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The (7, 4)-Conjecture in Finite Groups

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The first open case of the Brown–Erdős–Sós conjecture is equivalent to the following: for every $c > 0$, there is a threshold n_0 such that if a quasigroup has order $n \geqslant n_0$, then for every subset S of triples of the form (a, b, ab) with $|S| \geqslant cn^2$, there is a seven-element subset of the quasigroup which spans at least four triples of S. In this paper we prove the conjecture for finite groups.

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1. Introduction

This paper is about proving a special case of a famous conjecture in extremal combinatorics. The conjecture originates from Brown, Erdős and T. Sós [2]. Before we state it, let us introduce some notation we are going to use. *Triple systems* are families of three-element subsets of a finite set. In the theory of hypergraphs such systems are called 3-uniform hypergraphs. If a triple system has many triples, if it is dense in some sense, that is a *global* property. Usually it is hard to show that dense systems have *locally* dense subsystems. For example, Turán's conjecture states that if the number of triples is more than $\frac{5}{9}\binom{n}{3}$ in a triple system $\mathcal{T}$ on n elements, then there are four elements for which all four triples spanned by them are in $\mathcal{T}$. (The 3-uniform hypergraph contains a clique, $K_4^{(3)}$.) A more general question is the following: What can we say about the density of a triple system if one knows that no k elements span ℓ or more triples? Depending on the values of ℓ and k, the question might be a very hard one. Understanding how global properties induce local properties is a central problem in combinatorics.

The Brown–Erdős–Sós conjecture is that for any fixed $k \geqslant 3$, all triple systems on n elements in which no $k + 3$ elements span k triples should be sparse, *i.e.*, it has $o(n^2)$

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triples. Note that here sparseness is relative to the fact that such systems have $O(n^2)$ triples. Indeed, observe that if k triples have a common 2-element intersection, then $k+2$ elements span k triples. Therefore, if no $k+3$ elements carry k triples, then the number of triples is at most $(k-1)\binom{n}{2}/3$. In this paper we will suppose that our triple system is a linear hypergraph, that is, no triples share more than one element. If no $k+3$ elements carry k triples, then a constant fraction of the triples forms a linear hypergraph.

Lemma 1.1. *To prove or disprove the Brown–Erdős–Sós conjecture, it is enough to check it for linear* 3*-uniform hypergraphs.*

Let us order the m triples arbitrarily and check them one by one. We will select a subset of the edges, S, in such a way that the remaining hypergraph is linear. Following the order, add the next triple to S if it already has at most one common element with the triples in S. Using the property that no $k+2$ set contains k triples, it is clear that no triple can share two elements with $k-1$ other triples. Every selected triple has at most $k-2$ triples with two elements in common, so at the end of the selection $|S| \geqslant m/(k-1)$. □

In the other direction it was noted in [2] that a random construction shows that, for every $k \geqslant 3$, there is a $c_k > 0$ such that one can find triple systems with $c_k n^2$ triples on n elements for which no $k+2$ elements span k triples (n can be arbitrarily large). For the sake of completeness, we sketch the random construction here. The details can be found in [2].

Construction. Choose triples out of the possible $\binom{n}{3}$ triples in an n-element set independently at random, with probability δn^{-1}. If in this triple system there are $k+2$ elements which span k or more triples, then remove all such spanned triples from the system. There is a constant $c_k > 0$ such that, for any choice of $k+2$ elements, the probability that we have selected at least k triples out of the possible $\binom{k+2}{3}$ is less than $c_k \delta^k n^{-k}$. By the linearity of expectations, the expected number of the removed triples is less than

$$c_k \delta^k n^{-k} \binom{n}{k+2} \binom{k+2}{3} \leqslant c'_k \delta^k n^2,$$

for some $c'_k > 0$ depending only on k. If we choose δ sufficiently small that

$$c'_k \delta^k \leqslant \frac{\delta}{12},$$

then less than half of the selected triples have been removed, so there are still some $c''_k n^2$ triples remaining with positive probability.

