Nagoya Math. J., **238** (2020), 37–46
DOI 10.1017/nmj.2018.18

A CONJECTURE ON C-MATRICES OF CLUSTER ALGEBRAS

PEIGEN CAO, MIN HUANG AND FANG LI*

Abstract. For a skew-symmetrizable cluster algebra $\mathcal{A}_{t_0}$ with principal coefficients at t_0, we prove that each seed Σ_t of $\mathcal{A}_{t_0}$ is uniquely determined by its *C-matrix*, which was proposed by Fomin and Zelevinsky (Compos. Math. **143** (2007), 112–164) as a conjecture. Our proof is based on the fact that the positivity of cluster variables and sign coherence of c-vectors hold for $\mathcal{A}_{t_0}$, which was actually verified in Gross *et al.* (*Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31**(2) (2018), 497–608). Further discussion is provided in the sign-skew-symmetric case so as to obtain a weak version of the conjecture in this general case.

§1. Introduction and preliminaries

Cluster algebras with principal coefficients are important research objects in the theory of cluster algebras. The g-vectors and the related c-vectors are introduced to describe cluster variables and coefficients in some sense. As shown in [11], c-vectors and g-vectors are closely related to each other, where the c-vectors (respectively, g-vectors) are defined as the column vectors of *C-matrices* (respectively, *G-matrices*) of a cluster algebra with principal coefficients. In [4], the authors conjectured as follows.

Conjecture 1.1. [4, Conjecture 4.7] *Let $\mathcal{A}_{t_0}$ be the cluster algebra with principal coefficients at Σ_{t_0} (or say at t_0), and let $\Sigma_t = (X_t, \tilde{B}_t)$ be a seed of $\mathcal{A}_{t_0}$ obtained from Σ_{t_0} by iterated mutations. Then Σ_t is uniquely determined by C_t, where C_t is the lower part of $\tilde{B}_t = \binom{B_t}{C_t}$.*

Conjecture 1.1 is true in case $\mathcal{A}_{t_0}$ is of finite type [14], and in the skew-symmetric case [7] using the methods in [12, 13].

In this paper, we give an affirmation of Conjecture 1.1 for any skew-symmetrizable cluster algebra $\mathcal{A}_{t_0}$ with principal coefficients at t_0. The proof depends on the positivity of cluster variables and sign coherence of c-vectors

Received September 21, 2017. Revised May 10, 2018. Accepted May 10, 2018.
2010 Mathematics subject classification. 13F60, 05E40.
*Corresponding author.

of $\mathcal{A}_{t_0}$. Note that the positivity of cluster variables for skew-symmetric cluster algebra was proved first by Lee and Schiffler [8] using the elementary method and the coherence of c-vectors for skew-symmetric cluster algebra was first proved by Derksen *et al.* [2] using the quivers with potentials and their representations. Recently both properties have been proved for skew-symmetrizable cluster algebra by Gross *et al.* [5] using the methods from algebraic geometry.

This paper is organized as follows. In this section, some basic definitions are introduced. In Section 2, we give the main result, that is, Theorem 2.5, which gives an affirmation of Conjecture 1.1 in the skew-symmetrizable case. In Section 3, we discuss the weak version of Conjecture 1.1 for acyclic totally sign-skew-symmetric cluster algebra using the method in Section 2.

DEFINITION 1.2. Let A be an $(m+n)\times n$ integer matrix. The mutation of A in direction $k\in\{1,2,\ldots,n\}$ is the $(m+n)\times n$ matrix $\mu_k(A)=(a'_{ij})$, satisfying

$$a'_{ij}=\begin{cases}-a_{ij}, & i=k \text{ or } j=k;\\ a_{ij}+a_{ik}[-a_{kj}]_+ + [a_{ik}]_+a_{kj} & \text{otherwise,}\end{cases} \tag{1}$$

where $[a]_+=\max\{a,0\}$ for any $a\in\mathbb{R}$.

Recall that an $n\times n$ integer matrix $B=(b_{ij})$ is said to be *sign-skew-symmetric* if $b_{ij}b_{ji}<0$ or $b_{ij}=b_{ji}=0$ for any $i,j=1,2,\ldots,n$. A sign-skew-symmetric B is *totally sign-skew-symmetric* if any matrix B' obtained from B by a sequence of mutations is sign-skew-symmetric.

