2\,\mathrm{d}v_g\right)^{(n+2)/n} &\leqslant C\cdot\int_M|\nabla\eta_{\alpha,k}u_\alpha|_g^2\,\mathrm{d}v_g\left(\int_M(\eta_{\alpha,k}u_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))}\\ &\leqslant \frac{C}{A_\alpha B_\alpha^{4/(n(1+\epsilon_\alpha))}}\left(\int_M|\nabla\eta_{\alpha,k}u_\alpha|_g^2\,\mathrm{d}v_g A_\alpha\right)\\ &\qquad\times\left(B_\alpha\int_M(\eta_{\alpha,k}u_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))}.\end{aligned}$$

Using (2.21) and (2.22), we get

$$\left(\int_M(\eta_{\alpha,k}u_\alpha)^2\,\mathrm{d}v_g\right)^{(n+2)/n} \leqslant \frac{C}{A_\alpha B_\alpha^{4/(n(1+\epsilon_\alpha))}}\cdot m_\alpha^{(1+4/(n(1+\epsilon_\alpha)))\epsilon_k}.$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

By (2.2), $A_\alpha B_\alpha^{4/(n(1+\epsilon_\alpha))} \geqslant C > 0$. Since

$$\int_{B_{y_\alpha}(2^{-(k+1)}u_\alpha(y_\alpha)^{-2/n})} u_\alpha^2 \,\mathrm{d}v_g \leqslant \int_M (\eta_{\alpha,k}u_\alpha)^2 \,\mathrm{d}v_g,$$

(H_{k+1}) then follows. As a consequence, (H_k) is true for all k. Coming back to (2.22), we get that, for all k,

$$\lim m_\alpha^{-\epsilon_k} B_\alpha \int_{B_{y_\alpha}(2^{-k}u_\alpha(y_\alpha)^{-2/n})} u_\alpha^{1+\epsilon_\alpha} \,\mathrm{d}v_g = 0.$$

Using the fact that

$$\lim \frac{l_\alpha}{u_\alpha(y_\alpha)^{-2/n}} = 0$$

and choosing k such that $\epsilon_k \geqslant \frac{1}{2}n + 1$, we get $\lim \beta_\alpha = 0$, which ends the step.

Step 4 (Various estimates). This step proceeds in seven parts. Let c be a strictly positive number.

Part A. Let us prove that, $\forall k > 0$,

$$A_\alpha^{-k} \int_{M - B_{x_\alpha}(c)} u_\alpha^2 \,\mathrm{d}v_g \to 0. \tag{2.23}$$

Let $r_\alpha(x) = d_g(x, x_\alpha)$ and let $\delta \in \,]0, \frac{1}{4}n[$. Using step 3, we have

$$\begin{aligned} A_\alpha^{-\delta} \int_{M - B_{x_\alpha}(c)} u_\alpha^2 \,\mathrm{d}v_g &\leqslant C \cdot A_\alpha^{-\delta} \int_{M - B_{x_\alpha}(c)} u_\alpha^{1+\epsilon_\alpha} r_\alpha^{-(n/2)(1-\epsilon_\alpha)} \,\mathrm{d}v_g \\ &\leqslant C \cdot A_\alpha^{-\delta} \int_{M - B_{x_\alpha}(c)} u_\alpha^{1+\epsilon_\alpha} \,\mathrm{d}v_g. \end{aligned}$$

Recall the definition of A_α to get

$$A_\alpha^{-\delta} \int_{M - B_{x_\alpha}(c)} u_\alpha^2 \,\mathrm{d}v_g \to 0.$$

Mimicking what we did in the previous step, we prove by induction that

$$A_\alpha^{-((n+3)/(n+2))^k \delta} \int_{M - B_{x_\alpha}(2^k c)} u_\alpha^2 \,\mathrm{d}v_g \to 0. \tag{$\tilde{H}_k$}$$

Remark 2.4. As in the previous step, we have, for all $k > 0$,

$$\left. \begin{aligned} \lim A_\alpha^{-k} \int_{M - B_{x_\alpha}(c)} |\nabla u_\alpha|_g^2 \,\mathrm{d}v_g = 0, \\ \lim A_\alpha^{-k} \int_{M - B_{x_\alpha}(c)} u_\alpha^{1+\epsilon_\alpha} \,\mathrm{d}v_g = 0. \end{aligned} \right\} \tag{2.24}$$

Part B. Let us prove that

$$\exists\, t_0 > 0 \text{ such that } \Delta_g u_\alpha < 0 \text{ on } M - B_{x_\alpha}(t_0 {A_\alpha}^{1/2}). \tag{2.25}$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

Let $x \in M$ be such that $\Delta_g u_\alpha(x) \geqslant 0$. By equation (2.5), we have $u_\alpha(x) \geqslant C \cdot B_\alpha$. We also have, by (2.2), $B_\alpha \geqslant C \cdot A_\alpha^{-n/2}$, and, by step 3, $u_\alpha(x) \leqslant C \cdot r_\alpha(x)^{-n/2}$. We then have $r_\alpha(x)^{-n/2} \geqslant C \cdot A_\alpha^{-n/4}$.

Equation (2.25) then follows. We now let $\eta_\alpha = \eta((1/c)r_\alpha)$.

