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# ON OPTIMAL HETEROGENEOUS COMPONENTS GROUPING IN SERIES–PARALLEL AND PARALLEL–SERIES SYSTEMS

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In this paper, we consider optimal components grouping in series–parallel and parallel–series systems composed of $k$ subsystems. All components in each subsystem are drawn from a heterogeneous population consisting of $m$ different subpopulations. Firstly, we show that when one allocation vector is majorized by another one, then the series–parallel (parallel–series) system corresponding to the first (second) vector is more reliable than that of the other. Secondly, we study the impact of changes in the number of subsystems on the system reliability. Finally, we study the influence of the selection probabilities of subpopulations on the system reliability.

**Keywords:** heterogeneous population, majorization order, parallel–series system, series–parallel system, stochastic orders

## 1. INTRODUCTION

A system is a collection of components arranged in a specific design so as to achieve desired functions with acceptable performance and reliability. Components allocation has a direct effect on the system reliability. Kuo and Wan [10] described the state-of-the-art of redundancy allocation problem. Recently, several components allocation policies for the system have been developed in terms of various stochastic orders. Hu and Wang [9] studied the optimal allocation problem of a $k$-out-of-$n$ system and a series system with respect to stochastic orders. Misra, Dhariyal, and Gupta [15] studied the problem of allocating $k$ active spares to $n$ components of a series system in order to optimize the system reliability. Li and Ding [12] dealt with the allocation of i.i.d. active redundancies to a $k$-out-of-$n$ system with the usual stochastic order among its components. Cha [3] studied the optimal allocation policies for the combined stochastic risk processes and discussed its applications to various related fields. Di Crescenzo and Pellerey [6] considered components allocation in a series or parallel

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system and supposed that components are randomly selected from two different batches. Ding and Li [7] studied the allocation of active redundancies to a $k$-out-of-$n$ system with i.i.d. components in the sense of the hazard rate order. Li and Ding [13] reviewed the allocation of active redundancies to a coherent system and presented some recent theoretical results as well as some applications. Cha [4] studied the optimal allocation policy for the generalized combined risk processes by the stochastic comparisons of reliability functions. Zhuang and Li [20] studied the allocation of i.i.d. components to a $k$-out-of-$n$ system with non-i.i.d. components.

The series–parallel (parallel–series) system is composed of a certain number of subsystems in series (parallel), each subsystem includes one or several components arranged in parallel (series). Coit and Smith [5] demonstrated the use of a genetic algorithm to solve the redundancy allocation problem for a series–parallel system. Ramirez-Marquez, Coit, and Konak [16] studied the redundancy allocation problem in order to maximize the minimum subsystem reliability for a series–parallel system. Sarhan et al. [17] studied reliability equivalence of different designs of a series–parallel system consisting of four i.i.d. components. Billionnet [2] studied the redundancy allocation problem of a series–parallel system by using integer linear programming software. Levitin and Amari [11] presented an algorithm for determining an optimal loading of elements in series–parallel system. Sun, Li, and Zio [19] studied the redundancy allocation problem for multi-state series–parallel system.

In reality, lifetime distributions of manufactured components in different batches may be different due to many factors (e.g., supplied material, human factors, unstable conditions of production, etc.). Then, lots of batches of manufactured components constitute a heterogeneous population consisting of several different subpopulations. Since the lifetimes of components in the same batch are i.i.d., then it is reasonable to assume that components in a randomly selected batch belong to a fixed subpopulation of a heterogeneous population with a certain selection probability. We have known that some authors have studied redundancy allocation problems of different systems by means of stochastic orders. But most of the literatures studied the components allocation problem in a system under the condition that each allocative component is selected from a homogeneous population. Hazra, Finkelstein, and Cha [8] firstly studied the optimal components grouping problem in a series (parallel) system, where components of each group are selected from a subpopulation with a certain probability.

In this paper, we consider the components grouping problem in a series–parallel (parallel–series) system composed of $k$ subsystems. All components in each subsystem of the series–parallel (parallel–series) system are drawn from a heterogeneous population consisting of $m$ different subpopulations. Components grouping problems in this paper include the decision of the number of components in each subsystem and the number of subsystems such that the reliability of the series–parallel (parallel–series) system is maximized. We also study the effect of the selection probabilities of subpopulations on the reliability of the series–parallel (parallel–series) system.

Our model can be served as a valid formulation for a series–parallel (parallel–series) system consisting of $k$ subsystems, where these subsystems may be geographically separated (e.g., radar, meteorological observation device, airports security system, etc.). We need an optimal allocation policy to ensure that the overall system is maximally reliable subject to the constraints of the total number of allocative components. For example, meteorological observation devices are located in $k$ areas of a small country, and the zone weather information are periodically sent to a central command post. Each area includes several parallel (series) meteorological observation devices which are drawn from a randomly selected batch. The allocation policy, i.e., determining the number of allocative meteorological observation devices in each area, is to be made in order to maximize the reliability that the central command post obtains an accurate zone weather information.