An $n\times n$ integer matrix $B=(b_{ij})$ is said to be *skew-symmetrizable* if there is a positive integer diagonal matrix S such that SB is skew-symmetric, where S is called the *skew symmetrizer* of B.

It is well-known that skew symmetry and skew symmetrizability are invariant under mutation; thus skew-symmetrizable integer matrices are always totally sign-skew-symmetric.

For a sign-skew-symmetric matrix B, we can encode the sign pattern of entries of B by the directed graph $\Gamma(B)$ with the vertices $1,2,\ldots,n$ and the directed edges (i,j) for $b_{ij}>0$. A sign-skew-symmetric matrix B is said to be *acyclic* if $\Gamma(B)$ has no oriented cycles. As shown in [6], an acyclic sign-skew-symmetric integer matrix B is always totally sign-skew-symmetric.

Let $\tilde{B}=\binom{B_{n\times n}}{C_{m\times n}}=(\tilde{b}_{ij})$ be an $(m+n)\times n$ integer matrix such that B is totally sign-skew-symmetric. $\tilde{B}$ is said to be *skew-symmetric*, *skew-symmetrizable*, and *(totally) sign-skew-symmetric*, respectively if so is B.

DEFINITION 1.3. Let $\mathcal{F}$ be the field of rational functions in $m+n$ independent variables with coefficients $\mathbb{Q}$. A *(labeled) seed* Σ in $\mathcal{F}$ is a pair $(X, \tilde{B})$, where

(i) $\tilde{B} = \binom{B}{C}$ is given as above, called an *extended exchange matrix*. The $n \times n$ submatrix B is called an *exchange matrix* and the $m \times n$ submatrix C is called a *coefficient matrix*.

(ii) $X = (x_1, \ldots, x_{m+n})$ is an $(m+n)$-tuple, called an *extended cluster* with $x_1, \ldots, x_{m+n}$ forming a free generating set of $\mathcal{F}$. $x_1, \ldots, x_n$ are called *cluster variables* and $x_{n+1}, \ldots, x_{m+n}$ are called *frozen variables*. Sometimes we also write frozen variables as $y_1, \ldots, y_m$ in this paper, that is, $x_{n+i} = y_i$ for $i = 1, \ldots, m$.

DEFINITION 1.4. Let $k \in \{1, 2, \ldots, n\}$. The *mutation* of the seed $\Sigma = (X, \tilde{B})$ in the direction k is the seed $\Sigma' = (X', \tilde{B}')$ where $\tilde{B}' = \mu_k(\tilde{B})$ and $X' = (x'_1, \ldots, x'_{m+n})$ with $x'_i = x_i$ if $i \neq k$ and

$$x'_k x_k = \prod_{i=1}^{m+n} x_i^{[\tilde{b}_{ik}]_+} + \prod_{i=1}^{m+n} x_i^{[-\tilde{b}_{ik}]_+}.$$

We write $\Sigma' = \mu_k(\Sigma)$, where μ_k is called the *mutation map* in k.

It can be seen that the mutation map is always an involution, that is, $\mu_k \mu_k(\Sigma) = \Sigma$.

DEFINITION 1.5. The *cluster algebra* $\mathcal{A}(\Sigma)$ is the $\mathbb{Z}[y_1, \ldots, y_m]$-subalgebra of $\mathcal{F}$ generated by all cluster variables obtained from the seed $\Sigma = (X, \tilde{B})$ by iterated mutations. Σ is called an *initial seed* of $\mathcal{A}(\Sigma)$.

A cluster algebra is said to be *acyclic* if it admits an acyclic exchange matrix.

Denote by I_n an $n \times n$ identity matrix. If $\tilde{B} = \binom{B}{I_n}$, then $\mathcal{A}(\Sigma)$ is called the cluster algebra with *principal coefficients at* Σ. In this case, $m = n$.

