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Formulae and Asymptotics for Coefficients of Algebraic Functions

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We dedicate this article to the memory of Philippe Flajolet, who was and will remain a guide and a wonderful source of inspiration for so many of us. ❋❋❋

We study the coefficients of algebraic functions $\sum_{n\geqslant 0} f_n z^n$. First, we recall the too-little-known fact that these coefficients f_n always admit a closed form. Then we study their asymptotics, known to be of the type $f_n \sim CA^n n^\alpha$. When the function is a power series associated to a context-free grammar, we solve a folklore conjecture: the critical exponents α cannot be $1/3$ or $-5/2$; they in fact belong to a proper subset of the dyadic numbers. We initiate the study of the set of possible values for A. We extend what Philippe Flajolet called the Drmota–Lalley–Woods theorem (which states that $\alpha = -3/2$ when the dependency graph associated to the algebraic system defining the function is strongly connected). We fully characterize the possible singular behaviours in the non-strongly connected case. As a corollary, the generating functions of certain lattice paths and planar maps are not determined by a context-free grammar (*i.e.*, their generating functions are not $\mathbb{N}$-algebraic). We give examples of Gaussian limit laws (beyond the case of the Drmota–Lalley–Woods theorem), and examples of non-Gaussian limit laws. We then extend our work to systems involving non-polynomial entire functions (non-strongly connected systems, fixed points of entire functions with positive coefficients). We give several closure properties for $\mathbb{N}$-algebraic functions. We end by discussing a few extensions of our results (infinite systems of equations, algorithmic aspects).

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1. Introduction

Algebraic functions and their asymptotics are ubiquitous in combinatorics. In this article we review some of the main reasons for this.

Algebraic functions and language theory. The theory of context-free grammars and its relationship with combinatorics was initiated by the article of Noam Chomsky and Marcel-Paul Schützenberger in 1963 [39], where it is shown that the generating function of the number of words generated by a non-ambiguous context-free grammar is algebraic. Since then, there has been much use of context-free grammars in combinatorics. Several chapters of the Flajolet and Sedgewick book *Analytic Combinatorics* [65] are dedicated to what they called the 'symbolic method' (which is in large part isomorphic to the Joyal theory of species [73, 20] and, when restricted to context-free grammars, is sometimes called the 'DVS methodology', for Delest–Viennot–Schützenberger, as the Bordeaux combinatorics school made great use of it [117, 47] for the enumeration of polyominoes [48] and lattice paths [59], for example). Context-free grammars also allow one to enumerate trees [68, 89], sofic-Dyck automata [18], non-commutative identities [81, 98], pattern-avoiding permutations [80, 88, 3, 17], some types of planar maps, triangulations, Apollonian networks [26], non-crossing configurations, and dissections of polygons [62] (as studied by Euler in 1751, one of the founding problems of analytic combinatorics!); see also [110]. Links between asymptotics of algebraic functions and the (inherent) ambiguity of context-free languages was studied in [77, 61], and for prefixes of infinite words in [5]. Growth rates are studied in [36, 37], in connection with asymptotics of random walks [120, 38, 69, 84, 85]. Applications in bioinformatics or for patterns in RNA are given in [96, 118, 50, 42, 119].

Random generation of algebraic objects. The first efficient algorithm for uniform random generation of words of length n of a context-free grammar involved $n^2(\ln n)^2$ operations on average via fast Fourier transform, and is due to Hickey and Cohen [72]. Generating functions associated to context-free grammars are the key tool for improving this average complexity: the recursive method, coupled with the boustrophedon algorithm [66], led to a time-complexity $n \ln n$. Asymptotics of their coefficients are used in the implementation of this method with floating-point arithmetic [51]. The case of ambiguous grammars was also considered in [24]. A revolutionary change of paradigm on the generating functions arose in [60], where the authors introduced the 'Boltzmann method'; this led to $O(n)$ complexity for generating objects of size $n \pm \epsilon(n)$, benefitting new analytic investigations of the function. See also the applications of this method to algebraic objects in [46, 35] and to grammars with parameters in [27].

Other occurrences of algebraic functions. The link between algebraic functions and p-automatic sequences (numbers written in a given base via an automaton) is well illustrated by the Christol–Kamae–Mendès France–Rauzy theorem [40, 4].

More links with monadic second-order logic, tiling problems and vector addition systems appear in [92, 121].

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Algebraic functions via functional equations. Many algebraic functions pop up in combinatorics [32] via tools other than context-free grammars: quite often, they appear as the 'diagonal' of rational functions [13, 15], or as solutions of functional equations (solvable by the 'kernel method' and its variants [9, 33]), and the interplay with their asymptotics is crucial for analysis of lattice paths [10], walks with an infinite set of jumps [7, 14] (which are thus not coded by a grammar on a finite alphabet), or planar maps [11]. Sometimes, differential equations lead to algebraic solutions, as in some urn models [90].

Algebraicity of permutation related problems. Since Knuth enumerated permutations sortable by a stack, much attention has been drawn to permutations avoiding a given pattern, or counting the number of appearances of a given pattern. Many cases involved a Knuth-like approach (using the 'kernel method'), or an approach by (in)decomposable subclasses [80, 88, 3, 17], leading to algebraic functions.

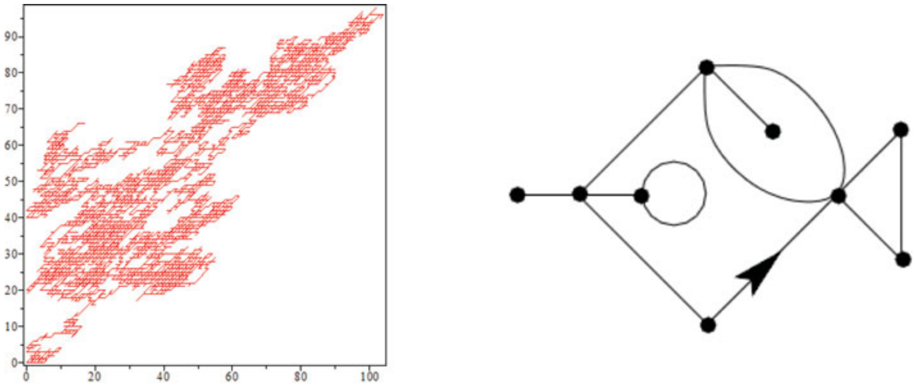
Algebraic functions and universality of critical exponents. When one thinks about the asymptotics of the coefficients of generating functions, one often gives the example of Catalan numbers,

$$\frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n^3}}$$

(this is a direct consequence of Stirling's formula for $n!$, or can also be obtained by the saddle point method or singularity analysis). One important fact with respect to the asymptotics of coefficients of algebraic generating functions is that it often involves the factor $1/\sqrt{\pi n^3}$. In [65] this was called the 'Drmota–Lalley–Woods theorem', due to independent similar results of these three authors [53, 84, 121], relying on a 'strong connectivity' assumption of the system implicitly defining the algebraic function. In this article, we extend the theorem by removing this restrictive assumption, to give all the possible asymptotics for the important class of context-free algebraic functions, and associated limit laws. In one sense, this could be considered a generalization of the Perron–Frobenius theorem to the algebraic case, also including a multivariate extension. We do this in a constructive way, which opens the door to full algorithmization.

Plan of this article.

- In Section 2 we give a few definitions, mostly illustrating the link between context-free grammars, solutions of positive algebraic systems, and $\mathbb{N}$-algebraic functions. We also prove some basic properties of such functions.
- In Section 3 we survey some closure properties of algebraic functions and give a closed form for their coefficients.
- In Section 4 we state our main theorem on the possible critical exponents of algebraic functions (associated to a context-free grammar with positive weights), and we give some of its consequences (see also Figure 1).
- In Section 5 we give finer results about asymptotics and the corresponding Puiseux expansions, which incidentally prove our main theorem.



(a) (b)

Figure 1. (Colour online) Many combinatorial objects are algebraic, sometimes for non-trivial reasons, *e.g.*, Gessel random walks (a) or planar maps (b). A nice consequence of our Theorem 4.2 is that these combinatorial structures (and many similar objects) cannot be generated by a context-free grammar, because their asymptotics have the critical exponent $-2/3$ and $-5/2$.

- In Section 6 we provide a complete picture of the asymptotic behaviour of the coefficients of functions that are solutions of positive systems of algebraic equations when 'periodic behaviour' occurs.
- In Section 7 we consider the mysterious set of all the radii of convergence of $\mathbb{N}$-algebraic functions, and some variants. We give some closure properties of this set.
- In Section 8 we prove that the associated limit laws are Gaussian for a broad variety of cases (thus extending the Drmota–Lalley–Woods theorem), and we also give an argument explaining the diversity of other possible limit laws.
- In Section 9 we give an analogue of our main theorem for systems involving entire functions.
- We end with a conclusion reading the reader to several extensions (algorithmic considerations, decidability questions, extensions to infinite systems, and to attribute grammars).

2. Definitions: $\mathbb{N}$-algebraic functions and well-defined systems

Coefficients of algebraic functions have very constrained asymptotics; we will focus on their 'critical exponents'.

Definition 2.1 (critical exponent of a sequence). For any sequence such that $f_n \sim Cn^{\alpha}\rho^{-n}$, α is called the critical exponent and ρ is the radius of convergence of the corresponding generating function

$$F(z) = \sum_{n\geqslant 0} f_n z^n.$$

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Moreover, if the sequence f_n has asymptotics depending on the modulo class of n, the definition extends naturally by saying that f_n has a (possibly different) critical exponent for each class. The singularities of F located on its radius of convergence are called the dominant singularities of F.

There is a dual point of view when one considers functions instead of sequences.

Definition 2.2 (Puiseux critical exponent of a function). For any function having a Puiseux expansion

$$F(z) = \sum_{k \geqslant k_0} a_k (z-\rho)^{rk}$$

(with $a_{k_0} \neq 0$), the number rk_0 is called the valuation of this series, while the smallest value of rk which is not a non-negative integer and for which $a_k \neq 0$ is called the Puiseux critical exponent of $F(z)$. In this article, ρ is by default assumed to be the radius of convergence of $F(z)$.

Examples include

$$F(z) = 1 + 2(1-z) + 5(1-z)^2 + (1-z)^{5/2} + (1-z)^3 + (1-z)^{7/2}(1+o(1)),$$

which has Puiseux critical exponent $5/2$, and

$$F(z) = 1/(1-2z) + (1-2z) + (1-2z)^{5/2}(1+o(1)),$$

which has Puiseux critical exponent -1.

The Flajolet–Odlyzko theory of singularity analysis [63] (which has roots in the works of Darboux, Hankel, Hardy and Littlewood) links the critical exponent α of a sequence (f_n) and the Puiseux critical exponent e of the corresponding function $F(z)$ via the relation $\alpha = -e-1$ (provided that ρ is the only singularity on the circle $|z| = \rho$).

We will consider algebraic functions defined via systems of algebraic equations. Such systems are ubiquitous in language theory. For the notions of automata, pushdown automata, and context-free grammars, we refer to the first three chapters of [116] (by Perrin on finite automata, by Berstel and Boasson on context-free languages, and by Salomaa [101] on formal languages and power series) or to the more recent survey [95] in [57]. Another excellent compendium on the subject is the *Handbook of Formal Languages* [100] and the Lothaire trilogy [87]. In language theory, it is natural to consider equations involving positive integer coefficients, whereas for other applications it is natural to consider positive real weights or probabilities. Accordingly, we will consider equations having various types of coefficients.

Definition 2.3 (well-posed systems, $\mathbb{N}$-algebraic functions, $\mathbb{Q}_+$-algebraic functions). A 'well-posed' system is a system[1]

$$\begin{cases} y_1 = P_1(z, y_1, \dots, y_d), \\ \quad \vdots \\ y_d = P_d(z, y_1, \dots, y_d), \end{cases} \tag{2.1}$$

where each polynomial P_i is such that $[y_i]P_i \neq 1$ and has coefficients in any set $\mathbb{S}$ of real numbers that is closed under addition and multiplication (in this article, we consider $\mathbb{S} = \mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}_+$, or $\mathbb{R}_+$)[2] and which has a tuple of solution $(f_1(z), \dots, f_d(z))$, where each $f_i(z)$ is a power series with coefficients in $\mathbb{S}$. Functional solutions of such a system are called $\mathbb{S}$-algebraic functions.

Example 1.

$$\begin{cases} y_1 = 3y_2y_1 + 4z \\ y_2 = 23y_1^2 + 5z \end{cases} \qquad \begin{cases} y_1 = \frac{1}{3}zy_2 + \frac{25}{4}zy_1 + z \\ y_2 = zy_1 + zy_2 \end{cases}$$

$$\begin{cases} y_1 = 3y_2y_1 + 4z \\ y_2 = 23y_1 - 5z \end{cases} \qquad \begin{cases} y_1 = 3.14zy_1y_2 + \pi z^3 y_1 y_3 \\ y_2 = \exp(-1)y_1y_2 + z^3 \\ y_3 = z + \cos(3)y_3^2 \end{cases}$$

The solutions of the above systems are examples of functions which are $\mathbb{N}$-algebraic (left, top), $\mathbb{Z}$-algebraic (left, bottom), $\mathbb{Q}_+$-algebraic (right, top), and $\mathbb{R}_+$-algebraic (right, bottom).

Remark. Requiring that the solutions have real coefficients is indeed a restriction: for example, the equation $y = 1 + z + y^2$ has two power series solutions, but each of them has complex coefficients. On the other hand, if the system has positive coefficients and is such that $[y_i]P_i = 0$ and $P_i(0, \mathbf{0}) = 0$ for all i, then it would be superfluous to require the y_i to have non-negative real coefficients, as this would result by iteration (while a system such as $y = z + 2y$ shows that the power series could elsewhere have negative coefficients as soon as there is one i such that $[y_i]P_i > 1$). Similarly, when one considers only $\mathbb{N}$-algebraic functions, one could also change the condition $[y_i]P_i \neq 1$ by the (apparently more restrictive) condition $[y_i]P_i = 0$; this would not actually change the set of $\mathbb{N}$-algebraic functions (this easily follows from our Proposition 2.6 hereafter).

We now give a definition which offers a fast 'combinatorial' way to ensure that a system is well-posed.

[1] In this article, we will often abbreviate the system (2.1) with the convenient short notation $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$, where bold fonts are used for vectors.

[2] We make use of the notations $\mathbb{Q}_+ = \{x \in \mathbb{Q}, x \geqslant 0\}$, $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geqslant 0\}$, and $[u^k]P(u)$, which stand for the coefficient of u^k in $P(u)$.

Definition 2.4 (well-defined systems). The above system (2.1), $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$, is called 'well-defined' if and only if the following conditions hold.

- *No monic production.* The coefficient of each y_j in each P_i is 0:

$$\forall i \forall j \quad [y_j]P_i(0, 0, \dots, 0, y_j, 0, \dots, 0) = 0.$$

- *No epsilon production.* For all i, $P_i(0, 0, \dots, 0) = 0$.
- *Terminating condition.* For any y_j, there is at least one y_k reachable from y_j (*i.e.*, $\exists t > 0 | \mathbf{A}^t_{jk} \neq 0$, where $\mathbf{A}$ is the adjacency matrix of the dependency graph associated to the system, as illustrated by Figure 2 in Section 5), such that $P_k(z, 0, \dots, 0) \neq 0$.

By design, any well-defined system is well-posed, but the converse is not true, as can be seen from systems of Example 2. We will discuss some analytic distinctions between these two notions in Section 5.1, because introducing both of them allows us to simplify the statements of our theorems without loss of generality concerning the combinatorial or analytical problems we want to consider.

Example 2.

$$\begin{cases} y_1 = \frac{1}{2}y_1 + \frac{1}{2}y_2 + z \\ y_2 = \frac{1}{2}y_1 + \frac{1}{2}y_2 + z \end{cases} \qquad \begin{cases} y = z + 2y \end{cases} \qquad \begin{cases} y_1 = zy_2 + zy_1 \\ y_2 = 1 + z + zy_2 + y_3 \\ y_3 = 5y_4 \\ y_4 = \frac{1}{10}y_3y_4 \end{cases}$$

$$\begin{cases} y = 1 + z + y^2 \end{cases} \qquad \begin{cases} y_1 = 1 + y_1y_3^4 \\ y_2 = 2y_3 \\ y_3 = \frac{1}{2}y_2 + y_3^2 + z \end{cases} \qquad \begin{cases} y_1 = zy_2 + zy_1 + z \\ y_2 = 3y_1 + zy_2 + z \\ y_3 = zy_2^2 + zy_1 \end{cases}$$

The above list shows systems that are not well-defined. Well-defined systems (our Definition 2.4) are just a convenient way to prevent pathological cases such as the examples in the first two columns. The systems from the right column are not well-defined, but are 'well-posed' (our Definition 2.3): they have power series solutions with non-negative real coefficients.

Remark. Several authors have introduced different notions with some minor distinctions: our 'well-defined' systems are thus related to the 'proper' or 'well-posed' or 'well-founded' notions of [39, 53, 97].

The *idée fixe* is that it corresponds to context-free grammars for which one has no 'infinite chain rules' (no infinite chain of monic productions, no infinite chain of epsilon productions).

In this article we need this to handle probability generating functions (in order to get general results on the limit laws), so we allow real weights. Therefore the situation is slightly more tricky than those considered by previous authors, who only needed to deal

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with systems having integer coefficients. Indeed, in the case of real coefficients, 'analytic convergence' can compensate 'formal divergence' (*e.g.*, for the grammar $A \to zA/2$). This is why (in the notion of a well-posed system), the coefficients of monic productions are allowed to be non-zero, as soon as the spectral radius of the Jacobian of the associated system is < 1, as this will guarantee a global contraction (we will also comment on this when discussing our notion of well-posed systems in Section 5.1; see also [114] for a more topological point of view). It is true that the two rules 'no monic productions' and 'no epsilon productions' are a little more restrictive than is necessary (as illustrated by the first case in the right column of our Example 2), but they make it easy to test the 'well-definedness' of a system. Moreover, the systems that do not fulfil all the conditions of Definition 2.4, but nonetheless have an analytic and combinatorial meaning, are easily transformed into a system fulfilling these conditions (essentially by replacing some coefficients of the polynomials P_i with some new formal parameters).