PART C. Let us prove that

$$\int_M (r_\alpha \eta_\alpha)^2 |\nabla u_\alpha|_g^2 \, dv_g \leqslant C. \tag{2.26}$$

Let

$$\gamma_\alpha = \int_M (r_\alpha \eta_\alpha)^2 |\nabla u_\alpha|_g^2 \, dv_g.$$

Integrating by parts, we compute

$$\gamma_\alpha = \int_M (\Delta_g u_\alpha) u_\alpha (r_\alpha \eta_\alpha)^2 \, dv_g - 2 \int_M u_\alpha r_\alpha \eta_\alpha \langle \nabla r_\alpha \eta_\alpha, \nabla u_\alpha \rangle_g \, dv_g,$$

and, by (2.25),

$$\gamma_\alpha \leqslant \int_{B_{x_\alpha}(t_0 A_\alpha^{1/2})} (\Delta_g u_\alpha) u_\alpha (r_\alpha \eta_\alpha)^2 \, dv_g + C \int_M u_\alpha r_\alpha \eta_\alpha |\nabla r_\alpha \eta_\alpha|_g |\nabla u_\alpha|_g \, dv_g.$$

Using (E_α), equation (2.5) and the following inequality (which comes from (2.9)),

$$u_\alpha(x) \leqslant \|u_\alpha\|_\infty \leqslant C \cdot A_\alpha^{-n/4},$$

one can prove that

$$\int_{B_{x_\alpha}(t_0 A_\alpha^{1/2})} (\Delta_g u_\alpha) u_\alpha (r_\alpha \eta_\alpha)^2 \, dv_g \leqslant C.$$

Moreover, by Hölder's inequality,

$$\int_M u_\alpha r_\alpha \eta_\alpha |\nabla r_\alpha \eta_\alpha|_g |\nabla u_\alpha|_g \, dv_g \\ \leqslant \left(\int_M u_\alpha^2 |\nabla r_\alpha \eta_\alpha|_g^2 \, dv_g \right)^{1/2} \left(\int_M |\nabla u_\alpha|_g^2 (r_\alpha \eta_\alpha)^2 \, dv_g \right)^{1/2}.$$

Finally,

$$\gamma_\alpha \leqslant C + C \cdot \gamma_\alpha^{1/2},$$

and (2.26) follows.

PART D. Let us prove that

$$\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g \leqslant C \sqrt{\alpha} A_\alpha. \tag{2.27}$$

Assume that, on the contrary,

$$\frac{\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g}{\sqrt{\alpha} A_\alpha} \to +\infty. \tag{H'}$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

Multiply (E_α) by

$$\frac{u_\alpha (r_\alpha \eta_\alpha)^2}{\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g},$$

and integrate over M. Then

$$\frac{2A_\alpha \int_M (\Delta_g u_\alpha) u_\alpha (r_\alpha \eta_\alpha)^2 \, dv_g}{\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g} + \frac{4B_\alpha \int_M u_\alpha^{1+\epsilon_\alpha} (r_\alpha \eta_\alpha)^2 \, dv_g}{n \int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g} = k_\alpha. \tag{2.28}$$

Integrating by parts, we clearly have (using (2.26))

$$\begin{aligned}\left| \int_M (\Delta_g u_\alpha) u_\alpha (r_\alpha \eta_\alpha)^2 \, dv_g \right| &\leqslant C \left| \int_M (r_\alpha \eta_\alpha)^2 |\nabla u_\alpha|_g^2 \, dv_g + \int_M u_\alpha^2 |\nabla r_\alpha \eta_\alpha|_g^2 \, dv_g \right| \\ &\leqslant C.\end{aligned}$$

Thus (H') implies

$$\frac{A_\alpha \int_M (\Delta_g u_\alpha) u_\alpha (r_\alpha \eta_\alpha)^2 \, dv_g}{\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g} \to 0. \tag{2.29}$$

From (2.3), (2.28) and (2.29), one easily gets that

$$\frac{B_\alpha \int_M u_\alpha^{1+\epsilon_\alpha} (r_\alpha \eta_\alpha)^2 \, dv_g}{\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g} \leqslant C.$$

We have already seen that $B_\alpha \geqslant C \cdot A_\alpha^{-n/4}$, and then

$$\frac{\int_M u_\alpha^{1+\epsilon_\alpha} (r_\alpha \eta_\alpha)^2 \, dv_g}{A_\alpha^{n/4} \int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g} \leqslant C. \tag{2.30}$$

Note that

$$\begin{aligned}\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} r_\alpha^2 \, dv_g - \int_M u_\alpha^{1+\epsilon_\alpha} (r_\alpha \eta_\alpha)^2 \, dv_g &\leqslant C \int_{M - B_{x_\alpha}(c)} u_\alpha^{1+\epsilon_\alpha} \, dv_g \\ &\leqslant C \cdot A_\alpha^k\end{aligned}$$

for all $k > 0$, by (2.24). Equations (H') and (2.30) then imply

$$\frac{\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} r_\alpha^2 \, dv_g}{A_\alpha^{n/4} \int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g} \leqslant C. \tag{2.31}$$

Let us show that

$$\int_M (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha} \, dv_g \leqslant C \frac{\int_M u_\alpha {r_\alpha}^2 (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} \, dv_g}{\alpha^{1/4} A_\alpha^{1/2}}. \tag{2.32}$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

We have

$$\begin{aligned}\int_M (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g &= \int_{B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g \\ &\quad + \int_{M-B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g \\ &\leqslant \int_{B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g \\ &\quad + \frac{C}{r_\alpha^{1-\epsilon_\alpha}} \int_{M-B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} r_\alpha^2\,\mathrm{d}v_g \\ &\leqslant \int_{B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g \\ &\quad + \frac{C}{{A_\alpha}^{1/2}\alpha^{1/4}} \int_{M-B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} r_\alpha^2\,\mathrm{d}v_g \qquad (2.33)\end{aligned}$$

(we have used the fact that $(A_\alpha^{1/2}\alpha^{1/4})^{1-\epsilon_\alpha} \leqslant A_\alpha^{1/2}\alpha^{1/4}$, which comes from (2.4)). Clearly,

$$\int_{B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g \leqslant C \cdot A_\alpha^{n/4+1/2}\alpha^{1/4}. \qquad (2.34)$$

