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# SPECTRAL INCLUSION AND C.N.E.

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**1.** An $n$-tuple $\mathbf{S} = (S_1, \ldots, S_n)$ of commuting bounded linear operators on a Hilbert space $H$ is said to have commuting normal extension if and only if there exists an $n$-tuple $\mathbf{N} = (N_1, \ldots, N_n)$ of commuting normal operators on some larger Hilbert space $K \supset H$ with the restrictions $N_i|_H = S_i$, $i = 1, \ldots, n$. If we take

$$K = \text{c.l.s. } \{\mathbf{N}^{*J}h : h \in H, J \geqq 0\},$$

the minimal reducing subspace of $\mathbf{N}$ containing $H$, then $\mathbf{N}$ is unique up to unitary equivalence and is called the c.n.e. of $\mathbf{S}$. (Here $J$ denotes the multi-index $(j_1, \ldots, j_n)$ of nonnegative integers and $\mathbf{N}^{*J} = N_1^{*j_1} \ldots N_n^{*j_n}$ and we emphasize that c.n.e. denotes minimal commuting normal extension.) If $n = 1$, then $S_1 = S$ is called subnormal and $N_1 = N$ its minimal normal extension (m.n.e.).

P. R. Halmos introduced subnormal operators and, along with J. Bram, developed much of their basic theory [**8**], [**4**], including a characterization of subnormality intrinsic to $S$. T. Ito considered the case of c.n.e., i.e., $n > 1$, [**12**] and extended many of the basic notions. Clearly $\mathbf{S}$ has c.n.e. implies $S_1, \ldots, S_n$ are commuting subnormal operators; examples of commuting subnormals without c.n.e. were first given independently in [**1**], [**13**], and subsequent examples [**14**], [**15**] exhibited even greater pathology. Thus, general commuting subnormal operators are very difficult to understand, but if there is c.n.e., natural analogs of the single operator case often hold.

Bram proved [**4**] that a subnormal operator $S$ satisfies the spectral inclusion relation $\partial\sigma(S) \subset \sigma_\perp(S) \subset \sigma(S)$, where $\sigma_\perp(S)$ denotes $\sigma(N)$, the spectrum of the m.n.e. of $S$. J. Bunce and J. Deddens [**5**], using a $C^*$-algebraic characterization of subnormality proved $\sigma_\perp(\pi(S)) \subset \sigma_\perp(S)$ for any *-representation $\pi$. W. Hastings extended the spectral inclusion theorem to the case of c.n.e. as follows [**10**]:

THEOREM A. *Let* $\mathbf{S}$ *have* c.n.e. $\mathbf{N}$, *and let* $\mathscr{S}$ *denote the closed algebra (in* $B(H)$*) generated by* $\{S_1, \ldots, S_n, I\}$ *and* $\mathscr{N}''$ *the double commutant of* $\{N_1, \ldots N_n\}$. *Then*

1) $\sigma_{\mathscr{N}''}(\mathbf{N}) \subset \sigma_{\mathscr{S}''}(\mathbf{S})$ *and*
2) $\sigma_{\mathscr{S}}(\mathbf{S})$ *is the polynomial convex hull of* $\sigma_{\mathscr{N}''}(\mathbf{N})$.

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We note that for $E \subset \mathbf{C}^n$, the polynomial convex hull

$$E\hat{} = \{\lambda = (\lambda_1, \ldots \lambda_n) \in \mathbf{C}^n: |p(\lambda)| \leqq \sup \{|p(\mathbf{z})|, \mathbf{z} \in E\}$$

for all $n$-variable polynomials $p$, and for an abelian algebra of operators $\mathscr{A}$ containing $\mathbf{A} = (A_1, \ldots, A_m)$, $\sigma_{\mathscr{A}}(\mathbf{A})$ denotes the joint spectrum of $\mathbf{A}$ in the Banach algebra $\mathscr{A}$, i.e.,

$$\sigma_{\mathscr{A}}(\mathbf{A}) = \{(\phi(A_1), \ldots, \phi(A_n)) \in \mathbf{C}^n: \phi \text{ is a multiplicative linear functional on } \mathscr{A}\}$$
$$= \{\lambda \in \mathbf{C}^n: \text{there is } \mathbf{B} \subset \mathscr{A} \text{ with } \sum B_i(A_i - \lambda_i) = I\}.$$

Thus, this notion of joint spectrum clearly depends on the algebra $\mathscr{A}$. The double commutant, being inverse closed, is a natural choice for $\mathscr{A}$, but the nature of the spectral inclusion theorem forces the use of two different algebras, $\mathscr{S}''$ and $\mathscr{N}''$.

