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Proof of the Van den Berg–Kesten Conjecture

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We prove the following conjecture of J. van den Berg and H. Kesten. For any events $\mathscr{A}$ and $\mathscr{B}$ in a product probability space, $\text{Prob}(\mathscr{A}\square\mathscr{B}) \leqslant \text{Prob}(\mathscr{A})\text{Prob}(\mathscr{B})$, where $\mathscr{A}\square\mathscr{B}$ is the event that $\mathscr{A}$ and $\mathscr{B}$ occur 'disjointly'.

1. Introduction

Consider the following cooperative card game between two players, Red and Yellow. At the start of the game, thirteen cards are dealt face up on the table. The cards contain one of each rank: one ace, one deuce and so on. Their suits are determined in some random fashion, independently for each rank. The players must choose hands (disjoint subsets) from the thirteen face-up cards: one hand for Red and one for Yellow. Each player has his or her own idea of what constitutes a good hand. Red may like many hands, perhaps varying greatly in size and content. Let R be the event that there is a hand Red likes in the thirteen cards and let Y be the event that there is a hand Yellow likes. We cannot tell if $\text{Prob}(R \cap Y)$ is bigger or smaller than $\text{Prob}(R)\text{Prob}(Y)$ without knowing something of the preferences of the two players. If their tastes are similar, we might expect $\text{Prob}(R \cap Y) \geqslant \text{Prob}(R)\text{Prob}(Y)$; and the opposite if their tastes conflict.

Since the hands must be disjoint, there may be in Red's hand some cards Yellow needs to form a good hand. For both players to be satisfied, there must be a pair of disjoint sets, one of which satisfies Red and the other which satisfies Yellow. Denote this event by $R\square Y$. It was conjectured in 1987 by Van den Berg and Kesten [1] that $\text{Prob}(R\square Y) \leqslant \text{Prob}(R)\text{Prob}(Y)$.

The above conjecture has a strong intuitive appeal. Even if Red and Yellow's tastes were similar, the fact that Red must remove a good hand from the table cannot possibly help Yellow, and we might expect $R\square Y$ to reflect a sort of negative correlation between R and Y. However, on deeper inspection, the conjecture is far from obvious. Many special cases of the conjecture have been proved. One result [1], used in percolation theory, states

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that the conjecture is true if both R and Y are increasing events. This would be the case in our card game if, for example, the suits were ordered and a hand could only improve (in either player's eyes) if the suits of some of its cards were upgraded.

In this paper we will prove the conjecture for all R and Y.

Let's formalize the above ideas. Let S_i be a finite set for $i \in [n] := \{1, 2, \ldots, n\}$ and ω_i be a probability measure on S_i. Let $\Omega = S_1 \times S_2 \times \cdots \times S_n$ and $\mu = \mu_1 \times \cdots \times \mu_n$. For $K \subseteq [n]$ and $\omega \in \Omega$, define the cylinder set $C(K, \omega) := \{\omega' \in \Omega : \omega_i' = \omega_i \text{ for all } i \in K\}$. Define

$$R \Box Y := \{\omega : \exists K \subseteq [n] \text{ such that } C(K, \omega) \subseteq R \text{ and } C([n] \setminus K, \omega) \subseteq Y\}.$$

The Van den Berg–Kesten conjecture can be stated as follows.

Theorem 1.1. $\mathrm{Prob}(Y \Box R) \leqslant \mathrm{Prob}(Y)\mathrm{Prob}(R)$ *for all events* Y *and* R.

Let $Q_n := \{0, 1\}^n$ for $n \in \{0, 1, 2, \ldots\}$. For $x \in Q_n$ write x_i for the ith entry of x. Intuitively, we can think of Q_n in two ways: either as a set of sequences, or as the set of vertices of an n-dimensional cube. For $x \in Q_n$ define x^c, the antipode of x, by $x_i^c := 1 - x_i$ for all i. For $S \subseteq Q_n$, define $S^c := \{x^c \in Q_n : x \in S\}$. Define $[x, y] \subseteq Q_n$ for $x, y \in Q_n$ as the set of all elements z such that, for every $i \in [n], z_i \in \{x_i, y_i\}$. In other words, $[x, y]$ is the smallest subcube of Q_n containing both x and y.