Now assume that

$$A_\alpha^{n/4+1/2}\alpha^{1/4} \geqslant \frac{t_\alpha}{A_\alpha^{1/2}\alpha^{1/4}} \int_{M-B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} r_\alpha^2\,\mathrm{d}v_g, \qquad (H'')$$

where $t_\alpha \to +\infty$. We would get from step 3 that

$$\begin{aligned}\int_{M-B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha \eta_\alpha r_\alpha)^2\,\mathrm{d}v_g &\leqslant \int_{M-B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha r_\alpha)^2 \eta_\alpha^{1+\epsilon_\alpha}\,\mathrm{d}v_g \\ &\leqslant C \int_{M-B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} r_\alpha^{2-n/2} (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g\end{aligned}$$

and, by (H'') and $r_\alpha^{-n/2} \leqslant A_\alpha^{-n/4}\alpha^{-n/8}$,

$$\int_{M-B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha \eta_\alpha r_\alpha)^2\,\mathrm{d}v_g \leqslant C\sqrt{\alpha}A_\alpha.$$

In addition, we clearly have

$$\int_{B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha \eta_\alpha r_\alpha)^2\,\mathrm{d}v_g \leqslant C\sqrt{\alpha}A_\alpha,$$

and thus

$$\int_M (u_\alpha \eta_\alpha r_\alpha)^2\,\mathrm{d}v_g \leqslant C\sqrt{\alpha}A_\alpha.$$

This assertion contradicts (H') and thus (H'') is false. Coming back to (2.33) and (2.34), this gives

$$\int_M (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g \leqslant \frac{C}{A_\alpha^{1/2}\alpha^{1/4}} \int_{M-B_{x_\alpha}(A_\alpha^{1/2}\alpha^{1/4})} (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} r_\alpha^2\,\mathrm{d}v_g,$$

which proves (2.32). Now write $N(A,B)(u_\alpha r_\alpha \eta_\alpha)$,

$$1 \leqslant A \cdot \frac{\int_M |\nabla u_\alpha r_\alpha \eta_\alpha|_g^2 \, dv_g (\int_M (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha} \, dv_g)^{4/(n(1+\epsilon_\alpha))}}{(\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g)^{1+2/n}} + B \cdot \frac{(\int_M (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha} \, dv_g)^{4/(n(1+\epsilon_\alpha))}}{(\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g)^{2/n}}. \tag{2.35}$$

Let us prove that

$$\lim \frac{(\int_M (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha} \, dv_g)^{4/(n(1+\epsilon_\alpha))}}{(\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g)^{2/n}} = 0. \tag{2.36}$$

Indeed,

$$\frac{(\int_M (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha} \, dv_g)^{4/(n(1+\epsilon_\alpha))}}{(\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g)^{n/2}} = \left(\frac{B_\alpha \int_M (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha} \, dv_g}{\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g} \right)^{4/(n(1+\epsilon_\alpha))} \frac{(\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g)^{4/(n(1+\epsilon_\alpha))-2/n}}{B_\alpha^{4/(n(1+\epsilon_\alpha))}}.$$

From (2.32) and the fact that $B_\alpha A_\alpha^{-n/4} \to C$ (by (2.2), (2.9) and (2.17)), we get

$$\frac{(\int_M (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha} \, dv_g)^{4/(n(1+\epsilon_\alpha))}}{(\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g)^{2/n}} \leqslant C \left(\frac{\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} r_\alpha^2 \, dv_g}{A_\alpha^{n/4} \int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g} \right)^{4/(n(1+\epsilon_\alpha))} \times \frac{A_\alpha^{1/(1+\epsilon_\alpha)(1-2/n)}}{\alpha^{1/(n(1+\epsilon_\alpha))}} \left(\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g \right)^{2/n}.$$

We clearly have

$$\frac{A_\alpha^{1/(1+\epsilon_\alpha)(1-2/n)}}{\alpha^{1/(n(1+\epsilon_\alpha))}} \to 0 \quad \text{and} \quad \left(\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g \right)^{2/n} \leqslant C.$$

Using (2.31), equation (2.36) then follows. Let us prove that

$$\frac{\int_M |\nabla u_\alpha r_\alpha \eta_\alpha|_g^2 \, dv_g (\int_M (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha} \, dv_g)^{4/(n(1+\epsilon_\alpha))}}{(\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g)^{1+2/n}} \to 0. \tag{2.37}$$

We have, by (2.32),

$$\frac{\int_M |\nabla u_\alpha r_\alpha \eta_\alpha|_g^2 \, dv_g (\int_M (u_\alpha r_\alpha \eta_\alpha)^{1+\epsilon_\alpha} \, dv_g)^{4/(n(1+\epsilon_\alpha))}}{(\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g)^{1+2/n}} \leqslant \frac{C}{\alpha^{1/(n(1+\epsilon_\alpha))}} \left(\frac{A_\alpha \sqrt{\alpha}}{\int_M (u_\alpha r_\alpha \eta_\alpha)^2 \, dv_g} \right)^{1-2/n} \times \int_M |\nabla u_\alpha r_\alpha \eta_\alpha|^2 \, dv_g \left(\frac{\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} r_\alpha^2 \, dv_g}{\int_M (u_\alpha r_\alpha \eta_\alpha)_g^2 \, dv_g A_\alpha^{n/4}} \right)^{4/(n(1+\epsilon_\alpha))}. \tag{2.38}$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

We remark that

$$\int_M |\nabla u_\alpha r_\alpha \eta_\alpha|_g^2 \, dv_g \leqslant \int_M |\nabla u_\alpha|_g^2 (r_\alpha \eta_\alpha)^2 \, dv_g + \int_M \langle \nabla u_\alpha, \nabla r_\alpha \eta_\alpha \rangle_g^2 u_\alpha r_\alpha \eta_\alpha \, dv_g + \int_M u_\alpha^2 |\nabla \eta_\alpha r_\alpha|_g^2 \, dv_g$$

and, by Hölder's inequality,

$$\int_M \langle \nabla u_\alpha, \nabla r_\alpha \eta_\alpha \rangle_g^2 u_\alpha r_\alpha \eta_\alpha \, dv_g \leqslant C \biggl(\int_M |\nabla u_\alpha|_g^2 (r_\alpha \eta_\alpha)^2 \, dv_g \biggr)^{1/2}.$$

By (2.26), it follows that

$$\int_M |\nabla u_\alpha r_\alpha \eta_\alpha|_g^2 \, dv_g \leqslant C. \tag{2.39}$$

Equations (2.31), (H'), (2.38) and (2.39) lead to (2.37). Equations (2.35), (2.36) and (2.37) prove that (H') is false.