DEFINITION 1.6. [9] A *cluster pattern* M in $\mathcal{F}$ is an assignment for each seed Σ_t to a vertex t of the n-regular tree $\mathbb{T}_n$, such that for any edge $t \overset{k}{\text{——}} t'$, $\Sigma_{t'} = \mu_k(\Sigma_t)$. The pair of Σ_t is written as $\Sigma_t = (X_t, \tilde{B}_t)$ with $X_t = (x_{1;t}, \ldots, x_{m+n;t})$, $\tilde{B}_t = \binom{B_t}{C_t}$, where $B_t = (b_{ij}^t)$, $C_t = (c_{ij}^t)$, $x_{n+i;t} = y_i$ for $i = 1, 2, \ldots, m$.

Let $\mathcal{A}_{t_0}$ be the cluster algebra with principal coefficients at Σ_{t_0} (or say at t_0). The authors [4] introduced a $\mathbb{Z}^n$-grading of $\mathbb{Z}[x_{1;t_0}^{\pm 1}, \ldots, x_{n;t_0}^{\pm 1}, y_1, \ldots, y_n]$ as follows:

$$\deg(x_{i;t_0}) = \boldsymbol{e}_i, \qquad \deg(y_j) = -\boldsymbol{b}_j,$$

where $\boldsymbol{e}_i$ is the ith column vector of I_n and $\boldsymbol{b}_j$ is the jth column vector of B_{t_0}, $i, j = 1, 2, \ldots, n$. As shown in [4], every cluster variable $x_{i;t}$ of $\mathcal{A}_{t_0}$ is homogeneous with respect to this $\mathbb{Z}^n$-grading. The *g-vector* of a cluster variable $x_{i;t}$ is defined to be its degree with respect to the $\mathbb{Z}^n$-grading and we write $\deg(x_{i;t}) = (g_{1i}^t, g_{2i}^t, \ldots, g_{ni}^t)^\top \in \mathbb{Z}^n$. Let X_t be a cluster of $\mathcal{A}_{t_0}$. We call the matrix $G_t = (\deg(x_{1;t}), \ldots, \deg(x_{n;t}))$ the *G-matrix* of the cluster X_t. In addition, C_t is called the *C-matrix* of Σ_t, whose column vectors are called *c-vectors* (see [4]).

PROPOSITION 1.7. *Let $\mathcal{A}_{t_0}$ be a cluster algebra with principal coefficients at t_0, let Σ_t be a seed of $\mathcal{A}_{t_0}$, and let $\Sigma_{t'} = \mu_k(\Sigma)$. Then*

(i) $c_{ij}^{t'} = \begin{cases} -c_{ij}^t, & j = k; \\ c_{ij}^t + b_{ik}^t[-b_{kj}^t]_+ + [c_{ik}^t]_+ b_{kj}^t & \textit{otherwise}. \end{cases}$

(ii) *[4, Proposition 6.6]*

$$g_{ij}^{t'} = \begin{cases} g_{ij}^t, & j \neq k; \\ -g_{ik}^t + \sum_{l=1}^n g_{il}^t [b_{lk}^t]_+ - \sum_{l=1}^n -b_{il}^{t_0} [c_{lk}^t], & j = k. \end{cases}$$

Proof. (i) It can be obtained from equality (1). ∎

Let $\mathcal{A}_{t_0}$ be the cluster algebra with principal coefficients at t_0. Each cluster variable $x_{i;t}$ can be expressed as a rational function

$$X_{i;t}^{t_0} \in \mathbb{Q}(x_{1;t_0}, \ldots, x_{n;t_0}, y_1, \ldots, y_n).$$

$X_{i;t}^{t_0}$ is called the *X-function* of cluster variable $x_{i;t}$.

DEFINITION 1.8. The *F-polynomial* of $x_{i;t}$ is defined by

$$F_{i;t}^{t_0} = X_{i;t}^{t_0}|_{x_{1;t_0} = \cdots x_{n;t_0} = 1} \in \mathbb{Z}[y_1, \ldots, y_n].$$

A vector $\boldsymbol{c} \in \mathbb{Z}^n$ is said to be *sign-coherent* [4] if any two nonzero entries of $\boldsymbol{c}$ have the same sign.