For $a, b \in Q_n$ we define the *butterfly* $B_{a,b}$ to be the ordered pair $\langle a, b\rangle$ with the following four associated subcubes of Q_n:

$$\mathrm{Red}(B_{a,b}) := [a, b], \quad \mathrm{Yellow}(B_{a,b}) := [a, b^c],$$
$$\mathrm{Body}(B_{a,b}) := \{a\}, \quad \mathrm{Tip}(B_{a,b}) := \{b\}.$$

A collection of butterflies, $\mathscr{B}$, on Q_n is called a *flock*. Set

$$\mathrm{Red}(\mathscr{B}) := \bigcup_{B_{a,b} \in \mathscr{B}} \mathrm{Red}(B_{a,b})$$

and define $\mathrm{Yellow}(\mathscr{B})$, $\mathrm{Body}(\mathscr{B})$ and $\mathrm{Tip}(\mathscr{B})$ in the same fashion. The main result of this paper is as follows.

Theorem 1.2. *For any flock of butterflies* $\mathscr{B}$ *with distinct bodies,*

$$|\mathscr{B}| \leqslant |\mathrm{Red}(\mathscr{B}) \cap \mathrm{Yellow}^c(\mathscr{B})|. \tag{1.1}$$

Note that $|\mathrm{Red}(\mathscr{B}) \cap \mathrm{Yellow}^c(\mathscr{B})| = |\{(r, y) \in \mathrm{Red}(\mathscr{B}) \times \mathrm{Yellow}(\mathscr{B}) : [r, y] = Q_n\}|$. So our theorem states that the number of Red–Yellow antipodal pairs of Q_n is at least the size of the flock.

Our statements will become a bit simpler if we slightly change our point of view. Notice that, if $\mathscr{B}$ is a flock of butterflies, then $\mathscr{B}' := \{B_{b,a} : B_{a,b} \in \mathscr{B}\}$ is also a flock of butterflies, where

$$\mathrm{Red}(\mathscr{B}') = \mathrm{Red}(\mathscr{B}) \text{ and } \mathrm{Yellow}(\mathscr{B}') = \mathrm{Yellow}^c(\mathscr{B});$$

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thus

$$|\mathrm{Red}(\mathscr{B}) \cap \mathrm{Yellow}^c(\mathscr{B})| = |\mathrm{Red}(\mathscr{B}') \cap \mathrm{Yellow}(\mathscr{B}')|.$$

Moreover, if the butterflies of $\mathscr{B}$ have distinct bodies, then those of $\mathscr{B}'$ have distinct tips. Thus, we can restate Theorem 1.2 as the following 'butterfly theorem'.

Theorem 1.3. *For any flock of butterflies* $\mathscr{B}$ *with distinct tips,*

$$|\mathscr{B}| \leqslant |\mathrm{Red}(\mathscr{B}) \cap \mathrm{Yellow}(\mathscr{B})|.$$

In Section 2, we will prove Theorem 1.3, or equivalently Theorem 1.2. Theorem 1.1 will be derived from Theorem 1.2 in Section 3.

2. Proof of the Butterfly Theorem

We will actually prove a stronger, linear-algebraic version of Theorem 1.3. Let $\bar{Y} := \{x \in Q_m : x \notin \mathrm{Yellow}(\mathscr{B})\}$ and $\bar{R} := \{x \in Q_m : x \notin \mathrm{Red}(\mathscr{B}) \cup \bar{Y}\}$. Since $\mathrm{Red}(\mathscr{B}) \cap \mathrm{Yellow}(\mathscr{B})$, $\bar{Y}$ and $\bar{R}$ partition the elements of Q_m,

$$|\mathrm{Red}(\mathscr{B}) \cap \mathrm{Yellow}(\mathscr{B})| + |\bar{Y}| + |\bar{R}| = |Q_m| = 2^m.$$

Our goal is to construct a one-to-one map, ψ, from $\mathscr{B} \cup \bar{Y} \cup \bar{R}$ to $\mathfrak{R}^{2^m}$ and show that the vectors in the image are linearly independent; and hence

$$|\mathscr{B}| + |\bar{Y}| + |\bar{R}| \leqslant \text{dimension}\left(\mathfrak{R}^{2^m}\right) = 2^m,$$

which implies the theorem.