PART E. Let us prove that

$$\frac{1 - (\int_M (u_\alpha \eta_\alpha)^2 \, dv_\xi)^{1+2/n}}{A_\alpha \sqrt{\alpha}} \leqslant C. \tag{2.40}$$

We first recall some results about the expansion of the metric. Let ξ denote the Euclidean metric. Since $(x_\alpha)_\alpha$ is convergent up to a subsequence, we may write, from the Cartan expansion of g in geodesic normal coordinates at $x_0 = \lim x_\alpha$, that, for α large,

$$|\nabla u_\alpha \eta_\alpha|_\xi^2(x) \leqslant |\nabla u_\alpha \eta_\alpha|_g^2(x)(1 + C \cdot r_\alpha^2)$$

and

$$(1 - C \cdot r_\alpha^2) \, dv_\xi \leqslant dv_g \leqslant (1 + C \cdot r_\alpha^2) \, dv_\xi. \tag{2.41}$$

Hence

$$\int_M |\nabla u_\alpha \eta_\alpha|_\xi^2 \, dv_\xi \leqslant \int_M |\nabla u_\alpha \eta_\alpha|_g^2 (1 + C \cdot r_\alpha^2) \, dv_g. \tag{2.42}$$

We now prove (2.40). We have, by (2.41),

$$\begin{aligned} 1 - \biggl(\int_M (u_\alpha \eta_\alpha)^2 \, dv_\xi \biggr)^{1+2/n} &\leqslant C \biggl(1 - \int_M (u_\alpha \eta_\alpha)^2 \, dv_\xi \biggr) \\ &\leqslant C \biggl(\int_M u_\alpha^2 \, dv_g - \int_M (u_\alpha \eta_\alpha)^2 \, dv_g + C \int_M (u_\alpha \eta_\alpha r_\alpha)^2 \, dv_g \biggr) \\ &\leqslant C \biggl(\int_M \Bigl(u_\alpha (1 - \eta_\alpha) \Bigr)^2 \, dv_g + C \int_M (u_\alpha \eta_\alpha r_\alpha)^2 \, dv_g \biggr). \end{aligned}$$

Equations (2.23) and (2.27) give (2.40).

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

PART F. Let us prove that

$$\left(\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_\xi\right)^{4/(n(1+\epsilon_\alpha))} \leqslant \left(\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))} + C\cdot A_\alpha^2\sqrt{\alpha}. \tag{2.43}$$

Multiply (E_α) by

$$\frac{u_\alpha(r_\alpha\eta_\alpha)^2}{A_\alpha\sqrt{\alpha}}$$

and integrate over M,

$$\frac{2}{\sqrt{\alpha}}\int_M (\Delta_g u_\alpha)u_\alpha(r_\alpha\eta_\alpha)^2\,\mathrm{d}v_g + \frac{4B_\alpha}{nA_\alpha\sqrt{\alpha}}\int_M u_\alpha^{1+\epsilon_\alpha}(r_\alpha\eta_\alpha)^2\,\mathrm{d}v_g = \frac{k_\alpha}{A_\alpha\sqrt{\alpha}}\int_M (u_\alpha r_\alpha\eta_\alpha)^2\,\mathrm{d}v_g. \tag{2.44}$$

As we did to get (2.39), we have, from (2.26),

$$\int_M (\Delta_g u_\alpha)u_\alpha(r_\alpha\eta_\alpha)^2\,\mathrm{d}v_g \leqslant C$$

and then, from (2.27), (2.44) and the fact that $B_\alpha \geqslant C\cdot A_\alpha^{-n/4}$,

$$\int_M u_\alpha^{1+\epsilon_\alpha}(r_\alpha\eta_\alpha)^2\,\mathrm{d}v_g \leqslant C\frac{A_\alpha\sqrt{\alpha}}{B_\alpha} \leqslant C\cdot\sqrt{\alpha}A_\alpha^{1+n/4}.$$

This implies

$$\int_M (\eta_\alpha u_\alpha)^{1+\epsilon_\alpha}{r_\alpha}^2\,\mathrm{d}v_g \leqslant C\cdot\sqrt{\alpha}A_\alpha^{1+n/4}. \tag{2.45}$$

Now

$$\begin{aligned}\left(\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_\xi\right)^{4/(n(1+\epsilon_\alpha))} &\leqslant \left(\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g + C\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}r_\alpha^2\,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))}\\ &\leqslant \left(\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))}\left(1+\frac{\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}r_\alpha^2\,\mathrm{d}v_g}{\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g}\right)^{4/(n(1+\epsilon_\alpha))}.\end{aligned}$$

Clearly,

$$\frac{\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}r_\alpha^2\,\mathrm{d}v_g}{\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}dv_g} \to 0.$$

Hence, developing, we check

$$\begin{aligned}\left(\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_\xi\right)^{4/(n(1+\epsilon_\alpha))} &\leqslant \left(\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))}\\ &\quad + C\cdot\left(\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))-1}\int_M (u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}r_\alpha^2\,\mathrm{d}v_g.\end{aligned}$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

From (2.45) and the fact that

$$\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g \leqslant C \cdot A_\alpha^{(n/4)(1+\epsilon_\alpha)},$$

we get (2.43) (we have used $A_\alpha^{\epsilon_\alpha} \to C > 0$).