A little caveat. We have mentioned the issue of formal convergence. Let us recall briefly that this is defined on the set of power series by the following ultrametric distance:

$$d(F(z), G(z)) := 2^{-\mathrm{val}(F(z)-G(z))},$$

where val gives the exponent of the first non-zero monomial. This distance extends to vectors of functions (and to multivariate series), and allows us to apply the Banach fixed-point theorem, since the system (2.1) is a contraction, implying existence and uniqueness of a solution of the system as a d-tuple of power series $(y_1, \ldots, y_d)$ (and they are analytic functions near 0, as we already know that they are power series and algebraic by nature). A common mistake is to forget that there exist situations for which the system (2.1) can admit several solutions as power series for y_1 (note that there is no contradiction to our previous claim, which considered *tuples*). We illustrate this point via the system

$$\begin{cases} y_1 = z(1 + y_2 + 2y_1^2), \\ y_2 = z(1 + y_1 + y_2^2). \end{cases} \tag{2.2}$$

If one eliminates y_2 in this system, this implies that y_1 is defined by the equation

$$4z^2y_1^4 - 4zy_1^3 + (2z + 1 + 4z^2)y_1^2 + (z^2 - 1 - 2z)y_1 + 2z^2 + z = 0, \tag{2.3}$$

which has four solutions for y_1:

$$\begin{array}{ll} z + z^2 + O(z^3), & 1 + 3\,z + 10\,z^2 + O(z^3), \\ \frac{1}{2}z^{-1} - \frac{3}{2}z - \frac{5}{4}z^2 + O(z^3), & \frac{1}{2}z^{-1} - 1 - \frac{5}{2}z - \frac{39}{4}z^2 + O(z^3). \end{array}$$

Here, several of these solutions are power series, and it can even be proved that two branches of equation (2.3) are power series with positive integer coefficients (this does not contradict unicity of the solution via the Banach fixed-point theorem, because equation (2.3) is not a contraction in $\mathbb{Q}[[z]]$). The analogous elimination of y_1 in the system leads to four solutions for y_2:

$$z + z^2 + O(z^3), \qquad 1 + 3\,z + 10\,z^2 + O(z^3),$$
$$\frac{1}{2}z^{-1} - \frac{3}{2}z - \frac{5}{4}z^2 + O(z^3), \quad \frac{1}{2}z^{-1} - 1 - \frac{5}{2}z - \frac{39}{4}z^2 + O(z^3).$$

However, because of the multidimensional fixed-point theorem, amongst the four pairs (y_1, y_2) of functions satisfying the system, there is only one pair of power series (namely the one with $y_1 \sim y_2 \sim z$):

$$\begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = \begin{pmatrix} 1 + 3z + 10z^2 + O(z^3) \\ \frac{1}{z} - 2z + 3z^2 + O(z^3) \end{pmatrix}, \qquad \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = \begin{pmatrix} z + z^2 + 3z^3 + O(z^4) \\ z + z^2 + 2z^3 + O(z^4) \end{pmatrix},$$
$$\begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = \begin{pmatrix} \frac{1}{2z} - 1 - \frac{5}{2}z + O(z^2) \\ \frac{1}{z} - \frac{1}{2} - \frac{1}{4}z + O(z^2) \end{pmatrix}, \text{ and } \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = \begin{pmatrix} \frac{1}{2z} - \frac{3}{2}z - \frac{5}{4}z^2 + O(z^3) \\ \frac{1}{2} - \frac{5}{4}z - \frac{1}{4}z^2 + O(z^3) \end{pmatrix}.$$

Different algebraic classes. By elimination theory (resultant or Gröbner bases: see the discussion in [111, 94]), $\mathbb{S}$-algebraic functions are algebraic functions: it is possible to transform a system of equations in the y_i into a single equation involving just y_1 and z. Now, we give a few trivial/folklore results. $\mathbb{N}$-algebraic functions correspond to generating functions of context-free grammars (this is the Chomsky–Schützenberger theorem: see [39]), or, equivalently, pushdown automata (via, *e.g.*, a Greibach normal form). They are closed with respect to sums, products and derivations. $\mathbb{Z}$-algebraic functions have no natural simple combinatorial structures associated to them, but they are related to $\mathbb{N}$-algebraic functions via the following proposition.

Proposition 2.5. *Any $\mathbb{Z}$-algebraic function is the difference of two $\mathbb{N}$-algebraic functions.* □

Proof. This can be seen by introducing two new sets of unknowns $\mathbf{a}$ and $\mathbf{b}$, and then by splitting the initial system into two:

$$\mathbf{y} = P(z, \mathbf{y}) \Longleftrightarrow \mathbf{a} - \mathbf{b} = P(z, \mathbf{a} - \mathbf{b}).$$

Indeed, expanding and collecting the positive terms gives an $\mathbb{N}$-algebraic equation for $\mathbf{a}$, and collecting the negative terms gives an $\mathbb{N}$-algebraic equation for $\mathbf{b}$: there are clearly no monic productions in right-hand side inherited from $P(z, \mathbf{y})$. □

For example, the $\mathbb{Z}$-algebraic function defined by $F = 1 + 3zF^2 - 3z^2F^3$ can be written as the difference of two $\mathbb{N}$-algebraic functions: $F = A - B$ with

$$A = 1 + 3zA^2 + 3zB^2 + 9z^2A^2B + 3z^2B^3 \quad \text{and} \quad B = 6zAB + 3z^2A^3 + 9z^2AB^2.$$

Proposition 2.6 (coefficients in $\mathbb{Z} \Rightarrow \mathbb{Z}$-algebraic). *Any algebraic power series $F(t)$ in $\mathbb{S}[[t]]$, where $\mathbb{S} = \mathbb{R}$, $\bar{\mathbb{Q}}$, $\mathbb{Q}$, or $\mathbb{Z}$, is in fact an $\mathbb{S}$-algebraic function in disguise. More precisely, after*

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a possible change of variable $F \to G = t^v F + a$, G is $\mathbb{S}$-algebraic, and the corresponding system has just one equation.

Proof. Let us start with an algebraic function $F(t)$ satisfying a general polynomial equation, namely $Q(t, F(t)) = 0$ (where Q is irreducible and has real coefficients). We would like to prove that we can also define it via an equation of the type of system (2.1), namely $F(t) = P(t, F(t))$.

To do this, we will use a little Newton polygon theory. To each monomial $F(t)^i t^j$ in $Q(t, F(t))$, one associates the point (i, j). The convex hull of the set of points associated to $Q(t, F(t))$ is called the Newton polygon of Q. Now imagine a source of light below this polygon, which thus illuminates all the lower segments of the polygon (and only these). Each illuminated segment has a slope (call it σ_k), and a length on the x-axis (call it λ_k). Newton polygon theory implies that one then has λ_k roots of valuation $-\sigma_k$. This can be checked by a 'plug and identify' process.

Now, if $F(t)$ is an algebraic power series of valuation v, the change of variable $F(t) \to t^v F(t)$ in its characteristic polynomial $Q(t, F(t)) = 0$ allows us to restrict without loss of generality to power series $F(t)$ having a non-zero constant term.

Equivalently, one of the points of the Newton polygon is $(0, 0)$, and another point is $(0, m)$ for some m (if not, there would be no root with integer valuation, *i.e.*, a power series solution). If $m = 1$, we obtain the shape of system (2.1). If $m > 1$, then making the change of variable $y \to y + a$ leads to a new equation, in which the coefficient of y is a polynomial in a, and any real value of a not cancelling this polynomial leads to an equation satisfying the shape of system (2.1).

Now, what about the coefficients of this newly found $P(t, F)$? Well, they are by design real, and if (at least) one of the coefficients (call it c) of this equation were not rational, then bootstrapping the equation $F = P(t, F)$ would imply that the power series $F(t)$ would have a coefficient (required to be rational) which would also be a linear relation between this irrational coefficient c and other rational numbers. The same holds for $(\mathbb{S} = \bar{Q})$: simply replace algebraic/transcendental with rational/irrational.

The same also holds with 'integer' ($\mathbb{S} = \mathbb{N}$) instead of 'rational'; it is, however, more tricky in this last case. Once we have obtained a polynomial $P(t, F)$ with rational coefficients, via the previous argument, just apply the Newton polygon theory in a 'p-adic way'. First make a simplification by the least common multiple (call it ℓ) of the denominators of the coefficients of $P(t, F)$ (call p a prime factor of this least common multiple), and then consider the Newton polygon (taking F and p as coordinates) of $\ell y = \ell P(t, F)$. There, the slope of the line corresponding to our power series solution implies that its valuation in p is negative, but since by hypothesis it is positive, this implies that no such prime p exists, and therefore there were no pure rational coefficients in $P(t, F)$. □

Remark. The above proposition does not hold for $\mathbb{S} = \mathbb{N}$. Indeed, we show later in Proposition 4.5 that $\mathbb{N}$-algebraic functions can intrinsically require the definition of more than one equation having non-negative coefficients, while they can always be defined with a single equation having positive *and* negative coefficients. A nice consequence of the algebraicity of a power series $F(z)$ with rational coefficients is the Eisenstein lemma, namely that there exists an integer b such that $F(bz)$ has integer coefficients only. This is obvious thanks to Proposition 2.6, as $F(z)$ is then $\mathbb{Q}$-algebraic, and thus $F(bz)$ (with b killing all the denominators of the coefficients) is then $\mathbb{Z}$-algebraic. This is a way to prove that exp or log type functions such as $\sum z^n/n!^k$ and $\sum z^n/n^k$ are transcendental (see example VII.37 in [65]).

There is a natural extension of $\mathbb{N}$-algebraic functions to a multivariate setting: the multivariate function $F(\mathbf{z})$ (for $\mathbf{z}$ a tuple of variables) is called $\mathbb{S}$-algebraic if there is a system such that $\mathbf{y} = \mathbf{P}(\mathbf{z}, \mathbf{y})$, where $\mathbf{z} = (z_1, \ldots, z_d)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ are tuples and where $\mathbf{P} = (P_1, \ldots, P_m)$ is a tuple of polynomials in $d + m$ variables, with coefficients in $\mathbb{S}$.

Proposition 2.7 (systems of $\mathbb{N}$-algebraic equations have $\mathbb{N}$-algebraic solutions). *Consider a system of equations $\mathbf{y} = \mathbf{N}(\mathbf{z}, \mathbf{y})$, where $\mathbf{y}$ and $\mathbf{z}$ are tuples, and where each N_i is a multivariate $\mathbb{S}$-algebraic function. Then the power series $y_i(\mathbf{z})$ solutions of this system are $\mathbb{S}$-algebraic functions.*

Proof. One has a system of d equations $y_j = N_j(\mathbf{z}, y_1, \ldots, y_d)$, where each $N_j(\mathbf{z}, \mathbf{y})$ is itself solution of a possibly large system $\mathbf{N}_j = \mathbf{P}_j(\mathbf{z}, \mathbf{y}, \mathbf{N}_j)$ (the reader will need good eyesight here to distinguish fonts: $\mathbf{N}_j$ is a tuple, of which N_j is the first component). By keeping track of all the intermediate variables, one gets a huge system and a polynomial A_j (with coefficients in $\mathbb{S}$) such that $y_j = A_j(\mathbf{z}, \mathbf{y}, \mathbf{N}_1, \ldots, \mathbf{N}_d)$; thus y_j is $\mathbb{S}$-algebraic. Alternatively, this can be seen with the context-free grammar approach, by doing a substitution: replacing some initial terminals with some non-terminals. □

There is actually a nice combinatorial example of this type of equation.

Corollary 2.8 ($\mathbb{N}$-algebraicity of classes of pattern-avoiding permutations). *If the generating function of the so-called 'simple' permutations (associated to a pattern P) is $\mathbb{N}$-algebraic, then the generating function of permutations avoiding the pattern P is also $\mathbb{N}$-algebraic.*

Proof. Permutations avoiding some patterns satisfy the functional equation

$$C = (1 + C)(z + S(C)) + C^2,$$

where S is the generating functions of the so-called 'simple' permutations (see [3, 17]), which is known to be $\mathbb{N}$-algebraic in many cases. For such cases, our Proposition 2.7 therefore implies that the generating function C is itself also $\mathbb{N}$-algebraic. Indeed, this is also trivially the case when there is a finite number of simple permutations. □

Now let us conclude this section by mentioning $\mathbb{N}$-rational functions: an $\mathbb{N}$-rational function is a function solution of a system (2.1) where each polynomial P_i has coefficients in $\mathbb{N}$ and total degree 1. Such functions correspond to generating functions of regular expressions or, equivalently, automata (a result attributed to Kleene [79]). Their coefficients satisfy a recurrence

$$f_{n+d} = \sum_{i=0}^{d-1} a_i f_{n+i},$$

with $a_i \in \mathbb{N}$. Much is known about $\mathbb{N}$-rational functions, and they are well characterized by their analytic and asymptotic properties [23, 115]. Our article aims at reaching a similar level of knowledge about $\mathbb{N}$-algebraic functions. However, before tackling them in Section 4, we continue with facts that hold for algebraic functions in full generality.

3. Closed form for the coefficients of algebraic functions

A first natural question is: How can we compute the nth coefficient f_n of an algebraic power series? The fastest way is to rely on the theory of D-finite functions. A function $F(z)$ is D-finite if it satisfies a linear differential equation with coefficients that are polynomials in z; equivalently, its coefficients f_n satisfy a linear recurrence with coefficients that are polynomials in n. There are numerous algorithms to deal with this important class of functions, which includes many special functions from physics, number theory and also combinatorics [110].

Proposition 3.1 (Abel–Tannery–Cockle–Harley–Comtet theorem). *Algebraic functions are D-finite. Moreover, for an algebraic function $F(z)$ that is the solution of a polynomial equation $Q(z, F(z)) = 0$ (where $Q(z, y)$ has degree d in y), the dimension of the space spanned by the derivatives $\partial_z^k F(z)$ is bounded by d.*

Proof. In combinatorics, Comtet popularized the fact that algebraic functions are D-finite (see [44, 45]). The key idea is to differentiate $Q(z, F(z)) = 0$ with respect to z, which shows that each derivative $\partial_z^k F(z)$ can be expressed as a rational fraction in z and $F(z)$. Taking Taylor expansions of these fractions and using the relation $Q(z, F) = 0$ then allows us to write them as polynomials in F of degree at most d, so they all live in the same space of dimension at most d, and thus we infer the relation claimed between the derivatives. It is amusing that this is in fact an old theorem rediscovered many times, *e.g.*, by Tannery [113], and Cockle and Harley [43, 70] in their method for solving quintic equations via ${}_4F_3$ hypergeometric functions. Last but not least, this theorem can also be found in an unpublished manuscript of Abel [1, p. 287]! □

The world of D-finite functions offers numerous closure properties. Let us mention some that are related to algebraic functions.

Proposition 3.2 (holonomy and closure properties for algebraic functions).

(1) *f and $1/f$ are simultaneously D-finite if and only if f'/f is algebraic.*
(2) *f and $\exp(\int f)$ are simultaneously D-finite if and only if f'/f is algebraic.*
(3) *Let g be algebraic of genus $\geqslant 1$. Then f and $g(f)$ are simultaneously D-finite if and only if f is algebraic.*

(4) *The Hadamard product of a rational and an algebraic function is algebraic.*
(5) *Let $\mathcal{A}$ be a context-free language and let $\mathcal{R}$ be a rational language. Then one has closure by intersection, Hadamard product, and set difference: $\mathcal{A} \cap \mathcal{R}$, $\mathcal{A} \odot \mathcal{R}$, $\mathcal{A} - \mathcal{R}$, and $\mathcal{R} - \mathcal{A}$ are context-free (and thus have algebraic generating functions).*
(6) *Each algebraic function is the diagonal of a bivariate rational function.*
(7) *In finite fields, Hadamard products of algebraic functions are algebraic.*
(8) *The set of generalized hypergeometric functions ${}_nF_{n-1}$ which are algebraic is well identified.*
(9) *It is possible to decide whether a D-finite equation has algebraic solutions.*

Proof. Property (1) is due to Harris and Sibuya [71], (2) and (3) to Singer [107], and (4) to Jungen [74]. Property (5) can be proved via the (pushdown) automata point of view (see also [22]). Property (6) is due to Denef and Lipshitz [49], (7) to Furstenberg [67], and (8) to Schwarz [105] and Beukers and Heckman [25]. It is still a challenge to find an efficient algorithm for (9), beyond the constructive approach given by Singer [106]. □

The linear recurrence satisfied by f_n allows us to compute in linear time all the coefficients $f_0, \ldots, f_n$. More precisely, it is proved in [28] that there exists an algorithm of complexity $O(nd^2 \ln d)$, where d is the degree of the function. If one just wants the nth coefficient f_n, it is possible to get it in $O(\sqrt{n})$ operations [41]. Many of these features (and a few others related to random generation and context-free grammars, and corresponding asymptotics) are implemented in the 'Algolib' library, a set of Maple packages developed by Flajolet, Salvy, Zimmermann, Chyzak and Mezzarobba (see `http://algo.inria.fr/libraries/`); see also the SageMath package by Kauers, Jaroschek and Johansson [75].

A less known fact is that these coefficients admit a closed-form expression in terms of a finite linear combination of weighted multinomial numbers. The multinomial number is the number of ways to divide m objects into d groups, of cardinality $m_1, \ldots, m_d$ (with $m_1 + \cdots + m_d = m$):

$$[{u_1}^{m_1} \cdots {u_d}^{m_d}](u_1 + \cdots + u_d)^m = \binom{m}{m_1, \ldots, m_d} = \frac{m!}{m_1! \cdots m_d!}.$$

More precisely, we have the following theorem.

Theorem 3.3 (Flajolet–Soria formula for coefficients of an algebraic function). *Consider a power series implicitly defined by a polynomial equation $Q(z, f(z)) = 0$ (plus initial conditions for $f(z)$, when the equation has several branches which are power series). Therefore (up to a change of variable as explained in Proposition 2.6), $f(z)$ can equivalently be defined by $f(z) = P(z, f(z))$, where $P(z, y)$ is bivariate polynomial such that $[y]P \neq 1$ and $P(z, 0) \neq 0$. Then the Taylor coefficients of $f(z)$ are given by the following finite sum:*

$$f_n = \sum_{m \geqslant 1} \frac{1}{m} [z^n y^{m-1}] P^m(z, y). \tag{3.1}$$

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Accordingly, applying the multinomial theorem to

$$P(z,y) = \sum_{i=1}^{d} a_i z^{b_i} y^{c_i}$$

leads to

$$f_n = \sum_{m\geqslant 1} \frac{1}{m} \sum_{\substack{m_1+\cdots+m_d=m\\ b_1m_1+\cdots+b_dm_d=n\\ c_1m_1+\cdots+c_dm_d=m-1}} \binom{m}{m_1,\ldots,m_d} {a_1}^{m_1}\cdots {a_d}^{m_d}, \tag{3.2}$$

where all the m_is are non-negative integers. This sum is finite, as is more easily seen via the equivalent formula

$$f_n = \sum_{m=0}^{n} \sum_{\substack{m_1+\cdots+m_d=m+1\\ b_1m_1+\cdots+b_dm_d=n\\ c_1m_1+\cdots+c_dm_d=m}} m!\, \frac{{a_1}^{m_1}}{m_1!}\cdots\frac{{a_d}^{m_d}}{m_d!}. \tag{3.3}$$

Proof. We consider $y = P(z,y)$ as the perturbation at $u = 1$ of the equation $y = uP(z,y)$, to which we apply (legitimate as $P(z,0) \neq 0$) the Lagrange inversion formula (considering u as the main variable, and z as a fixed parameter). This gives

$$[u^m]y = \frac{1}{m}[y^{m-1}]P(z,y)^m.$$

Then, summing for all m (the sum converges to $y(z)$, as it is well-defined) and extracting $[z^n]$ on both sides leads to a non-trivial equality (and therefore to (3.1)), because $[y]P(z,y) \neq 1$.