2.1. The maps

Before defining the map ψ from $\mathscr{B} \cup \bar{Y} \cup \bar{R}$ to $\mathfrak{R}^{2^m}$, we need the following vector definitions. In what follows let $\oplus$ represent concatenation of vectors. For example $\langle a, b\rangle \oplus \langle c, d\rangle = \langle a, b, c, d\rangle$. For $a \in \mathfrak{R}^2$ and $x \in \mathfrak{R}^n$, define the tensor product of a and x by $a \otimes x := a_1 x \oplus a_2 x$.

For x, $y \in Q_m$, define the vectors e_x, f_x and $g_{x,y}$ as follows. Set

$$\begin{aligned} e_0 &:= \langle 1,1\rangle, & e_1 &:= \langle 0,1\rangle, & f_0 &:= \langle 1,0\rangle, & f_1 &:= \langle 1,-1\rangle, \\ g_{0,0} &:= \langle 1,0\rangle, & g_{0,1} &:= \langle 1,1\rangle, & g_{1,0} &:= \langle 0,1\rangle, & g_{1,1} &:= \langle 1,-1\rangle, \end{aligned}$$

$$e_x := \bigotimes_{i=1}^{n} e_{x_i}, \quad f_x := \bigotimes_{i=1}^{n} f_{x_i}, \quad \text{and} \quad g_{x,y} := \bigotimes_{i=1}^{n} g_{x_i,y_i}.$$

We define the map ψ from $\mathscr{B} \cup \bar{Y} \cup \bar{R}$ to $\mathfrak{R}^{2^m}$ as follows. If $x \in \bar{R}$ then $\psi_{\bar{R}}(x) = e_x$. If $x \in \bar{Y}$ then $\psi_{\bar{Y}}(x) = f_x$. If $B_{x,y} \in \mathscr{B}$ then $\psi_{\mathscr{B}}(B_{a,b}) = g_{a,b}$. Finally, ψ is the union of $\psi_{\bar{R}}$, $\psi_{\bar{Y}}$ and $\psi_{\mathscr{B}}$. We will show that the image vectors of ψ are independent, thus proving the theorem.

2.2. Independence

As usual for A, $B \in \mathfrak{R}^n$ we write $A \perp B$ when every element in A is perpendicular to every element in B. We will write (x, y) for the standard inner product in $\mathfrak{R}^n$, recalling (see, *e.g.*,

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MacLane and Birkhoff [4]) that

$$\left(\bigotimes_{i=1}^{n} u_i, \bigotimes_{i=1}^{n} v_i\right) = \prod_{i=1}^{n}(u_i, v_i) \tag{2.1}$$

where $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are arbitrary lists of vectors.

To show that the images of $\bar{R}, \bar{Y}$ and $\mathscr{B}$ form an independent set we must prove the following six assertions:

(1) $\psi_{\bar{R}}(\bar{R}) \perp \psi_{\bar{Y}}(\bar{Y})$,
(2) $\psi_{\bar{R}}(\bar{R}) \perp \psi_{\mathscr{B}}(\mathscr{B})$,
(3) $\psi_{\bar{Y}}(\bar{Y}) \perp \psi_{\mathscr{B}}(\mathscr{B})$,
(4) $\psi_{\bar{R}}(\bar{R})$ is independent,
(5) $\psi_{\bar{Y}}(\bar{Y})$ is independent,
(6) $\psi_{\mathscr{B}}(\mathscr{B})$ is independent.

where $\psi(X)$ is the image of X under our map, ψ. In the argument we will treat $\psi(X)$ as a multiset, that is, a set with possible repetitions of elements, but, of course, once we show that its elements are pairwise perpendicular it will follow that they are distinct.

(1) From their definitions, it is clear that $\bar{R} \cap \bar{Y} = \phi$. The definitions of e_i and f_i were selected so that $(e_0, f_1) = (e_1, f_0) = 0$. This implies, via (2.1), that $(e_x, f_y) = 0$ since it is the product of terms one of which is zero.

The proofs of the next two parts are similar.