Part G (Conclusion). By [3], we get

$$\left(\int_M (u_\alpha \eta_\alpha)^2\,\mathrm{d}v_\xi\right)^{1+2/n} \leqslant A_0(n) \int_M |\nabla u_\alpha \eta_\alpha|_\xi^2\,\mathrm{d}v_\xi \left(\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_\xi\right)^{4/(n(1+\epsilon_\alpha))}.$$

Clearly, by (2.26) and (2.42),

$$\int_M |\nabla u_\alpha \eta_\alpha|_\xi^2\,\mathrm{d}v_\xi \leqslant \int_M |\nabla u_\alpha|_g^2 \eta_\alpha^2\,\mathrm{d}v_g + C.$$

From (2.1), (2.43),

$$\left(\int_M (u_\alpha \eta_\alpha)^2\,\mathrm{d}v_\xi\right)^{1+2/n} \leqslant A_0(n) \int_M |\nabla u_\alpha|_g^2 \eta_\alpha^2\,\mathrm{d}v_g \left(\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))} + C \cdot \sqrt{\alpha} A_\alpha. \tag{2.46}$$

From the very first definition of u_α, we get

$$1 = \left(\frac{1}{\mu_\alpha} \int_M |\nabla u_\alpha|_g^2\,\mathrm{d}v_g + \frac{\alpha}{\mu_\alpha}\right) A_\alpha. \tag{2.47}$$

We now compute (cf. (2.46), (2.47)) $(A_\alpha \sqrt{\alpha})^{-1}$,

$$\frac{1 - (\int_M (u_\alpha \eta_\alpha)^2\,\mathrm{d}v_\xi)^{1+2/n}}{A_\alpha \sqrt{\alpha}} \geqslant -\frac{A_0(n)}{\sqrt{\alpha}} \int_M |\nabla u_\alpha|_g^2 {\eta_\alpha}^2\,\mathrm{d}v_g \frac{(\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g)^{4/(n(1+\epsilon_\alpha))}}{A_\alpha} + \frac{1}{\mu_\alpha \sqrt{\alpha}} \int_M |\nabla u_\alpha|_g^2\,\mathrm{d}v_g + \frac{\sqrt{\alpha}}{\mu_\alpha} - C.$$

Note that

$$\frac{1}{\mu_\alpha} \geqslant A_0(n), \qquad \frac{(\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha}\,\mathrm{d}v_g)^{4/(n(1+\epsilon_\alpha))}}{A_\alpha} \leqslant 1.$$

It follows that

$$\frac{1 - (\int_M (u_\alpha \eta_\alpha)^2\,\mathrm{d}v_\xi)^{1+2/n}}{A_\alpha \sqrt{\alpha}} \geqslant \frac{A_0(n)}{\sqrt{\alpha}} \int_M |\nabla u_\alpha|_g^2 (1 - \eta_\alpha^2)\,\mathrm{d}v_g + A_0(n)\sqrt{\alpha} - C.$$

The first member of this inequality is bounded from below (by (2.40)), while the second member goes to $+\infty$. This contradiction ends the proof of the theorem.

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

The case of manifolds with boundary and complete manifolds

Theorem 1.1 is also true on compact Riemannian manifolds with boundary. The proof may easily be modified when the sequence x_α goes to the boundary. Moreover, note that the result is always true when g is not a fixed metric. More precisely, we consider $(g_\alpha)_\alpha$, a bounded family of Riemannian metrics on M (in $C^2(M)$); this bound allows us to control each constant that appears in the proof. We are then able to get the following result for the complete case.

THEOREM 2.5. *Let (M, g) be a smooth complete Riemannian n-manifold, $n \geqslant 1$. We let $|Rm_g|$ and $|\nabla Rm_g|_g$ be the norms of the Riemannian curvature of g and of its gradient. We note r_g, the injectivity radius of g. Let C, C', $\delta > 0$ and assume that $|Rm_g| \leqslant C$, $|\nabla Rm_g|_g \leqslant C'$ and $r_g \geqslant \delta$. Then there exists $B > 0$ such that*

$$\left(\int_M u^2 \, \mathrm{d}v_g\right)^{1+2/n} \leqslant \left(A_0(n) \int_M |\nabla u|_g^2 \, \mathrm{d}v_g + B \int_M u^2 \, \mathrm{d}v_g\right)\left(\int_M |u| \, \mathrm{d}v_g\right)^{4/n}.$$

Moreover, B depends only on n, C, C' and δ.

To prove this theorem, we use a covering of M with balls and a smooth partition of unity. The results in the case of manifolds with boundary applied on these balls gives the theorem. A complete study of this problem is done in [6] for the sharp Sobolev inequality.

3. Proof of theorem 1.2

As in theorem 1.1, we may assume that $\mathrm{Vol}(M) = 1$. We write $N(A_0(n), B_0)(1)$, to get $B_0 \geqslant 1$. Now let $x \in M$. As in theorem 1.4 of [4], we let u ($u \not\equiv 0$ and radially symmetric) be an eigenfunction associated to λ_1. We also set

$$u_\epsilon = \begin{cases} u(r/\epsilon) - u(1) & \text{if } r < \epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

where $r = d_g(x, \cdot)$, and d_g stands for the distance with respect to g. Let

$$I_{B_0}(v) = \frac{A_0(n)(\int_M |\nabla v|_g^2 \, \mathrm{d}v_g + B_0 \int_M v^2 \, \mathrm{d}v_g)(\int_M |v| \, \mathrm{d}v_g)^{4/n}}{(\int_M v^2 \, \mathrm{d}v_g)^{2/n}}.$$