Note that formula (3.1) still holds even if P is not a polynomial but more generally a power series $\in \mathbb{C}[[z,y]]$. □

This *Flajolet–Soria formula* was first published in the habilitation thesis of Michèle Soria in 1990, and then in 1998 in the INRIA proceedings of the Algorithms Seminar; it is also mentioned in [65, Section VII.34, p. 495]. It was also found by Gessel (as published in 1999 in [110, exercise 5.39, p. 148]), and it was finally also rediscovered in 2009 by Sokal [109]. It is worth observing that Lagrange [83] initially presented his inversion formula in order to solve algebraic equations of any degree (considering the coefficients of the equations as parameters).

It is noteworthy that if P has 3 terms or less, then the multiple sum in formula (3.2) reduces to a single term, and it then remains just a simple sum on m. For instance, if we consider the equation

$$f(z) = z + z^2 f^2(z) + z^3 f^3(z), \tag{3.4}$$

then the coefficients have the nice form (although this is a matter of taste!)

$$f_{n+1} = \sum_{m=\lceil 3/5n\rceil}^{\lfloor 2/3n\rfloor} \frac{m!}{(n+1-m)!(5m-3n)!(2n-3m)!}. \tag{3.5}$$

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It is not possible to get this nice formula via a naive application of Lagrange's inversion formula, but it is a direct application of Theorem 3.3.

If one consider the case of the equation for the generating functions of d-ary trees, namely $y = 1 + zy^d$, then the formula simplifies greatly: each nested sum involves just one term. This gives the classical result

$$f_n = \frac{1}{(d-1)n+1}\binom{dn}{n}.$$

More generally, the coefficients of an algebraic function defined by $y = P(z, y)$ are therefore given by $d-2$ nested sums of binomials, where d is the number of terms of $P(z, y)$. Let d_1 be the z degree of P and let d_2 be the y degree of P. The worst-case number of nested sums in equation (3.3) is therefore $(1+d_1)(1+d_2)-3$. For example, if the y and z degree are bounded by 2, we will have six nested sums at most, as is the case for $P(z, y) = 1 + y^2 + z + zy + zy^2 + z^2 + z^2y + z^2y^2$, while $P(z, y) = 1 + zy^2 + z^2y^2$ will lead to just one sum.

This can lead to impressive identities such as several thousands of nested sums which actually simplify in a non-trivial way to a single factorial-like product. An example of such a phenomenon follows from the observation of Rodriguez-Villegas that

$$F(z) = \sum_{n\geqslant 0} \frac{(30n)!n!}{(15n)!(10n)!(6n)!} z^n$$

is a (generalized hypergeometric) algebraic function of minimal degree 483 840. More generally, it is an interesting algorithmic question to get the minimal number of nested sums giving the coefficients f_n (see [103] for an approach related to Karr's $\Pi\Sigma$-fields). Also, the set of sequences which can be expressed as nested sums of multinomials is exactly the set of diagonals of rational functions (see [31, 93]).

All these 'closed forms' are nice for arithmetical/combinatorial properties, but they are not the right way to access any form of universal asymptotics for the coefficients f_n. In the next section we use a completely different approach to tackle these questions.

4. Critical exponents for coefficients of algebraic function

It would at least be desirable to determine directly, from a positive (but reducible) system, the type of singular behaviour of the solution, but the systematic research involved in such a programme is yet to be carried out.

Philippe Flajolet and Robert Sedgewick [65, p. 493]

In this section we will characterize the singular behaviours of such systems, thus answering the wish of Flajolet and Sedgewick. Our approach relies on the theory of Puiseux expansions, which implies that the critical exponents are pure rational numbers for pure algebraic functions. (Pure algebraic means algebraic but not rational, pure rational means rational but not integer.) The full question is: Which subset of rational number do we get? We first start with the following proposition, which shows that all rational numbers

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are obtained if we do not constrain the algebraic function to satisfy a positive system of the type $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$.

Theorem 4.1. *For any rational number r that is not a negative integer, there exists an algebraic power series $F(z) = \sum_{n \geqslant 0} f_n z^n$ with positive integer coefficients f_n which have critical exponent r, i.e., $f_n \sim C n^r A^n$ (for some positive constants A, C). Moreover, this power series is $\mathbb{Z}$-algebraic.*

Proof. First, consider

$$F(z) := \frac{1 - (1 - a^2 z)^{1/a}}{az},$$

where a is any positive or negative integer. Accordingly, its coefficients are given by

$$f_n = \binom{1/a}{n+1} a^{2n+1} (-1)^n.$$

The fact that the f_n are positive integers was proved in [86], via a link with a variant of Stirling numbers. We give here another shorter proof. First, the Newton binomial theorem applied on $(1 - azF)^a = (1 - a^2 z)$ leads to an algebraic equation for $F(z)$:

$$F(z) = 1 + \sum_{k=2}^{a} \binom{a}{k} a^{k-2} (-1)^k z^{k-1} (F(z))^k.$$

Then, if one sees this equation as a fixed-point equation (as a rewriting rule in the style of context-free grammars), it is obvious that the f_n belong to $\mathbb{Z}$. But as

$$f_{n+1} = a \frac{(a(n+1) - 1) f_n}{n+2},$$

it is also clear that the f_n are indeed positive integers. Thus, we have obtained any Puiseux critical exponent $1/a$, and we now want to get any Puiseux critical exponent b/a, where b is any positive integer (not a multiple of a). Indeed, it is not possible to take F^b directly, as it does not have Puiseux critical exponent b/a (but $1/a$), so we consider $G(z) = e(azF(z) - 1)^b$ (where $e = 1$ if $a > b \bmod (2a)$ and $e = -1$ elsewhere), which gives a series with integer coefficients (because of the integrality of the coefficients of F), positive coefficients (except for a few of its first coefficients, as seen via the Newton binomial expansion). Removing these negative coefficient terms gives a power series with only positive integer coefficients, with a Puiseux expansion $G(z) = e(-1)^b (1 - a^2 z)^{b/a}$, and consequently its coefficients have the asymptotics $C n^{-1-b/a} a^{2n}$ for some $C > 0$. □

One may then wonder if there is something stronger. For example, is it the case that for any radius of convergence, any critical exponent is possible? It happens not to be the case, as can be seen via a result of Fatou: a power series with integer coefficients and radius of convergence 1 is either rational or transcendental (in fact the transcendental case necessarily involves a natural boundary: this was a conjecture of Pólya proved by Carlson). However, we have the following neat generic behaviour.

Theorem 4.2 (main result: dyadic critical exponents for $\mathbb{R}_+$-algebraic function). *Any power series*

$$F(z) = \sum_{n \geqslant 0} f_n z^n,$$

which is a solution of a well-defined positive polynomial system of equations $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ (*i.e., any* $\mathbb{R}_+$*-algebraic function, and* a fortiori *any* $\mathbb{N}$*-algebraic function*) *has the following asymptotic behaviour.*

- *If* $F(z)$ *has a single singularity* ρ *on its radius of convergence* $|z| = \rho$, *then we have*

  $$f_n \sim C \frac{1}{\Gamma(1+\alpha)} n^{\alpha} \rho^{-n}$$

  for some positive (*algebraic if* P *or* F *have algebraic coefficients*) *constants* C *and* ρ, *and a dyadic critical exponent* α *which belongs to the set*

  $$\mathbb{D}_2 = \left\{-1 - \frac{1}{2^k} : k \geqslant 1\right\} \cup \left\{\frac{m}{2^k} - 1 : m \geqslant 1,\, k \geqslant 0\right\}. \tag{4.1}$$

  This set $\mathbb{D}_2$ *of possible critical exponents is sparse on* $[-3/2, -1)$ (*starting with the values* $\{-3/2, -5/4, -9/8, -17/16, \ldots\}$) *and dense on* $(-1, +\infty)$, *where it contains all dyadic numbers.*
- *If* $F(z)$ *has several singularities on its radius of convergence* $|z| = \rho$, *then there exists an integer* $p \geqslant 1$ *such that for every residue class* $\ell \in \{0, 1, \ldots, p-1\}$ *we have either* $f_n = 0$ *for sufficiently large* n *with* $n \equiv \ell \bmod p$, *or*

  $$f_n \sim C_\ell \frac{1}{\Gamma(1+\alpha_\ell)} n^{\alpha_\ell} A_\ell^n, \quad n \equiv \ell \bmod p,$$

  where A_ℓ *and* C_ℓ *are positive* (*algebraic if* P *or* F *have algebraic coefficients*) *constants and the critical exponent* α_ℓ *belongs to the set* $\mathbb{D}_2$ *defined in* (4.1).

Proof. In the case of a single singularity on the radius of convergence, this theorem is the consequence of our stronger theorem on the Puiseux expansion of $\mathbb{R}_+$-algebraic functions (see Theorem 5.3). The periodic case (this essentially means that $F(z)$ has several dominant singularities, as may be seen from $F(z) = 1/(1-9z^2) + 1/(1-8z^3)$, which is 6 periodic) involves some additional care, and we consider the full details of such cases in Section 6 (Theorem 6.3); this even gives a proof which works in a more general setting than well-defined polynomial systems. The algebraicity of the constants C, A_ℓ, ρ follows from our Proposition 2.6, while coupling it with our results on the Puiseux expansions (in the following sections) with the so-called 'transfer' Theorem VI.3 from [65]. □

Accordingly, we have the following two propositions.

Proposition 4.3 (dyadic critical exponents for $\mathbb{N}$-algebraic functions). *All the critical exponents of Theorem 4.2 are indeed obtained, even for the subclass of* $\mathbb{R}_+$*-algebraic functions made up of* $\mathbb{N}$*-algebraic functions.*

Proof. By singularity analysis [65], there is a direct link between the singular behaviour of $F(z)$ and the critical exponent of its coefficients f_n, namely, if $F(z) \sim (1-Az)^\alpha$ then

$$f_n \sim \frac{1}{\Gamma(-\alpha)} n^{-1-\alpha} A^n$$

(when $\rho = 1/A$ is the only singularity on the radius of convergence $|z| = \rho$). We will therefore show that all the singular behaviours corresponding to our dyadic set $\mathbb{D}_2$ of critical exponents are indeed obtained. The system of equations

$$y_1 = z(y_2 + y_1^2), y_2 = z(y_3 + y_2^2), y_3 = z(1 + y_3^2)$$

has the following (explicit) solution:

$$y_1(z) = \frac{1-(1-2z)^{1/8}\sqrt{2z\sqrt{2z\sqrt{1+2z}+\sqrt{1-2z}}+(1-2z)^{3/4}}}{2z}$$
$$y_2(z) = \frac{1-(1-2z)^{1/4}\sqrt{2z\sqrt{1+2z}+\sqrt{1-2z}}}{2z}, \quad \text{and}$$
$$y_3(z) = \frac{1-\sqrt{1-4z^2}}{2z}.$$

Here $y_1(z)$ has dominant singularity $(1-2z)^{1/8}$, and it is clear that this example can be generalized. Indeed, consider the system $y_i = z(y_{i+1} + y_i^2)$ for $i = 1, \ldots, k-1$, and $y_k = z(1 + y_k^2)$, which leads to behaviour $(1-2z)^{2^{-k}}$ for each $k \geqslant 1$. Now, taking the system of equations $y = z(y_0^m + y)$, $y_0 = z(1 + 2y_0y_1)$ leads to behaviour $(1-2z)^{-m2^{-k}}$ for each $m \geqslant 1$ and $k \geqslant 0$. See also [78, 112] for another explicit combinatorial structure (a family of coloured trees related to a critical composition) exhibiting all these critical exponents. □

Proposition 4.4 (non-ℕ-algebraicity). *Planar maps and several families of lattice paths* (*such as Gessel walks*) *are not* ℕ*-algebraic* (*i.e., they cannot be generated by an unambiguous context-free grammar*). *The Franel numbers* (*and other sequences counting some tuples of integer compositions having the same numbers of parts*) *are not algebraic.*

Proof. This comes as a nice consequence of our Theorem 4.2: none of the families of planar maps of [11] can be generated by an unambiguous context-free grammar, because of their critical exponent $-5/2$, see also Figure 1. Also, the tables [29] of lattice paths in the quarter plane and their asymptotics (where some of the connection constants are guessed, but all the critical exponents are proved, and this is enough for our point) allow us to prove that many sets of jumps give a non-algebraic generating function, as they lead to a critical exponent which is a negative integer or involving $1/3$. One very neat example is Gessel walks (their algebraicity was a nice surprise [30]), where the hypergeometric formula for their coefficients leads to an asymptotic $4^n/n^{2/3}$ that is not compatible with ℕ-algebraicity.[3]

[3] The fact that critical exponents involving $1/3$ were not possible was an informal conjecture in the community for years (see, *e.g.*, [61, 32] and Note 2 in [64]). We thank Philippe Flajolet, Mireille Bousquet-Mélou and

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The Franel number of order m, defined by

$$\sum_{k=0}^{n} \binom{n}{k}^m,$$

and other sequences counting some tuples of integer compositions having the same numbers of parts [13], *e.g.*, the sequence

$$\sum_{k=0}^{n} \binom{n-k}{k}^m,$$

do not have an algebraic generating function (for $m \geqslant 3$), as their asymptotics involve a non-algebraic multiplicative constant $C =$ algebraic number$/\pi^{(m-2)/2}$ (for even $m > 2$) or a negative integer critical exponent (for odd $m > 2$), both cases being incompatible with Theorem 4.2. □

The critical exponents $-3/4, -1/4, 1/4$ which appear for walks on the slit plane [34] and other lattice path questions [30] are compatible with $\mathbb{N}$-algebraicity, but these lattice paths are in fact not $\mathbb{N}$-algebraic (one can use Ogden's pumping lemma to prove that these walks cannot be generated by a context-free grammar). To get a constructive method to decide $\mathbb{N}$-algebraicity (input, a polynomial equation; output, a context-free specification, when it exists) is a challenging task.

It is well known that any $\mathbb{N}$-rational function has star height at most 2, *e.g.*, the regular expression $(x(x(xx^*)^*)^*)^*$ involves 3 nested stars but can also be written as $1 + x + x^*(3x)^*x^2 + x^2x^*$. For context-free grammars, one could consider the Chomsky and Greibach normal forms as a 'similar flavour' result. On the other hand, one consequence of our Theorem 5.3 is that there exist context-free languages with unbounded 'non-terminal height', as shown more precisely in the following result.

Proposition 4.5 (unbounded number of non-terminals for context-free grammars). *For all $k \in \mathbb{N}$, there exists a context-free language requiring inherently at least k non-terminals for any grammar generating it, and there exists a context-free language requiring inherently at least k non-strongly connected components for any grammar generating it.*

Proof. Indeed, the integer k (in Theorem 4.2) is the number of 'nested dominant critical components' (as is transparent from our proof in Section 5.4), and each of these components requires at least one non-terminal. Multicoloured supertrees are an example of a structure requiring k non-terminals: they are a generalization of Example VI.10 in [65], *i.e.*, they are 'trees of trees of trees . . . ' with nodes of two colours, defined via $\mathcal{T}_{k+1} = \mathcal{T}_k[2\mathcal{Z}\mathcal{T}_k]$ and $\mathcal{T}_0 = \mathcal{Z} \times \mathrm{Seq}(\mathcal{T}_0)$. □

In this section we have characterized the critical exponents of coefficients of algebraic functions. Can we also characterize the subdominant critical exponents? Well, for algebraic

Gilles Schaeffer, who encouraged us to work on this question. Non-algebraicity of Franel numbers was another folklore conjecture; see, *e.g.*, [13]. We thank Alin Bostan for pointing out this example to us. He also has a nice proof of their non-algebraicity via a p-Lucas property satisfied by this sequence.

functions, it is a consequence of their local Puiseux expansions and of singularity analysis that their coefficients behave, for example, like

$$f_n = A^n n^\alpha \left(\sum_{k\geqslant 0} a_k \frac{1}{n^k} \right) + B^n n^\beta \left(\sum_{k\geqslant 0} b_k \frac{1}{n^k} \right) + O(C^n),$$

where $C < B$, with $A > B$ or $A = B$ and $\alpha > \beta$. Moreover, we have proved in Theorem 4.2 that for $\mathbb{N}$-algebraic functions (and more generally for $\mathbb{Q}_+$-algebraic functions), α belongs to a specific subset of dyadic numbers. It is thus natural to ask what can be said for β. In fact, the union of Proposition 2.5 ($\mathbb{Z}$-algebraic functions are the difference of two $\mathbb{N}$-algebraic functions) and Theorem 4.1 ($\mathbb{Z}$-algebraic functions can have any rational critical exponent) implies that subdominant critical exponents β of $\mathbb{N}$-algebraic functions can be any rational number that is not a negative integer.

5. Finer asymptotics for $\mathbb{R}_+$-algebraic functions

The main goal of this section is to obtain a theorem on the Puiseux expansion of $\mathbb{R}_+$-algebraic functions.

Section 5.1 gives precise conditions on the system. Section 5.2 gives our fundamental result, Theorem 5.3, which implies Theorem 4.2. Section 5.3 introduces the notion of dependency graph and a few preliminary lemmas, while the last subsection, Section 5.4, gives the proof of Theorem 5.3.

5.1. Well-defined versus well-posed systems of functional equations

In Definitions 2.3 and 2.4, we have described the so-called well-posed and well-defined systems of algebraic equations $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$, and by definition it is clear that every positive well-defined system is also well-posed. (We just have to start with $\mathbf{y}_0 = \mathbf{0}$ and consider the iteration $\mathbf{y}_{k+1} = \mathbf{P}(z, \mathbf{y}_k)$, which converges formally and analytically to a tuple of power series

$$(f_1(z), \ldots f_d(z)) = \lim_{k\to\infty} \mathbf{y}_k$$

with non-negative coefficients and the property $f_i(0) = 0$.) However, as already indicated, there are also meaningful systems with power series solutions that are not well-defined in the sense of Definition 2.4. The essential point is that such a *meaningful* system has power series solutions $y_j = f_j(z)$ with non-negative coefficients. (Of course, if the algebraic system is positive then we can expect non-negative coefficients, in particular if the iteration from above converges.)

Let us make this more precise by formulating an analytic condition for systems to be meaningful.

Definition 5.1 (analytically well-defined system). A positive system of polynomial equations $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ will be called *analytically well-defined* if $\mathbf{P}(0, \mathbf{0}) = \mathbf{0}$, if the Jacobian

$$\mathbf{P}_\mathbf{y}(0, \mathbf{0}) = \left(\frac{\partial P_i}{\partial y_j}(0, \mathbf{0}) \right)$$

has spectral radius smaller than 1, and if all solution functions $y_j = f_j(z)$ (with $f_j(0) = 0$) are neither zero nor polynomials.