(2) If $B_{y,z} \in \mathscr{B}$ and $x \in \bar{R}$, then $x \notin \mathrm{Red}(B_{y,z}) = [y, z]$. Now, $x \notin [y, z]$ implies for some $i \in [n]$, $x_i \neq y_i = z_i$. Since $(e_0, g_{1,1}) = (e_1, g_{0,0}) = 0$ it follows that $(e_x, g_{y,z}) = 0$.

(3) If $B_{y,z} \in \mathscr{B}$ and $x \in \bar{Y}$, then $x \notin \mathrm{Yellow}(B_{y,z}) = [y, z^c]$. Now, $x \notin [y, z^c]$ implies for some $i \in [n]$, $x_i = z_i \neq y_i$. Since $(f_0, g_{1,0}) = (f_1, g_{0,1}) = 0$ it follows that $(f_x, g_{y,z}) = 0$.

We will now show that each of the three image sets is itself independent. To prove independence we will use the fact that if $\{a, b\}$ is a basis of $\mathfrak{R}^2$ then $\bigotimes_{i=1}^{m}\{a, b\}$ is a basis of $\mathfrak{R}^{2^m}$.

(4) Clearly $\psi_{\bar{R}}(\bar{R}) \subseteq \psi_{\bar{R}}(Q_m) = \bigotimes_{i=1}^{m}\{e_0, e_1\}$, which is a basis of $\mathfrak{R}^{2^m}$ since $\{e_0, e_1\}$ is a basis of $\mathfrak{R}^2$.

(5) Similarly $\psi_{\bar{Y}}(\bar{Y}) \subseteq \psi_{\bar{Y}}(Q_n) = \bigotimes_{i=1}^{n}\{f_0, f_1\}$, which is a basis of $\mathfrak{R}^{2^m}$ since $\{f_0, f_1\}$ is a basis of $\mathfrak{R}^2$.

(6) Recall that the only stipulation on $\mathscr{B}$ is that it has unique tips. So for each $y \in Q_n$ there is at most one $g_{x,y}$ in $\psi_{\mathscr{B}}(\mathscr{B})$. It is thus enough to show that any set of vectors $\{g_{x(y),y}\}_{y \in Q_m}$ is independent. We will actually prove independence even in $\mathbf{Z}_2^{2^m}$. Since $g_{0,0}$ and $g_{0,1}$n are independent in $\mathbf{Z}_2^2$, we know that $\bigotimes_{i=1}^{m}\{g_{0,0}, g_{0,1}\}$ is a basis of $\bigotimes_{i=1}^{m}\mathbf{Z}_2^2$ $(= \mathbf{Z}_2^{2^m})$. Moreover, noting that $g_{1,0} = g_{0,0} + g_{0,1}$ and $g_{1,1} = g_{0,1}$, we have, for every $i \in [m]$,

$$g_{x_i,y_i} = g_{0,y_i} + \epsilon_i g_{0,1},$$

$$\text{where } \epsilon_i := \begin{cases} 1, & x_i = 1, y_i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

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Thus

$$g_{x,y} = \bigotimes_{i=1}^{m} g_{x_i,y_i} = \bigotimes_{i=1}^{m} (g_{0,y_i} + \epsilon_i g_{0,1}).$$

Expanding, we find that $g_{x,y}$ is the sum of $\bigotimes g_{0,y_i}$ and terms of the form $\bigotimes_{i=1}^{n} g_{0,z_i}$, where $z > y$ (in the usual product order on Q_m). We see that, for a suitable ordering of indices, the change-of-basis matrix is upper triangular, with 1 s on the diagonal. Thus our new vector set is also a basis, and in particular an independent set.

We have demonstrated all six parts, thus proving Theorem 1.2. □

3. Proof of the Van den Berg–Kesten conjecture

Our goal is to show that Theorem 1.2 implies the Van den Berg–Kesten conjecture. In 1987 Van den Berg and Fiebig [2] demonstrated that Theorem 1.1 is equivalent to the case $S_i = \{0,1\}$, $\mu_i(\{0\}) = \mu_i(\{1\}) = \frac{1}{2}$; that is, to the following theorem.

Theorem 3.1. *Let R and Y be subsets of $\{0,1\}^n$; then $|R||Y| \geqslant |R \square Y|2^n$.*

We need to translate from the notation $C(K,x)$ to the two-vector representation used for butterflies.

