Canad. Math. Bull. Vol. 63 (4), 2020 pp. 787–801
http://dx.doi.org/10.4153/S0008439519000821

Slow Continued Fractions and Permutative Representations of $\mathcal{O}_N$

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Abstract. Representations of the Cuntz algebra $\mathcal{O}_N$ are constructed from interval dynamical systems associated with slow continued fraction algorithms introduced by Giovanni Panti. Their irreducible decomposition formulas are characterized by using the modular group action on real numbers, as a generalization of results by Kawamura, Hayashi, and Lascu. Furthermore, a certain symmetry of such an interval dynamical system is interpreted as a covariant representation of the C^*-dynamical system of the "flip-flop" automorphism of $\mathcal{O}_2$.

1 Introduction

Permutative representations of the Cuntz algebras $\mathcal{O}_N$ are a special class of representations arising from branching function systems. Bratteli and Jorgensen [6] classified irreducible permutative representations of $\mathcal{O}_N$ up to unitary equivalence. Kawamura, Hayashi, and Lascu [11] studied the permutative representation arising from the Gauss map, a well-studied dynamical system related to continued fractions. They showed that unitary equivalence classes of irreducible permutative representations of $\mathcal{O}_\infty$ correspond to $\mathrm{PGL}_2(\mathbb{Z})$ equivalence classes of irrational numbers. Moreover, representations labeled by solutions to quadratic equations with integer coefficients are characterized by the existence of certain eigenvectors. This establishes a correspondence between the number theoretic properties of the label and the properties of the representation. If this connection is further developed, the rich combintorial and algebraic structure of continued fractions can be used to study the representation theory of Cuntz algebras.

In this note, we study permutative representations associated with so-called slow continued fraction algorithms (hereafter SCFAs) recently introduced by Panti [14]. This is a broad class of continued fractions including regular continued fractions, Zagier's ceiling continued fractions [18], even and odd continued fractions [5], and "backwards" continued fractions [1]. Our main result is a correspondence between unitary equivalence classes of irreducible permutative representations of $\mathcal{O}_N$ and $\mathrm{PGL}_2(\mathbb{Z})$ equivalence classes of real numbers for finite N (Theorem 6.3). In contrast to the $N = \infty$ case, it is insufficient to consider irrational numbers. The dynamics of rational numbers under the iteration of SCFAs is more complicated and less studied than that of irrational numbers, and is the source of several technical difficulties that we must overcome.

Received by the editors October 21, 2019.
Published online on Cambridge Core January 3, 2020.
AMS subject classification: **46LXX**, 11J70, 39B12.
Keywords: Cuntz algebra, continued fractions, permutatative representation, projective linear group, Möbius transformation.

As the name suggests, SCFAs can be thought of as slow downs of analogues of the Gauss map. These "Gauss maps" can then be realized as jump transformations of SCFAs. We relate permutative representations associated to SCFAs and those associated with their jump transformations. Precisely, we show that the permutative representation of $\mathcal{O}_N$ associated with an SCFA can be precomposed with an embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_N$ to obtain the representation of $\mathcal{O}_\infty$ associated with a jump transformation (Theorem 6.5). In a similar vein, we translate combinatorial relationships between different SCFAs into embeddings of $\mathcal{O}_N$ into $\mathcal{O}_2 \rtimes \mathbb{Z}_2$. We show that the representation of $\mathcal{O}_N$ associated with any SCFA is the composition of this embedding and a representation of $\mathcal{O}_2 \rtimes_\theta \mathbb{Z}_2$ (Theorem 6.8).

The paper is organized in the following way. In Section 2, we review Cuntz algebras and their permutative representations. In Section 3, we discuss regular continued fractions and conclude with an outline of the argument of Kawamura, Hayashi, and Lascu [11]. In Section 4, we introduce the basic definitions and facts for SCFAs, largely following Panti [14]. In Section 5 we consider the symbolic dynamics of SCFAs, paying special attention to rational numbers. In Section 6 we state and prove our main results connecting slow continued fractions and permutative representations of Cuntz algebras. In Section 7 we consider a few examples of Theorem 6.3.

2 Permutative Representations

In this section, we review Cuntz algebras and their permutative representations. For $N = 2, 3, \dots, \infty$, the Cuntz algebra $\mathcal{O}_N$ is the universal C^*-algebra generated by $\{S_i\}_{i=1}^N$ satisfying [8]:

$$S_i^* S_j = \delta_{ij} 1, \qquad \sum_{i=1}^{N} S_i S_i^* = 1. \tag{2.1}$$

For $N = \infty$, the second equality is replaced with $\sum_{i=1}^{n} S_i S_i^* \le 1$ for all $n \in \mathbb{N}$. Throughout this section, we treat the finite and $N = \infty$ cases simultaneously. For convenience, we will write $\mathbb{N}_N$ for $\{1, 2, \dots N\}$, with the understanding that $\mathbb{N}_\infty = \mathbb{N}$.

Definition 2.1

(i) A *representation* of a C^*-algebra A on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism from A into $B(\mathcal{H})$, the set of all bounded linear operators on $\mathcal{H}$.

(ii) A subspace $V \subset \mathcal{H}$ is *invariant* for a representation $\pi\colon A \to B(\mathcal{H})$ if $\pi(a)v \in V$ for any $a \in A$ and $v \in V$.

(iii) For a representation π of A on $\mathcal{H}$ with a closed invariant subspace V, the restriction $\pi|_V$ of π to V is defined as the restriction of the operator $\pi(a)$ to V for each $a \in A$. We call $\pi|_V : A \to B(V)$ a *subrepresentation* of π.

(iv) A representation $\pi\colon A \to B(\mathcal{H})$ is *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only closed invariant subspaces for π.

Any collection of isometries satisfying the relations (2.1) determines a representation of $\mathcal{O}_N$, because $\mathcal{O}_N$ is simple [8]. All C^*-algebras, representations, and embeddings that we consider are unital.

Definition 2.2 ([6, Chapter 2]) For $N = 2, 3, \ldots, \infty$, a *branching function system* (BFS) of order N on a set Ω is a collection of injective transformations $\{f_i\}_{i=1}^N$ on Ω with pairwise disjoint ranges whose union is Ω.

We will refer to such a system by the tuple $\{\Omega, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$, where $\Delta_i = f_i(\Omega)$, and F is the piecewise function on Ω defined by f_i^{-1} on Δ_i.