The organization of this paper is as follows. In Section 2, we describe our notations of models and provide some definitions and lemmas which will be used in the sequel. In Sections 3 and 4, we use the theory of stochastic orders and majorization order to present the optimal components grouping of the series–parallel system and the parallel–series system, respectively. In Section 5, we conclude this paper.

Throughout this paper, all random variables under consideration are nonnegative, increasing and decreasing properties of a function are not used in the strict sense. All expectations are assumed to exist whenever they appear.

## 2. PRELIMINARIES

### 2.1. Notations

Let $n$ components be grouped as $k$ groups. We draw $a_1$ components from one of the $m$ subpopulations, and use these $a_1$ components to form the first subsystem; then we draw $a_2$ components from one of the $m$ subpopulations, and use these $a_2$ components to form the second subsystem, and so on. Suppose that we select one subpopulation randomly with probabilities $\boldsymbol{p} = (p_1, p_2, \ldots, p_m)$, where $p_i$ is the selection probability of the $i$th subpopulation, $i = 1, 2, \ldots, m$.

Denote by

$$\mathbb{A} = \Bigg\{ (k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) : n \in \mathbb{N}_+, m \in \mathbb{N}_+; \boldsymbol{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{N}_+^k \text{ s.t. } \sum_{i=1}^{k} a_i = n, \\ k = 1, 2, \ldots, n; \boldsymbol{p} \in [0,1]^m \text{ s.t. } \sum_{i=1}^{m} p_i = 1 \text{ and } \boldsymbol{X} \in \Omega \Bigg\}$$

the set of all admissible models, where

$$\Omega = \{ \boldsymbol{X} = (X_1, X_2, \ldots, X_m) : X_i \text{ is the lifetime of i.d.d. components in the } i\text{th} \\ \text{subpopulation}, \quad i = 1, 2, \ldots, m \}.$$

Further, let $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X})$ $(H(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}))$ be the random variable representing the lifetime of a series–parallel (parallel–series) system corresponding to the model $(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \in \mathbb{A}$.

### 2.2. Concepts and lemmas

For ease of reference, let us recall some important concepts and lemmas closely related to our study. Let $X_i$ be a non-negative random variable with probability density function $f_i(x)$, distribution function $F_i(x)$, reliability function $\bar{F}_i(x)$, hazard rate function $h_i(x) = f_i(x)/\bar{F}_i(x)$, and reversed hazard rate function $r_i(x) = f_i(x)/F_i(x)$, respectively, $i = 1, 2$.

Definition 2.1: The random variable $X_1$ is said to be smaller than $X_2$ in the

(i) usual stochastic order (denoted by $X_1 \le_{\text{st}} X_2$) if $\bar{F}_1(x) \le \bar{F}_2(x)$ for all $x \ge 0$.

(ii) hazard rate order (denoted by $X_1 \le_{\text{hr}} X_2$) if $\bar{F}_2(x)/\bar{F}_1(x)$ is increasing in $x \ge 0$, or equivalently, $X_1 \le_{\text{hr}} X_2$ if, and only if, $h_1(x) \ge h_2(x)$ for all $x \ge 0$.

(iii) reversed hazard rate order (denoted by $X_1 \le_{\rm rh} X_2$) if $F_2(x)/F_1(x)$ is increasing in $x \ge 0$, or equivalently, $X_1 \le_{\rm rh} X_2$ if, and only if, $r_1(x) \le r_2(x)$ for all $x \ge 0$.

(iv) likelihood ratio order (denoted by $X_1 \le_{\rm lr} X_2$) if $f_2(x)/f_1(x)$ is increasing in $x \ge 0$.

For more details on stochastic orders, one may refer to Shaked and Shanthikumar [18].

The majorization order enables us to compare the diversity of two real vectors. The following definition of the majorization order can be found in Marshall, Olkin, and Arnold [14].

Definition 2.2: Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ be any two vectors, and $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$ and $y_{(1)} \le y_{(2)} \le \cdots \le y_{(n)}$ be the increasing arrangements of the components of $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. The vector $\boldsymbol{x}$ is said to majorize the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \succeq_{\rm m} \boldsymbol{y}$) if for any $j = 1, 2, \ldots, n-1$, we have

$$\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)} \text{ and } \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

Definition 2.3: $D_{\rm sym} = \{\boldsymbol{X} \in \Omega : X_1 \ge_{\rm sym} X_2 \ge_{\rm sym} \cdots \ge_{\rm sym} X_m\}$ and $E_{\rm sym} = \{\boldsymbol{X} \in \Omega : X_1 \le_{\rm sym} X_2 \le_{\rm sym} \cdots \le_{\rm sym} X_m\}$, where 'sym' could be 'st', 'hr', 'rh', 'lr' and $D_Q = \{\boldsymbol{x} \in Q^m : x_1 \ge x_2 \ge \cdots \ge x_m\}$ and $E_Q = \{\boldsymbol{x} \in Q^m : x_1 \le x_2 \le \cdots \le x_m\}$, where $Q \subseteq \mathbb{R}$.