J. Janas [**11**] also considered the problem of spectral inclusion and R. Curto [**6**] has recently proved a spectral inclusion theorem using the Taylor spectrum. In Section 3 below, we prove a spectral inclusion theorem using the Waelbroeck-Arveson spectrum for commuting operators. The Bunce-Deddens result extends easily in this context. The major tool we use is a result of Bruce Abrahamse which appears in an unpublished alternate version of [**1**]; we therefore present this interesting result here. We note that all Hilbert spaces considered below are assumed to be sparable.

**2.** We first state some well-known theorems from which the lemma and theorem of Abrahamse follow by a clever observation. The basic facts concerning direct integrals as well as proofs of Theorems B and C can be found in [**7**, Chapter II]; Theorem D is due to J. Bastian [**3**], and Theorems E, F, G are due to Abrahamse. We use the notation $\int \oplus H_x d\mu(x)$ for the direct integral of Hilbert spaces $H_x$ over a compact set $X$ supporting $\mu$ in the complex plane. $M_x$ denotes the operator on $\int \oplus H_x d\mu(x)$ defined by $(M_x f)(x) = xf(x)$ and $A$ on $\int \oplus H_x d\mu(x)$ is called *decomposable* if and only if for each $x$ there is an operator $A_x$ on $H_x$ such that the function $x \to \|A_x\|$ is bounded and Borel measurable and $(Af)(x) = A_x f(x)$ a.e. $[\mu]$; Such an operator $A$ will be denoted $\int \oplus A_x d\mu(x)$.

THEOREM B (Spectral Theorem). *Any normal operator with spectrum $X$ is unitarily equivalent to $M_x$ on some direct integral space $\int \oplus H_x d\mu(x)$. Further, $M_x$ on $\int \oplus H_x d\mu(x)$ is unitarily equivalent to $M_x$ on $\int \oplus K_x dv(x)$ if and only if $\mu$ and $v$ are mutually absolutely continuous and the dimensions of $H_x$ and $K_x$ are equal a.e. $[\mu]$.*

THEOREM C. *An operator on $\int \oplus H_x d\mu(x)$ commutes with $M_x$ if and only if it is decomposable.*

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THEOREM D. *A decomposable operator* $S = \int \oplus S_x d\mu(x)$ *is subnormal if and only if* $S_x$ *is subnormal on* $H_x$ *a.e.* $[\mu]$.

THEOREM E. *In Theorem D, the* m.n.e. *of* $S$ *is* $N = \int \oplus N_x d\mu(x)$ *on* $\int \oplus K_x d\mu(x)$ *where* $N_x$ *on* $K_x$ *is the* m.n.e. *of* $S_x$.

*Proof.* Since $\|N_x\| = \|S_x\|$ for all $x$, $N$ is easily seen to be normal on a direct integral space and also a minimal extension of $\mathscr{S}$. (The standard technicalities of fundamental sets and measurability are handled using the fundamental set of $\int \oplus H_x d\mu(x)$, the structure of the m.n.e. spaces $K_x$ with respect to $H_x$, and the separability of the various spaces.)

LEMMA F. *Let* $S$ *be subnormal on* $H$ *with* m.n.e. $N$ *on* $K$. *If* $A$ *is normal in* $H$ *and* $SA = AS$, *then* $A$ *has normal extension* $B$ *on* $K$ *commuting with* $N$ *and* $B$ *is unitarily equivalent to* $A$.

*Proof.* Except for the unitary equivalence, this result is well-known. By Theorem B, there is a unitary $U$ such that $UAU^* = M_x$ on $\int \oplus H_x d\mu(x)$. By C and D, there is a decomposable operator $\int \oplus S_x d\mu(x)$ such that each $S_x$ is subnormal on $H_x$ and $USU^* = \int \oplus S_x d\mu(x)$. By E, the m.n.e. $N$ is unitarily equivalent to $\int \oplus N_x d\mu(x)$ on $\int \oplus K_x d\mu(x)$ where $N_x$ is the m.n.e. of $S_x$. Clearly, $M_x$ on $\int \oplus K_x d\mu(x)$ is a normal extension of $UAU^*$ and commutes with $\int \oplus S_x d\mu(x)$. If $H_x$ is finite dimensional, then $S_x$ is normal [**9**] and hence $H_x = K_x$. If the dimension of $H_x$ is infinite, then dim $H_x$ = dim $K_x$, so by again using Theorem B, $A$ is unitarily equivalent to $M_x$ on $\int K_x d\mu(x)$.