Definition 2.3 A *permutative representation* of $\mathcal{O}_N$ on $\mathcal{H}$ is a representation π for which there is an orthonormal basis $\{e_k : k \in K\}$ for $\mathcal{H}$ such that

$$\pi(S_i)e_n \in \{e_k : k \in K\} \qquad (n \in K, i \in \mathbb{N}_N).$$

Proposition 2.4 ([6, p. 7]) *Let $\ell^2(\Omega)$ denote the Hilbert space with orthonormal basis $\{e_\omega : \omega \in \Omega\}$. Any BFS $\{\Omega, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$ induces a permutative representation $\pi_F\colon \mathcal{O}_N \to B(\ell^2(\Omega))$ defined by*

$$\pi_F(S_i)e_\omega = e_{f_i(\omega)} \qquad (\omega \in \Omega, i \in \mathbb{N}_N). \tag{2.2}$$

Lemma 2.5 *If $\{\Omega, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$ and $\{\Omega', G, \{g_i\}_{i=1}^N, \{\Delta_i'\}_{i=1}^N\}$ are conjugate, then the representations π_F and π_G are unitarily equivalent.*

Proof The function systems are conjugate if there exists a bijection $C\colon \Omega \to \Omega'$ such that

$$C \circ f_i = g_i \circ C \qquad i \in \mathbb{N}_N. \tag{2.3}$$

Define the unitary $U\colon \ell^2(\Omega) \to \ell^2(\Omega')$ by $Ue_\omega = e_{C(\omega)}$. From (2.3), we obtain

$$U\pi_F(S_i) = \pi_G(S_i)U \quad (i \in \mathbb{N}_N). \qquad \blacksquare$$

Definition 2.6 ([6, Chapter 4])

(i) For a finite word $w = w_1 w_2 \cdots w_k$ in the alphabet $\mathbb{N}_N$, we denote $S_{w_1} S_{w_2} \cdots S_{w_k}$ by S_w. Let $\mathcal{F}_N$ denote the C^*-subalgebra of $\mathcal{O}_N$ generated by all elements of the form $S_w S_{w'}^*$, where w and w' are finite words with equal length in the alphabet $\mathbb{N}_N$.

(ii) We call an irreducible permutative representation of $\mathcal{O}_N$ a *cycle*.

(iii) We call an irreducible component of the restriction of a cycle to $\mathcal{F}_N$ an *atom*.

We recall a construction of the shift representation π_S^N of $\mathcal{O}_N$. Let Ω_N denote the set $N^{\mathbb{N}}$ of all infinite sequences in the alphabet $\{1, \ldots, N\}$. Define the BFS $\{\sigma_i\}_{i=1}^N$ on Ω_N by

$$\sigma_i\big((x_1, x_2, x_3, \ldots)\big) = (i, x_1, x_2, x_3, \ldots) \qquad (i \in \mathbb{N}_N).$$

From Proposition 2.4, we obtain a representation π_S^N of $\mathcal{O}_N$ on $\ell^2(\Omega_N)$ as

$$\pi_S^N(S_i)e_{(x_n)} = e_{\sigma_i((x_n))} \qquad \big(i \in \mathbb{N}_N \quad (x_n) \in \Omega_N\big).$$

For (x_n) and (y_n) in $\Omega_N := N^{\mathbb{N}}$, write $(x_n) \sim (y_n)$ if there exist $z \in \mathbb{Z}$, $m \in \mathbb{N}$ such that $x_{n+z} = y_n$ for $n \geq m$. Write $(x_n) \approx (y_n)$ if z can be taken to be 0. These are equivalence relations, which we call *tail equivalence* and *eventual equivalence*, respectively.

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We denote the equivalence class of (x_n) under $\sim$ by $[(x_n)]$ and the equivalence class of x under $\approx$ by $[\![(x_n)]\!]$.

Proposition 2.7 ([6, Chapter 6])

(i) *The decomposition of π_S^N into cycles corresponds to the decomposition of $\ell^2(\Omega_N)$ into subspaces*

$$\ell^2(\Omega_N) = \bigoplus_{[(x_n)]\in\Omega_N/\sim} \mathcal{H}_{[(x_n)]},$$

where $\mathcal{H}_{[(x_n)]}$ is the (separable) subspace of $\ell^2(\Omega_N)$ with basis $\{e_{(y_n)} : (y_n) \in [(x_n)]\}$.

(ii) *Any irreducible permutative representation of $\mathcal{O}_N$ is unitarily equivalent to exactly one such a representation.*

(iii) *The decomposition of π_S^N (restricted to $\mathcal{F}_N$) into atoms corresponds to the decomposition of $\ell^2(\Omega_N)$ into subspaces*

$$\ell^2(\Omega_N) = \bigoplus_{[\![(x_n)]\!]\in\Omega_N/\approx} \mathcal{H}_{[\![(x_n)]\!]},$$

where $\mathcal{H}_{[\![(x_n)]\!]}$ is the subspace of $\ell^2(\Omega_N)$ with basis $\{e_{(y_n)} : (y_n) \in [\![(x_n)]\!]\}$.

3 Regular Continued Fractions

In this section, we review a few useful definitions and facts about regular continued fractions. The regular continued fraction expansion of $x \in \mathcal{I} := [0,1] \setminus \mathbb{Q}$ is written as

$$x = [a_1, a_2, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{\ddots}}}, \tag{3.1}$$

where the partial quotients a_i are positive integers. Continued fractions can be understood in terms of the Gauss and Farey maps, $G_R, F_R : \mathcal{I} \to \mathcal{I}$ defined as

$$G_R(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \qquad F_R(x) = \begin{cases} \frac{x}{1-x} & \text{if } x \in (0,1/2), \\ \frac{1-x}{x} & \text{if } x \in (1/2,1), \end{cases}$$

where $\lfloor x \rfloor$ is the floor function. For $[a_1, a_2, \dots]$ as in (3.1) G_R and F_R act as:

$$G_R([a_1, a_2, a_3, \dots]) = [a_2, a_3, a_4, \dots],$$

$$F_R([a_1, a_2, a_3, \dots]) = \begin{cases} [a_1 - 1, a_2, a_3, \dots] & \text{if } a_1 \geq 2, \\ [a_2, a_3, a_4, \dots] & \text{if } a_1 = 1. \end{cases} \tag{3.2}$$

Let $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. We have the relation

$$G_R(x) = {F_R}^{r(x)+1}(x) \qquad r(x) = \inf\{n \in \mathbb{N}_0 : {F_R}^n(x) \in (1/2,1)\}. \tag{3.3}$$

For x in $\mathcal{I}$, (3.1) implies that $r(x) + 1 = a_1$, and, in particular, $r(x)$ is finite. We say that the Gauss map is the *jump transformation* of the Farey map obtained by inducing on the interval $(1/2,1)$. From (3.3), the Gauss map G_R can be regarded as an acceleration of the Farey map F_R. Conversely, F_R can be regarded as a slow-down of G_R.