Lemma 2.4: Let $B \subseteq \mathbb{R}^m$, and $\varphi : B \to \mathbb{R}$ be a continuously differentiable function. Then, for $\boldsymbol{x}, \boldsymbol{y} \in B$,

$$\boldsymbol{x} \succeq_{\rm m} \boldsymbol{y} \Longrightarrow \varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y}),$$

if, and only if, the following two conditions hold:

(i) $\varphi$ is symmetric on $B$,

(ii) for all $\boldsymbol{z} \in B$ and $1 \le i \ne j \le m$,

$$(z_i - z_j)\left[\frac{\partial \varphi(\boldsymbol{z})}{\partial z_i} - \frac{\partial \varphi(\boldsymbol{z})}{\partial z_j}\right] \le 0.$$

Lemma 2.5: Let $X$ and $Y$ be two independent random variables. Then

(i) $X \le_{\rm rh} Y$ if, and only if, $E[\alpha(X)]E[\beta(Y)] \ge E[\alpha(Y)]E[\beta(X)]$ for all functions $\alpha$ and $\beta$ such that $\beta$ is nonnegative and $\alpha/\beta$ and $\beta$ are decreasing.

(ii) $X \le_{\rm hr} Y$ if, and only if, $E[\alpha(X)]E[\beta(Y)] \le E[\alpha(Y)]E[\beta(X)]$ for all functions $\alpha$ and $\beta$ such that $\beta$ is nonnegative and $\alpha/\beta$ and $\beta$ are increasing.

The above two lemmas can be found in Marshall et al. [14] and Shaked and Shanthikumar [18], respectively.

## 3. SERIES–PARALLEL SYSTEM

We consider the series–parallel system consisting of $k$ independent subsystems in series, and the $j$th subsystem has $a_j$ components connected in parallel, $j = 1, 2, \ldots, k$. All components in each subsystem are selected from the $i$th subpopulation with probability $p_i$, $i = 1, 2, \ldots, m$. Figure 1 presents the structural diagram of the series–parallel system. Note

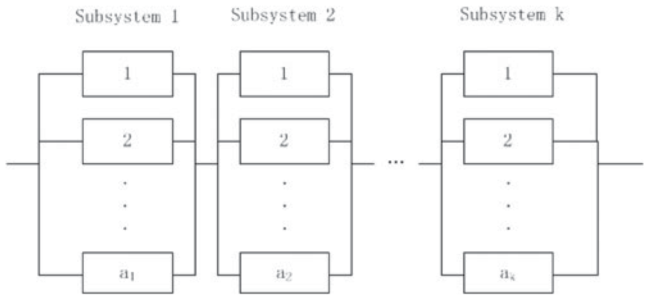


FIGURE 1. Series–parallel system

that subsystems lifetimes are independent and the $j$th subsystem has distribution function $\sum_{i=1}^{m} p_i F_i^{a_j}(t)$, $j = 1, 2, \ldots, k$. Hence the reliability function of $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X})$ is given by

$$\bar{F}_{S(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t) = \prod_{j=1}^{k} \left[1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t)\right], \quad t \geq 0.$$

One of the important problems is how to draw $n$ components from $m$ different subpopulations such that the resulting system will be optimal in some stochastic sense. Let the number of subsystems be equal to $k$, the following result shows that if the allocation vector is majorized by another allocation vector, then the series–parallel system corresponding to the first vector is more reliable than that of the other. It means that the system's reliability can be increased by balancing the allocation of components.

THEOREM 3.1: Let $(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \in \mathbb{A}$ and $(k, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{X}) \in \mathbb{A}$. If $\boldsymbol{a} \preceq_{\mathrm{m}} \boldsymbol{b}$, then $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\mathrm{st}} S(k, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{X})$.