PROPOSITION 1.9. [4, 5] *Let $\mathcal{A}_{t_0}$ be a cluster algebra with principal coefficients at t_0, and let $x_{i;t}$ be a cluster variable of $\mathcal{A}_{t_0}$. Then*

(i) $x_{i;t} = (\prod_{j=1}^{n} x_{j;t_0}^{g_{ji}^{t}}) F_{i;t}^{t_0}|_{\mathcal{F}}(\hat{y}_{1;t_0}, \ldots, \hat{y}_{n;t_0})$, *where* $\hat{y}_{k;t_0} = y_k \prod_{i=1}^{n} x_{i;t_0}^{b_{ik}^{t_0}}$ *for* $k \in \{1, 2, \ldots, n\}$.

(ii) *The following are equivalent:*

   (a) *The column vectors of C_t are sign-coherent.*
   (b) *Each F-polynomial has constant term 1.*
   (c) *Each F-polynomial has a unique monomial of maximal degree, which has coefficient 1 and is divisible by all other occurring monomials.*

§2. Affirmation of Conjecture 1.1 in the skew-symmetrizable case

In this section, we give an affirmation of Conjecture 1.1 (see Theorem 2.5) in the skew-symmetrizable case depending on the results in [5], that is, Theorem 2.1 and Proposition 2.2.

THEOREM 2.1. (Positivity, [5]) *Any cluster variable $x_{i;t}$ can be expressed as a Laurent polynomial in $\mathbb{Z}_{\geqslant 0}[y_1, \ldots, y_m][x_{1;t_0}^{\pm 1}, \ldots, x_{n;t_0}^{\pm 1}]$ in a skew-symmetrizable cluster algebra $\mathcal{A}(\Sigma_{t_0})$ with initial seed Σ_{t_0}.*

PROPOSITION 2.2. [5] *Let $\mathcal{A}_{t_0}$ be a skew-symmetrizable cluster algebra with principal coefficients at t_0, and let $x_{i;t}$ be a cluster variable of $\mathcal{A}_{t_0}$. Then the column vectors of C_t are sign-coherent.*

PROPOSITION 2.3. [1, 10, 11] *Let $\mathcal{A}_{t_0}$ be a skew-symmetrizable cluster algebra with principal coefficients t_0 and with skew symmetrizer S, and let Σ_t be a seed of $\mathcal{A}_{t_0}$. Then $G_t B_t S^{-1} G_t^{\top} = B_{t_0} S^{-1}$ and $S C_t S^{-1} G_t^{\top} = I_n$ and $\det(G_t) = \pm 1$.*

LEMMA 2.4. *Let $\mathcal{A}_{t_0}$ be a cluster algebra with principal coefficients at t_0, and let $\Sigma_t = (X_t, \tilde{B}_t)$ be a seed of $\mathcal{A}_{t_0}$. If $G_t = I_n$, then $\Sigma_t = \Sigma_{t_0}$.*

Proof. By Proposition 2.3 and $G_t = I_n = G_{t_0}$, we obtain $B_t = B_{t_0}$ and $C_t = I_n$. So we can also view $\mathcal{A}_{t_0}$ as a cluster algebra with principal coefficients at t. Let Σ_{t_1} be a seed of $\mathcal{A}_{t_0}$; the C-matrix of Σ_{t_1} is always the lower part of $\tilde{B}_{t_1}$, no matter which seed (Σ_{t_0} or Σ_t) is chosen as the initial seed. By Proposition 2.3 again, we know the G-matrices of Σ_{t_1} with

https://doi.org/10.1017/nmj.2018.18 Published online by Cambridge University Press

respect to Σ_{t_0} and Σ_t are the same. By Proposition 1.9(i), we know

$$x_{i;t} = x_{i;t_0} F_{i;t}^{t_0}\left(y_1 \prod_{l=1}^{n} x_{l;t_0}^{b_{l1}^{t_0}}, \ldots, y_n \prod_{l=1}^{n} x_{l;t_0}^{b_{ln}^{t_0}}\right), \tag{2}$$

by viewing Σ_{t_0} as an initial seed and

$$x_{j;t_0} = x_{j;t} F_{j;t_0}^{t}\left(y_1 \prod_{l=1}^{n} x_{l;t}^{b_{l1}^{t}}, \ldots, y_n \prod_{l=1}^{n} x_{l;t}^{b_{ln}^{t}}\right), \tag{3}$$