Definition 3.1 Let $\mathrm{PGL}_2(\mathbb{Z})$ be the group of two by two integer matrices of determinant ±1, with the matrices M and $-M$ identified. We will regard $\mathrm{PGL}_2(\mathbb{Z})$ as the group of fractional linear transformations with integer coefficients by identifying the matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PGL}_2(\mathbb{Z})$ with the function $x \mapsto \frac{ax+b}{cx+d}$.

Definition 3.2 Let Σ be a subgroup of $\mathrm{PGL}_2(\mathbb{Z})$. We say $x, y \in \mathbb{R}$ are Σ-equivalent and write $x \sim_\Sigma y$ if there exists a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Sigma$ such that $y = \frac{ax+b}{cx+d}$. This is an equivalence relation, and we denote the equivalence class of x by $[x]_\Sigma$.

We recall the following well-known facts about continued fractions.

Proposition 3.3 ([10, Theorems 2.3, 6.1, and 5.3 in Chapter 10])

(i) *Let Ω_∞ denote the set of all sequences of positive integers. The map $\mathcal{I} \ni x \mapsto (a_1, a_2, \dots) \in \Omega_\infty$ defined by the correspondence in (3.1) is bijective.*
(ii) (Lagrange) *An irrational number has an eventually periodic continued fraction expansion if and only if it is the solution of a quadratic equation with integer coefficients.*
(iii) (Serret, [15, p. 34]) *Two irrational numbers x and y have tail equivalent continued fraction expansions if and only if $x \sim_{\mathrm{PGL}_2(\mathbb{Z})} y$.*

We end this section with a brief sketch of the argument of Kawamura, Hayashi, and Lascu in our terminology. The (regular) Gauss map G_R is a branching function system on $\mathcal{I} = [0,1] \setminus \mathbb{Q}$, which the correspondence in Proposition 3.3(i) conjugates to the full shift on $\Omega_\infty = \mathbb{N}^{\mathbb{N}}$. By Lemma 2.5, the representation π_{G_R} of $\mathcal{O}_\infty$ associated with the Gauss map by Proposition 2.4 is unitarily equivalent to the shift representation π_S^∞. The conjugacy sends $\mathrm{PGL}_2(\mathbb{Z})$-equivalence classes of irrational numbers to tail equivalence classes of sequences, as per Proposition 3.3(iii). In light of Proposition 2.7, we obtain a correspondence between unitary equivalence classes of irreducible permutative representations of $\mathcal{O}_\infty$ and $\mathrm{PGL}_2(\mathbb{Z})$-equivalence classes of irrational numbers. Proposition 3.3(ii) implies that an irreducible permutative representation of $\mathcal{O}_\infty$ has finitely many atoms if and only if it is labeled by a class of solutions to quadratic equations with integer coefficients.

4 Slow Continued Fraction Algorithms

In this section, we introduce SCFAs and discuss a few useful combinatorial properties.

Definition 4.1

(i) A subinterval $[\frac{p}{q}, \frac{p'}{q'}] \subset [0,1]$ with rational endpoints is said to be *unimodular* if $\frac{p}{q}$ and $\frac{p'}{q'}$ are reduced fractions such that $pq' - p'q = -1$.
(ii) A *unimodular partition* is a finite collection unimodular intervals I_i whose union is $[0,1]$, such that for $i \neq j$, $I_i \cap I_j$ contains at most one point.

Definition 4.2 An SCFA is a finite collection of functions $h_i\colon [0,1] \to [0,1]$ such that

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(i) each function h_i is a fractional linear transformation in $\mathrm{PGL}_2(\mathbb{Z})$;
(ii) the images $\{h_i([0,1])\}_{i=1}^N$ form a unimodular partition.

The most important example is the regular Farey map with inverse branches $\frac{x}{x+1}$ and $\frac{1}{x+1}$. In general, the fractional linear transformation $x \mapsto \frac{ax+b}{cx+d}$ is continuous and monotone except at the singular point $-\frac{d}{c}$. The assumption $h_i([0,1]) \subset [0,1]$ ensures that this singularity does not occur on the interval $[0,1]$. Hence, $h_i\colon [0,1] \to [0,1]$ is continuous and strictly monotone. By assumption, $h_i([0,1])$ is a unimodular interval, which we denote $[\frac{p_i}{q_i}, \frac{p_i'}{q_i'}]$. If h_i is increasing, then $h_i(\frac{0}{1}) = \frac{p_i}{q_i}$ and $h_i(\frac{1}{1}) = \frac{p_i'}{q_i'}$. We, therefore, have the formula

$$h_i(x) = \begin{bmatrix} p_i' - p_i & p_i \\ q_i' - q_i & q_i \end{bmatrix}(x).$$

If h_i is decreasing, the situation is reversed and

$$h_i(x) = \begin{bmatrix} p_i - p_i' & p_i' \\ q_i - q_i' & q_i' \end{bmatrix}(x).$$

In general, denoting the determinant of h_i (equivalently, the sign of its derivative), by $\epsilon_i \in \{\pm 1\}$, h_i is given by the formula

$$h_i(x) = \begin{bmatrix} p_i' - p_i & p_i \\ q_i' - q_i & q_i \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}^{(1-\epsilon_i)/2}(x). \tag{4.1}$$

Hence, the data of a unimodular partition $\{[\frac{p_i}{q_i}, \frac{p_i'}{q_i'}]\}_{i=1}^N$ and signs $\{\epsilon_i\}_{i=1}^N$, $\epsilon_i \in \{-1,1\}$ specifies an SCFA. Our convention will be to order the unimodular partition (and hence the h_i) such that $0 = \frac{p_1}{q_1}$ and $\frac{p_i'}{q_i'} = \frac{p_{i+1}}{q_{i+1}}$ for $1 \le i < N$.