PROOF: For all $t \geq 0$, let

$$\phi(\boldsymbol{a}) = \bar{F}_{S(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t) = \prod_{j=1}^{k} \left[1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t)\right].$$

Differentiating $\phi(\boldsymbol{a})$ with respect to $a_\alpha$, then we have

$$\frac{\partial \phi(\boldsymbol{a})}{\partial a_\alpha} = -\sum_{i=1}^{m} p_i F_i^{a_\alpha}(t) \ln F_i(t) \prod_{1 \leq j \neq \alpha \leq k} \left[1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t)\right],$$

hence

$$\begin{aligned}
&(a_\alpha - a_\beta)\left[\frac{\partial \phi(\boldsymbol{a})}{\partial a_\alpha} - \frac{\partial \phi(\boldsymbol{a})}{\partial a_\beta}\right] \\
&= (a_\alpha - a_\beta)\left\{\left[-\sum_{i=1}^{m} p_i F_i^{a_\alpha}(t) \ln F_i(t)\right] \prod_{1 \leq j \neq \alpha \leq k} \left[1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t)\right]\right. \\
&\quad \left. - \left[-\sum_{i=1}^{m} p_i F_i^{a_\beta}(t) \ln F_i(t)\right] \prod_{1 \leq j \neq \beta \leq k} \left[1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t)\right]\right\}
\end{aligned}$$

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$$= (a_\alpha - a_\beta) \prod_{1\le j\ne\alpha,\beta\le k} \left[1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t)\right]$$
$$\times \left\{ \sum_{i=1}^{m} p_i F_i^{a_\beta}(t) \ln F_i(t) \left[1 - \sum_{i=1}^{m} p_i F_i^{a_\alpha}(t)\right] \right.$$
$$\left. - \sum_{i=1}^{m} p_i F_i^{a_\alpha}(t) \ln F_i(t) \left[1 - \sum_{i=1}^{m} p_i F_i^{a_\beta}(t)\right]\right\}.$$

If $a_\alpha < a_\beta$, then

$$1 - \sum_{i=1}^{m} p_i F_i^{a_\alpha}(t) \le 1 - \sum_{i=1}^{m} p_i F_i^{a_\beta}(t)$$

and

$$\sum_{i=1}^{m} p_i F_i^{a_\beta}(t) \ln F_i(t) \ge \sum_{i=1}^{m} p_i F_i^{a_\alpha}(t) \ln F_i(t).$$

Thus

$$\sum_{i=1}^{m} p_i F_i^{a_\beta}(t) \ln F_i(t) \left[1 - \sum_{i=1}^{m} p_i F_i^{a_\alpha}(t)\right] - \sum_{i=1}^{m} p_i F_i^{a_\alpha}(t) \ln F_i(t) \left[1 - \sum_{i=1}^{m} p_i F_i^{a_\beta}(t)\right] \ge 0.$$

Hence

$$(a_\alpha - a_\beta)\left[\frac{\partial\phi(\boldsymbol{a})}{\partial a_\alpha} - \frac{\partial\phi(\boldsymbol{a})}{\partial a_\beta}\right] \le 0.$$

Similarly, if $a_\alpha > a_\beta$, then

$$(a_\alpha - a_\beta)\left[\frac{\partial\phi(\boldsymbol{a})}{\partial a_\alpha} - \frac{\partial\phi(\boldsymbol{a})}{\partial a_\beta}\right] \le 0.$$

In conclusion,

$$(a_\alpha - a_\beta)\left[\frac{\partial\phi(\boldsymbol{a})}{\partial a_\alpha} - \frac{\partial\phi(\boldsymbol{a})}{\partial a_\beta}\right] \le 0$$

holds for all $1 \le \alpha \ne \beta \le k$. It is easy to see that $\phi(\boldsymbol{a})$ is symmetric with respect to $\boldsymbol{a}$. Therefore, by Lemma 2.4, we have that $\boldsymbol{a} \preceq_{\mathrm{m}} \boldsymbol{b}$ implies $\phi(\boldsymbol{a}) \ge \phi(\boldsymbol{b})$, i.e., $\bar{F}_{S(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t) \ge \bar{F}_{S(k,\boldsymbol{b},\boldsymbol{p},\boldsymbol{X})}(t)$. ■

The following example shows that the condition of Theorem 3.1 cannot be dropped.

*Example 3.2*: Let $n = 109$, $k = 8$, $m = 5$, $\boldsymbol{a} = (2, 2, 5, 6, 8, 10, 32, 44)$, $\boldsymbol{b} = (1, 4, 4, 20, 20, 20, 20, 20)$, $\boldsymbol{p} = (0.15, 0.25, 0.15, 0.35, 0.1)$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (0.2, 0.5, 0.5, 0.3, 0.9)$, and $\bar{F}_i(t) = e^{-t^{\gamma_i}}$, $i = 1, 2, \ldots, 5$. It is easy to see that $\boldsymbol{a} \preceq_{\mathrm{m}} \boldsymbol{b}$ does not hold. From Figure 2, we can see that $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \ge_{\mathrm{st}} S(k, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{X})$ is not established. Therefore, the condition $\boldsymbol{a} \preceq_{\mathrm{m}} \boldsymbol{b}$ of Theorem 3.1 cannot be dropped.