The condition on the spectral radius of $\mathbf{P_y}$ ensures that the matrix $\mathbf{I} - \mathbf{P_y}(0, \mathbf{0})$ (which is the Jacobian of the system $\mathbf{y} - \mathbf{P}(z, \mathbf{y}) = \mathbf{0}$) is invertible, so the implicit function theorem implies that there is a unique tuple of analytic solution $(f_1(z), \ldots, f_d(z))$ with $f_j(0) = 0$. Furthermore, this solution is obtained as the limit $(f_1(z), \ldots f_d(z)) = \lim_{k\to\infty} \mathbf{y}_k$ of the iteration $\mathbf{y}_{k+1} = \mathbf{P}(z, \mathbf{y}_k)$ with $\mathbf{y}_0 = \mathbf{0}$ (actually the iteration is uniform for $|z| \leqslant \eta$, where $\eta > 0$ is sufficiently small). Since all the iterates $\mathbf{y}_k$ are polynomials with non-negative coefficients, the (uniform) limit has the same non-negativity property. Note that this convergence need not be formal, as the example $y = z + \frac{1}{2}y$ shows.

As mentioned above, the condition $\mathbf{P}(0, \mathbf{0}) = \mathbf{0}$ is not a real restriction. If $\mathbf{P}(0, \mathbf{0}) \neq \mathbf{0}$ and if there exists a non-negative vector $\mathbf{y}_0$ with $\mathbf{P}(0, \mathbf{y}_0) = \mathbf{y}_0$ such that the spectral radius of $\mathbf{P_y}(0, \mathbf{y}_0)$ is smaller than 1, then the same argument as in the preceding paragraph shows that there exists a unique tuple of analytic solution $(f_1(z), \ldots, f_d(z))$ with $(f_1(0), \ldots, f_d(0)) = \mathbf{y}_0$. Furthermore, we apply a shift to reduce it to the case $\mathbf{P}(0, \mathbf{0}) = \mathbf{0}$. We set $\tilde{\mathbf{y}} = \mathbf{y} + \mathbf{y}_0$ so that we obtain a system for $\tilde{\mathbf{y}}$ of the form $\tilde{\mathbf{y}} = \tilde{\mathbf{P}}(z, \tilde{\mathbf{y}})$ with $\tilde{\mathbf{P}}(z, \tilde{\mathbf{y}}) = \mathbf{P}(z, \tilde{\mathbf{y}} + \mathbf{y}_0) - \mathbf{y}_0$. Since $\mathbf{P}$ has non-negative coefficients, the same holds for $\tilde{\mathbf{P}}$. Consequently, there is no loss of generality in assuming that we have a system $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ with $\mathbf{P}(0, \mathbf{0}) = \mathbf{0}$.

Finally, it is very easy to detect whether a meaningful system has some zero or polynomial solutions $f_j(z)$. In any case, if zero or polynomial solutions $f_j(z)$ appear, we simply replace all appearances of y_j with $f_j(z)$ and remove the jth equation. This leads to a subsystem where no solution $f_j(z)$ is zero or a polynomial. The Jacobian of the new (and smaller) system is just a submatrix of the original Jacobian, and thus has a spectral radius that is not larger than that of the original one. So if the spectral radius of the Jacobian of the original system is smaller than 1, we get the same property for the new system.

Summing up, it is no loss of generality to consider analytically well-defined systems. Furthermore, well-defined systems are also analytically well-defined.

Lemma 5.2. *Every well-defined system* $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ *is analytically well-defined. And every analytically well-defined system is well-posed.*

Proof. By definition, a well-defined system satisfies $\mathbf{P}(0, \mathbf{0}) = \mathbf{0}$. Furthermore, the condition $[y_j]P_i(0, 0, \ldots, y_i, \ldots) = 0$ implies that $\mathbf{P_y}(0, \mathbf{0})$ is the zero matrix, which has spectral radius 0. Finally, the terminating condition ensures that there are no zero or polynomial solutions $f_j(z)$.

We have already discussed that every analytically well-defined system has a tuple of power series solutions $f_j(z)$ with non-negative coefficients. This completes the proof of the lemma. □

There are several reasons why we distinguish between well-defined and analytically well-defined systems of equations. From a formal point of view, well-defined systems are very easy to describe, since we just have to look at the polynomial system. On the

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other hand, it excludes some meaningful systems (in particular systems having epsilon production or loops of monic productions with a total weight < 1). When it comes to proofs, it is easier to work with analytic conditions such as the condition on the spectral radius, and therefore we will mainly rely on analytically well-defined systems. A good motivation for this approach is that the analysis of the behaviour of the spectral radius $r(\mathbf{P}_{\mathbf{y}}(z,\mathbf{f}(z)))$ actually plays a dominant role in the proof of Theorem 5.3.

5.2. Critical exponents of $\mathbb{R}_+$-algebraic functions

Our first main goal is to characterize the Puiseux critical exponents of the singular expansions of $f_j(z)$ at the radius of convergence ρ_j, when we are considering power series solutions of analytically well-defined positive polynomial systems of equations. The main observation is that these exponents are special dyadic rational numbers, in contrast to general algebraic functions (see Theorem 4.1).

Theorem 5.3. *Let $\mathbf{y} = \mathbf{P}(z,\mathbf{y})$ be an analytically well-defined positive polynomial system of functional equations.*

Then the solutions $f_j(z)$ have positive and finite radii of convergence ρ_j and the Puiseux critical exponents are either of the form 2^{-k_j} for some integer $k_j \geqslant 1$ or of the form $-m_j 2^{-k_j}$ for some integer $m_j \geqslant 1$ and $k_j \geqslant 0$. In particular, the singular behaviour of $f_j(z)$ around ρ_j is either of type

$$f_j(z) = f_j(\rho_j) + c_j(1 - z/\rho_j)^{2^{-k_j}} + c_j'(1 - z/\rho_j)^{2\cdot 2^{-k_j}} + \cdots , \tag{5.1}$$

where $c_j \neq 0$ (and an integer $k_j \geqslant 1$), or of type

$$f_j(z) = \frac{d_j}{(1 - z/\rho_j)^{m_j 2^{-k_j}}} + \frac{d_j'}{(1 - z/\rho_j)^{(m_j-1)2^{-k_j}}} + \cdots , \tag{5.2}$$

where $d_j \neq 0$ (and integers $m_j \geqslant 1$ and $k_j \geqslant 0$).

This theorem already gives a partial result for the asymptotic structure of the coefficients $f_{j;n}$ of $f_j(z)$. If we assume that ρ_j is the only singularity on the circle of convergence $|z| = \rho_j$ (which we call the *aperiodic case*), then the transfer theorem of Flajolet and Odlyzko [63] implies that $f_{j;n}$ is asymptotically given by

$$f_{j;n} \sim C_j n^{\alpha_j} \rho_j^{-n} \quad (n \to \infty), \tag{5.3}$$

where $C_j > 0$, $\rho_j > 0$, and α_j is either of the form $\alpha_j = -2^{-k_j} - 1$ for some integer $k_j \geqslant 1$ or of the form $\alpha_j = m_j/2^{k_j} - 1$ for some integers $k_j \geqslant 0$ and $m_j \geqslant 1$.

In fact we will provide a complete answer to the problem in the periodic case too: see Theorems 6.2 and 6.3. In all cases we obtain asymptotic properties as stated in (5.3). However, we have to distinguish between residue classes modulo some positive integer p, and the asymptotic scale might be different in each residue class. In order to make the presentation more transparent, we first deal with Theorem 5.3 and then consider the more involved question of *periodicities*.

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5.3. Dependency graph and auxiliary results

A main ingredient of the proof of Theorem 5.3 is the analysis of the *dependency graph* G of the system $y_j = P_j(z, y_1, \ldots, y_d)$, $1 \leqslant j \leqslant d$. The vertex set is $\{1, \ldots, d\}$, and there is a directed edge from i to j if P_j depends on y_i (see Figure 2). If the dependency graph is strongly connected then we are in a very special case of Theorem 5.3, for which we have one of the following two situations (see [53]).

Lemma 5.4 (rational singular behaviour). *Let* $\mathbf{y} = \mathbf{A}(z)\mathbf{y} + \mathbf{B}(z)$ *be an affine analytically well-defined system of equations, where the dependency graph is strongly connected. Then the functions* $f_j(z)$ *have a joint polar singularity* ρ *of order one as the dominant singularity:*

$$f_j(z) = \frac{c_j(z)}{1 - z/\rho},$$

where $c_j(z)$ *is non-zero and analytic at* $z = \rho$.

Proof. In the affine case the Jacobian of the system equals $\mathbf{A}(z)$. Hence, by assumption, the spectral radius of $\mathbf{A}(0)$ is smaller than 1; this implies that $\mathbf{f}(z)$ can be represented as

$$\mathbf{f}(z) = (\mathbf{I} - \mathbf{A}(z))^{-1}\mathbf{B}(z)$$

if $|z|$ is sufficiently small. Since the dependency graph is strongly connected, it follows that the matrix $\mathbf{A}(z)$ is a positive and irreducible matrix if $z > 0$. Consequently, by the Perron–Frobenius theorem, the spectral radius $r(\mathbf{A}(z))$ is a strictly increasing and continuous function in $z > 0$. Hence, there exists a unique $\rho > 0$ with $r(\mathbf{A}(\rho)) = 1$. Again by the Perron–Frobenius theorem, the spectral radius is the dominant eigenvalue of $\mathbf{A}(\rho)$ and is also simple. This also implies that the function

$$z \mapsto \det(\mathbf{I} - \mathbf{A}(z))$$

has a simple root at $z = \rho$. Of course, this leads to a simple polar singularity for $\mathbf{f}(z)$. Note that this singularity has to appear for all functions $f_j(z)$, $1 \leqslant j \leqslant d$, since the system is strongly connected. □

Lemma 5.5 (algebraic singular behaviour). *Let* $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ *be an affine analytically well-defined polynomial system of equations that is not affine and where the dependency graph is strongly connected. Then the functions* $f_j(z)$ *have a joint radius of convergence* ρ *and Puiseux singular exponent* $1/2$ *at* $z = \rho$*, that is, they can be locally represented as*

$$f_j(z) = g_j(z) - h_j(z)\sqrt{1 - \frac{z}{\rho}},$$

where $g_j(z)$ *and* $h_j(z)$ *are non-zero and analytic at* $z = \rho$.

Proof. Since the system is positive and well-posed, there exists a unique solution $\mathbf{f}(z)$ with $\mathbf{f}(0) = \mathbf{0}$ which has non-negative coefficients. By assumption the spectral radius of the Jacobian $\mathbf{P_y}(0, \mathbf{0})$ is smaller than 1. Since the dependency graph is strongly connected, the

$$\begin{cases} y_1 = P_1(z, y_1, y_2, y_5) \\ y_2 = P_2(z, y_2, y_3, y_5) \\ y_3 = P_3(z, y_3, y_4) \\ y_4 = P_4(z, y_3, y_4) \\ y_5 = P_5(z, y_5, y_6) \\ y_6 = P_6(z, y_5, y_6) \end{cases}$$

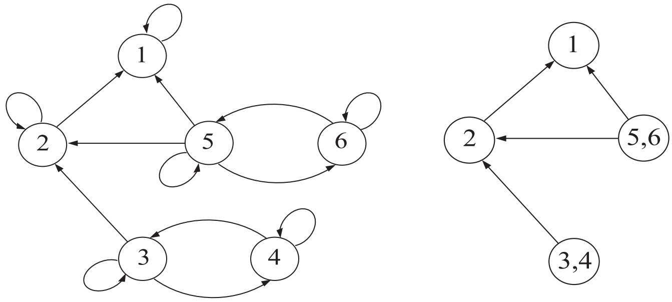

Figure 2. A positive system, its dependency graph G and its reduced dependency graph $\widetilde{G}$. None of these graphs are strongly connected. For example, the state 1 is a sink; it is thus a typical example of a system not covered by the Drmota–Lalley–Woods theorem but covered by our new result implying dyadic critical exponents.

matrix $\mathbf{P_y}(z, \mathbf{f}(z))$ is a positive and irreducible matrix for $z > 0$ (as long as $\mathbf{f}(z)$ is regular). Furthermore the spectral radius $r(\mathbf{P_y}(z, \mathbf{f}(z))))$ is a strictly increasing and continuous function in $z > 0$. Recall that (by the implicit function theorem) $\mathbf{f}(z)$ is certainly regular if $r(\mathbf{P_y}(z, \mathbf{f}(z)))) < 1$. Hence, it follows that there exists $\rho > 0$ with the property that $\mathbf{f}(\rho)$ exists (although it will not be a regular point) and $r(\mathbf{P_y}(\rho, \mathbf{f}(\rho)))) = 1$. Thus, we can now apply the Drmota–Lalley–Woods theorem [54, 65], which implies that ρ is the dominant singularity for $f_j(z)$ and they are all of square-root type.[4] □

In the proof of Theorem 5.3 we will use extended versions of Lemmas 5.4 and 5.5, where we introduce additional parameters, that is, we consider systems of functional equations of the form

$$\mathbf{y} = \mathbf{P}(z, \mathbf{y}, \mathbf{u}),$$

where $\mathbf{P}$ is now a polynomial in $z, \mathbf{y}, \mathbf{u}$ with $\mathbf{P}(0, \mathbf{0}, \mathbf{0}) = \mathbf{0}$ and non-negative coefficients and where the dependency graph (with respect to $\mathbf{y}$) is strongly connected. We also assume that $r(\mathbf{P_y}(0, \mathbf{0}, \mathbf{0})) < 1$ so that we can consider the solution that we denote by $\mathbf{f}(z, \mathbf{u})$. We also consider situations where $\mathbf{u}$ is strictly positive from the very beginning. In this case we restrict ourselves to situations where $\mathbf{f}(0, \mathbf{u})$ exists and where the spectral radius of the Jacobian $\mathbf{P_y}(0, \mathbf{f}(0, \mathbf{u}), \mathbf{u})$ is smaller than 1.

If we are in the affine setting ($\mathbf{y} = \mathbf{A}(z, \mathbf{u})\mathbf{y} + \mathbf{B}(z, \mathbf{u})$) it follows that $\mathbf{f}(z, \mathbf{u})$ has a polar singularity:

$$f_j(z, \mathbf{u}) = \frac{c_j(z, \mathbf{u})}{1 - z/\rho(\mathbf{u})}, \tag{5.4}$$

where the functions $\rho(\mathbf{u})$ and $c_j(z, \mathbf{u})$ are non-zero and analytic (see Lemma 5.6). Note that we have to distinguish two cases. If $\mathbf{A}(z, \mathbf{u}) = \mathbf{A}(z)$ does not depend on $\mathbf{u}$, then $\rho(\mathbf{u}) = \rho$ is constant and the dependency from $\mathbf{u}$ just comes from $\mathbf{B}(z, \mathbf{u})$. Of course, if $\mathbf{A}(z, \mathbf{u})$ depends on $\mathbf{u}$ then $\rho(\mathbf{u})$ is not constant. More precisely, it depends exactly on those coordinates of $\mathbf{u}$ that appear in $\mathbf{A}(z, \mathbf{u})$.

[4] The proofs of Lemma 5.4 and 5.5 could be simplified, since we work only with algebraic functions. However, in Section 9 we will also consider entire systems of functional equations, and the present proof generalizes to this situation.

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Similarly, in the non-affine setting we obtain representations of the form

$$f_j(z,\mathbf{u}) = g_j(z,\mathbf{u}) - h_j(z,\mathbf{u})\sqrt{1-\frac{z}{\rho(\mathbf{u})}}, \tag{5.5}$$

where the functions $\rho(\mathbf{u})$, $g_j(z,\mathbf{u})$, and $h_j(z,\mathbf{u})$ are non-zero and analytic. In this case $\rho(\mathbf{u})$ is always non-constant and depends on all coordinates of $\mathbf{u}$ (see Lemma 5.6).

In fact we have to be careful with the property that $\rho(\mathbf{u})$ is analytic. By looking at the above proofs it immediately follows that $\rho(\mathbf{u})$ exists but analyticity is not immediate. For notational convenience we will denote by D_0 the set of positive real vectors $\mathbf{u}$ for which

$$r(\mathbf{P}_{\mathbf{y}}(0,\mathbf{f}(0,\mathbf{u}),\mathbf{u})) < 1.$$

Lemma 5.6. *The function $\rho(\mathbf{u})$ that appears in the representations* (5.4) *and* (5.5) *is analytic in a proper complex neighbourhood of D_0. Moreover, if $\mathbf{u} \in D_0$ is real, then $\rho(\mathbf{u})$ tends to* 0 *when $\mathbf{u}$ approaches the boundary of D_0 in such a way that all coordinates of $\mathbf{u}$ are non-decreasing.*

Proof. We recall a general method for reducing the system $\mathbf{y} = \mathbf{P}(z,\mathbf{y},\mathbf{u})$ to a single equation if the dependency graph is strongly connected. We split the first equation into

$$y_1 = P_1(z,y_1,\overline{\mathbf{y}},\mathbf{u}) \quad \text{and} \quad \overline{\mathbf{y}} = \overline{\mathbf{P}}(z,y_1,\overline{\mathbf{y}},\mathbf{u}),$$

where

$$\overline{\mathbf{y}} = (y_2,\dots,y_d) \quad \text{and} \quad \overline{\mathbf{P}} = (P_2,\dots,P_d).$$

Suppose that $r(\mathbf{P}_{\mathbf{y}}(\rho,\mathbf{y}_0,\mathbf{u}_0)) = 1$ for some $\mathbf{u}_0 \in D_0$ (and $\rho = \rho(\mathbf{u}_0)$, $\mathbf{y}_0 = \mathbf{y}_0(\mathbf{u}_0)$). Then, by the Perron–Frobenius theorem,

$$r(\overline{\mathbf{P}}_{\overline{\mathbf{y}}}(\rho,y_{1,0},\overline{\mathbf{y}}_0,\mathbf{u}_0)) < 1,$$

since $\overline{\mathbf{P}}_{\overline{\mathbf{y}}}$ is the submatrix that results from $\mathbf{P}_{\mathbf{y}}$ by deleting the first row and column. Hence, by the implicit function theorem, there is an analytic solution $\overline{\mathbf{f}}(z,y_1,\mathbf{u})$ of the subsystem (locally around $\rho, y_{1,0}, \mathbf{u}_0$) that we can insert into the first equation, so that we are left with a single equation:

$$y_1 = P_1(z,y_1,\overline{\mathbf{f}}(z,y_1,\mathbf{u}),\mathbf{u}) = Q(z,y_1,\mathbf{u}).$$

In the affine case this rewrites to $y_1 = a(z,\mathbf{u})y_1 + b(z,\mathbf{u})$ or $y_1 = b(z,\mathbf{u})/(1-a(z,\mathbf{u}))$. Since we are in the well-posed case, $a(z,\mathbf{u})$ depends on z and $\mathbf{u}$. Furthermore $a(\rho(\mathbf{u}_0,\mathbf{u}_0)) = 1$. Since we certainly have $a_z(\rho(\mathbf{u}_0),\mathbf{u}_0) > 0$, the implicit function theorem implies that $\rho(\mathbf{u})$ is analytic locally at $\mathbf{u}_0 \in D_0$.