One might think that the $(k+2, k)$ case is solved, since the triple systems without k elements carrying $k+2$ triples cannot have more than $C_k n^2$ triples on n elements, and as the previous construction shows, there are such systems with $c_k n^2$ triples. But there is an interesting question that still remains open: the two constants are far apart. In the previous arguments $C_k \to \infty$ and $c_k \to 0$ as $k \to \infty$.

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Problem 1.2. Is it true that, for every integer $k \geqslant 100$, if a triple system on $n \geqslant k+2$ elements contains at least $n^2/100$ triples then it contains $k+2$ points carrying at least k triples? (Of course, 100 is just an arbitrary number here. Does the statement hold for some constant?)

I first heard the problem above from Nati Linial, but others have probably had similar questions too. In a related conjecture of Erdős – which would imply a negative answer to the previous problem – the question is formulated as follows.

Conjecture 1.3 (Erdős's Steiner Triple System Conjecture). *For every $r \geqslant 4$ there are arbitrary large Steiner triple systems where no $r+2$ elements carry at least r triples.*

There exist partial results on Erdős's Steiner Triple System Conjecture. We refer to the papers [5] and [7] for further details.

2. Main result

We will reformulate the Brown–Erdős–Sós conjecture as a statement in quasigroups. Our hope is that some tools from algebra can be used to tackle this notoriously hard problem. The informal definition of quasigroups is that they are groups without associativity. More formally, a quasigroup $(Q, \circ)$ is a set Q with a binary operation '$\circ$' so that, for any $a, b \in Q$, there exist unique elements $u, w \in Q$ such that $a \circ u = b$ and $w \circ a = b$. If the quasigroup has an identity element then it is called a loop.

Every Steiner triple system defines a commutative loop in a natural way: for any two distinct elements a and b, the product, $c = a \circ b$, is the third element of the triple spanned by a and b. We add an identity element, e, so that $a \circ a = e$ for any element a. These quasigroups are called Steiner quasigroups. One can also see that the triples of a quasigroup $(a, b, a \circ b)$ form a Steiner triple system. For more details we refer the reader to [10].

It was conjectured by Lindner [9] that any partial Steiner triple system of order u can be embedded in a Steiner triple system of order $2u$. This conjecture has been proved by Bryant and Horsley [3], so the following is equivalent to the original Brown–Erdős–Sós conjecture.

Conjecture 2.1 (Brown, Erdős and Sós). *For every $c > 0$, there is a threshold n_0 such that if a quasigroup has order $n \geqslant n_0$, then for every subset S of triples of the form (a, b, ab) with $|S| \geqslant cn^2$, there is a seven-element subset of the quasigroup which spans at least four triples of S.*

Now the question is: For which families of quasigroups can one prove the conjecture? The main result of this paper is to show that the (7,4)-conjecture holds for finite groups.

Theorem 2.2 (Brown–Erdős–Sós conjecture for groups). *For every $c > 0$, there is a threshold n_0 such that if a group has order $n \geqslant n_0$, then for every subset S of triples of the form*

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(a,b,ab) with $|S| \geqslant cn^2$, there is a seven-element subset of the group which spans at least four triples of S.

In addition to the algebraic techniques there are some combinatorial tools which can be used when one works with triple systems. The most powerful one is the so-called Hypergraph Removal Lemma [8, 12], which we are going to apply here. The simplest case, the Triangle Removal Lemma, states that for every dense subset of triples of the form (a,b,ab) there is a six-element subset of the quasigroup which spans at least three triples from the selected subset. This is called the (6,3)-theorem, and was proved by Ruzsa and Szemerédi [15]. In search of a proof of the (7,4)-conjecture, Frankl and Rödl [6] proved the Removal Lemma for 3-uniform hypergraphs.