Under certain conditions, the next theorem shows that if the number of subsystems becomes smaller, then the system becomes more reliable. The proof is easy and is omitted.

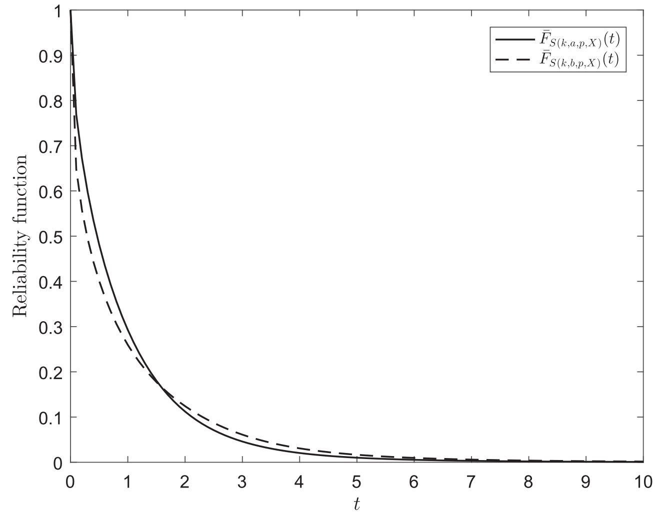


FIGURE 2. Reliability function

THEOREM 3.3: Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_{k-1})$, where $b_j \geq a_j$ for all $j = 1, 2, \ldots, k-1$. Then $S(k-1, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\text{st}} S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X})$.

It is well known that the reliability function of a coherent system is bounded below by the reliability function of series system and above by the reliability function of parallel system (Barlow and Proschan [1, p. 35]). Let $(1, n, \boldsymbol{p}, \boldsymbol{X})$ denote that the series–parallel system has only one subsystem which has $n$ components connected in parallel and $(n, 1, \boldsymbol{p}, \boldsymbol{X})$ denote that the series–parallel system has $n$ subsystems and each subsystem includes only one component. The next theorem shows that $(1, n, \boldsymbol{p}, \boldsymbol{X})$ is the best model and $(n, 1, \boldsymbol{p}, \boldsymbol{X})$ is the worst model among all admissible models in the sense of the hazard rate order.

THEOREM 3.4: For any $(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \in \mathbb{A}$, $S(1, n, \boldsymbol{p}, \boldsymbol{X}) \geq_{\text{hr}} S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\text{hr}} S(n, 1, \boldsymbol{p}, \boldsymbol{X})$.

PROOF: Note that the hazard rate function of $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X})$ is given by

$$h_{S(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t) = \sum_{l=1}^{k} \frac{\sum_{i=1}^{m} p_i a_l F_i^{a_l}(t) r_i(t)}{1 - \sum_{i=1}^{m} p_i F_i^{a_l}(t)}.$$

Since $n = \sum_{i=1}^{k} a_i$, we see that

$$h_{S(1,n,\boldsymbol{p},\boldsymbol{X})}(t) = \frac{\sum_{i=1}^{m} p_i n F_i^{n}(t) r_i(t)}{1 - \sum_{i=1}^{m} p_i F_i^{n}(t)} = \sum_{l=1}^{k} \frac{\sum_{i=1}^{m} p_i a_l F_i^{n}(t) r_i(t)}{1 - \sum_{i=1}^{m} p_i F_i^{n}(t)}$$

and

$$h_{S(n,1,\boldsymbol{p},\boldsymbol{X})}(t) = \frac{n\sum_{i=1}^{m} p_i F_i(t) r_i(t)}{1 - \sum_{i=1}^{m} p_i F_i(t)} = \sum_{l=1}^{k} \frac{\sum_{i=1}^{m} p_i a_l F_i(t) r_i(t)}{1 - \sum_{i=1}^{m} p_i F_i(t)}.$$

Hence, $S(1, n, \boldsymbol{p}, \boldsymbol{X}) \geq_{\text{hr}} S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\text{hr}} S(n, 1, \boldsymbol{p}, \boldsymbol{X})$ holds if

$$\frac{\sum_{i=1}^{m} p_i a_l F_i^{n}(t) r_i(t)}{1 - \sum_{i=1}^{m} p_i F_i^{n}(t)} \leq \frac{\sum_{i=1}^{m} p_i a_l F_i^{a_l}(t) r_i(t)}{1 - \sum_{i=1}^{m} p_i F_i^{a_l}(t)} \leq \frac{\sum_{i=1}^{m} p_i a_l F_i(t) r_i(t)}{1 - \sum_{i=1}^{m} p_i F_i(t)} \qquad \textbf{(3.1)}$$