In [4], it is shown that

$$\frac{A_0(n) \int_M |\nabla u_\epsilon|_g^2 \, \mathrm{d}v_g (\int_M |u_\epsilon| \, \mathrm{d}v_g)^{4/n}}{(\int_M u_\epsilon^2 \, \mathrm{d}v_g)^{1+2/n}} = 1 + \frac{1}{6n} X S_g(x) \epsilon^2 + o(\epsilon^2),$$

where

$$X = -\frac{2}{n+2} - \frac{n-2}{\lambda_1}.$$

From the explicit computations made in [4], we get easily that

$$B_0 \frac{(\int_M |u_\epsilon| \, \mathrm{d}v_g)^{4/n}}{(\int_M u_\epsilon^2 \, \mathrm{d}v_g)^{2/n}} = B_0 Y \epsilon^2 + o(\epsilon^2),$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

where

$$Y = \left(\int_{\mathcal{B}} |u(x) - u(1)| \,\mathrm{d}x\right)^{4/n} \left(\int_{\mathcal{B}} (u(x) - u(1))^2 \,\mathrm{d}x\right)^{-2/n}.$$

We then have

$$I_{B_0}(u_\epsilon) = 1 + \epsilon^2\left(\frac{1}{6n} X S_g(x) + B_0 Y\right) + o(\epsilon^2).$$

Hence, since $I_{B_0}(u) \geqslant 1$ for every $u \in H_1^2(M)$, we get that

$$B_0 \geqslant -\frac{X}{6nY} \max_{x\in M} S_g(x).$$

We now compute Y. As in [4], we have

$$Y = (-|\mathcal{B}|u(1))^{4/n}(\tfrac{1}{2}(n+2)u(1)^2|\mathcal{B}|)^{-2/n} = |\mathcal{B}|^{2/n}(\tfrac{1}{2}(n+2))^{-2/n}.$$

This gives the theorem.

4. Proof of theorem 1.3

First, let

$$\tilde{B}_0 = \inf\{B > 0 \mid \tilde{N}(A_0(n), B) \text{ is valid}\}.$$

Also let $\alpha_0 = B_0 A_0(n)^{-1}$ and $\tilde{\alpha}_0 = \tilde{B}_0 A_0(n)^{-1}$. In the following, the limits are taken as $\alpha \to 0$. Proceeding as in the proof theorem 1.1, one can find $(\epsilon_\alpha)_\alpha$ that goes to 0 such that, if we define,

$$J_\alpha(u) = \left(\int_M |\nabla u|_g^2 \,\mathrm{d}v_g + (\alpha_0 - \alpha)\right)\left(\int_M |u|^{1+\epsilon_\alpha} \,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))},$$

$$\tilde{J}_\alpha(u) = \left(\int_M |\nabla u|_g^2 \,\mathrm{d}v_g + (\tilde{\alpha}_0 - \alpha)\left(\int_M |u|^{1+\epsilon_\alpha} \,\mathrm{d}v_g\right)^{2/(1+\epsilon_\alpha)}\right) \times \left(\int_M |u|^{1+\epsilon_\alpha} \,\mathrm{d}v_g\right)^{4/(n(1+\epsilon_\alpha))},$$

then

$$\mu_\alpha < A_0(N)^{-1} \quad \text{and} \quad \mu_\alpha \to A_0(N)^{-1}, \tag{4.1}$$

$$\tilde{\mu}_\alpha < A_0(N)^{-1} \quad \text{and} \quad \tilde{\mu}_\alpha \to A_0(N)^{-1}, \tag{4.2}$$

where $\mu_\alpha = \inf_{u\in\varLambda} J_\alpha(u)$ and $\tilde{\mu}_\alpha = \inf_{u\in\varLambda} \tilde{J}_\alpha(u)$ ($\varLambda$ is defined as in the proof of theorem 1.1). Moreover, there exists u_α and $\tilde{u}_\alpha$, two non-negative functions in $C^2(M)$, such that

$$\int_M u_\alpha^2 \,\mathrm{d}v_g = 1 \quad \text{and} \quad \mu_\alpha = J_\alpha(u_\alpha),$$

$$\int_M \tilde{u}_\alpha^2 \,\mathrm{d}v_g = 1 \quad \text{and} \quad \tilde{\mu}_\alpha = \tilde{J}_\alpha(\tilde{u}_\alpha).$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

To prove the theorem, it is enough to show that

$$\liminf \int_M |u_\alpha|^{1+\epsilon_\alpha}\,\mathrm{d}v_g > 0.$$

Indeed, this implies that

$$\int_M |\nabla u_\alpha|_g^2\,\mathrm{d}v_g \leqslant C$$

and, by classical methods, the theorem follows. Suppose that, on the contrary,

$$\int_M |u_\alpha|^{1+\epsilon_\alpha}\,\mathrm{d}v_g \to 0.$$

We can then find $a > 0$ small enough such that

$$(\alpha_0 - a) - (\tilde{\alpha}_0 - \alpha)\biggl(\int_M |u_a|^{1+\epsilon_a}\,\mathrm{d}v_g\biggr)^{2/(1+\epsilon_a)} > 0. \tag{4.3}$$

Now let $\alpha \in\,]0, a[$. We have, using Hölder's inequality,

$$\begin{aligned} J_a(u_a) - \tilde{J}_\alpha(u_a) &= (\alpha_0 - a)\biggl(\int_M |u_a|^{1+\epsilon_a}\,\mathrm{d}v_g\biggr)^{4/(n(1+\epsilon_a))} \\ &\qquad - (\tilde{\alpha}_0 - \alpha)\biggl(\int_M |u_a|^{1+\epsilon_\alpha}\,\mathrm{d}v_g\biggr)^{1/(1+\epsilon_\alpha)(2+4/n)} \\ &\geqslant (\alpha_0 - a)\biggl(\int_M |u_a|^{1+\epsilon_a}\,\mathrm{d}v_g\biggr)^{4/(n(1+\epsilon_a))} \\ &\qquad - (\tilde{\alpha}_0 - \alpha)\biggl(\int_M |u_a|^{1+\epsilon_a}\,\mathrm{d}v_g\biggr)^{1/(1+\epsilon_a)(2+4/n)} \\ &= \biggl((\alpha_0 - a) - (\tilde{\alpha}_0 - \alpha)\biggl(\int_M |u_a|^{1+\epsilon_a}\,\mathrm{d}v_g\biggr)^{2/(1+\epsilon_a)}\biggr) \\ &\qquad \times \biggl(\int_M |u_a|^{1+\epsilon_a}\,\mathrm{d}v_g\biggr)^{4/(n(1+\epsilon_a))}. \end{aligned}$$