Proposition 4.3 *Fix an SCFA $\{h_i\}_{i=1}^N$, and let f_i be the restriction of h_i to $\mathfrak{I} := [0,1] \smallsetminus \mathbb{Q}$. The functions $\{f_i\}_{i=1}^N$ form a BFS on $\mathfrak{I}$.*

Proof As we have already remarked, the condition $h_i([0,1]) \subset [0,1]$ guarantees that h_i is continuous and strictly monotone on $[0,1]$. Hence, f_i is injective. Since $[0,1] \subset \cup_{i=1}^N h_i([0,1])$ and h_i maps irrational numbers to irrational numbers, $\mathfrak{I} \subset \cup_{i=1}^N f_i(\mathfrak{I})$. For $i \neq j$, $h_i([0,1]) \cap h_j([0,1])$ is either empty or a rational singleton. Hence, $f_i(\mathfrak{I}) \cap f_j(\mathfrak{I}) = \varnothing$. ∎

Definition 4.4 With the notation introduced in Definition 2.2(i), let $\{\mathfrak{I}, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$ be the BFS associated with an SCFA. Suppose $E \subset \mathfrak{I}$ is of the form $\cup_{i=j}^k \Delta_i$ for some $1 \le j \le k \le N$. Define

$$\mathfrak{I}_E := \{x \in \mathfrak{I} : F^n(x) \in E \text{ for infintely many } n \in \mathbb{N}\},$$
$$r(x) := \inf\{n \in \mathbb{N}_0 : F^n(x) \in E\} \text{ for } x \in \mathfrak{I}_E.$$

The *jump transformation* of F induced on E is $G(F,E)\colon \mathfrak{I}_E \to \mathfrak{I}_E$,

$$G(F,E)(x) = F^{r(x)+1}(x).$$

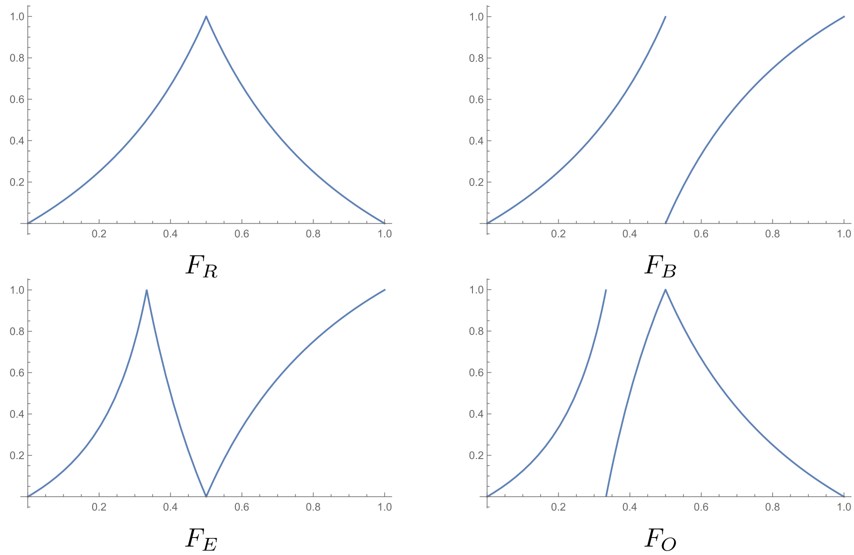

Figure 1: The SCFAs of Example 4.5.

If E is a proper subset of $\mathcal{I}$, then G will have countable many inverse branches $g_j\colon \mathcal{I}_E \to \mathcal{I}_E$, which also form a BFS.

With the above generalization of the relationship between the Farey and Gauss maps in hand, we can now describe several motivating examples of SCFAs and their jump transformations.

Example 4.5 (See also Figure 1.)

(i) The classical Farey map F_R in (3.2) is the SCFA associated with the partition $[0,1/2],[1/2,1]$ and signs $1,-1$. From (3.3), inducing on $\Delta_2 = [1/2,1] \cap \mathcal{I}$ yields the classical Gauss map G_R as its acceleration [2,7,9].

(ii) An important SCFA, which we denote F_B is the SCFA associated with the partition $[0,1/2],[1/2,1]$ and signs $1,1$. Inducing F_B, on $\Delta_2 = [1/2,1] \cap \mathcal{I}$ yields Zagier's ceiling algorithm $G(x) = \lceil \frac{1}{x} \rceil - \frac{1}{x}$, where $\lceil x \rceil$ is the ceiling function [18]. Inducing on $\Delta_1 = [0,1/2] \cap \mathcal{I}$ yields the "backwards" continued fractions [1].

(iii) The even and odd Farey maps F_E and F_O are SCFAs associated with the partition $[0,1/3],[1/3,1/2],[1/2,1]$ and signs $1,-1,1$ and $1,1,-1$, respectively. Inducing F_E and F_O on $\Delta_2 \cup \Delta_3 = [1/3,1] \cap \mathcal{I}$ yields the even and odd Gauss maps [4,5,16].

The following lemma provides a useful description of the inverse branches of an arbitrary SCFA as compositions of those of F_B. Let $b_1 = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $b_2 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right]$.

Lemma 4.6 *Fix an SCFA* $\{h_i\}_{i=1}^N$.

(i) *Each* $h_i = \left[\begin{smallmatrix} p_i'-p_i & p_i \\ q_i'-q_i & q_i \end{smallmatrix}\right]\left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right]^{(1-\epsilon_i)/2}$ *can be written as* $b_{v_i} T^{(1-\epsilon_i)/2}$ *where* b_{v_i} *is a word in* $\{b_1, b_2\}$, $T(x) = 1 - x$.

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(ii) *The words $\{b_{\nu_i}\}_{i=1}^N$ are the leaves of a finite, rooted binary tree. In particular, none of the words are left factors of another. The word $b_\mu b_1$ is a left factor of a word in $\{b_{\nu_i}\}_{i=1}^N$ if and only if $b_\mu b_2$ is a left factor of a word in $\{b_{\nu_i}\}_{i=1}^N$.*