by viewing Σ_t as an initial seed for $i, j \in \{1, 2, \ldots, n\}$. Replacing $x_{1;t_0}, \ldots, x_{n;t_0}$ in (2) with those in (3), we obtain that

$$\begin{aligned} x_{i;t} = x_{i;t} F_{i;t_0}^{t}&\left(y_1 \prod_{l=1}^{n} x_{l;t}^{b_{l1}^{t}}, \ldots, y_n \prod_{l=1}^{n} x_{l;t}^{b_{ln}^{t}}\right) \\ &\cdot F_{i;t}^{t_0}\left(y_1 \prod_{l=1}^{n}\left(x_{l;t} F_{l;t_0}^{t}\left(y_1 \prod_{k=1}^{n} x_{k;t}^{b_{k1}^{t}}, \ldots, y_n \prod_{k=1}^{n} x_{k;t}^{b_{kn}^{t}}\right)\right)^{b_{l1}^{t_0}},\right. \\ &\left.\ldots, y_n \prod_{l=1}^{n}\left(x_{l;t} F_{l;t_0}^{t}\left(y_1 \prod_{k=1}^{n} x_{k;t}^{b_{k1}^{t}}, \ldots, y_n \prod_{k=1}^{n} x_{k;t}^{b_{kn}^{t}}\right)\right)^{b_{ln}^{t_0}}\right), \end{aligned}$$

which implies that

$$\begin{aligned} 1 = F_{i;t_0}^{t}&\left(y_1 \prod_{l=1}^{n} x_{l;t}^{b_{l1}^{t}}, \ldots, y_n \prod_{l=1}^{n} x_{l;t}^{b_{ln}^{t}}\right) \\ &\cdot F_{i;t}^{t_0}\left(y_1 \prod_{l=1}^{n}\left(x_{l;t} F_{l;t_0}^{t}\left(y_1 \prod_{k=1}^{n} x_{k;t}^{b_{k1}^{t}}, \ldots, y_n \prod_{k=1}^{n} x_{k;t}^{b_{kn}^{t}}\right)\right)^{b_{l1}^{t_0}},\right. \\ &\left.\ldots, y_n \prod_{l=1}^{n}\left(x_{l;t} F_{l;t_0}^{t}\left(y_1 \prod_{k=1}^{n} x_{k;t}^{b_{k1}^{t}}, \ldots, y_n \prod_{k=1}^{n} x_{k;t}^{b_{kn}^{t}}\right)\right)^{b_{ln}^{t_0}}\right). \end{aligned}$$

Take $x_{1;t} = x_{2;t} = \cdots = x_{n;t} = 1$ in the above equality; then we have

$$\begin{aligned} 1 = {} & F_{i;t_0}^{t}(y_1, \ldots, y_n) \\ & \cdot F_{i;t}^{t_0}\left(y_1 \prod_{l=1}^{n}(F_{l;t_0}^{t}(y_1, \ldots, y_n))^{b_{l1}^{t_0}}, \ldots, y_n \prod_{l=1}^{n}(F_{l;t_0}^{t}(y_1, \ldots, y_n))^{b_{ln}^{t_0}}\right). \end{aligned} \tag{4}$$

By Theorem 2.1 and the definition of F-polynomial, we have $F_{j;t}^{t_0}$, $F_{j;t_0}^{t} \in \mathbb{Z}_{\geqslant 0}[y_1, \ldots, y_n]$. Then we know each $y_j \prod_{l=1}^{n}(F_{j;t_0}^{t}(y_1, \ldots, y_n))^{b_{lj}^{t_0}}$ has the form of h_j/g_j, where $h_j, g_j \in \mathbb{Z}_{\geqslant 0}[y_1, \ldots, y_n]$, $j = 1, \ldots, n$. We claim that $F_{i;t}^{t_0}(y_1, \ldots, y_n) = 1$. If $F_{i;t}^{t_0}(y_1, \ldots, y_n) \neq 1$, then by Propositions 2.2 and 1.9(ii), we can assume $F_{i;t}^{t_0} = y_1^{k_{1i}} y_2^{k_{2i}} \cdots y_n^{k_{ni}} + u_i(y_1, \ldots, y_n) + 1$, where $(k_{1i}, \ldots, k_{ni}) \neq 0$, $u_i \in \mathbb{Z}_{\geqslant 0}[y_1, \ldots, y_n]$, and each monomial in u_i is a factor of $y_1^{k_{1i}} y_2^{k_{2i}} \cdots y_n^{k_{ni}}$. We assume $F_{i;t_0}^{t} = w_i(y_1, \ldots, y_n) + 1$, where $w_i \in \mathbb{Z}_{\geqslant 0}[y_1, \ldots, y_n]$.