From (4.3), we get $J_a(u_a) \geqslant \tilde{J}_\alpha(u_a)$. From the definition of $\tilde{u}_\alpha$, $\tilde{J}_\alpha(u_\alpha) \leqslant \tilde{J}_\alpha(u_a)$. Therefore,

$$\tilde{J}_\alpha(u_\alpha) \leqslant J_a(u_a). \tag{4.4}$$

Equation (4.4), together with (4.1) and (4.2), shows that the assumption we made is false. This gives the theorem.

5. Examples

Let (M, g) be a smooth compact Riemannian n-manifold. Studying how the best second constant B_0 in the sharp L^2-Nash inequality depends on the geometry of (M, g) is a difficult problem. At the moment, we explicitly know B_0 only for the circle S^1. Even in this case, the result is not trivial (see below). The example of the

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

torus $S^1(R) \times S^{n-1}(1)$, where the radius of the first circle is going to 0, shows that the naive idea that

$$B_0 = \max\biggl(\mathrm{Vol}(M)^{-2/n}, \frac{|\mathcal{B}|^{-2/n}}{6n}\biggl(\frac{2}{n+2} + \frac{n-2}{\lambda_1}\biggr)\biggl(\frac{n+2}{2}\biggr)^{2/n} \max_{x\in M} S_g(x)\biggr)$$

is false. This paragraph is devoted to the study of these examples. First, we prove a general result.

PROPOSITION 5.1. *Suppose that the L^1-Nash inequality (1.1) is true on M, with $A = A_0(n)$. Let u be the extremal function given by theorem 1.3. If the set $\{x \in M \mid u(x) = 0\}$ is negligible, then $B_0 = \mathrm{Vol}(M)^{-2/n}$.*

Without loss of generality, we may assume that $\mathrm{Vol}(M) = 1$. We use the same notations than those used in the proof of theorem 1.3. First, let us prove that $u \in C^1(M)$ and $u_\alpha \to u$ in $C^1(M)$. Suppose that $\lim \|u_\alpha\|_\infty = +\infty$. Then let $v_\alpha = u_\alpha/\|u_\alpha\|_\infty$. Note that u_α satisfies the same equation than that involved in theorem 1.1. Hence we easily get that $\|\Delta_g v_\alpha\|_\infty \leqslant C$. As we did in theorem 1.1, we can find $v \in C^0(M)$ such that $v_\alpha \to v$ in $C^0(M)$ (up to a subsequence). Let x_α be a maximum of u_α. There exists x_0 such that $x_\alpha \to x_0$ up to a subsequence. We clearly have $v(x_0) = 1$. Therefore,

$$\int_M v \,\mathrm{d}v_g > 0.$$

However,

$$\int_M v \,\mathrm{d}v_g = \lim \frac{\int_M u_\alpha^{1+\epsilon_\alpha} \,\mathrm{d}v_g}{\|u_\alpha\|_\infty^{1+\epsilon_\alpha}} = 0.$$

This shows that $\lim \|u_\alpha\|_\infty < +\infty$. We then have $\|\Delta_g u_\alpha\|_\infty \leqslant C$. The result then easily follows. Now we prove the proposition. Since $\mathrm{Vol}(\{x \in M \mid u(x) = 0\}) = 0$, we get

$$\lim \int_M u_\alpha^{\epsilon_\alpha} \,\mathrm{d}v_g = 1.$$

Set

$$l_1 = \int_M u \,\mathrm{d}v_g, \qquad l_\nabla = \int |\nabla u|^2 \,\mathrm{d}v_g.$$

Integrating (E_α) (where (E_α) is as in theorem 1.1), we have

$$\frac{4}{n} \lim B_\alpha = \lim k_\alpha l_1.$$

We then get, from the definitions of B_α, k_α and from $\lim I_\alpha(u_\alpha) = A_0(n)^{-1}$,

$$\frac{4A_0(n)^{-1}}{nl_1} = \biggl(2l_\nabla l_1^{4/n} + \frac{4}{n} A_0(n)^{-1}\biggr) l_1.$$

Since $\lim I_\alpha(u_\alpha) = A_0(n)^{-1}$, we also have

$$1 = (A_0(n) l_\nabla + B_0) {l_1}^{4/n}.$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

As one easily checks, it follows that

$$B_0 = \left(\left(1+\frac{2}{n}\right) - \frac{2}{n} l_1^{-2}\right) l_1^{-4/n}.$$

Now let, for $0 \leqslant x \leqslant 1$,

$$f(x) = \left(\left(1+\frac{2}{n}\right) - \frac{2}{n} x^{-2}\right) x^{-4/n}.$$

A simple study of f gives that $f(x) \leqslant 1$, with equality if and only if $x = 1$. As $B_0 \geqslant 1$, we have necessarily $l_1 = 1$ and $B_0 = 1$. An application of this result is the following.

Corollary 5.2. *On the standard circle of radius* 1, $B_0 = (2\pi)^{-2}$.