Proof Any unimodular partition can be obtained uniquely from the interval $[\frac{0}{1}, \frac{1}{1}]$ by repeatedly splitting an interval $[\frac{p}{q}, \frac{p'}{q'}]$ into two subintervals $[\frac{p}{q}, \frac{p+p'}{q+q'}]$ and $[\frac{p+p'}{q+q'}, \frac{p'}{q'}]$. Equation (4.1) associates the intervals $[\frac{p}{q}, \frac{p'}{q'}]$, $[\frac{p+p'}{q+q'}, \frac{p'}{q'}]$, and $[\frac{p}{q}, \frac{p+p'}{q+q'}]$ to the matrices $\begin{bmatrix} p'-p & p \\ q'-q & q \end{bmatrix}$, $\begin{bmatrix} -p & p+p' \\ -q & q+q' \end{bmatrix}$, and $\begin{bmatrix} p' & p \\ q' & q \end{bmatrix}$. The lemma follows from the observabtion that sp an interval corresponds to right multiplication of its associated matrix with b_1 and b_2:

$$\begin{bmatrix} p'-p & p \\ q'-q & q \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} p' & p \\ q' & q \end{bmatrix},$$

$$\begin{bmatrix} p'-p & p \\ q'-q & q \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -p & p+p' \\ -q & q+q' \end{bmatrix}. \qquad \blacksquare$$

5 Symbolic Dynamics for SCFAs

The analogue of continued fractions are *itineraries* with respect to an SCFA. Although only irrational numbers have infinite continued fraction expansions, every real number will have an infinite itinerary. For the same reason that the (terminating) continued fraction expansions of rational numbers are not unique, each rational number in $(0,1)$ will have two itineraries. Instead of a bijection, we therefore work separately with a surjective decoding map and an injective encoding map.

Definition 5.1 Fix an SCFA $\{h_i\}_{i=1}^N$ with associated BFS $\{\mathcal{I}, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$.

(i) A sequence $(x_n) \in \Omega_N$ is an F*-itinerary* for x if $x \in \cap_{n=1}^\infty h_{x_n} \circ h_{x_{n-1}} \circ \cdots \circ h_{x_1}([0,1])$.

(ii) We write $x \sim_F y$ if there exist tail equivalent F-itineraries for x and y. This is an equivalence relation, and we write $[x]_F$ for the $\sim_F$-equivalence class of x.

(iii) The Panti–Serret group Σ_F associated with $\{\mathcal{I}, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$ is the subgroup of $\mathrm{PGL}_2(\mathbb{Z})$ generated by the matrices h_i.

(iv) The SCFA $\{\mathcal{I}, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$ is said to satisfy the Serret theorem if for irrational numbers, the relation $\sim_F$ coincides with $\sim_{\Sigma_F}$ as in Definition 3.2.

The Serret theorem in Definition 5.1(iv) holds for some, but not all SCFAs. In general, each Σ_F equivalence class is the union of $\sim_F$ equivalence classes. A practical criterion for checking the validity of the Serret theorem is established in [14]. We recall the following facts from [13].

Proposition 5.2 (Panti) (i) ([13, Observation 3]) *The intersection in Definition 5.1(i) is always a singleton.*

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(ii) (Generalized Lagrange Theorem [13, Section 3]) *An irrational number has an eventually periodic itinerary with respect to every SCFA if and only if it is a solution to a quadratic equation with integer coefficients.*

Definition 5.3 Fix an SCFA $\{h_i\}_{i=1}^N$ with BFS $\{\mathcal{I}, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$.

(i) For $(x_n) \in \Omega_N$, let x be the singleton $\cap_{n=1}^{\infty} h_{x_n} \circ h_{x_{n-1}} \circ \cdots \circ h_{x_1}([0,1])$, as in Proposition 5.2(i). Define the *decoding map* $\mathrm{Dec}_F : \Omega_N \to [0,1]$ by $\mathrm{Dec}_F((x_n)) = x$.

(ii) For $x \in \mathcal{I}$, let $(x_n) \in \Omega_N$ be the sequence such that $F^{n-1}(x) \in \Delta_{x_n}$. Define the *encoding map* $\mathrm{Enc}_F : \mathcal{I} \to \Omega_N$ by $\mathrm{Enc}_F(x) = (x_n)$.

Remark 5.4 Since $\cup_{i=1}^N h_i([0,1]) = [0,1]$, Dec_F is surjective. The injectivity of Enc_F is [13, Observation 3(i)].

Proposition 5.5 *The encoding map* Enc_F *conjugates* $\{\mathcal{I}, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$ *to a subshift* $\{\mathrm{Enc}_F(\mathcal{I}), \sigma, \{\sigma_i\}_{i=1}^N, \{\sigma_i(\mathrm{Enc}_F(\mathcal{I}))\}_{i=1}^N\}$.

Proof Since Enc_F is injective, it is enough to observe that

$$\mathrm{Enc}_F \circ f_i = \sigma_i \circ \mathrm{Enc}_F \qquad i \in \{1, \dots, N\}. \qquad \blacksquare$$

The encoding map in the above proposition is never surjective; its image is always Ω_N minus the itineraries of rational numbers. For example, denoting the constant sequences by $\overline{1} = 1,1,1,\dots$ and $\overline{2} = 2,2,2,\dots$,

$$\mathrm{Enc}_{F_R}(\mathcal{I}) = \Omega_2 \smallsetminus \{(x_n) : (x_n) \sim \overline{1}\},$$
$$\mathrm{Enc}_{F_B}(\mathcal{I}) = \Omega_2 \smallsetminus \{(x_n) : (x_n) \sim \overline{2} \text{ or } (x_n) \sim \overline{1}\}.$$

We conclude this section by establishing a strong version of the Serret theorem, which includes rational numbers for a specific family of SCFAs, which we denote F_N.

Definition 5.6 For $2 \le N < \infty$, let F_N be the SCFA associated with the unimodular partition $[0, \frac{1}{N}], [\frac{1}{N}, \frac{1}{N-1}], [\frac{1}{N-1}, \frac{1}{N-2}], \dots, [\frac{1}{2}, \frac{1}{1}]$ and signs $1, -1, -1, \dots, -1$.

Proposition 5.7

(i) *The subgroup* Σ_{F_N} *of* $\mathrm{PGL}_2(\mathbb{Z})$ *generated by* $\{h_i\}_{i=1}^N$ *coincides with* $\mathrm{PGL}_2(\mathbb{Z})$.

(ii) *The Serret theorem holds for* F_N *in the sense of Definition* 5.1(iv).

(iii) *A sequence* $(x_n) \in \Omega_N$ *is an* F_N*-itinerary of a rational number if and only if* $(x_n) \sim \overline{1} = 1,1,1,\dots$.

Corollary 5.8 *Let* $x, y \in [0,1]$ *and* $(x_n), (y_n) \in \Omega_N$ *such that* $\mathrm{Dec}_{F_N}((x_n)) = x$ *and* $\mathrm{Dec}_{F_N}((y_n)) = y$. *Then* $(x_n) \sim (y_n)$ *if and only if* $x \sim_{\mathrm{PGL}_2(\mathbb{Z})} y$.