Then equality (4) has the form

$$1 = (w_i + 1)\left(\left(\frac{h_1}{g_1}\right)^{k_{1i}} \cdots \left(\frac{h_n}{g_n}\right)^{k_{ni}} + u_i\left(\frac{h_1}{g_1}, \ldots, \frac{h_n}{g_n}\right) + 1\right),$$

and thus

$$\begin{aligned} & g_1^{k_{1i}} g_2^{k_{2i}} \cdots g_n^{k_{ni}} \\ & \quad = (w_i + 1)\Bigg(h_1^{k_{1i}} \cdots h_n^{k_{ni}} + g_1^{k_{1i}} \cdots g_n^{k_{ni}} u_i\left(\frac{h_1}{g_1}, \ldots, \frac{h_n}{g_n}\right) + g_1^{k_{1i}} \cdots g_n^{k_{ni}}\Bigg). \end{aligned}$$

We obtain that

$$\begin{aligned} 0 = {} & w_i\left(h_1^{k_{1i}} \cdots h_n^{k_{ni}} + g_1^{k_{1i}} g_2^{k_{2i}} \cdots g_n^{k_{ni}} u_i\left(\frac{h_1}{g_1}, \ldots, \frac{h_n}{g_n}\right) + g_1^{k_{1i}} \cdots g_n^{k_{ni}}\right) \\ & + \left(h_1^{k_{1i}} \cdots h_n^{k_{ni}} + g_1^{k_{1i}} g_2^{k_{2i}} \cdots g_n^{k_{ni}} u_i\left(\frac{h_1}{g_1}, \ldots, \frac{h_n}{g_n}\right)\right). \end{aligned} \tag{5}$$

Note that $g_1^{k_{1i}} g_2^{k_{2i}} \cdots g_n^{k_{ni}} u_i(h_1/g_1, \ldots, h_n/g_n) \in \mathbb{Z}_{\geqslant 0}[y_1, \ldots, y_n]$, since each monomial occurring in $u_i(y_1, \ldots, y_n)$ is a factor of $y_1^{k_{1i}} y_2^{k_{2i}} \cdots y_n^{k_{ni}}$. Then equality (5) is a contradiction, since each term in equality (5) is an element in $\mathbb{Z}_{\geqslant 0}[y_1, \ldots, y_n]$ and $h_1^{k_{1i}} \cdots h_n^{k_{ni}} \neq 0$ by the fact that $h_j \neq 0$, $j = 1, \ldots, n$. Thus we must have $F_{i;t}^{t_0}(y_1, \ldots, y_n) = 1$, which implies $x_{i;t} = x_{i;t_0}$ by the equality (2). Then $\Sigma_t = \Sigma_{t_0}$. □

Now, we obtain the main result as follows.

THEOREM 2.5. *Let $\mathcal{A}_{t_0}$ be the cluster algebra with principal coefficients at t_0, and let Σ_{t_1} and Σ_{t_2} be two seeds of $\mathcal{A}_{t_0}$. If $C_{t_1} = C_{t_2}$, then $\Sigma_{t_1} = \Sigma_{t_2}$.*