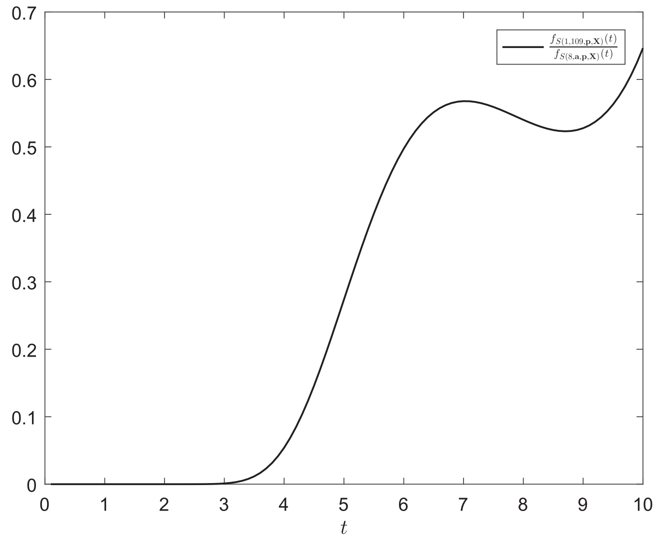


FIGURE **3.** The ratio of two density functions

for all $l = 1, 2, \ldots, k$. Define

$$\psi(x) = \frac{\sum_{i=1}^m p_i a_l F_i^x(t) r_i(t)}{1 - \sum_{i=1}^m p_i F_i^x(t)},$$

then we have

$$\begin{aligned}\psi'(x) &= \frac{\begin{array}{c}[\sum_{i=1}^m p_i a_l F_i^x(t) r_i(t) \ln F_i(t)]\,[1 - \sum_{i=1}^m p_i F_i^x(t)] \\ + [\sum_{i=1}^m p_i F_i^x(t) \ln F_i(t)] \sum_{i=1}^m p_i a_l F_i^x(t) r_i(t)\end{array}}{[1 - \sum_{i=1}^m p_i F_i^x(t)]^2} \\ &\leq 0.\end{aligned}$$

That is, $\psi(x)$ is decreasing in $x$. Hence the eq. (3.1) holds. ■

The following example shows that the hazard rate order of Theorem 3.4 cannot be strengthened to the likelihood ratio order.

*Example 3.5*: Let $n = 109$, $k = 8$, $m = 5$, $\boldsymbol{a} = (13, 13, 13, 14, 14, 14, 14, 14)$, $\boldsymbol{p} = (0.15, 0.25, 0.15, 0.35, 0.1)$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (0.2, 0.5, 0.5, 0.3, 0.9)$ and $\bar{F}_i(t) = e^{-t^{\gamma_i}}$, $i = 1, 2, \ldots, 5$. From Figure 3, we see that $S(1, 109, \boldsymbol{p}, \boldsymbol{X}) \geq_{\rm lr} S(8, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X})$ is invalid.

Let $P$ and $Q$ be two discrete random variables with probability mass functions $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively. The next theorem shows how some of the well-known stochastic orders between $P$ and $Q$ translate into the stochastic orders between $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X})$ and $S(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$. It states that the reliability of series–parallel system will become bigger with the increase of selection probabilities of reliable subpopulations.

THEOREM 3.6: Let $(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \in \mathbb{A}$ and $(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X}) \in \mathbb{A}$.

(i) If $P \leq_{\rm rh} Q$ and $\boldsymbol{X} \in D_{\rm lr}$, then $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\rm hr} S(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$.

(ii) If $P \leq_{\rm hr} Q$ and $\boldsymbol{X} \in E_{\rm lr}$, then $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \leq_{\rm hr} S(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$.

(iii) If $P \leq_{\rm st} Q$ and $\boldsymbol{X} \in E_{\rm st}(D_{st})$, then $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \leq_{\rm st} (\geq_{\rm st}) S(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$.

PROOF: (i) Note that

$$1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t) = E[1 - F_P^{a_j}(t)] \text{ and } 1 - \sum_{i=1}^{m} q_i F_i^{a_j}(t) = E[1 - F_Q^{a_j}(t)].$$

Let $\alpha(i) = 1 - F_i^{a_j}(t_2)$ and $\beta(i) = 1 - F_i^{a_j}(t_1)$ for any $t_2 \geq t_1$. Suppose $X_i(a_j)$ is a random variable with reliability function $1 - F_i^{a_j}(t)$, $i = 1, 2, \ldots, m$. It is easy to verify that $X_1 \geq_{\rm lr} X_2 \geq_{\rm lr} \cdots \geq_{\rm lr} X_m$ implies $X_1(a_j) \geq_{\rm lr} X_2(a_j) \geq_{\rm lr} \cdots \geq_{\rm lr} X_m(a_j)$. Hence $X_1(a_j) \geq_{\rm hr} X_2(a_j) \geq_{\rm hr} \cdots \geq_{\rm hr} X_m(a_j)$. Then, we obtain that

$$\frac{\alpha(1)}{\beta(1)} \geq \frac{\alpha(2)}{\beta(2)} \geq \cdots \geq \frac{\alpha(m)}{\beta(m)}.$$