Proof of Proposition 5.7

(i) Note that F_2 is simply the Farey map F_R. Denote the inverse branches of F_R by r_1 and r_2. The inverse branches of F_N are $r_1^{N-1}, r_1^{N-2}r_2, r_1^{N-3}r_2, \dots, r_1^2 r_2, r_1 r_2, r_2$. Proposition 5.7(i) then follows from the fact that $\{r_1, r_2\}$ generates $\mathrm{PGL}_2(\mathbb{Z})$.

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(ii) We construct the transducer (finite state automaton) considered in [14, Lemma 5.5]. Let b_1 and b_2 be the branches of F_B and $T(x) = 1 - x$, as in Lemma 4.6. Given $h \in \{h_i\}_{i=1}^N$ and $v \in V$, there is a unique $w \in V$ for which there exists a (possibly empty) word $\mu = \mu_1\mu_2\cdots\mu_n$ in the alphabet $\{1,\dots,N\}$ such that

$$vh = h_{\mu_1}h_{\mu_2}\cdots h_{\mu_n}w.$$

The transducer in question has state set $V = \{b_1^k T^e : 0 \le k \le N-2, e \in \{0,1\}\}$. For v and w as in (ii), it has a directed edge from v to w labeled with input h and output $h_{\mu_1}h_{\mu_2}\cdots h_{\mu_n}$. To construct the edge set, we consider the following cases:

(a) If $v = b_1^k$ and $h = b_1^{N-1}$: We obtain a self loop, labeled with both input and output h.

(b) If $v = b_1^k$ and $h = b_1^j b_2 T$: We obtain an edge to 1. If $k = 0$ (*i.e.*, $v = 1$) this self loop is labeled with both input and output h.

(c) If $v = b_1^k T$ and $h = b_1^{N-1}$: We obtain an edge to b_1^{N-2}.

(d) If $v = b_1^k T$ and $h = b_1^j b_2 T$ for $j \ne 0$: We obtain an edge to 1.

(e) If $v = b_1^k T$ and $h = b_2 T$, for $k \ne N-2$: We obtain an edge to b_1^{k+1}.

(f) If $v = b_1^{N-2} T$ and $h = b_2 T$: We obtain an edge to 1.

An infinite path in the transducer constructed above eventually consists of an infinitely repeated self loop, labeled with the same input as output. Hence, the output of the transducer is always tail equivalent to its input. By [14, Corollary 5.6], the Serret theorem holds.

(iii) This is a special case of the following Lemma 5.9. ∎

Lemma 5.9 *Suppose the SCFA $\{h_i\}_{i=1}^N$ satisfies $\epsilon_1 = 1$ and $\epsilon_N = -1$. (Recall our ordering convention that $0 \in h_1([0,1])$ and $1 \in h_N([0,1])$.) A sequence $(x_n) \in \Omega_N$ is an itinerary of a rational number if and only if $(x_n) \sim \bar{1} = 1,1,1,\dots$.*

Proof By the proof of [13, Observation 3], for $r \in \mathbb{Q}$ and $(x_n) \in \Omega_N$, there is $M \in \mathbb{N}$ such that r is not in the topological interior of $h_{x_m} \circ \cdots \circ h_{x_1}([0,1])$ for $m \ge M$. If (x_n) is an itinerary for $x \in \mathbb{Q} \cap [0,1]$, this implies that x is an endpoint of $h_{x_m} \circ \cdots \circ h_{x_1}([0,1])$ for $m \ge M$. If the determinant of $h_{x_m} \circ \cdots \circ h_{x_1}$ is 1, then $h_{x_m} \circ \cdots \circ h_{x_1}([0,1])$ shares its right endpoint with $h_N \circ h_{x_m} \circ \cdots \circ h_{x_1}([0,1])$ and left endpoint with $h_1 \circ h_{x_m} \circ \cdots \circ h_{x_1}([0,1])$. If the determinant is -1, the situation is reversed. In this way, the shared endpoint and the determinant of $h_{x_m} \circ \cdots \circ h_{x_1}$ inductively determine x_{m+1}.

If $x_m = N$, the determinants of $h_{x_m} \circ \cdots \circ h_{x_1}$ and $h_{x_{m-1}} \circ \cdots \circ h_{x_1}$ differ by the assumption $\det(h_N) = -1$. By the above, $x_{m+1} = 1$. If $x_m = 1$, the determinants of $h_{x_m} \circ \cdots \circ h_{x_1}$ and $h_{x_{m-1}} \circ \cdots \circ h_{x_1}$ coincide by the assumption $\det(h_1) = 1$. Again, $x_{m+1} = 1$. We conclude that $x_m = 1$ for $m > M$, proving the forward implication.

Conversely, if (x_n) is tail equivalent to $\bar{1}$, then for n and m sufficiently large, the intervals $h_{x_n} \circ \cdots \circ h_{x_1}([0,1])$ and $h_{x_m} \circ \cdots \circ h_{x_1}([0,1])$ share a rational endpoint. The common endpoint is therefore the unique point in the intersection $\cap_{n=1}^{\infty} h_{x_n} \circ \cdots \circ h_{x_1}([0,1])$. We conclude that (x_n) is the itinerary of a rational number, proving the lemma. ∎

6 Main Results

In this section, we state and prove our main results. We begin with the following corollary of Lemma 2.5 and Propositions 2.7 and 5.5.

Proposition 6.1 *Let $\{\mathcal{I}, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$ be the BFS associated with an SCFA and let π_F be the permutative representation of $\mathcal{O}_N$ on $\ell^2(\mathcal{I})$ in Proposition 2.4. Then the irreducible decomposition of π_F is given as*

$$\ell^2(\mathcal{I}) = \bigoplus_{[x]_F \in \mathcal{I}/\sim_F} \mathcal{H}_{[x]_F}$$

where $\mathcal{H}_{[x]_F}$ is the subspace of $\ell^2(\mathcal{I})$ with basis $\{e_y : y \sim_F x\}$.

Remark 6.2 If the SCFA satisfies the Serret theorem as in Definition 5.1(iv), then the sets $[x]_F$ coincide with Σ_F-orbits of irrational numbers.

We now consider the SCFAs F_N introduced in Definition 5.6 to produce a bijection between equivalence classes of irreducible permutative representations of $\mathcal{O}_N$ and $\mathrm{PGL}_2(\mathbb{Z})$-equivalence classes of real numbers.