Proof. By Proposition 2.3 and due to $C_{t_1} = C_{t_2}$, we have $B_{t_1} = B_{t_2}$ and $G_{t_1} = G_{t_2}$. Since Σ_{t_0} can be obtained from Σ_{t_1} by a sequence of

https://doi.org/10.1017/nmj.2018.18 Published online by Cambridge University Press

mutations, we write $\Sigma_{t_0} = \mu_{s_k} \cdots \mu_{s_2}\mu_{s_1}(\Sigma_{t_1})$. Let $\Sigma_t = \mu_{s_k} \cdots \mu_{s_2}\mu_{s_1}(\Sigma_{t_2})$. Since $(G_{t_1}, C_{t_1}, B_{t_1}) = (G_{t_2}, C_{t_2}, B_{t_2})$, by Proposition 1.7, we know that $(G_t, C_t, B_t) = (G_{t_0}, C_{t_0}, B_{t_0}) = (I_n, I_n, B_{t_0})$. By Lemma 2.4, we know $\Sigma_t = \Sigma_{t_0}$ and thus $\mu_{s_1}\mu_{s_2} \cdots \mu_{s_k}(\Sigma_t) = \mu_{s_1}\mu_{s_2} \cdots \mu_{s_k}(\Sigma_{t_0})$, that is, $\Sigma_{t_2} = \Sigma_{t_1}$. $\square$

As readers can see, the proof of this result depends mainly on the positivity of cluster variables and sign coherence of c-vectors of a cluster algebra. Positivity and sign coherence are two deep phenomena in cluster algebras. It is known in [11] that sign coherence can relate to many other properties of cluster algebras. Also, we believe that positivity can be used to explain some other properties of cluster algebras. Through the method of the proof of Theorem 2.5, we attempt to provide an evidence for the essentiality of positivity and sign coherence.

It was proved that Σ_t is uniquely determined by G_t for a skew-symmetric matrix B_{t_0} in [2]. By Proposition 2.3, $C_{t_1} = C_{t_2}$ if and only if $G_{t_1} = G_{t_2}$. So, the result of Theorem 2.5 means that Σ_t is uniquely determined by G_t, which gives a generalization of the result in [2] in the skew-symmetrizable case.

§3. How to determine seed in the sign-skew-symmetric case

Finally, we discuss how to determine a seed using the C-matrix or G-matrix in the sign-skew-symmetric case. The obtained result may be thought of as the weak version of Conjecture 1.1.

In this section, we always assume that $\mathcal{A}_{t_0}$ is an acyclic sign-skew-symmetric cluster algebra with principal coefficients at t_0. Here, the "acyclic" condition is assumed since we need the following conclusion.

PROPOSITION 3.1. [6, Theorems 7.11 and 7.13] *Let $\mathcal{A}_{t_0}$ be an acyclic sign-skew-symmetric cluster algebra with principal coefficients at t_0. Then we have the following.*

(i) *Each cluster variable $x_{i;t} \in \mathbb{Z}_{\geqslant 0}[y_1, \ldots, y_m][x_{1;t_0}^{\pm 1}, \ldots, x_{n;t_0}^{\pm 1}]$.*
(ii) *Each $\boldsymbol{F}$-polynomial has constant term 1.*

LEMMA 3.2. *Let $\mathcal{A}_{t_0}$ be an acyclic sign-skew-symmetric cluster algebra with principal coefficients at t_0. If $\Sigma_t = (X_t, \tilde{B}_t)$ is a seed of $\mathcal{A}_{t_0}$ satisfying $(G_t, C_t, B_t) = (G_{t_0}, C_{t_0}, B_{t_0})$, then $\Sigma_t = \Sigma_{t_0}$.*

Sketch of Proof. Since $C_t = C_{t_0} = I_n$ and $G_t = G_{t_0} = I_n$, $\mathcal{A}_{t_0}$ can be also seen as a cluster algebra with principal coefficients at t. Based on

Proposition 3.1, we can use the same method with that of Lemma 2.4 so as to show $X_t = X_{t_0}$, which implies $\Sigma_t = \Sigma_{t_0}$ since $\tilde{B}_t = \tilde{B}_{t_0}$ is already known. □

Note that since we do not have the skew symmetrizer S in this case, Proposition 2.3 cannot be used here as in the proof of Lemma 2.4. This is the reason we need the condition $(G_t, C_t, B_t) = (G_{t_0}, C_{t_0}, B_{t_0})$.

Following Lemma 3.2, using the same method with the proof of Theorem 2.5, we have the following proposition.