In the non-affine case, the situation is similar but slightly more involved. Since the equation $y_1 = Q(z,y_1,\mathbf{u})$ is singular for $z = \rho, y_1 = y_{1,0}, \mathbf{u} = \mathbf{u}_0$, we have

$$y_{1,0} = Q(\rho,y_{1,0},\mathbf{u}_0), \quad 1 = Q_{y_1}(\rho,y_{1,0},\mathbf{u}_0).$$

Furthermore, this small system can be used to calculate $\rho(\mathbf{u})$ (locally near $\mathbf{u}_0$). Here $\mathbf{u}$ is the variable and $\rho = \rho(\mathbf{u})$, $y_{1,0} = y_{1,0}(\mathbf{u}_0)$ are the unknown functions. By the implicit

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function theorem we just have to observe that the corresponding Jacobian

$$\begin{pmatrix} Q_z & Q_{y_1}-1 \\ Q_{zy_1} & Q_{y_1y_1} \end{pmatrix} = \begin{pmatrix} Q_z & 0 \\ Q_{zy_1} & Q_{y_1y_1} \end{pmatrix}$$

is regular. We certainly have $Q_z > 0$ and $Q_{y_1y_1} > 0$ (if the system is non-affine). Hence, the determinant of the Jacobian is non-zero. This implies that $\rho(\mathbf{u})$ is analytic in a complex neighbourhood of $\mathbf{u}_0 \in D_0$.

Finally, if $\mathbf{u}$ increases and gets close to the boundary of D_0, then the spectral radius

$$r(\mathbf{P}_\mathbf{y}(0, \mathbf{f}(0,\mathbf{u}), \mathbf{u}))$$

is close to 1. This implies that the radius of convergence $\rho(\mathbf{u})$ has to be close to zero. □

5.4. Proof of Theorem 5.3 on possible Puiseux expansions

In order to give a flavour of the proof of Theorem 5.3 in the general case, we first discuss a simple example. Suppose we are dealing with the system of equations depicted in Figure 2. The first step is to consider the reduced dependency graph $\widetilde{G}$, which is obtained by the following procedure. The vertices of $\widetilde{G}$ are the strongly connected components of G; these are the maximal strongly connected subgraphs. In our example, these components are $\{1\}$, $\{2\}$, $\{3,4\}$, $\{5,6\}$. Next, two different components C_1, C_2 in $\widetilde{G}$ are linked by a directed edge if there exist vertices $v_1 \in C_1$ and $v_2 \in C_2$ that are linked in G. The resulting graph $\widetilde{G}$ (also depicted in Figure 2) is acyclic and comprises precisely the connectivity relation in G. Furthermore, this directed acyclic graph (DAG) $\widetilde{G}$ indicates how the system of equations $y_j = P_j(z, y_1, \ldots, y_d)$ can be solved. First, one considers all components in G (vertices in $\widetilde{G}$) with zero in-degree. (Since $\widetilde{G}$ is acyclic such vertices have to exist.) In our example, these are the components $\{3,4\}$ and $\{5,6\}$, which correspond to the subsystems

$$\begin{aligned} y_3 &= P_3(z, y_3, y_4) \\ y_4 &= P_4(z, y_3, y_4) \end{aligned} \quad \text{and} \quad \begin{aligned} y_5 &= P_3(z, y_5, y_6) \\ y_6 &= P_4(z, y_5, y_6). \end{aligned}$$

These subsystems can be independently solved and their solutions $f_3(z), f_4(z)$ and $f_5(z), f_6(z)$, respectively, can be put into the remaining equations:

$$\begin{aligned} y_1 &= P_1(z, y_1, y_2, f_5(z)), \\ y_2 &= P_2(z, y_2, f_3(z), f_5(z)). \end{aligned}$$

This resulting system of equation for the unknown $y_1 = f_1(z)$, $y_2 = f_2(z)$ corresponds to a dependency graph, where the corresponding vertices $3,4$ and $5,6$ are deleted. This is depicted in Figure 2. As above, we can solve all equations that correspond to components in the reduced dependency graph with zero in-degree. In our example, this is only the component $\{2\}$. With this solution $f_2(z)$ (where we already use the previous solutions $f_3(z)$, $f_5(z)$), we can finally obtain $y_1 = f_1(z)$ by solving the remaining equation

$$y_1 = P_1(z, y_1, f_2(z), f_5(z)).$$

Of course, this procedure generalizes easily to any system of functional equations of the form $y_j = P_j(z, y_1, \ldots, y_d)$.

In fact we will use a two-step procedure. First, for each component of the dependency graph we solve the corresponding system, where the input functions are considered as

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additional parameters $\mathbf{u}$ (as discussed in Section 5.3). For example, for the component $\{2\}$, one should consider the solution $f_2(z; y_3, y_5)$.

According to the generalizations of Lemmas 5.4 and 5.5, these functions have either a polar singularity or a square-root singularity $\rho(\mathbf{u})$ (which depends on the parameters). In the second step we then insert step by step the solutions of the subsystems and get the solution of the system. The main problem is to trace the leading singularity. For example, in the first example in the proof of Proposition 4.3, the square-root singularities coalesce and give rise to singularities of fourth and eighth roots. Similarly, in the second example in the proof of Proposition 4.3, a polar and a square-root singularity give rise to a singularity of the form $1/\sqrt{1-2z}$.

The main problem in the proof of Theorem 5.3 is to show that this insertion process does not create other singularities than stated.

We fix some notation. Let G denote the dependency graph of the system and $\widetilde{G}$ the reduced dependency graph. Its vertices are the strongly connected components $C_1, \dots C_L$ of G. We can then *reduce* the dependency graph to its components (see Figure 2).

Let $\mathbf{y}_1, \dots, \mathbf{y}_L$ denote the system of vectors with coordinates corresponding to the components $C_1, \dots C_L$, and let $\mathbf{u}_1, \dots, \mathbf{u}_L$ denote the input vectors related to these components. In the above example we have

$$\begin{array}{llll} C_1 = \{1\}, & C_2 = \{2\}, & C_3 = \{3,4\}, & C_4 = \{5,6\}, \\ \mathbf{y}_1 = y_1, & \mathbf{y}_2 = y_2, & \mathbf{y}_3 = (y_3, y_4), & \mathbf{y}_4 = (y_5, y_6), \\ \mathbf{u}_1 = (y_2, y_5), & \mathbf{u}_2 = (y_3, y_5), & \mathbf{u}_3 = \emptyset, & \mathbf{u}_4 = \emptyset. \end{array}$$

As mentioned above in the first step, for each strongly connected component C_ℓ we solve the corresponding subsystem in the variables z and $\mathbf{u}_\ell$ and obtain solutions $\mathbf{f}(z, \mathbf{u}_\ell)$, $1 \leqslant \ell \leqslant L$. In our example these are the functions

$$\begin{array}{ll} \mathbf{f}_1(z, \mathbf{u}_1) = f_1(z; y_2, y_5), & \mathbf{f}_2(z, \mathbf{u}_2) = f_2(z; y_3, y_5), \\ \mathbf{f}_3(z, \mathbf{u}_3) = (f_3(z), f_4(z)), & \mathbf{f}_4(z, \mathbf{u}_4) = (f_5(z), f_6(z)). \end{array}$$

Finally, for each component C_ℓ we define the set D_ℓ of real vectors $\mathbf{u}_\ell$ for which the spectral radius of the Jacobian of the ℓth subsystem evaluated at $z = 0$, $\mathbf{y}_\ell = f_\ell(0, \mathbf{u}_\ell$ is smaller than 1.

Since the dependency graph $\widetilde{G}$ is acyclic, there are components $C_{\ell_1}, \dots, C_{\ell_m}$ with no input, that is, the corresponding functions $\mathbf{f}_{\ell_1}(z), \dots, \mathbf{f}_{\ell_m}(z)$ can be computed without any further information. By Lemmas 5.4 and 5.5, they either have a polar singularity or a square-root singularity, that is, they are of the types that are included in the statement of Theorem 5.3.

Now, we proceed inductively. We consider a strongly connected component C_ℓ together with its function $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ and assume that all the functions $f_j(z)$ that correspond to the input coordinates $\mathbf{u}_\ell$ are already known, and that their leading singularities are of the two types stated in Theorem 5.3. By the discussion following Lemmas 5.4 and 5.5, it follows that coordinate functions in $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ have either a common polar singularity or a common square-root singularity $\rho(\mathbf{u}_\ell)$. We distinguish between three cases.

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Case 1. First, let us assume that $\mathbf{f}_\ell(z,\mathbf{u}_\ell)$ comes from an affine system and thus has a polar singularity. Since all functions in $\mathbf{f}_\ell(z,\mathbf{u}_\ell)$ have the same form, we just consider one of these functions and denote it by $f(z,\mathbf{u}_\ell)$:

$$f(z,\mathbf{u}_\ell) = \frac{c(z,\mathbf{u}_\ell)}{1-z/\rho(\mathbf{u}_\ell)}. \tag{5.6}$$

If $\rho(\mathbf{u}_\ell) = \rho'$ is constant then the only dependency from $\mathbf{u}_\ell$ comes from the numerator $c(z,\mathbf{u}_\ell)$. Since this solution comes from an affine system, $c(z,\mathbf{u}_\ell)$ is just a linear combination of the polynomials of $\mathbf{B}(z,\mathbf{u}_\ell)$ with coefficient functions that depend only on z (this follows from the expansion of $(\mathbf{I}-\mathbf{A}(z))^{-1}\mathbf{B}(z,\mathbf{u}_\ell)$). Furthermore, since $f(z,\mathbf{u}_\ell)$ is (in principle) a power series in z and $\mathbf{u}_\ell$ with non-negative coefficients, the coefficients of this polynomial (if z is some positive real number) have to be non-negative, too.

When we substitute $\mathbf{u}_\ell$ with the functions $f_j(z)$ that correspond to $\mathbf{u}_\ell$, we obtain the functions $f(z)$ that correspond to the component C_ℓ. We have to consider the following subcases.

Case 1.1. The dominating singularities ρ_j of the $f_j(z)$ are larger than ρ'. In this case the resulting dominating singularity ρ_ℓ is ρ' and we just get a polar singularity for $f(z)$.

Case 1.2. At least one of the dominating singularities ρ_j of the functions $f_j(z)$ is smaller than ρ'. Let ρ'' denote the smallest of these singularities. If all of the functions $f_j(z)$ with $\rho_j = \rho''$ have a singular behaviour of the form (5.1), then we just make a local expansion of $c(z,\mathbf{u}_\ell)$ at the corresponding points $f_j(\rho'')$ (for u_j) and observe again an expansion of this form. Note that the largest appearing k_j reappears in the expansion of $f(z)$.

Second, if at least one of the functions $f_j(z)$ with $\rho_j = \rho''$ is of type (5.2), then we use the property that $c(z,\mathbf{u}_\ell)$ is just a polynomial in $\mathbf{u}_\ell$ (with non-negative coefficients). It is clear that the leading singular behaviour comes from these functions. In fact they have to be multiplied and added. However, since functions of the type (5.2) are closed under multiplication and addition, this again gives a function of type (5.2). Note that the coefficient functions that depend just on z have to be expanded at ρ'', too, and do not disturb the overall structure.

Case 1.3. The smallest dominating singularities ρ_j of the functions $f_j(z)$ equals ρ'. Here we can argue similarly to the previous case. If all of the functions $f_j(z)$ with $\rho_j = \rho'$ have a singular behaviour of the form (5.1) then we perform a local expansion in the numerator. Let $\tilde{k}$ be the largest k_j that appears. Then we interpret the polar singularity $(1-z/\rho')^{-1}$ as $(1-z/\rho')^{-m2^{-\tilde{k}}}$ with $m = 2^{\tilde{k}}$ and obtain a singular expansion of the form (5.2). If at least one of the functions $f_j(z)$ with $\rho_j = \rho'$ is of type (5.2), then we use the polynomial structure of the numerator as above and obtain an expansion of the form (5.2). By combining this with the factor $(1-z/\rho')^{-1}$, we finally obtain an expansion of the form (5.2) for $f(z)$ too.

Case 2. Second, let us (again) assume that $\mathbf{f}_\ell(z,\mathbf{u}_\ell)$ comes from an affine system (and has a polar singularity) of the form (6.1). However, we now assume that $\rho(\mathbf{u}_\ell)$ is not constant but depends on some of the u_j (not necessarily on all of them). In this case we first study the behaviour of the denominator when $\mathbf{u}_j$ is replaced with the corresponding functions $f_j(z)$. For the sake of simplicity we will work with the difference $\rho(\mathbf{u}_j) - z$. Of course, this is equivalent to the discussion of the denominator $1 - z/\rho(\mathbf{u}_j)$, since the factor $\rho(\mathbf{u}_j)$ can also be put to the numerator. Finally, let J'_ℓ denote the set of indices of functions u_j on which the function $\rho(\mathbf{u}_\ell)$ actually depends.

Let ρ' denote the smallest radius of convergence of the functions $f_j(z)$, $j \in J'_\ell$. Then we consider the difference $\delta(z) = \rho((f_j(z))_{j\in J'_\ell}) - z$. We have to consider the following subcases for the denominator.

Case 2.1. $\delta(\rho'') = 0$ for some $\rho'' < \rho'$ such that $(f_j(\rho''))_{j\in J'_\ell} \in D_\ell$. First, we note that $\delta(z)$ has at most one positive zero since $\rho((f_j(z))_{j\in J'_\ell})$ is decreasing and z is increasing. Furthermore, the derivative satisfies $\delta'(\rho'') > 0$. Consequently, we have a simple zero ρ'' in the denominator.

Case 2.2. We have $\delta(\rho') = 0$ such that $(f_j(\rho'))_{j\in J'_\ell} \in D_\ell$. In this case all functions $f_j(z)$, $j \in J'_\ell$, with $\rho_j = \rho'$ have to be of type (5.1). Consequently $\delta(z)$ behaves like

$$c(1 - z/\rho')^{2^{-\tilde{k}}} + \cdots,$$

where $c > 0$ and $\tilde{k}$ is the largest appearing k_j (among those functions $f_j(z)$ with $\rho_j = \rho'$).

Case 2.3. We have $\delta(\rho') > 0$ such that $(f_j(\rho'))_{j\in J'_\ell} \in D_\ell$. In this case all functions $f_j(z)$, $j \in J'_\ell$, with $\rho_j = \rho'$ have to be (again) of type (5.1). Consequently $\delta(z)$ behaves like

$$c_0 - c_1(1 - z/\rho')^{2^{-\tilde{k}}} + \cdots,$$

where $c_0 > 0$ and $c_1 > 0$ and $\tilde{k}$ is the largest appearing k_j (among those functions $f_j(z)$ with $\rho_j = \rho'$). Hence, $1/\delta(z)$ is of type (5.1).

Note that there are no other subcases. This follows from the fact that $\rho(\mathbf{u}_\ell) \to 0$ if $\mathbf{u}_\ell$ approaches the boundary of D_ℓ. This means that if we trace the function $z \to \delta(z)$ for $z > 0$ then we either meet a singularity of $\delta(z)$ or we pass a zero of $\delta(z)$ before $(f_j(z))_{j\in J'_\ell})$ leaves D_ℓ.

Finally, we have to discuss the numerator (as in the above case). Note that there might occur u_j with $j \notin J'_\ell$, so that more functions $f_j(z)$ than in the denominator are involved. Nevertheless, in all possible subcases we can combine the expansions of the numerator and denominator and obtain for $f(z)$ either type (5.1) or (5.2).

Case 3. Finally, let us assume that $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ comes from a non-affine system and thus has a square-root singularity. Again, since all functions in $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ have the same form, we simply consider one of these functions and denote it by $f(z, \mathbf{u}_\ell)$:

$$f(z, \mathbf{u}_\ell) = g(z, \mathbf{u}_\ell) - h(z, \mathbf{u}_\ell)\sqrt{1 - \frac{z}{\rho(\mathbf{u}_\ell)}}. \tag{5.7}$$

In this case $\rho(\mathbf{u}_\ell)$ depends on all components of $\mathbf{u}_\ell$, which makes the analysis slightly easier. As above, we will study the behaviour of the square-root $\sqrt{\rho(\mathbf{u}_\ell) - z}$ instead of $\sqrt{1 - z/\rho(\mathbf{u}_\ell)}$, since the non-zero factor $\sqrt{\rho(\mathbf{u}_\ell)}$ can be put to $h(z, \mathbf{u}_\ell)$.

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Let ρ' denote the smallest radius of convergence of the functions $f_j(z)$ that correspond to $\mathbf{u}_\ell$. Here we have to consider the following subcases.

Case 3.1. $\delta(\rho'') = 0$ for some $\rho'' < \rho'$ such that $(f_j(\rho'')) \in D_\ell$. This means that $\rho((f_j(z)) - z$ has a simple zero. Thus, we can represent this function as

$$\rho((f_j(z)) - z = (\rho'' - z)H(z),$$

where $H(z)$ is non-zero and analytic at ρ''. Consequently

$$\sqrt{\rho((f_j(z)) - z} = \sqrt{\rho'' - z}\sqrt{H(z)},$$

and we observe that $f(z)$ has a (simple) square-root singularity.

Case 3.2. We have $\delta(\rho') = 0$ such that $(f_j(\rho')) \in D_\ell$. In this case all functions $f_j(z)$ with $\rho_j = \rho'$ have to be of type (5.1). Hence, the square-root of $\delta(z)$ behaves like

$$\sqrt{c(1 - z/\rho')^{2^{-\tilde{k}}} + \cdots} = \sqrt{c}(1 - z/\rho')^{2^{-\tilde{k}-1}} + \cdots,$$

where $\tilde{k}$ equals the largest appearing k_j plus 1. Thus, $f(z)$ is of type (5.1).

Case 3.3. We have $\delta(\rho') > 0$ such that $(f_j(\rho'))_{j \in J'_\ell} \in D_\ell$. In this case all functions $f_j(z)$, with $\rho_j = \rho'$ have to be (again) of type (5.1). Consequently the square-root of $\delta(z)$ behaves like

$$\sqrt{c_0 - c_1(1 - z/\rho')^{2^{-\tilde{k}}} + \cdots} = \sqrt{c_0}\left(1 - \frac{c_1}{2c_0}(1 - z/\rho')^{2^{-\tilde{k}}} + \cdots\right),$$

where $c_0 > 0$ and $c_1 > 0$ and $\tilde{k}$ is the largest appearing k_j (among those functions $f_j(z)$ with $\rho_j = \rho'$). Hence, $f(z)$ is of type (5.1).