Theorem 6.3 *For $2 \le N < \infty$, the decoding map Dec_{F_N} in Definition 5.3(i) provides a bijection between unitary equivalence classes of irreducible permutative representations of $\mathcal{O}_N$ and $\mathrm{PGL}_2(\mathbb{Z})$-equivalence classes of real numbers. Moreover, an equivalence class of representations corresponds to an equivalence class of solutions to quadratic equations with integer coefficients if and only if it has finitely many atoms.*

Proof By Proposition 2.7, there is a bijection between unitary equivalence classes of irreducible permutative representations of $\mathcal{O}_N$ and the subspaces $\mathcal{H}_{[(x_n)]}$ of $\ell^2(\Omega_N)$. We consider the bijection

$$\mathcal{H}_{[(x_n)]} \longmapsto [\mathrm{Dec}_{F_N}((x_n))]_{\mathrm{PGL}_2(\mathbb{Z})} \in [0,1]/\sim_{\mathrm{PGL}_2(\mathbb{Z})} .$$

This is well defined by the forward implication in Corollary 5.8. It is injective by the backwards implication in Corollary 5.8, and surjective by Remark 5.4. By Proposition 5.2(ii) for irrationals and Proposition 5.7(iii) for rationals, $[(x_n)]$ is labeled by a class of solutions to quadratic equations with integer coefficients if and only if $[(x_n)]$ contains a periodic sequence. This is equivalent to consisting of finitely many eventual equivalence classes, which by Proposition 2.7(iii) is equivalent to the associated irreducible permutative representation of $\mathcal{O}_N$ having finitely many atoms. ■

Remark 6.4 If $x \in [0,1]$ has F_N-itinerary (x_n) and w_n is the word $x_1x_2\cdots x_n$, then irreducible permutative representations of $\mathcal{O}_N$ labeled by $[x]_{\mathrm{PGL}_2(\mathbb{Z})}$ are characterized by the existence of a vector ξ such that $S_{w_n}^*\xi \neq 0$ for $n \in \mathbb{N}$. For $\pi_S^N|_{\mathcal{H}_{[(x_n)]}}$, ξ is simply $e_{(x_n)}$. We consider examples in Section 7.

Theorem 6.5 *Let $\{\mathcal{I}, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$ be the BFS associated with an SCFA, and $G = G(F, E)$ be the jump transformation of F induced on E. There is a unital embedding $\varphi\colon \mathcal{O}_\infty \to \mathcal{O}_N$ that is compatible with the representations π_F and π_G (as defined in*

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Proposition 2.4) in the sense that $\ell^2(\mathcal{I}_E)$ sits inside $\ell^2(\mathcal{I})$ as a closed, invariant subspace on which

(6.1) $$\pi_F \circ \varphi_F = \pi_G.$$

Proof From Definition 4.4, $\mathcal{I}_E = \{x \in \mathcal{I} : F^n(x) \in E \text{ for infinitely many } n \in \mathbb{N}\}$, and hence $\ell^2(\mathcal{I}_E)$ is invariant under the action of $\pi_F(\mathcal{O}_N)$. Let $\{T_j : j \in \mathbb{N}\}$ be the generators of $\mathcal{O}_\infty$ and let $\{S_i : i = 1, \dots, N\}$ be the generators of $\mathcal{O}_N$. Let $f = \{f_i : \text{range}(f_i) \subset E\}$ and $f_{E^c} = \{f_i : \text{range}(f_i) \cap E = \varnothing\}$. The inverse branches g_j of $G(F, E)$ are of the form $g_j = f_{j_1} \circ f_{j_2} \circ \cdots \circ f_{j_k}$ where $f_{j_k} \in f_E$ and $f_{j_1}, \dots, f_{j_{k-1}} \in f_{E^c}$. We denote the word $j_1 j_2 \cdots j_k$ by μ_j. Define $\varphi(1) = 1$ and $\varphi(T_j) = S_{\mu_j}$, from which (6.1) immediately follows. To show that φ is an embedding, it suffices to verify (2.1). For $j, j' \in \mathbb{N}$, only f_{j_k} and $f_{j'_{k'}}$ belong to f_E. Hence, μ_j is a left factor of $\mu_{j'}$ only if $j = j'$. This verifies the left-hand equality of (2.1):

$$\varphi(T_{j'}^*)\varphi(T_j) = S_{\mu_{j'}}^* S_{\mu_j} = \delta_{j,j'}1.$$

For $j \neq j'$, μ_j is not a left factor of $\mu_{j'}$ nor vice versa, so $\varphi(T_j)\varphi(T_j)^*$ and $\varphi(T_{j'})\varphi(T_{j'})^*$ are projections with disjoint ranges, verifying the right-hand inequality of (2.1). ∎

Remark 6.6 A theorem of a similar flavor appears in [12], relating representations associated with the regular Gauss and Farey maps.

The "flip-flop" [3] automorphism θ of $\mathcal{O}_2$ is defined by $\theta(S_1) = S_2$ and $\theta(S_2) = S_1$. Since θ is an involution, it determines an action of the group $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ on $\mathcal{O}_2$. We write $\mathcal{O}_2 \rtimes_\theta \mathbb{Z}_2$ for the associated crossed product. We refer to [17, Chapter 2, Section 2.3] for a treatment of crossed products of C^*-algebras by finite groups. Recall the SCFA F_B introduced in Example 4.5(ii).

Proposition 6.7 *The representation π_{F_B} of $\mathcal{O}_2$ on $\ell^2(\mathcal{I})$ extends to a representation $\widetilde{\pi}_{F_B}$ of $\mathcal{O}_2 \rtimes_\theta \mathbb{Z}_2$ on $\ell^2(\mathcal{I})$.*

Proof Let $U_\theta \in \mathcal{O}_2 \rtimes_\theta \mathbb{Z}_2$ be the unitary that implements θ, *i.e.,*

$$U_\theta A U_\theta^* = \theta(A), \qquad A \in \mathcal{O}_2.$$

Since S_1 and S_2 generate $\mathcal{O}_2$ and $U_\theta = U_\theta^*$, this is equivalent to

$$U_\theta S_1 = S_2 U_\theta.$$

Elements of $\mathcal{O}_2 \rtimes_\theta \mathbb{Z}_2$ can be written in the form $A + U_\theta B$, where $A, B \in \mathcal{O}_2$. Therefore, any self-adjoint unitary $U\colon \ell^2(\mathcal{I}) \to \ell^2(\mathcal{I})$ that satisfies

$$U\pi_{F_B}(S_1) = \pi_{F_B}(S_2)U$$

defines an extension $\widetilde{\pi}_{F_B}$ by setting $\widetilde{\pi}_{F_B}(U_\theta) = U$. Define the self-adjoint unitary $Ue_x = e_{1-x}$. Equations (2.2) and (4.1) yield

$$\pi_{F_B}(S_1)e_x = e_{\frac{x}{1+x}}, \qquad \pi_{F_B}(S_2)e_x = e_{\frac{1}{2-x}}.$$