That is, $\alpha(i)/\beta(i)$ is decreasing in $i$. On the other hand, $X_1(a_j) \geq_{\rm hr} X_2(a_j) \geq_{\rm hr} \cdots \geq_{\rm hr} X_m(a_j)$ implies $X_1(a_j) \geq_{\rm st} X_2(a_j) \geq_{\rm st} \cdots \geq_{\rm st} X_m(a_j)$. Then we have

$$\beta(1) \geq \beta(2) \geq \cdots \geq \beta(m),$$

that is, $\beta(i)$ is decreasing in $i$. By Lemma 2.5 (i), if $P \leq_{\rm rh} Q$, then

$$\frac{1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t_2)}{1 - \sum_{i=1}^{m} q_i F_i^{a_j}(t_2)} = \frac{E[\alpha(P)]}{E[\alpha(Q)]} \geq \frac{E[\beta(P)]}{E[\beta(Q)]} = \frac{1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t_1)}{1 - \sum_{i=1}^{m} q_i F_i^{a_j}(t_1)}$$

for any $t_2 \geq t_1$, which means that

$$\frac{1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t)}{1 - \sum_{i=1}^{m} q_i F_i^{a_j}(t)}$$

is increasing in $t$. Thus,

$$\frac{\bar{F}_{S(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t)}{\bar{F}_{S(k,\boldsymbol{a},\boldsymbol{q},\boldsymbol{X})}(t)} = \frac{\Pi_{j=1}^{k}\left[1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t)\right]}{\Pi_{j=1}^{k}\left[1 - \sum_{i=1}^{m} q_i F_i^{a_j}(t)\right]}$$

is increasing in $t$, that is, $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\rm hr} S(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$.

(ii) Note that $X_1 \leq_{\rm lr} X_2 \leq_{\rm lr} \cdots \leq_{\rm lr} X_m$ implies $X_1(a_j) \leq_{\rm hr} X_2(a_j) \leq_{\rm hr} \cdots \leq_{\rm hr} X_m(a_j)$, $j = 1, 2, \ldots, k$. By Theorem 1.B.14 of Shaked and Shanthikumar [18], $P \leq_{\rm hr} Q$ implies that

$$\frac{1 - \sum_{i=1}^{m} q_i F_i^{a_j}(t)}{1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t)}$$

is increasing in $t$. Hence

$$\frac{\bar{F}_{S(k,\boldsymbol{a},\boldsymbol{q},\boldsymbol{X})}(t)}{\bar{F}_{S(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t)} = \frac{\Pi_{j=1}^{k}\left[1 - \sum_{i=1}^{m} q_i F_i^{a_j}(t)\right]}{\Pi_{j=1}^{k}\left[1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t)\right]}$$

is increasing in $t$, that is, $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \leq_{\rm hr} S(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$.

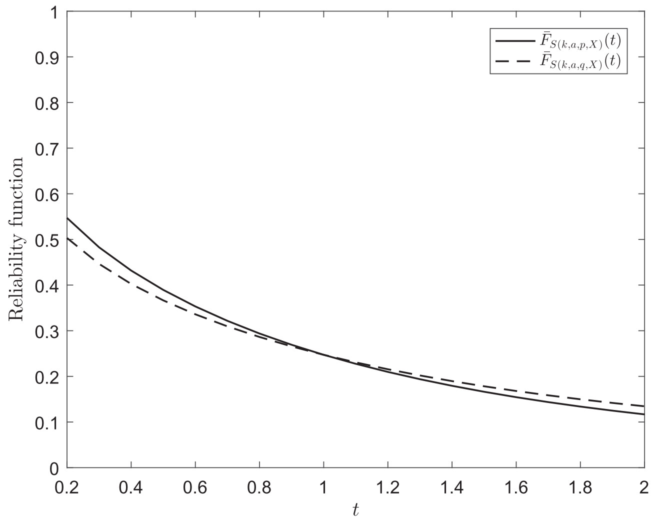


FIGURE 4. Reliability function

(iii) Note that $X_1 \le_{\rm st} X_2 \le_{\rm st} \cdots \le_{\rm st} X_m (X_1 \ge_{\rm st} X_2 \ge_{\rm st} \cdots \ge_{\rm st} X_m)$ implies $X_1(a_j) \le_{\rm st} X_2(a_j) \le_{\rm st} \cdots \le_{\rm st} X_m(a_j) (X_1(a_j) \ge_{\rm st} X_2(a_j) \ge_{\rm st} \cdots \ge_{\rm st} X_m(a_j))$, $j = 1, 2, \ldots, k$. That is, $1 - F_i^{a_j}(t)$ is increasing (decreasing) in $i$. By Theorem 1.A.3(a) of Shaked and Shanthikumar [18], $P \le_{\rm st} Q$ implies $E[1 - F_P^{a_j}(t)] \le (\ge) E[1 - F_Q^{a_j}(t)]$, i.e.,

$$1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t) \le (\ge) 1 - \sum_{i=1}^{m} q_i F_i^{a_j}(t).$$