Theorem 2.3 (Frankl–Rödl). *Let $H_n^{(3)}$ be a 3-uniform hypergraph on n vertices with the property that every edge is contained in exactly one clique, $K_4^{(3)}$. Then the number of edges in $H_n^{(3)}$ is $o(n^3)$.*[1]

Our application of the above theorem is similar to the technique we used in [16]. Theorem 2.3 is enough to prove the $(7,4)$-conjecture in groups, but for some quantitative results the following stronger statement is useful.

Theorem 2.4 (Frankl–Rödl). *For every real number $c > 0$, there is a $c' > 0$ such that the following holds. If $H_n^{(3)}$ is a 3-uniform hypergraph on n vertices with the property that it has at least cn^3 edge-disjoint $K_4^{(3)}$-cliques, then it contains at least $c'n^4$ distinct (but not necessarily edge-disjoint) $K_4^{(3)}$-cliques.*

From the theory of groups our main tool is a classical result of Erdős and Straus [4], which states that every finite group contains a large abelian subgroup. The best – and asymptotically optimal – bound is due to Pyber [13].

Theorem 2.5 (Pyber). *There is a universal constant $\nu > 0$ such that every group of order n contains an abelian subgroup of order at least $e^{\nu\sqrt{\log n}}$.*

Pyber's theorem was also used in a predecessor of this paper [17]. Here we prove a stronger statement which was stated as a conjecture in [17].

Theorem 2.6. *For every $\kappa > 0$ there is a threshold $n_0 \in \mathbb{N}$ such that if G is a finite group of order $|G| \geqslant n_0$ then the following holds. Any set $H \subset G \times G$ with $|H| \geqslant \kappa|G|^2$ contains four elements (α,β), (α,γ), (δ,γ), and (δ,β) such that $\alpha\beta = \delta\gamma$.*

It is easy to see that Theorem 2.6 implies Theorem 2.2. Every triple (a,b,ab) is uniquely determined by $(a,b) \in G \times G$. The triples $(a,b,ab), (a,c,ac), (d,c,dc)$, and (d,b,db) determine

[1] For every $\varepsilon > 0$ there is a threshold $n_0 = n_0(\varepsilon)$ such that if $n \geqslant n_0$ and $H_n^{(3)}$ has the above property then it has at most εn^3 edges.

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at most seven elements of G, which are $a,b,c,d,ab=dc,ac$, and db. (The last two elements might coincide.)

Proof of Theorem 2.6. Let A be the largest abelian subgroup of G. By Pyber's theorem we know that $|A|\geqslant e^{\nu\sqrt{\log n}}$. There are elements $\ell,r\in G$ such that H has at least average density in the product of the left and right cosets $\ell A\times Ar$, that is, $|H\cap(\ell A\times Ar)|\geqslant\kappa|A|^2$.

Let us define a 4-partite 3-uniform hypergraph using H,ℓ,r, and A. The four vertex partitions are $\ell A=V_1, Ar=V_2, \ell Ar=V_3, A=V_4$. Every triple (a,b,c) where $(a,b)\in H\cap(\ell A\times Ar)$ and $c\in A$ defines four edges forming a $K_4^{(3)}$-clique on the four vertices $g_i\in V_1$, $g_j\in V_2$, $g_k\in V_3$, and $g_l\in V_4$, where

(1) $ac=g_i$,
(2) $cb=g_j$,
(3) $acb=g_k$,
(4) $c=g_l$.

With this definition every edge belongs to a unique (a,b,c) triple. From the three vertices of an edge one can recover the values of a,b, and c. (For the inverse of an element $g\in G$ we use the usual g^{-1} notation.)