This completes the induction proof of Theorem 5.3. □

6. Periodicities

When we are interested in the asymptotic properties of the coefficients of $\mathbb{R}_+$-algebraic equations, we need the structure of all singularities z on the radius of convergence $|z| = \rho$. When there are several such singularities, periodic behaviour can appear, justifying the following definition.

Definition 6.1 (strong (a)periodicity). A function $f(z)$ that is the solution of a positive system of algebraic equations will be called *strongly aperiodic* if $z = \rho$ is the only singularity on the circle $|z| = \rho$.

Similarly, we call such a function $f(z)$ *strongly periodic* with period $p > 1$ if $f(z)$ is not strongly aperiodic but can be represented as

$$f(z) = \sum_{j=0}^{p} z^j f_j(z^p)$$

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such that all functions $f_j(z)$ are either polynomials or strongly aperiodic functions and at least one of these functions is strongly aperiodic.

The main purpose of this section is to prove the following property.

Theorem 6.2. *Every function $f(z)$ that is the solution of an analytically well-defined positive polynomial system of equations (see Definition 5.1) is either strongly aperiodic or strongly periodic (with some period $p > 1$).*

In particular, this implies the following asymptotic relations for the coefficients of solutions of a positive polynomial system.

Theorem 6.3. *Suppose that $f(z)$ is a solution of an analytically well-defined positive polynomial system of equations (see Definition 5.1). Then there exists an integer $p \geqslant 1$ such that for all $j = 0, 1, \ldots, p-1$, we either have $f_n = 0$ for almost all $n \geqslant n_{0,j}$ with $n \equiv j \bmod p$, or*

$$f_n \sim C_j n^{\alpha_j} \rho_j^{-n} \quad (n \to \infty,\ n \equiv j \bmod p),$$

where $C_j > 0$, $\rho_j > 0$, and α_j is either of the form $\alpha_j = -2^{-k_j} - 1$ for some integer $k_j \geqslant 1$ or of the form $\alpha_j = m_j/2^{k_j} - 1$ for some integers $k_j \geqslant 0$ and $m_j \geqslant 1$.

Proof. If $f(z)$ is strongly aperiodic, then the radius of convergence ρ is the only singularity on the circle $|z| = \rho$, and the possible singularities are given by Theorem 5.3. Furthermore, since $f(z)$ is an algebraic function, it can be analytically continued to a region of the form

$$\{z \in \mathbb{C} : |z| < \rho + \eta\} \setminus [\rho, \infty) \quad \text{for some } \eta > 0.$$

Consequently we can apply the transfer principle of Flajolet and Odlyzko [63] and obtain the proposed asymptotic expansion for the coefficients.

In the periodic case we just apply this for $f_j(z)$, $0 \leqslant j < p$. □

The proof of Theorem 6.2 runs along similar lines to the proof of Theorem 5.3, that is, we partition the dependency graph into strongly connected components and solve the system step by step. The core of the proof is to check in every step that each solution is strongly aperiodic or strongly periodic.

For this purpose we will have to *split* the solution functions into several parts.

Lemma 6.4. *Suppose that $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ is an analytically well-defined positive polynomial system of equations and $\mathbf{y} = \mathbf{f}(z) = (f_1(z), \ldots, f_d(z))$ is the solution. Then for every $p \geqslant 1$ we can represent $f_k(z)$, $1 \leqslant k \leqslant d$, as*

$$f_k(z) = \sum_{j=0}^{p-1} z^j f_{k,j}(z^p)$$

and the functions $f_{k,j}(z) - f_{k,j}(0)$, $1 \leqslant k \leqslant d$, $0 \leqslant j < p$, that are not polynomials are again solutions of an analytically well-defined positive polynomial system of equations $\tilde{\mathbf{y}} = \tilde{\mathbf{P}}(z, \tilde{\mathbf{y}})$.

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Proof. Let $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ be an analytically well-defined positive system of polynomial equations that has $y_1 = f(z)$ as one of its solutions. By substituting

$$\mathbf{y} = \sum_{j=0}^{p-1} z^j \mathbf{y}_j,$$

and expanding the polynomials of $\mathbf{P}(z, \mathbf{y})$, it follows that we can represent it as

$$\mathbf{P}(z, \mathbf{y}) = \sum_{j=0}^{p-1} z^j \mathbf{P}_j(z^p, \mathbf{y}_0, \dots, \mathbf{y}_{p-1}).$$

Hence, if we consider the $p \times d$-dimensional system

$$\mathbf{y}_j = \mathbf{P}_j(z, \mathbf{y}_0, \dots, \mathbf{y}_{p-1}), \quad 1 \leqslant j < p,$$

then with $\tilde{\mathbf{y}} = (\mathbf{y}_0, \dots, \mathbf{y}_{p-1})$ and $\tilde{\mathbf{P}} = (\mathbf{P}_0, \dots, \mathbf{P}_{p-1})$ we obtain a proper positive polynomial system $\tilde{\mathbf{y}} = \tilde{\mathbf{P}}(z, \tilde{\mathbf{y}})$, where the functions $f_{k,j}(z)$, defined by

$$f_k(z) = \sum_{j=0}^{p-1} z^j f_{k,j}(z^p),$$

are solutions. Of course, if $f_{k,j}(0) > 0$ we can shift the system to have solutions $f_{k,j}(z) - f_{k,j}(0))$, and we can remove polynomial solutions from the system.

The final step is to show that the spectral radius of the Jacobian $\tilde{\mathbf{P}}_{\tilde{\mathbf{y}}}$ (for $z = 0$ and $y_{k,j} = f_{k,j}(0))$) is again smaller than 1. In fact it is an easy exercise to show that the spectral radii are the same. More precisely, if λ is a positive eigenvalue of $\tilde{\mathbf{P}}_{\tilde{\mathbf{y}}}$ with a positive eigenvector, then it is also a positive eigenvalue of $\mathbf{P}_{\mathbf{y}}$ (with a corresponding positive eigenvector). We illustrate the idea of the proof in a slightly simplified situation. Suppose we have the system $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$, and we write $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ and

$$\mathbf{P}(z, \mathbf{y}_1 + \mathbf{y}_2) = \mathbf{P}_1(z, \mathbf{y}_1, \mathbf{y}_2) + \mathbf{P}_1(z, \mathbf{y}_1, \mathbf{y}_2)$$

and consider the extended system

$$\mathbf{y}_1 = \mathbf{P}_1(z, \mathbf{y}_1, \mathbf{y}_2), \quad \mathbf{y}_2 = \mathbf{P}_2(z, \mathbf{y}_1, \mathbf{y}_2).$$

Let $\lambda > 0$ be an eigenvalue of

$$\begin{pmatrix} \mathbf{P}_{1,\mathbf{y}_1} & \mathbf{P}_{1,\mathbf{y}_2} \\ \mathbf{P}_{2,\mathbf{y}_1} & \mathbf{P}_{2,\mathbf{y}_2} \end{pmatrix},$$

with a positive eigenvector $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, that is,

$$\begin{pmatrix} \mathbf{P}_{1,\mathbf{y}_1} & \mathbf{P}_{1,\mathbf{y}_2} \\ \mathbf{P}_{2,\mathbf{y}_1} & \mathbf{P}_{2,\mathbf{y}_2} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}.$$

By multiplying from the left with $(\mathbf{I}, \mathbf{I})$ and by observing that

$$\mathbf{P}_{\mathbf{y}} = \mathbf{P}_{\mathbf{y}_1} = \mathbf{P}_{1,\mathbf{y}_1} + \mathbf{P}_{2,\mathbf{y}_1} = \mathbf{P}_{\mathbf{y}_2} = \mathbf{P}_{1,\mathbf{y}_2} + \mathbf{P}_{2,\mathbf{y}_2},$$

we obtain

$$\mathbf{P}_{\mathbf{y}}(\mathbf{x}_1 + \mathbf{x}_2) = \lambda(\mathbf{x}_1 + \mathbf{x}_2).$$

It is now clear how we can adapt this example to the original situations. □

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Lemma 6.5. *Suppose that*

$$f(z) = \sum_{n \geqslant 0} a_n z^n$$

is a strongly aperiodic function with non-negative coefficients a_n and radius of convergence ρ. Then, for every $p \geqslant 1$ we can represent $f(z)$ as

$$f(z) = \sum_{j=0}^{p-1} z^j f_j(z^p),$$

where the functions

$$f_j(z) = \sum_{n \geqslant 0} a_{j+pn} z^n, \quad 0 \leqslant j < m,$$

are strongly aperiodic and have the same kind of dominating singularity.

Proof. By Lemma 6.4, we already know that $f_j(z)$ is a solution of an analytically well-defined positive polynomial system of equations. Furthermore, $f_j(z)$ can be represented as

$$f_j(z) = \frac{1}{p} \sum_{\ell=0}^{p-1} e^{-2\pi i j \ell / p} f(z^{1/p} e^{2\pi i \ell / p}).$$

Since $f(z)$ is strongly aperiodic, the radius of convergence ρ is the unique singularity on the circle of convergence $|z| = \rho$. Hence, $\rho^{1/p}$ is the radius of convergence of $f_j(z)$ and (again) the only singularity of $f_j(z)$ on the circle of convergence $|z| = \rho^{1/p}$. Since the coefficients $(a_{j+pn})_{n \geqslant 0}$ have the same kind of asymptotic expansion as $(a_n)_{n \geqslant 0}$, it also follows that $f_j(z)$ (for $z \sim \rho^{1/p}$) and $f(z)$ (for $z \sim \rho$) have the same kind of singularity. □

We start our considerations concerning the proof of Theorem 6.2 with a strongly connected affine system. In order to make the statements (and proofs) simpler, we assume that we have already reduced the system to a single equation of the form $y = a(z)y + b(z)$, where $a(z)$ and $b(z)$ are rational functions with non-negative coefficients that are regular for $|z| < \rho + \varepsilon$ for some $\varepsilon > 0$, where $\rho > 0$ is given by $a(\rho) = 1$ and is the radius of convergence of $f(z) = b(z)/(1 - a(z))$.

Lemma 6.6. *Suppose that $a(z)$ and $b(z)$ are non-zero rational functions with non-negative coefficients that are regular for $|z| \leqslant \rho$, where $\rho > 0$ is given by $a(\rho) = 1$. Furthermore, we assume that $a'(z) \neq 0$. Then $f(z)$ is strongly aperiodic or strongly periodic (with period p for some integer $p > 1$) such that all singularities on the circle of convergence are poles of order* 1.

We note that this lemma can be generalized to functions $f(z)$ that are solutions of not necessarily strongly connected affine systems; see [102, Theorem 10.1]. The only difference is that the order of poles might be larger than 1 in the non-strongly connected case, but the order of ρ is the maximum order appearing.

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Proof. Let

$$a(z) = \sum_{n \geqslant 0} a_n z^n.$$

Since $a_n \geqslant 0$ and $a(z) \neq 0$, it is clear that there exists a unique $\rho > 0$ with $a(\rho) = 1$ which is (by assumption) the radius of convergence and also a polar singularity of $f(z)$. Now, suppose that $z = \rho\zeta$ is also a singularity of $f(z)$, where $|\zeta| = 1$. Then we certainly have $a(\rho\zeta) = 1$. On the other hand, we have

$$|a(\rho\zeta)| = \left| \sum_{n \geqslant 0} a_n \rho^n \zeta^n \right| \leqslant \sum_{n \geqslant 0} a_n \rho^n = 1,$$

which implies that all inequalities have to be equalities. In particular we have $a_p \rho^p \zeta^p = a_p \rho^p$ for some $p > 0$ for which $a_p > 0$. Consequently $\zeta^m = 1$. Thus, we are certainly in the strongly aperiodic or strongly periodic case. □

The example

$$f(z) = \frac{1}{1-z} + \frac{1}{1-z^2} = \frac{2+z}{1-z^2}$$

shows that even a single equation of the form $y = z^2 y + 2 + z$ can lead to a (strongly) periodic case with period $p = 2 > 1$, where the behaviour in both residue classes is different and non-zero. However, if we use the method of Lemma 6.4, we can reduce this equation to a system with only strongly aperiodic solutions. If we set $y = y_0 + zy_1$, then we have

$$z^2 y + 2 + z = z^2(y_0 + zy_1) + 2 + z = (2 + z^2 y_0) + z(1 + z^2 y_1).$$

Hence, if we consider the system $\{y_0 = 2 + zy_0, y_1 = 1 + zy_1\}$, then we have as solutions $f_0(z) = 2/(1-z)$ and $f_1(z) = 1/(1-z)$, which are strongly aperiodic and give back the original solutions $f(z)$ as

$$f(z) = f_0(z^2) + zf_1(z^2).$$

It is interesting to observe that in non-affine and strongly connected systems there is only one residue class modulo p for which the coefficients are non-zero. This is proved in the next lemma. As in the affine case we assume that we have already reduced the system of equations to a single equation; as in the affine case the right-hand side of the equation is no longer a polynomial but an algebraic function. In fact, the reduction procedure (compare also with the proof of Lemma 5.6) leads to an equation that satisfies the following regularity conditions.

Lemma 6.7. *Suppose that*

$$P(z, y) = \sum_{k,\ell} a_{k\ell} z^k y^\ell$$

is an algebraic function with non-negative coefficients such that $a_{k\ell} > 0$ *for some* $\ell \geqslant 2$ *and* $a_{01} < 1$. *Furthermore, let* $\rho > 0$ *denote the radius of convergence of the solution* $f(z)$ *of* *the equation $y = P(z, y)$ and suppose that there exist $\varepsilon > 0$ such that $P(z, y)$ is regular for $|z| < \rho + \varepsilon$ and $|y| < f(\rho) + \varepsilon$.*

Let p be the largest positive integer for which there exists an integer $r \geqslant 0$ such that $P(z, 0)$ can be represented as $P(z, 0) = z^r q(z^p)$ for a proper function $q(z)$ with non-negative coefficients, and that p divides $k + r(\ell - 1)$ for all $a_{k\ell} > 0$ with $\ell > 0$. If $p = 1$, then $f(z)$ is strongly aperiodic, and if $p > 1$, then $f(z)$ is strongly periodic with period p and can be represented as $f(z) = z^r \tilde{f}(z^p)$, where $\tilde{f}(z)$ is strongly aperiodic.

Proof. Let $\rho > 0$ be the radius of convergence of $f(z)$ and $\eta = f(\eta) > 0$. Then we have $P(\rho, \eta) = \eta$ and $P_y(\rho, \eta) = 1$. If $|z'| = \rho$ then we have $|f(z')| \leqslant f(|z'|) = \eta$, and consequently

$$|P_y(z', f(z'))| \leqslant P_y(|z'|, |f(z')|) \leqslant P_y(|z'|, f(|z'|)) = 1.$$

Hence, if z' is a singularity, that is, we certainly have $P_y(z', f(z')) = 1$, then all these inequalities have to be equalities. From $|f(z')| = f(|z'|)$ it follows (similarly to the proof of Lemma 6.6) that $f(z)$ can be written as $f(z) = z^r \tilde{f}(z^p)$ (for some integers $r \geqslant 0$ and $p \geqslant 1$) and z' is of the form $z' = \rho e^{2\pi j/p}$ (for some integer j that is coprime to m). Consequently, from $P_y(z', f(z')) = 1$ and $f(z') = \eta e^{2\pi i j r/p}$ it follows that p divides $k + r(\ell - 1)$ for all pairs (k, ℓ) for which $a_{k\ell} > 0$ and $\ell > 0$. Finally, we also have $f(z') = P(z', f(z'))$, which implies that $P(z, 0)$ can be represented as $P(z, 0) = z^r q(z^p)$ for a polynomial q.

Conversely, we can search for the largest positive integer p for which there exists an integer $r \geqslant 0$ such that $P(z, 0)$ can be represented as $P(z, 0) = z^r q(z^p)$ and that p divides $k + r(\ell - 1)$ for all pairs (k, ℓ) with $a_{k\ell} > 0$ and $\ell > 0$. It is clear that the power series of $f(z)$ is divisible by z^r. Furthermore, we can represent P as

$$P(z, y) = z^r q(z^p) + z^r \sum_{k \geqslant 0, \ell \geqslant 1} a_{k,\ell} z^{k + r(\ell - 1)} (y/z^r)^\ell = z^r Q(z^p, y/z^r)$$

for some proper function Q. Hence, $\overline{y} = y/z^r$ solves the equation $\overline{y} = Q(z^p, \overline{y})$ and can be represented as $\overline{f}(z) = \tilde{f}(z^p)$. This implies that $f(z) = z^r \tilde{f}(z^p)$. Finally, since p was chosen to be the largest integer satisfying the above-mentioned properties, it follows that $\tilde{f}(z)$ is strongly aperiodic. Otherwise, we could iterate the procedure and obtain a contradiction. □

In order to complete the proof of Theorem 6.2, we now follow the proof of Theorem 5.3 and observe step by step that all functions are strongly aperiodic or strongly periodic. For this purpose we will frequently use the set $\mathcal{F}$ of algebraic functions f for which there exists $p \geqslant 1$ and (algebraic) functions f_j, $0 \leqslant j < p$, with

$$f(z) = \sum_{j=0}^{p} z^j f_j(z^p)$$

and the functions f_j, $0 \leqslant j < p$, have positive radii of convergence which are the only singularities on the circles of convergence and all dominating singularities are of the types described in Theorem 5.3. This set of functions has the following property.

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Lemma 6.8. *The set $\mathcal{F}$ is closed under taking sums and products. Furthermore, suppose that $f \in \mathcal{F}$ has only one singularity ρ on the circle of convergence and $f(\rho)$ is finite. Let $c(z,u)$ be a power series*

$$c(z,u) = \sum_{n,k} a_{n,k} z^n u^k$$

with non-negative coefficients $a_{n,k}$ that is analytic at $(\rho, f(\rho))$, and there is some n and some $k > 0$ with $a_{n,k} > 0$. Then the function $g(z) = c(z, f(z))$ is also in $\mathcal{F}$, and again has the property that ρ is the only singularity on the circle of convergence.

Proof. Suppose that f and g are in $\mathcal{F}$ and have radii of convergence ρ_1 and ρ_2 and *periods* p_1 and p_2. Of course, we only have to consider the case $\rho_1 = \rho_2$.

If $p_1 = p_2 = 1$, it is immediate to see that $f + g$ and $f \cdot g$ are in $\mathcal{F}$.

If $p_1 > 1$ or $p_2 > 1$, then let p denote the least common multiple of p_1 and p_2. With the help of Lemma 6.5 it follows that we can represent f and g as

$$f(z) = \sum_{j=0}^{p-1} z^j f_j(z^p) \quad \text{and} \quad g(z) = \sum_{j=0}^{p-1} z^j g_j(z^p),$$

where f_j and g_j have the property that there is only one singularity on the circle of convergence. Hence, it follows similarly to the case $p_1 = p_2 = 1$ that $f + g$ and $f \cdot g$ are in $\mathcal{F}$.