Proof. We keep the same notations. It is sufficient to prove that

$$\mathrm{Vol}(\{x \in M \mid u(x) = 0\}) = 0.$$

Suppose that there exists $x \in S^1$ such that $u(x) = 0$. Clearly, by the works of Carlen and Loss [3], we would get

$$\left(\int_{S^1} u^2 \,\mathrm{d}v_\xi\right)^3 \leqslant A_0(1) \int_{S^1} |\nabla u|_\xi^2 \,\mathrm{d}v_\xi \left(\int_{S^1} u \,\mathrm{d}v_\xi\right)^4.$$

Moreover, since u is extremal,

$$\left(\int_{S^1} u^2 \,\mathrm{d}v_\xi\right)^3 = \left(A_0(1) \int_{S^1} |\nabla u|_\xi^2 \,\mathrm{d}v_\xi + B_0\right) \left(\int_{S^1} u \,\mathrm{d}v_\xi\right)^4.$$

We get a contradiction. The result then follows. □

We give another example. As a remark, taking $M = T^{n-1}$ in the following result, we see that B_0 may be as large as we want while the metric is kept Euclidean.

Proposition 5.3. *Let (M, g) be a smooth compact Riemannian $(n-1)$-manifold with $n \geqslant 2$. Let G_k be the group of rotations in $\mathbb{R}^2$ of centre 0 and angle $2\pi/k$. Let $S_k = S^1/G_k$ and $M_k = S_k \times M$, with the standard product metric g_k. Then $B_0(M_k)$ is as large as we want in the sense that $\forall C > 0$, $\exists\ k_0 \in \mathbb{N}$ such that $\forall k \geqslant k_0$, $B_0(M_k) \geqslant C \cdot \mathrm{Vol}(M_k)^{-2/n}$.*

Proof. Assume that, on the contrary, there exists $C > 0$ and $(k_i)_i$ such that $\lim_i k_i = +\infty$ and

$$B_0(M_{k_i}) \leqslant C \cdot \mathrm{Vol}(M_{k_i})^{-2/n}.$$

Note that $\mathrm{Vol}(M_{k_i}) = \mathrm{Vol}(M_1)/k$. It is clear that we can find $u \in C^\infty(M)$ such that

$$\left(\int_M u \,\mathrm{d}v_g\right)^2 < C \cdot \mathrm{Vol}(M) \int_M u^2 \,\mathrm{d}v_g.$$

Let $\tilde{u} \in C^\infty(M_{k_i})$ be defined by

$$\forall (t, x) \in S_k \times M, \quad \tilde{u}(t, x) = u(x).$$

https://doi.org/10.1017/S0308210501000270 Published online by Cambridge University Press

We have

$$\left(\int_{M_{k_i}} \tilde{u}^2 \, \mathrm{d}v_{g_{k_i}}\right)^{1+2/n} \leqslant \left(A_0(n) \int_{M_{k_i}} |\nabla \tilde{u}|^2_{g_{k_i}} \, \mathrm{d}v_{g_{k_i}} + \left(C \cdot \frac{\mathrm{Vol}(M_1)}{k_i}\right)^{-2/n} \int_{M_{k_i}} \tilde{u}^2 \, \mathrm{d}v_{g_{k_i}}\right) \left(\int_{M_{k_i}} \tilde{u} \, \mathrm{d}v_{g_{k_i}}\right)^{4/n}.$$

Clearly, on M_1,

$$\left(\int_{M_1} \tilde{u}^2 \, \mathrm{d}v_{g_1}\right)^{1+2/n} \leqslant (k_i)^{-2/n}\left(A_0(n) \int_{M_1} |\nabla \tilde{u}|^2_{g_1} \, \mathrm{d}v_{g_1} + \left(C \cdot \frac{\mathrm{Vol}(M_1)}{k_i}\right)^{-2/n} \int_{M_1} \tilde{u}^2 \, \mathrm{d}v_{g_1}\right) \left(\int_{M_1} \tilde{u} \, \mathrm{d}v_{g_1}\right)^{4/n},$$

and on M (using the definition of $\tilde{u}$),

$$\left(2\pi \int_{M} u^2 \, \mathrm{d}v_{g}\right)^{1+2/n} \leqslant (k_i)^{-2/n}\left(A_0(n) 2\pi \int_{M} |\nabla u|^2_{g} \, \mathrm{d}v_{g} + \left(C \cdot \frac{\mathrm{Vol}(M_1)}{k_i}\right)^{-2/n} 2\pi \int_{M} u^2 \, \mathrm{d}v_{g}\right) \left(2\pi \int_{M} u \, \mathrm{d}v_{g}\right)^{4/n}.$$

When $k_i \to +\infty$, we get (using $\mathrm{Vol}(M_1) = 2\pi \, \mathrm{Vol}(M)$)

$$\left(\int_M u \, \mathrm{d}v_g\right)^2 \geqslant C \cdot \mathrm{Vol}(M) \int_M u^2 \, \mathrm{d}v_g.$$

Recall that u has been chosen such that the previous inequality is false. We then get the proposition. □

References

1 T. Aubin. Nonlinear analysis on manifolds—Monge–Ampère equations. *Grundlehren der Mathematischen Wissenschaften* **252** (1982).

2 T. Aubin and Y. Y. Li. On the best Sobolev inequality. *J. Math. Pures Appl.* **78** (1999), 353–382.

3 E. A. Carlen and M. Loss. Sharp constant in Nash's inequality. *Int. Math. Res. Notices* **7** (1993), 213–215.

4 O. Druet, E. Hebey and M. Vaugon. Optimal Nash's inequalities on Riemannian manifolds. *Int. Math. Res. Notices* **14** (1999), 735–779.

5 O. Druet. The best constant problem in Sololev inequalities. *Math. Annln* **314** (1999), 327–346.

6 E. Hebey and M. Vaugon. Meilleures constantes dans le théorème d'inclusion de Sobolev. *Annls Henri Poincaré* **13** (1996), 57–93.

7 E. Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*. Lecture Notes of the Courant Institute, vol. 5 (1999).

8 E. Humbert. Extremal functions for the L^2-Nash inequality. Preprint, University of Cergy-Pontoise (1999).

(*Issued 15 June 2001*)