PROPOSITION 3.3. *Let $\mathcal{A}_{t_0}$ be an acyclic sign-skew-symmetric cluster algebra with principal coefficients at t_0. If Σ_{t_1} and Σ_{t_1} are two seeds of $\mathcal{A}_{t_0}$ with $(G_{t_1}, C_{t_1}, B_{t_1}) = (G_{t_2}, C_{t_2}, B_{t_2})$, then $\Sigma_{t_1} = \Sigma_{t_2}$.*

REMARK 3.4. When B_{t_0} is skew-symmetrizable, by Proposition 2.3, either $G_{t_1} = G_{t_2}$ or $C_{t_1} = C_{t_2}$ implies $(G_{t_1}, C_{t_1}, B_{t_1}) = (G_{t_2}, C_{t_2}, B_{t_2})$. So, Proposition 3.3 can be thought of as a weak version of Theorem 2.5 in the acyclic sign-skew-symmetric case.

Acknowledgments. The authors thank T. Nakanishi for his valuable comments on the earlier version of this paper. This project is supported by the National Natural Science Foundation of China (No. 11671350 and No. 11571173).

REFERENCES

[1] P. G. Cao and F. Li, *Some conjectures on generalized cluster algebras via the cluster formula and D-matrix pattern*, J. Algebra **493** (2018), 57–78.

[2] H. Derksen, J. Weyman and A. Zelevinsky, *Quivers with potentials and their representations II: applications to cluster algebras*, J. Amer. Math. Soc. **23**(3) (2010), 749–790.

[3] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15**(2) (2002), 497–529 (electronic).

[4] S. Fomin and A. Zelevinsky, *Cluster algebras. IV. Coefficients*, Compos. Math. **143** (2007), 112–164.

[5] M. Gross, P. Hacking, S. Keel and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31**(2) (2018), 497–608.

[6] M. Huang and F. Li, *Unfolding of acyclic sign-skew-symmetric cluster algebras and applications to positivity and F-polynomials*, submitted, arXiv:1609.05981v2.

[7] R. Inoue, O. Iyama, B. Keller, A. Kuniba and T. Nakanishi, *Periodicities of T and Y-systems, dilogarithm identities, and cluster algebras I: type B_r*, Publ. Res. Inst. Math. Sci. **49** (2013), 1–42; arXiv:1001.1880, 35 pp.

[8] K. Lee and R. Schiffler, *Positivity for cluster algebras*, Ann. of Math. **182** (2015), 73–125.

[9] R. J. Marsh, *Lecture Notes on Cluster Algebras*, Zurich Lectures in Advanced Mathematics, European Mathematical Society Publishing House, Zürich, 2013.

https://doi.org/10.1017/nmj.2018.18 Published online by Cambridge University Press

[10] T. Nakanishi, *Structure of seeds in generalized cluster algebras*, Pacific J. Math. **277**(1) (2015), 201–218.

[11] T. Nakanishi and A. Zelevinsky, "*On tropical dualities in cluster algebras*", in *Algebraic Groups and Quantum Groups*, Contemporary Mathematics **565**, American Mathematical Society, Providence, RI, 2012, 217–226.

[12] P.-G. Plamondon, *Cluster algebras via cluster categories with infinite-dimensional morphism spaces*, Compos. Math. **147**(6) (2011), 1921–1954.

[13] P.-G. Plamondon, *Cluster characters for cluster categories with infinite-dimensional morphism spaces*, Adv. Math. **227**(1) (2011), 1–39.

[14] N. Reading and D. E. Speyer, *Combinatorial frameworks for cluster algebras*, Int. Math. Res. Not. **2016**(1) (2016), 109–173.

Peigen Cao
School of Mathematical Sciences
Zhejiang University (Yuquan Campus)
Hangzhou
Zhejiang 310027
PR China
peigencao@126.com

Min Huang
Département de mathématiques
Université de Sherbrooke
Sherbrooke
Québec
J1K 2R1
Canada
minhuang1989@hotmail.com

Fang Li
School of Mathematical Sciences
Zhejiang University (Yuquan Campus)
Hangzhou
Zhejiang 310027
PR China
fangli@zju.edu.cn