Finally, in order to handle the function $g(z) = c(z, f(z))$ we just have to observe (by a local expansion) that $g(z)$ has the same kind of singularity as $f(z)$ (at $z = \rho$) and that there are no other singularities for $|z| \leqslant \rho$ other than ρ. □

In order to prove Theorem 6.2, we now show inductively that all appearing solution functions of an analytically well-defined system are contained in $\mathcal{F}$.

Suppose that we are considering a strongly connected component C_ℓ and the corresponding system $\mathbf{y}_\ell = \mathbf{P}_\ell(z, \mathbf{y}_\ell, \mathbf{u}_\ell)$, where $\mathbf{u}_\ell$ denotes the input vector that corresponds to those components that have already been solved (and for which we can assume by induction that they are in $\mathcal{F}$).

As in the proof of Theorem 5.3, we distinguish between three cases.

Case 1. First, let us assume that $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ comes from an affine system of the form

$$\mathbf{y}_\ell = \mathbf{A}_\ell(z)\mathbf{y}_\ell + \mathbf{B}_\ell(z, \mathbf{u}_\ell),$$

that is, the matrix $\mathbf{I} - \mathbf{A}_\ell(z)$ does not depend on $\mathbf{u}$ or equivalently

$$f(z, \mathbf{u}_\ell) = \frac{c(z, \mathbf{u}_\ell)}{1 - z/\rho'}, \tag{6.1}$$

where $c(z, \mathbf{u}_\ell)$ is a polynomial in $\mathbf{u}_\ell$ with non-negative coefficients. Note that the *coefficient functions* $c_j(z)/(1 - z/\rho')$ are in $\mathcal{F}$.

When we substitute $\mathbf{u}_\ell$ by the functions $f_j(z)$ that correspond to $\mathbf{u}_j$ (and are thus in $\mathcal{F}$), then it follows from Lemma 6.8 that the resulting function is in $\mathcal{F}$.

Case 2. Second, let us (again) assume that $\mathbf{f}_\ell(z,\mathbf{u}_\ell)$ comes from an affine system

$$\mathbf{y}_\ell = \mathbf{A}_\ell(z,\mathbf{u}_\ell)\mathbf{y}_\ell + \mathbf{B}_\ell(z,\mathbf{u}_\ell),$$

where $\mathbf{A}_\ell(z,\mathbf{u}_\ell)$ depends on some of the u_j. As above, let J'_ℓ denote the set of indices of functions u_j on which $\mathbf{A}_\ell(z,\mathbf{u}_\ell)$ really depends. Of course, if we represent $f(z,\mathbf{u}_\ell)$ as

$$f(z,\mathbf{u}_\ell) = \frac{c(z,\mathbf{u}_\ell)}{1-z/\rho(\mathbf{u}_\ell)},$$

then J'_ℓ is precisely the set of indices of functions u_j on which the function $\rho(\mathbf{u}_\ell)$ depends. Note that $c(z,\mathbf{u}_\ell)$ is a polynomial in the u_j with $j \notin J'_\ell$. (In what follows we will denote the coefficients of this polynomial by $c_r(z,(u_j)_{j\in J'_\ell})$. We note that for all r the function

$$c_r(z,(u_j)_{j\in J'_\ell})/(1-z/\rho(u_j)_{j\in J'_\ell})$$

is a power series with non-negative coefficients in z and $(u_j)_{j\in J'_\ell}$.)

Let ρ' denote the smallest radius of convergence of the functions $f_j(z)$, $j \in J'_\ell$. Then we consider the difference $\delta(z) = \rho((f_j(z))_{j\in J'_\ell}) - z$. We have to consider the following subcases.

Case 2.1. $\delta(\rho'') = 0$ for some $\rho'' < \rho'$ such that $(f_j(\rho''))_{j\in J'_\ell} \in D_\ell$. Here we have a polar singularity, and with the help of Lemma 6.6 we deduce that

$$c_r(z,(f_j(z))_{j\in J'_\ell})/(1-z/\rho((f_j(z))_{j\in J'_\ell}))$$

is in $\mathcal{F}$. Hence, it follows (again) from Lemma 6.8 that the resulting function is in $\mathcal{F}$.

Case 2.2. We have $\delta(\rho') = 0$ such that $(f_j(\rho'))_{j\in J'_\ell} \in D_\ell$. In this case, we first replace the system $\mathbf{y}_\ell = \mathbf{A}_\ell(z,\mathbf{u}_\ell)\mathbf{y} + \mathbf{B}_\ell(z,\mathbf{u}_\ell)$ with another one. By assumption, we know that all functions $f_j(z)$ are strongly aperiodic (with *periods* p_j). We define p as the least common multiple of the periods p_j and by Lemma 6.5 we can represent them all as

$$f_j(z) = \sum_{i=0}^{p-1} z^i f_{j,i}(z^p),$$

with strongly aperiodic functions $f_{j,i}(z)$. Formally, this means that we replace the *parameters* u_j with

$$\sum_{i=0}^{p-1} z^i u_{j,i}.$$

Furthermore, by following the proof method of Lemma 6.4 we split the (originally d-dimensional) system into a $(p \times d)$-dimensional system, where the solutions $f_i(z)$ correspond to the original one by

$$f(z) = \sum_{i=0}^{p-1} z^i f_i(z^p).$$

Let us denote this new system by

$$\tilde{\mathbf{y}}_\ell = \tilde{\mathbf{A}}_\ell(z,\tilde{\mathbf{u}}_\ell)\tilde{\mathbf{y}}_\ell + \tilde{\mathbf{B}}_\ell(z,\tilde{\mathbf{u}}_\ell).$$

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It is easy to check that this new system is either strongly connected or the corresponding dependency graph decomposes into several strongly connected components without any link between these components. For the sake of simplicity, we assume that we have only one component (in the other case we have to deal with each component separately, but this does not cause any difficulty).

The advantage of this construction is that all functions $f_{j,i}(z)$ (that have to be substituted for $u_{j,i}$) are strongly aperiodic, which implies that $|f_{j,i}(z)| < f_{j,i}(|z|)$ if z is not a positive real number.

It might occur that the new system (or one of the new systems) falls into Cases 2.1 or 2.3. In these cases, we proceed as explained there. Thus, we can assume that we are again in Case 2.2. We note that the function $\tilde{\rho}(\tilde{\mathbf{u}}_\ell)$ is determined by the property that the positive matrix $\tilde{\mathbf{A}}_\ell(z, \tilde{\mathbf{u}}_\ell)$ has spectral radius 1. Since this matrix is irreducible, it follows that the spectral radius is strictly smaller than 1 if at least one of the entries decreases in modulus. Hence, if we substitute the $u_{j,i}$ by $f_{j,i}(z)$, there is certainly no singularity if $|z| = \tilde{\rho}'$ but $z \neq \tilde{\rho}'$. Consequently, the only singularity of the resulting function is on the circle of convergence $z = \tilde{\rho}'$, and we know already from the proof of Theorem 5.3 which type of singularity will appear.

Summing up, it follows that the solution of the *expanded system* is strongly aperiodic, and thus the solution of the original system is either strongly aperiodic or strongly periodic.

Case 2.3. We have $\delta(\rho') > 0$ such that $(f_j(\rho'))_{j\in J'_\ell} \in D_\ell$. We first apply the same transformation of the system as in Case 2.2 and are led to one (or several) strongly dependent new system. Of course, some of them might be in Cases 2.1 or 2.2. But then we just apply the procedure there. However, it is important to note that it is not necessary to apply the transformation again, since we have already reached the goal that all functions $f_{j,i}(z)$ are strongly aperiodic.

In order to simplify the notation, we stick with the original notation of the system $\mathbf{y}_\ell = \mathbf{A}_\ell(z, \mathbf{u}_\ell)\mathbf{y}_\ell + \mathbf{B}_\ell(z, \mathbf{u}_\ell)$.

In this case, we can represent the function $f(z)$ that results after substitution as a polynomial in $f_j(z)$ with $j \notin J'$ with coefficient functions that are analytic in z and $(f_j(z))_{j\in J'}$ at $z = \rho'$ and $(f_j(\rho'))_{j\in J'}$. Of course, all coefficients in the power series expansions are non-negative. By extending the last part of Lemma 6.8 to several (strongly aperiodic) functions, it follows that the coefficient functions are in $\mathcal{F}$, which implies (via Lemma 6.8) that the resulting function is in $\mathcal{F}$.

Case 3. Finally, let us assume that $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ comes from a non-affine system and thus has a square-root singularity of the form (5.7). In this case $\rho(\mathbf{u}_\ell)$ depends on all components of $\mathbf{u}_\ell$. As above, we set $\delta(z) = \rho(\mathbf{u}_\ell) - z$ and let ρ' denote the smallest radius of convergence of the functions $f_j(z)$ that correspond to $\mathbf{u}_\ell$.

Case 3.1. $\delta(\rho'') = 0$ for some $\rho'' < \rho'$ such that $(f_j(\rho'')) \in D_\ell$. Here, we are precisely in the situation of Lemma 6.7. From the very beginning, we can substitute u_j with the functions $f_j(z)$ and thus obtain either a strongly aperiodic function or a strongly periodic function, where only one residue class modulo m appears.

Case 3.2. We have $\delta(\rho') = 0$ such that $(f_j(\rho')) \in D_\ell$. As in Case 2.2, we first replace the original system $\mathbf{y}_\ell = \mathbf{P}_\ell(z, \mathbf{y}_\ell, \mathbf{u}_\ell)$ with a new one, $\tilde{\mathbf{y}} = \tilde{\mathbf{P}}_\ell(z, \tilde{\mathbf{y}}_\ell, \tilde{\mathbf{u}}_\ell)$, which might decompose into several strongly connected but non-affine systems. As above, we assume that it is just one system and that it is again in Case 3.2. Here, the function $\tilde{\rho}(\tilde{\mathbf{u}}_\ell)$ is given by the property that the Jacobian $\tilde{P}_{\tilde{\mathbf{y}}}(z, \tilde{\mathbf{y}}_\ell, \tilde{\mathbf{u}}_\ell)$ has spectral radius 1. As in the proof of Case 2.2, the spectral radius is smaller than 1 if at least one entry of this matrix decreases in modulus. This again implies that there are no singularities on the cycle of convergence $|z| = \rho'$ other than ρ'. And the type of singularity that appears for $z = \rho'$ is already known from the proof of Theorem 5.3.

Summing up, we again get that the solution of the *expanded system* is strongly aperiodic, and thus the solution of the original system is either strongly aperiodic of strongly periodic.

Case 3.3. We have $\delta(\rho') > 0$ such that $(f_j(\rho'))_{j \in J'_\ell} \in D_\ell$. We proceed similarly to Case 2.3. We first expand the system as in Case 3.2 and assume (without loss of generality) that the new system is again in Case 3.3 (and for the sake of simplicity we stick with the original notation).

In this case the spectral radius of $\mathbf{P}_{\ell,\mathbf{y}_\ell}(z, \mathbf{y}_\ell, \mathbf{u}_\ell)$ is smaller than 1 for $\mathbf{u}_\ell = (f_j(\rho'))_{j \in J'_\ell}$. Thus, it follows that we can invert the system of equations $\mathbf{y}_\ell = \mathbf{P}_\ell(z, \mathbf{y}_\ell, \mathbf{u}_\ell)$ locally to $\mathbf{y}_\ell = \mathbf{Q}(z, \mathbf{u}_\ell)$, which implies that the resulting function $f(z)$ is singular at $z = \rho'$ (and we know from the proof of Theorem 5.3 that the types of singularities of $y_j(z)$ are inherited). Furthermore, it follows that there are no other singularities on the circle of convergence $|z| = \rho'$ other than ρ'. This follows from the fact that the spectral radius of the Jacobian stays smaller than one and that the functions $f_j(z)$ are not singular for $|z| = \rho'$, $z \neq \rho'$.

This completes the proof of Theorem 6.2. □

7. Possible radius of convergence of $\mathbb{Q}_+$-algebraic functions

In this section we briefly discuss the radius of convergence ρ that can appear in a positive algebraic system with rational coefficients. Of course, ρ has to be a positive algebraic number, but it is not immediate whether all positive algebraic numbers actually appear. Let us begin with an assertion for more restricted systems of equations.

Conjecture 7.1 (radius of $\mathbb{N}$-rational functions). *All positive algebraic numbers $\leqslant 1$ appear as a radius of convergence of solutions of a positive affine system of equations with integer coefficients.*

Note that the Berstel–Soittola theorem [108] implies that dominant roots have to differ by root of unity factors (this is why, for example, $(z + 5z^2)/(1 + z - 5z^2 - 125z^3)$ is not $\mathbb{N}$-rational). It is not trivial to see how this theorem affects Conjecture 1, but this conjecture (which we do not believe to be true, *e.g.*, it is a challenge to find an $\mathbb{N}$-rational function with $\rho = (1 + 2^{1/3})/3$) implies the following weaker conjecture.

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Conjecture 7.2 (radius of $\mathbb{N}$-algebraic functions). *All positive algebraic numbers $\leqslant 1$ appear as a radius of convergence of solutions of a positive algebraic system of equations with integer coefficients.*

If $F(z)$ is $\mathbb{N}$-algebraic, then $F(az)$ (for any rational a) is $\mathbb{Q}_+$-algebraic, so it is clear that Conjecture 7.2 implies the following conjecture.

Conjecture 7.3 (radius of $\mathbb{Q}_+$-algebraic functions). *All positive algebraic numbers appear as a radius of convergence of solutions of a positive algebraic system of equations with rational coefficients.*

For each of these conjectures, it is also natural to ask: What are the properties of the set of corresponding radii of convergence, *e.g.*, to what extent are they closed under sum or product? In what follows we present some properties of the set of these algebraic numbers which led us to the above conjecture.

Theorem 7.4. *The set R of radii of convergence of $\mathbb{Q}_+$-algebraic functions has the following properties.*

(i) *All positive roots of equations of the form $p(z) = 1$, where $p(z)$ is a polynomial with non-negative rational coefficients, are in R, in particular all rational numbers and all roots of rational numbers.*
(ii) *If $\rho_1 \in R$ and ρ_2 is a radius of convergence that appear in positive rational systems, then $\rho_1\rho_2 \in R$.*
(iii) *All positive quadratic irrational numbers are in R.*

Proof. **(i)** Suppose that $p(z)$ is a polynomial with non-negative rational coefficients and z_0 a positive solution of the equation $p(z) = 1$. Then it is certainly the radius of convergence of $y = f(z)$ that satisfies the equation $y = z + p(z)y$. Since $p(0) < 1$, this equation is well-posed. By setting $p(z) = z/\rho$ or $p(z) = z^m/\alpha$, it follows that rational numbers ρ and roots $\rho = \alpha^{1/m}$ are in R.

(ii) The fact that the Hadamard product of an algebraic function with a rational function is algebraic [74] has a non-commutative version [104]. This implies that the Hadamard product of an $\mathbb{N}$-algebraic function with an $\mathbb{N}$-rational function is $\mathbb{N}$-algebraic. The same holds for $\mathbb{Q}_+$ instead of $\mathbb{N}$.

We first assume that we are in the aperiodic case. Then the asymptotic expansion for the coefficients a_n and b_n of the $\mathbb{Q}$-algebraic function and the $\mathbb{Q}$-rational function are of the form $a_n \sim An^\alpha\rho_1^{-n}$ and $b_n \sim Bn^\beta\rho_2^{-n}$, so we can directly consider the Hadamard product $a_nb_n \sim ABn^{\alpha+\beta}(\rho_1\rho_2)^{-n}$ and observe that the radius of convergence ρ of the Hadamard product is just the product $\rho_1\rho_2$. Since the Hadamard product is $\mathbb{Q}_+$-algebraic, it follows that $\rho_1\rho_2 \in R$.

Now, suppose that $a(z)$ is $\mathbb{Q}_+$-algebraic but not aperiodic. Then we can represent $a(z)$ as

$$a(z) = \sum_{j=0}^{p-1} z^j a_j(z^p),$$

where the functions $a_j(z)$ are $\mathbb{Q}_+$-algebraic, too, and there is at least one function, say $a_{j_0}(z^p)$, that has the same radius of convergence as $a(z)$. Hence, if we consider the function

$$\tilde{a}(z) = \left(1 + z/\rho + (z/\rho)^2 + \cdots + (z/\rho)^{p-1}\right) a_{j_0}(z^p),$$

then $\tilde{a}(z)$ is again $\mathbb{Q}_+$-algebraic and the coefficients $\tilde{a}_n$ have an asymptotic expansion of the form $\tilde{a}_n \sim \tilde{A} n^\alpha \rho_1^{-n}$ (for all n and not only in a residue class). A similar procedure works for an aperiodic $\mathbb{Q}_+$-rational function, and we can proceed as above.

(iii) Finally, we suppose that $\rho = \alpha + \beta\sqrt{m}$ is a positive quadratic irrational number (where α and β are rational numbers and m is a square-free positive integer). We have to distinguish several cases. First, suppose that $\alpha < 0$ and $\beta > 0$. Since $\rho = \alpha + \beta\sqrt{m} > 0$, this implies $\alpha^2 - \beta^2 m < 0$. If we set

$$p(z) = \frac{2\alpha}{\alpha^2 - \beta^2 m} z + \frac{1}{\beta^2 m - \alpha^2} z^2,$$

then $p(z)$ has positive coefficients and we also have $p(\alpha + \beta\sqrt{m}) = 0$. Consequently, ρ is in R. Moreover we certainly have $c_n \sim d\rho^n$ for the coefficients of the solution of $y = z + p(z)y$. Next, suppose that $\alpha > 0$ and $\beta < 0$. In this case we have $\alpha^2 - \beta^2 m > 0$ so that ρ is root of the polynomial $z^2 - 2\alpha z + (\alpha^2 - \beta^2 m) = 0$, which cannot be written in an equivalent form $p(z) = 1$, where $p(z)$ has non-negative coefficients. Here, we consider the (rational) system of equations

$$y_1 = z + (az + y_2)y_1, \quad y_2 = bz + czy_2,$$

where $y_1 = f_1(z)$ has the solution

$$f_1(z) = \frac{z}{1 - az - \frac{bz}{1-cz}} = \frac{z(1-cz)}{1 - (a+b+c)z + acz^2}.$$

Consequently, if there are non-negative rational numbers a, b, c with

$$a + b + c = 2\alpha/(\alpha^2 - \beta^2 m) \quad \text{and} \quad ac = 1/(\alpha^2 - \beta^2 m),$$

then we are done. For the moment set

$$a = c = 1/\sqrt{\alpha^2 - \beta^2 m}.$$

Then the trivial inequality $\alpha > \sqrt{\alpha^2 - \beta^2 m}$ implies

$$\frac{2\alpha}{\alpha^2 - \beta^2 m} > \frac{2}{\sqrt{\alpha^2 - \beta^2 m}}.$$

Hence, by setting

$$b = \frac{2\alpha}{\alpha^2 - \beta^2 m} - \frac{2}{\sqrt{\alpha^2 - \beta^2 m}},$$

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we obtain $a+b+c = 2\alpha/(\alpha^2-\beta^2 m)$ and $ac = 1/(\alpha^2-\beta^2 m)$ with non-negative a,b,c. The only *problem* is that a,b,c are not rational (in general). However, we can choose a to be a proper rational approximation of $1/\sqrt{\alpha^2-\beta^2 m}$ and then set $c = 1/(a(\alpha^2-\beta^2 m))$ and $b = 2\alpha/(\alpha^2-\beta^2 m) - a - c$. By continuity, we can choose this rational approximation in such a way that a,b,c are all positive. Consequently, ρ is in R. Moreover, we again have $c_n \sim d\rho^n$ for the coefficients of the solution of $f_1(z)$.