THEOREM G. *Let* $\mathbf{S} = (S_1, S_2)$ *have* c.n.e. $\mathbf{N} = (N_1, N_2)$ *on* $K$. *Then for* $i = 1, 2$, $N_i$ *is unitarily equivalent to* m.n.e. $S_i$.

*Proof.* We emphasize the obvious fact that in general $N_i$ and the m.n.e. of $S_i$ are not equal. We let $M_1$ = m.n.e. $(S_1)$ be defined on $K_1$, and we can clearly assume that $K_1 \subset K$. Then $T_2 = N_2|_{K_1}$ is a subnormal extension of $S_2$ on $K_1$ commuting with $M_1$. By F, $M_1$ extends to a normal operator $M_1'$ unitarily equivalent to $M_1$ commuting with m.n.e. $(T_2)$. Since $N_2$ on $K$ is a normal extension of $T_2$ and $\mathbf{N}$ is a minimal extension of $\mathbf{S}$, we have $N_2$ = m.n.e. $(T_2)$ and hence $M_1' = N_1$. Thus, $N_1$ is unitarily equivalent to m.n.e. $(S_1)$ and symmetrically for $i = 2$.

COROLLARY 1. *Let* $\mathbf{S} = (S_1, \dots, S_n)$ *have* c.n.e. $\mathbf{N} = (N_1, \dots, N_n)$. *Then for* $i = 1, \dots n$, $N_i$ *is unitarily equivalent to* m.n.e. $(S_i)$.

*Proof.* Suppose $n > 2$ and let $\mathbf{M} = (M_1, \dots, M_{n-1})$ on $K'$ be the c.n.e. of $(S_1, \dots, S_{n-1})$. Then we may assume $K' \subset K$ and $M_i = N_i|_{K'}$, and by induction $M_i$ is unitarily equivalent to m.n.e. $(S_i)$, $i = 1, \dots, n - 1$. Letting $T_n = N_n|_{K'}$, we have m.n.e. $(T_n) = N_n$, and by G for $i = 1, \dots, n - 1$, $M_i$ extends to a unitarily equivalent operator, clearly $N_i$, commuting with $N_n$. Thus, $N_i$ and m.n.e. $(S_i)$ are unitarily equivalent, and similarly $M_n$ and m.n.e. $(S_n)$.

COROLLARY 2. *Let* $\mathbf{S} = (S_1, \ldots, S_n)$ *have* c.n.e. $\mathbf{N} = (N_1, \ldots N_n)$. *Then for any n-variable polynomial* $p$, $p(\mathbf{N})$ *is unitarily equivalent to* m.n.e. $p(\mathbf{S})$.

*Proof*. $(S_1, \ldots, S_n, p(\mathbf{S}))$ has c.n.e. $(N_1, \ldots, N_n, p(\mathbf{N}))$.

Corollary 2 has the following easy but somewhat curious application.

COROLLARY 3. *Let* $K = L^2(\mu)$ *where* $\mu$ *is a Borel measure on* $\mathbf{C}^n$ *with compact support. Let* $H = H^2(\mu)$ *be the* $L^2(\mu)$ *closure of the n-variable polynomials and let* $p(\mathbf{z}) = p(z_1, \ldots, z_n)$ *be such a polynomial. Let* $K_1$ *be the closed linear span of*

$$\{\overline{p(\mathbf{z})}^m f(z): m = 0, 1, \ldots; f \in H\} \quad \textit{and} \quad K_2 = K \ominus K_1.$$

*Let* $M$ *be the multiplication operator on* $K$ *defined by* $Mf = pf$ *and* $N_1$ *and* $N_2$ *be the normal operators* $M|_{K_1}$ *and* $M|_{K_2}$ *respectively. Then*

$$\sigma(N_2) \subset \sigma(N_1) = \sigma(M) = \{p(\mathbf{z}): \mathbf{z} \in \text{supp }(\mu)\}.$$

**3.** Our spectral inclusion theorem uses a notion of joint spectrum due to L. Waelbroeck and used by W. Arveson [**16**], [**2**]. For $\mathbf{T}$ a commuting $n$-tuple of operators, we define sp $(\mathbf{T})$ to be the set of all complex $n$-tuples $\boldsymbol{\lambda}$ such that $p(\boldsymbol{\lambda}) \in \sigma(p(\mathbf{T}))$ for every $n$-variable polynomial $p$. (Note that $\sigma$ denotes the ordinary spectrum.) We state the following lemma due to Arveson [**2**, 1.1.2].