(1) If $g_i\in V_1, g_j\in V_2, g_k\in V_3$ spans an edge defined by (a_1,b_1,c_1), then

$$
\begin{aligned}
a_1 &= g_kg_j^{-1},\\
b_1 &= g_i^{-1}g_k,\\
c_1 &= a^{-1}g_i = g_jg_k^{-1}g_i.
\end{aligned}
$$

(2) If $g_i\in V_1, g_j\in V_2, g_l\in V_4$ spans an edge defined by (a_2,b_2,c_2), then

$$
\begin{aligned}
a_2 &= g_ig_l^{-1},\\
b_2 &= g_l^{-1}g_j,\\
c_2 &= g_l.
\end{aligned}
$$

(3) If $g_i\in V_1, g_k\in V_3, g_l\in V_4$ spans an edge defined by (a_3,b_3,c_3), then

$$
\begin{aligned}
a_3 &= g_ig_l^{-1},\\
b_3 &= g_i^{-1}g_k,\\
c_3 &= g_l.
\end{aligned}
$$

(4) If $g_j\in V_2, g_k\in V_3, g_l\in V_4$ spans an edge defined by (a_4,b_4,c_4), then

$$
\begin{aligned}
a_4 &= g_kg_j^{-1},\\
b_4 &= g_l^{-1}g_j,\\
c_4 &= g_l.
\end{aligned}
$$

If two generating triples of the edges of a $K_4^{(3)}$-clique are given, then they determine the vertices and therefore the remaining two edges uniquely. So, if a clique is not generated by a single triple then all four edges have distinct generators, (a_1,b_1,c_1), (a_2,b_2,c_2), (a_3,b_3,c_3), and (a_4,b_4,c_4). By Theorem 2.3 we know that such a $K_4^{(3)}$ exists if $|A|$ is large enough.

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Note that the four pairs $(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4) \in H$ will satisfy the requirements of Theorem 2.6.

Set

$$\delta = a_1 = a_4,$$
$$\beta = b_1 = b_3,$$
$$\alpha = a_2 = a_3,$$
$$\gamma = b_2 = b_4.$$

It remains to check that $\alpha\beta = \delta\gamma$. The two elements c_1 and c_2 are from the abelian subgroup A, so

$$c_1^{-1}c_2^{-1} = c_2^{-1}c_1^{-1},$$
$$g_i^{-1}g_kg_j^{-1}g_l^{-1} = g_l^{-1}g_i^{-1}g_kg_j^{-1},$$
$$g_kg_j^{-1}g_l^{-1}g_j = g_ig_l^{-1}g_i^{-1}g_k,$$
$$a_1b_2 = a_2b_1,$$
$$\delta\gamma = \alpha\beta.$$

□

Finally, we briefly bound the number of (7, 4)-configurations our calculation finds in a group. By the quantitative version of the Frankl–Rödl theorem, Theorem 2.4, the number of $K_4^{(3)}$-cliques for the selected ℓ and r elements is at least $c'|A|^4$. That guarantees at least $c'|A|^3$ (α, β), (α, γ), (δ, γ), (δ, β) quadruples from S such that $\alpha\beta = \delta\gamma$. Set S has high density in a positive fraction of the left and right cosets:

$$\left|\left\{(r', \ell') : |H \cap (\ell' A \times Ar')| \geqslant \frac{\kappa}{2}|A|^2\right\}\right| \geqslant c''\frac{n^2}{|A|^2}.$$

For these r', ℓ' pairs one can repeat the calculations as we did before, so in each case there are at least $c'''|A|^3$ (α, β), (α, γ), (δ, γ), (δ, β) quadruples such that $\alpha\beta = \delta\gamma$.

Theorem 2.7. *There is a constant $\mu > 0$ depending only on κ, the density of S, such that the number of (α, β), (α, γ), (δ, γ), (δ, β) quadruples from S such that $\alpha\beta = \delta\gamma$ in the group is at least*

$$\mu|A|n^2 \geqslant \mu e^{\nu\sqrt{\log n}}n^2.$$

It might be that the right magnitude is ξn^3 for some universal constant $\xi > 0$ independent of the group.

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