If $\alpha > 0$ and $\beta > 0$, then we write $\rho = \alpha + \beta\sqrt{m}$ in the form

$$\alpha + \beta\sqrt{m} = \frac{1}{\alpha^2-\beta^2 m}(\alpha - \beta\sqrt{m})$$

if $\alpha^2 - \beta^2 m > 0$, and in the form

$$\alpha + \beta\sqrt{m} = \frac{1}{\beta^2 m - \alpha^2}(-\alpha + \beta\sqrt{m})$$

if $\alpha^2 - \beta^2 m < 0$. In both cases, ρ equals the product $\rho_1\rho_2$, where ρ_1 and ρ_2 are radii of convergence of (proper) positive rational systems. Consequently, ρ is in R, too. □

8. Limit laws

8.1. The limit law version of the Drmota–Lalley–Woods theorem

In several applications in combinatorics, we are not only interested in a univariate situation where z is the *counting variable*, but we are also interested in a second parameter that we *count* with the help of another variable (say u). Hence, we are led to consider systems of equations of the form $\mathbf{y} = \mathbf{P}(z,\mathbf{y},u)$. Of course, if we set $u = 1$, we come back to the *original* counting problem. The next theorem (from [53]) shows that the limiting distribution of the additional parameter is always Gaussian if the system is strongly connected.

Theorem 8.1 (Drmota–Lalley–Woods, limiting distribution version: Gaussian limit law for strongly connected systems). *Suppose that $\mathbf{y} = \mathbf{P}(z,\mathbf{y},u)$ is a strongly connected and analytically well-defined entire or polynomial system of equations that depends on u and has a solution $\mathbf{f}(z,u)$ that exists in a neighbourhood of $u = 1$. Furthermore, let $h(z,u)$ be given by*

$$h(z,u) = \sum_{n\geqslant 0} h_n(u) z^n = H(z,\mathbf{f}(z,u),u),$$

where $H(z,y,u)$ is entire or a polynomial function with non-negative coefficients that depends on $\mathbf{y}$, and suppose that $h_n(u) \neq 0$ for all $n \geqslant n_0$ (for some $n_0 \geqslant 0$).

Let X_n be a random variable whose distribution is defined by

$$\mathbb{E}[u^{X_n}] = \frac{h_n(u)}{h_n(1)}.$$

Then X_n has a Gaussian limiting distribution. More precisely, we have $\mathbb{E}[X_n] = \mu n + O(1)$ and $\mathbb{V}\mathrm{ar}[X_n] = \sigma^2 n + O(1)$ for constants $\mu > 0$ and $\sigma^2 \geqslant 0$ and

$$\frac{1}{\sqrt{n}}(X_n - \mathbb{E}[X_n]) \to N(0,\sigma^2).$$

8.2. More Gaussian examples, beyond the Drmota–Lalley–Woods case

If the system of equations is not strongly connected, then we can still define a random variable X_n, but it is not necessarily Gaussian, as we will see in the next section. Nevertheless, it is possible to state sufficient conditions where a Gaussian limiting distribution is present.

Theorem 8.2 (Gaussian limit law for non-strongly connected systems). *Let* $\mathbf{y} = \mathbf{P}(z, \mathbf{y}, u)$ *be a system of equations as in Theorem 8.1, the only difference being that it is not strongly connected. Furthermore, we assume that the function* $h(z, 1)$ *is strongly aperiodic. For every strongly connected component* C_ℓ *of the dependency graphs G, let* ρ_ℓ *denote the radius of convergence of those functions* $f_j(z, 1)$ *that correspond to* C_ℓ*. If all* ρ_ℓ *are different, then* X_n *(defined as in Theorem 8.1) has a Gaussian limiting distribution.*

Proof. We start by setting $u = 1$ and check the proof of Theorem 5.3. If all the ρ_ℓ are different, then the only cases that can appear are Cases 2.1 and 3.1. In all other cases, the new radius of convergence is inherited from another function $f_j(z)$. In fact the same situation holds if u varies in a sufficiently small neighbourhood of 1. By the implicit function theorem, it follows that there exists $\rho''(u)$ with $\rho(f_j(\rho''(u))) = \rho''(u)$, that is, we get the same singularity structure with a *small perturbation* because of u. This is precisely the situation that is needed for the proof of the central limit theorem for X_n (see [65]). □

8.3. Non-Gaussian limit laws

This section illustrates the wide variety of distributions followed by a parameter in a non-strongly connected grammar. What is a 'limiting distribution'? There is no universal answer to this, but roughly speaking, we say that a random variable X_n has a limiting distribution (or a limit law) if the curve $(k, \mathrm{Prob}(X_n = k))$ (possibly renormalized) has a limit when n goes to infinity. This leads to continuous distributions as well as discrete distributions, even leading to less common limiting distributions, such as multi-valued functions. There is thus a large zoo of limiting distributions, and the following theorem shows that they even occur for simple models.

Theorem 8.3 (diversity of possible limit laws for context-free systems). *Let* X_n *be the number of occurrences of any given pattern – this pattern could be a given letter! – in a word of length n generated by a grammar (or even by a simpler model of a Markov chain, with an alphabet of 2 letters, each letter having an integer weight). Then* X_n *can follow 'any limit law', in the sense that there exist some patterns and some grammars for which the limit curve (for large n) of* $(k/n, \mathrm{Prob}(X_n = k))$ *can, be arbitrarily close to any* càdlàg *multi-valued curve in* $[0, 1]^2$.

Proof. This is a consequence of the fact that one can get any piecewise affine function, as proved in [8], so by the Weierstrass theorem one gets any continuous (or *càdlàg*) distribution. Due to the (possible) periodic behaviour of the coefficients of the distribution functions, there is also a (possible) periodic behaviour of the limiting distribution, that is,

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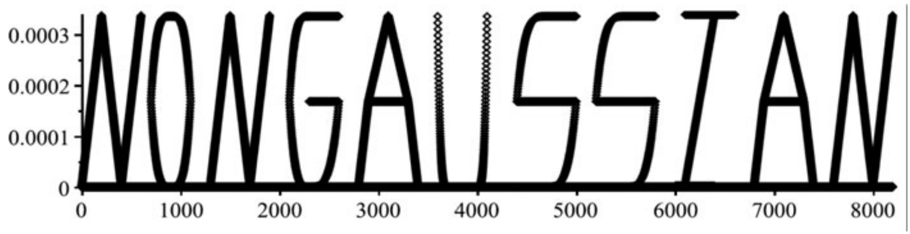

Figure 3. This figure, taken from [8], gives the distribution of the letter 'b' in words of length $n = 8200$ in a language generated by an *ad hoc* regular expression of a few lines. It gives a (discrete) probability distribution which looks like the word 'NONGAUSSIAN'. The key point of this example is that it is designed such that if one rescales the plot by dividing its width by n, then the distribution converges towards a curve, which still looks like 'NONGAUSSIAN'. Note that this curve is, *at the limit*, a curve of a *multi-valued* functional (as can be seen in the letters O, G, A, S, I). However, we achieve it for *finite-length words* via a *single-valued* function, by interlacing two sequences mod 2. This figure illustrates the huge diversity of possible limit laws, even for the distribution of a single letter. It is possible to play the same game starting from continuous distribution instead of discrete distribution.

for every fixed residue class mod m we get different laws. Putting these finitely many limit laws into one figure leads to a multi-valued curve, as illustrated in Figure 3. □

9. Beyond the algebraic case: positive systems of entire functions

In this section we will see that most parts of the analysis of positive polynomial systems of equations also work for positive entire systems. However, we cannot expect the same universal algebraic behaviour as for pure polynomial systems, as the following example shows.

Example 3. The system of equations

$$\begin{cases} y_1 = z(e^{y_2} + y_1), \\ y_2 = z(1 + 2y_2y_3), \\ y_3 = z(1 + y_3^2) \end{cases} \tag{9.1}$$

has the following explicit solutions:

$$\begin{cases} f_1(z) = \dfrac{z}{1-z} \exp\left(\dfrac{z}{\sqrt{1-4z^2}}\right), \\ f_2(z) = \dfrac{z}{\sqrt{1-4z^2}}, \\ f_3(z) = \dfrac{1-\sqrt{1-4z^2}}{2z}. \end{cases} \tag{9.2}$$

So, while the subsystem for y_2, y_3 is just polynomial (and they behave as stated in Theorem 5.3), the solution for y_1 clearly has a non-algebraic singularity.

As we have seen in the proof of Theorem 5.3, the main difficulties arise from the interactions with the affine case, which has to be treated with care. The good news is that

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if we just require that our system has no affine subsystem, then we can obtain a universal algebraic behaviour with a singularity of the form (5.1).

Theorem 9.1 (dyadic exponents for entire systems). *Let* $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ *be an analytically well-defined positive system of functional equations, where* $\mathbf{P}$ *consists of entire functions and we have*

$$\frac{\partial^2 P_j}{\partial y_j^2} \neq 0, \quad 1 \leqslant j \leqslant d. \tag{9.3}$$

Then the solutions $f_j(z)$ *have positive and finite radii of convergence* ρ_j *and a Puiseux critical exponent of the form* 2^{-k_j} *with integers* $k_j \geqslant 1$*, that is, the singular behaviour of* $f_j(z)$ *around* ρ_j *is of type*

$$f_j(z) = f_j(\rho_j) + c_j(1 - z/\rho_j)^{2^{-k_j}} + c_j'(1 - z/\rho_j)^{2 \cdot 2^{-k_j}} + \cdots, \tag{9.4}$$

where $c_j \neq 0$ *and where* k_j *is a positive integer.*

Remark. Instead of assuming condition (9.3), it is also sufficient that the subsystems $\mathbf{y}_\ell = \mathbf{P}_\ell(z, \mathbf{y}_\ell, \mathbf{u}_\ell)$ (corresponding to the strongly connected component C_ℓ) are not affine in $\mathbf{y}_\ell$. Indeed, both assumptions are sufficient to obtain a singular expansion of the form (9.4).

Proof. First of all, Lemmas 5.5 and 5.6 hold for entire systems of equations. Furthermore, the condition (9.3) ensures that no subsystem is affine. Hence, by checking the proof of Theorem 5.3, only singularities of type (9.4) occur. Actually, we only have to go through Case 3, and in all these cases we obtain solutions of type (9.4). □

It is possible to cover some cases where affine subsystems occur. For example, the following theorem ensures that non-algebraic singularities (as in Example 3) do not occur.

Theorem 9.2. *Let* $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ *be an analytically well-defined positive system of functional equations, where* $\mathbf{P}$ *consists of entire functions. Furthermore, we assume that for each* $j = 1, \ldots, d$ *we either have*

$$\frac{\partial^2 P_j}{\partial y_j^2} \neq 0$$

or, if P_j *is affine in* y_j*, then we have*

$$\frac{\partial^2 P_j}{\partial y_j \partial y_i} \neq 0 \quad \text{for all } i \neq j \text{ with } \frac{\partial P_j}{\partial y_i} \neq 0. \tag{9.5}$$

Then the solutions $f_j(z)$ *have positive and finite radii of convergence* ρ_j*. Furthermore, the singular behaviour of* $f_j(z)$ *around* ρ_j *has a Puiseux critical exponent of the form* 2^{-k_j}*, with integers* $k_j \geqslant 1$*, or of the form* -2^{-k_j}*, with integers* $k_j \geqslant 0$*. Thus the singular behaviour of* $f_j(z)$ *around* ρ_j *is of type*

$$f_j(z) = f_j(\rho_j) + c_j(1 - z/\rho_j)^{2^{-k_j}} + c_j'(1 - z/\rho_j)^{2 \cdot 2^{-k_j}} + \cdots, \tag{9.6}$$

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where $c_j \neq 0$ and where k_j is a positive integer, or of type

$$f_j(z) = \frac{d_j}{(1 - z/\rho_j)^{2^{-k_j}}} + d_j' + d_j''(1 - z/\rho_j)^{2^{-k_j}} + \cdots, \tag{9.7}$$

where $d_j \neq 0$ and k_j are non-negative integers.

Instead of assuming condition (9.5) it is also sufficient that only the affine subsystems

$$\mathbf{y}_\ell = \mathbf{P}_\ell(z, \mathbf{y}_\ell, \mathbf{u}_\ell) = \mathbf{A}_\ell(z, \mathbf{u}_\ell)\mathbf{y}_\ell + \mathbf{B}_\ell(z, \mathbf{u}_\ell)$$

(corresponding to the strongly connected component C_ℓ) have the property that $\mathbf{A}_\ell(z, \mathbf{u}_\ell)$ depends on all components of $\mathbf{u}_\ell$.

Proof. By checking the proof of Theorem 5.3, we observe that we only have to consider Cases 2 and 3. More precisely, in Case 2, $\delta(z)$ depends on all y_j (that correspond to $\mathbf{u}_j$). If we are in Case 3, then we obtain a singularity of type (9.6) as in the proof of Theorem 9.1.

Let us discuss Case 2 in more detail.

- For Case 2.1 we obtain a polar singularity ρ'' that is smaller than all singularities of the functions $f_j(z)$. Hence we obtain a singular expansion of type (9.7) (with $k = 0$).
- For Case 2.2 we obtain a singularity of type (9.7), where the exponent 2^{-k} is inherited from the functions f_j. Note that the numerator is of type (9.6) since the denominator depends on all possible functions f_j, and thus no new singularity can appear in the numerator.
- Finally, for Case 2.3 we obtain a singularity of type (9.6) that is inherited from the functions $f_j(z)$ (of smallest radius of convergence). □

This concludes our investigations of the numerous variants of systems leading to algebraic behaviour.

10. Conclusions

Now that we have a better picture of the behaviour of positive systems of equations and of the asymptotics of the coefficients of the corresponding solutions, several extensions are possible, and we plan to say more in future works on the following questions.

Algorithmic aspects. In order to automate finding the asymptotics, one has to follow the correct branch of the algebraic equations; this is doable by a disjunction of cases following the proof of our main theorem, coupled with an inspection of the associated spectral radii. This leads to a more 'algebraic' approach suitable for computer algebra, bypassing some numerical methods, *e.g.*, the Flajolet–Salvy ACA (analytic continuation of algebraic) algorithm [65]. With respect to the Pisot problem (*i.e.*, deciding if one, or an infinite number of f_n are zero), finding the best equivalent for $\mathbb{N}$-algebraic functions of the Skolem–Mahler–Lech theorem for $\mathbb{N}$-rational functions is also a nice question [2, 19]. It is also of interest to get algorithms to decide if $f_n \geqslant 0$ for all n [76]. The binomial formula of Section 3 leads to many identities; it is not always easy to predict when the nested sums can be simplified. This has some links with diagonals of rational generating functions.

Decidability of ℕ-algebraicity. The converse by Soittola [108, 23] of a theorem of Berstel [21] shows that it is possible to decide if a rational function is in fact an ℕ-rational function. There is an effective version of this decidability result [16]; Koutschan [82] completed the details in order to get the first implementation of the algorithm. Giving an algorithm to decide if a function is ℕ-algebraic, in a constructive way, would be nice.

Extension to differential systems. It is possible to follow a similar approach for linear systems of differential equations, where there is, however, a broader type of behaviour.

Extension to infinite systems. If one considers systems having an infinite (countable) number of unknowns $y_i(z)$, it is proved in [91] that strongly connected systems also lead to a square-root behaviour. It is proved in [55] that the limit law is Gaussian (as soon as a Jacobian operator associated to the system is compact). When the conditions of strong connectivity or of compactness are dropped, many different behaviours may appear, but it is possible to describe interesting subclasses having a regular behaviour.

Extension to attributed grammars. Attribute grammars were introduced by Knuth. Many interesting parameters (*e.g.*, internal path length in trees or area below lattice paths [12, 58, 99]) are captured by such grammars. They lead to statistics with a mean which is no longer linear. For a large class of strongly connected positive systems (with a Jacobian condition), it leads to the Airy function, and it is expected that it will also be the case for a class of functional equations allowing negative coefficients.

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The genesis of this article is due to Philippe Flajolet. The topic of the paper is in fact very much linked to Philippe Flajolet. The story goes as follows. In the early 1990s, Philippe tried to organize those communities of people interested in mathematical tools for the average-case analysis of algorithms at a more international level. This led to a series of meetings, which soon became the annual international conference ‘AofA’. The first meeting was organized at Dagstuhl in 1993 by Philippe, Helmut Prodinger, and Rainer Kemp; it had many fascinating talks (including, for example, Knuth’s presentation of his ‘giant paper on the giant component’, with Janson, Łuczak and Pittel), and many discussions over a glass of beer, as it should be with Philippe. During one of these discussions, in the next meeting in 1995, Philippe suggested to Michael Drmota the topic of positive and strongly connected systems of equations, and convinced him to work on limit laws as well. This led to the article [53], which together with [52] and [56] were the beginning of Drmota’s life in analytic combinatorics. Philippe kindly promoted the main

theorem from [53] by calling it the *Drmota–Lalley–Woods theorem*, since similar results were obtained independently by Lalley [84] and Woods [121].

In his PhD thesis [6] under the supervision of Philippe Flajolet, Cyril Banderier considered some enumerative and asymptotic properties of lattice paths and planar maps. The corresponding articles [9, 10, 11] were the beginning of Banderier's life in analytic combinatorics. Many families of maps and lattice paths have algebraic generating functions. It was then natural to ask if they can be generated by a context-free grammar, and to ask what kind of asymptotics context-free grammars can lead to, beyond the Drmota–Lalley–Woods theorem. Encouraged by Mireille Bousquet-Mélou and Gilles Schaeffer, who also tried to tackle this question [32], he further investigated the problem with his student Hanane Tafat Bouzid, who gave a constructive list of critical exponents for $\mathbb{N}$-algebraic functions in Chapter 3 of her PhD thesis [112].

Philippe is also partly *responsible* for the present general result (dyadic exponents for Puiseux expansion around the radius of convergence). When Michael Drmota presented an incomplete pre-version of this general result in the ANR Boole meeting in May 2010 in Paris, it was again Philippe who asked for an unconditional result for positive polynomial systems. Drmota and Banderier then realized that they were both working on the same problem, so they joined forces for the present article, which is thus naturally dedicated to the memory of Philippe!

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