Lemma 3.2.

$$C(K,x) = [x, b_{K,x}], \text{ where } b_{K,x} := \begin{cases} x_i, & i \in K, \\ x_i^c, & i \notin K. \end{cases}$$

Proof.

$$\begin{aligned} \omega \in C(K,x) &\iff \forall i \in K, \omega_i = x_i \\ &\iff \forall i, \omega_i \in \{x_i, (b_{K,x})_i\} \\ &\iff \omega \in [x, b_{K,x}]. \end{aligned}$$ □

Thus $R \square Y = \{a \in Q_n : \exists b \in Q_n$ such that $[a,b] \subseteq R$ and $[a,b^c] \subseteq Y\} = \{\text{Body}(B_{a,b}) : B_{a,b}$ is a butterfly on Q_n with $\text{Red}(B_{a,b}) \subseteq R$ and $\text{Yellow}(B_{a,b}) \subseteq Y\}$.

Conversely, given a flock of butterflies $\mathscr{B}$ we clearly have $\text{Body}(\mathscr{B}) \subseteq \text{Red}(\mathscr{B}) \square \text{Yellow}(\mathscr{B})$. Thus Theorem 3.1 is equivalent to the statement that, for any flock of butterflies $\mathscr{B}$,

$$|\text{Body}(\mathscr{B})|2^n \leqslant |\text{Red}(\mathscr{B})||\text{Yellow}(\mathscr{B})|.$$

Proof of Theorem 3.1. It is clearly enough to show the above when $\mathscr{B}$ has distinct bodies. (The Van den Berg–Kesten conjecture was earlier considered in this form by Fishburn and Shepp [3].) We will partition the elements, (r,y), of $\text{Red}(\mathscr{B}) \times \text{Yellow}(\mathscr{B})$ according to the subcube $[r,y]$ of Q_n, and apply Theorem 1.2 to the subcubes. Notice that, if Q is a subcube and B a butterfly on Q_n with $\text{Body}(B) \in Q$, then we have a butterfly on Q, denoted B_Q, with $\text{Red}(B_Q) = \text{Red}(B) \cap Q$, $\text{Yellow}(B_Q) = \text{Yellow}(B) \cap Q$

and $\text{Body}(B_Q) = \text{Body}(B)$. It is easy to see that B_Q is a butterfly on the cube Q. We can then define for each Q a flock of butterflies $\mathscr{B}_Q = \{B_Q : B \in \mathscr{B}, \text{Body}(B) \in Q\}$.

$$\text{Set Body}(\mathscr{B}_Q) := \bigcup_{B_Q \in \mathscr{B}_Q} \text{Body}(B_Q).$$

Define $\text{Red}(\mathscr{B}_Q)$ and $\text{Yellow}(\mathscr{B}_Q)$ in the same fashion. As we let Q range over the subcubes of Q_n, we have

$$\begin{aligned}
|\text{Red}(\mathscr{B})|\,|\text{Yellow}(\mathscr{B})| &= |\{(r,y) \in \text{Red}(\mathscr{B}) \times \text{Yellow}(\mathscr{B})| \\
&= \sum_{Q \in Q_n} |\{(r,y) \in \text{Red}(\mathscr{B}) \times \text{Yellow}(\mathscr{B}) : [r,y] = Q\}| \\
&\geqslant \sum_{Q \in Q_n} |\{(r,y) \in \text{Red}(\mathscr{B}_Q) \times \text{Yellow}(\mathscr{B}_Q) : [r,y] = Q\}| \qquad (3.1) \\
&\geqslant \sum_{Q \in Q_n} |\text{Body}(\mathscr{B}_Q)| \qquad (3.2) \\
&= 2^n\,|\text{Body}(\mathscr{B})|\,. \qquad (3.3)
\end{aligned}$$

Inequality (3.1) follows from $\text{Red}(\mathscr{B}_Q) \subseteq \text{Red}(\mathscr{B})$ and $\text{Yellow}(\mathscr{B}_Q) \subseteq \text{Yellow}(\mathscr{B})$. Notice that $[r,y] = Q$ means that r and y are complements relative to Q, so (3.2) follows from Theorem 1.2; and (3.3) comes from the fact that each $x \in Q_n$ lies in exactly 2^n subcubes. □

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