LEMMA 1. $\sigma(p(\mathbf{T})) = p(\text{sp }(\mathbf{T}))$ *for every n-variable rational function* $p$ *with poles off* sp $(\mathbf{T})$

2). sp $(\mathbf{T}) = \sigma_{\mathscr{R}}(\mathbf{T})$, *where* $\mathscr{R}$ *is the smallest inverse-closed Banach algebra containing* $\{I, T_1, \ldots, T_n\}$.

THEOREM 1. *Let* $\mathbf{S} = (S_1, \ldots, S_n)$ *have* c.n.e. $\mathbf{N}$. *Then*

$$\text{sp }(\mathbf{N}) \subset \text{sp }(\mathbf{S}) \subset \text{sp }(\mathbf{N})\hat{}.$$

*Proof*. Let $\boldsymbol{\lambda} \in \text{sp }(\mathbf{N})$, $p$ be an $n$-variable polynomial, and $N_p =$ m.n.e. $p(\mathbf{S})$. Then

$$p(\boldsymbol{\lambda}) \in \sigma(p(\mathbf{N})) = \sigma(N_p) \subset \sigma(p(\mathbf{S}))$$

by Corollary 2 and the spectral inclusion theorem. Thus, sp $(\mathbf{N}) \subset$ sp $(\mathbf{S})$.

Let $\boldsymbol{\lambda} \in \text{sp }(\mathbf{S})$ and fix $p$. Since $p(\boldsymbol{\lambda}) \in \sigma(p(\mathbf{S}))$, we have

$$\begin{aligned} |p(\boldsymbol{\lambda})| &\leqq \sup \{|z|: z \in \sigma(p(\mathbf{S}))\} = \sup\{|z|: z \in \sigma(N_p)\} \\ &= \sup \{|z|: z \in \sigma(p(\mathbf{N}))\} = \sup \{|z|: z \in p(\text{sp }(\mathbf{N}))\} \\ &= \sup \{|p(\mathbf{z})|: \mathbf{z} \in \text{sp }(\mathbf{N})\}. \end{aligned}$$

Thus, sp $(\mathbf{S}) \subset$ sp $(\mathbf{N})\hat{}$.

We now show sp $(\mathbf{N})$, the normal joint spectrum of $\mathbf{S}$, is invariant under *-representations. The following theorems are extensions of the single

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operator results of Bunce and Deddens; the proofs from [**5**] extend immediately to this case.

THEOREM 2. $\mathbf{S} = (S_1, \ldots, S_n)$ *has* c.n.e. *if and only if*

$$\sum_{I,J} B_I^* \mathbf{S}^{*J} \mathbf{S}^I B_J \geqq 0 \quad \text{for every finite set } \{B_I\} \subset C^*(\mathbf{S}),$$

*where* $C^*(\mathbf{S})$ *is the* $C^*$*-algebra generated by* $\{I, S_1, \ldots, S_n\}$.

THEOREM 3. *Let* $\mathbf{S}$ *have* c.n.e. *Then* $\lambda \in \operatorname{sp}(\mathbf{S})$ *if and only if there exists* $\alpha > 0$ *and polynomial* $p$ *such that*

$$\sum_{I,J} (B_I^* \mathbf{S}^{*J} (p(\mathbf{S}) - p(\lambda))^* (p(\mathbf{S}) - p(\lambda)) \mathbf{S}^I B_J) \geqq \alpha \sum_{I,J} B_I^* \mathbf{S}^{*J} \mathbf{S}^I B_J$$

*for every finite set* $\{B_I\} \subset C^*(\mathbf{S})$.

COROLLARY. *If* $\mathbf{S}$ *has* c.n.e. *and* $\pi$ *is a *-representation, then* $\pi(\mathbf{S})$ *has* c.n.e. *and* $\operatorname{sp}(\pi(\mathbf{S})) \subset \operatorname{sp}(\mathbf{S})$.

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