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Applying these,

$$U\pi_{F_B}(S_1) = \pi_{F_B}(S_2)U,$$

and hence, $\widetilde{\pi}_{F_B}(U_\theta)e_x = e_{1-x}$ gives the claimed extension. ■

Theorem 6.8 *Let $\{\mathcal{I}, F, \{f_i\}_{i=1}^N, \{\Delta_i\}_{i=1}^N\}$ be the BFS associated with an SCFA. There is a unital embedding $\psi_F \colon \mathcal{O}_N \to \mathcal{O}_2 \rtimes_\theta \mathbb{Z}_2$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_2 \rtimes_\theta \mathbb{Z}_2 & \xrightarrow{\widetilde{\pi}_{F_B}} & B(\ell^2(\mathcal{I})) \\ {\scriptstyle \psi_F}\uparrow & \nearrow{\scriptstyle \pi_F} & \\ \mathcal{O}_N. & & \end{array}$$

Proof Let $\{S_i\}_{i=1}^N$ be the generators of $\mathcal{O}_N$, and B_1, B_2, and U_θ the generators of $\mathcal{O}_2 \rtimes_\theta \mathbb{Z}_2$. Applying Lemma 4.6(i), write $f_i = b_{\nu_i}T^{e_i}$ and define

$$\psi_F(S_i) = B_{\nu_i}U_\theta^{e_i},$$

which immediately satisfies $\widetilde{\pi}_{F_B} \circ \psi_F = \pi_F$. Consider

$$\psi(S_{i'}^*)\psi(S_i) = U_\theta^{e_{i'}}B_{\mu_{i'}}^*B_{\nu_i}U_\theta^{e_i}.$$

This expression is nonzero only if μ_i is a left factor of $\mu_{i'}$ or vice versa. By Lemma 4.6(ii), this occurs only when $i = i'$. Hence,

$$\psi(S_{i'}^*)\psi(S_i) = \delta_{i,i'}1,$$

verifying the left-hand side of (2.1). Now consider

$$\sum_{1\le i\le N} \psi_F(S_i)\psi_F(S_i)^* = \sum_{1\le i\le N} B_{\nu_i}U_\theta^{p_i}U_\theta^{p_i}B_{\nu_i}^* = \sum_{1\le i\le N} B_{\nu_i}B_{\nu_i}^*.$$

Repeated application of $B_1B_1^* + B_2B_2^* = 1$, together with the binary tree structure described in Lemma 4.6(ii), shows that

$$\sum_{1\le i\le N} B_{\nu_i}B_{\nu_i}^* = 1,$$

verifying the right-hand side of (2.1). ■

Remark 6.9 If the signs ϵ_i of the SCFA are all positive, Theorem 6.8 gives an embedding into $\mathcal{O}_2$.

7 Examples

In this section we give examples of Theorem 6.3. We choose a representative of a $\mathrm{PGL}_2(\mathbb{Z})$-equivalence class from $[0,1]$ and characterize the associated unitary equivalence class of irreducible permutative representations of $\mathcal{O}_N$ for $N = 2, 3, \ldots$.

Example 7.1 For $N = 2, 3, \ldots$, the F_N-itinerary of 0 is $\overline{1} = 1, 1, 1, \ldots$. The irreducible permutative representations of $\mathcal{O}_N$ labeled by 0 are characterized by the existence of a vector ξ such that $S_1\xi = \xi$, and consist of a single atom.

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Example 7.2 Similarly, for $N = 2, 3, \ldots$, the F_N-itinerary of $\frac{\sqrt{5}-1}{2}$ is $\overline{N}$, and the irreducible permutative representations of $\mathcal{O}_N$ labeled by $\frac{\sqrt{5}-1}{2}$ are characterized by the existence of a vector ξ such that $S_N \xi = \xi$, and consist of a single atom.

Example 7.3 For $N = 2$, $\sqrt{2}-1$ has F_2-itinerary $\overline{12}$, so the corresponding irreducible permutative representations of $\mathcal{O}_2$ are characterized by the existence of a vector ξ such that $S_2 S_1 \xi = \xi$. There are two atoms, corresponding to the two eventual equivalence classes. For $2 < N < \infty$, $\sqrt{2} - 1$ has F_N-itinerary $\overline{N-1}$, so the corresponding irreducible permutative representations of $\mathcal{O}_N$ are characterized by the existence of a vector ξ such that $S_{N-1}\xi = \xi$, and consist of a single atom.

Remark 7.4 The existence of an eigenvector for a finite composition of the generating isometries is how Hayashi, Kawamura, and Lascu characterize representations associated with quadratic irrationals for $\mathcal{O}_\infty$. This is equivalent to having finitely many atoms, which is the characterization established in Theorem 6.3 for finite N.

Finally, we consider a label that is not a quadratic root, for which the corresponding irreducible permutative representations must have countably many atoms.

Example 7.5 Let e be the base of the natural logarithm. The regular continued fraction expansion of $e - 2$ is $[1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots]$. The F_2-itinerary of $e - 2$ is

$$2, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 1, 1, 2, 2, \ldots.$$

For each N, the itinerary is aperiodic. The F_3, F_4, and F_5-itineraries of $e - 2$ are respectively

$$3, 2, 3, 3, 1, 1, 2, 3, 3, 1, 1, 1, 1, 2, 3, 3, 1, 1, 1, 1, 1, 1, 2, 3, \ldots,$$
$$4, 3, 4, 4, 1, 2, 4, 4, 1, 1, 1, 2, 4, 1, 1, 1, 1, 1, 2, 4, \ldots,$$
$$5, 4, 5, 5, 2, 5, 5, 1, 1, 2, 5, 1, 1, 1, 1, 2, 5, \ldots.$$

Every finite composition of the generating isometries of the associated representation of $\mathcal{O}_N$ is a pure isometry. In particular, there are no eigenvectors as in the previous examples. The F_N-itineraries nevertheless characterize the representations as discussed in Remark 6.4.

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