Thus,

$$\bar{F}_{S(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t) = \prod_{j=1}^{k} \left[1 - \sum_{i=1}^{m} p_i F_i^{a_j}(t)\right] \le (\ge) \prod_{j=1}^{k} \left[1 - \sum_{i=1}^{m} q_i F_i^{a_j}(t)\right] = \bar{F}_{S(k,\boldsymbol{a},\boldsymbol{q},\boldsymbol{X})}(t),$$

that is, $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \le_{\rm st} (\ge_{\rm st}) S(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$.

■

*Example 3.7*: Let $n = 9$, $k = 3$, $m = 6$, $\boldsymbol{a} = (3, 1, 5)$, $\boldsymbol{p} = (0.05, 0.1, 0.15, 0.2, 0.24, 0.26)$, $\boldsymbol{q} = (0.05, 0.07, 0.08, 0.1, 0.3, 0.4)$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) = (0.9, 0.8, 0.6, 0.5, 0.3, 0.2)$, and $\bar{F}_i(t) = e^{-t^{\gamma_i}}$, $i = 1, 2, \ldots, 5$. It is easy to see that $P \le_{\rm st} Q$ is established, but $\boldsymbol{X} \in E_{\rm st} (\in D_{\rm st})$ does not hold. From Figure 4, we can see that $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \le_{\rm st} (\ge_{\rm st}) S(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$ is invalid. Hence, the condition $\boldsymbol{X} \in E_{\rm st} (\in D_{\rm st})$ of Theorem 3.6(iii) cannot be dropped.

Let $\boldsymbol{p} \in D_{[0,1]}$ and $\boldsymbol{q} \in D_{[0,1]}$. It is easy to see that $\boldsymbol{p} \succeq_{\rm m} \boldsymbol{q}$ is equivalent to $P \le_{\rm st} Q$. By Theorems 3.1 and 3.6(iii), we have the following result.

THEOREM 3.8: Let $\boldsymbol{p} \in D_{[0,1]}, \boldsymbol{q} \in D_{[0,1]}, \boldsymbol{X} \in D_{\rm st} (\in E_{\rm st})$, $(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \in \mathbb{A}$ and $(k, \boldsymbol{b}, \boldsymbol{q}, \boldsymbol{X}) \in \mathbb{A}$. If $\boldsymbol{a} \preceq_{\rm m} (\succeq_{\rm m}) \boldsymbol{b}$ and $\boldsymbol{p} \succeq_{\rm m} \boldsymbol{q}$, then $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \ge_{\rm st} (\le_{\rm st}) S(k, \boldsymbol{b}, \boldsymbol{q}, \boldsymbol{X})$.

The next theorem shows that if components are drawn from the subpopulations with more reliable components, then the corresponding system is more reliable as well.

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Theorem 3.9: Let $(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \in \mathbb{A}$ and $(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{Y}) \in \mathbb{A}$.

(i) If $X_i \geq_{\rm st} Y_i$ for all $i = 1, 2, \ldots, m$, then $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\rm st} S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{Y})$.

(ii) If $X_i \geq_{\rm lr} Y_l$ for all $i = 1, 2, \ldots, m$ and $l = 1, 2, \ldots, m$, then $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\rm hr} S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{Y})$.

Proof: (i) The proof is trivial and is omitted.

(ii) Let $X_i(a_j)$ and $Y_l(a_j)$ be two random variables with distribution functions $F_{X_i}^{a_j}(t)$ and $F_{Y_l}^{a_j}(t)$, respectively. Then, it is easy to verify that $X_i \geq_{\rm lr} Y_l$ implies $X_i(a_j) \geq_{\rm lr} Y_l(a_j)$, and hence $X_i(a_j) \geq_{\rm hr} Y_l(a_j)$. By Theorem 1.B.8 of Shaked and Shanthikumar [18], we obtain that

$$\frac{\sum_{i=1}^m p_i a_j F_{X_i}^{a_j}(t) r_{X_i}(t)}{1 - \sum_{i=1}^m p_i F_{X_i}^{a_j}(t)} \leq \frac{\sum_{l=1}^m p_l a_j F_{Y_l}^{a_j}(t) r_{Y_l}(t)}{1 - \sum_{l=1}^m p_l F_{Y_l}^{a_j}(t)},$$

$j = 1, 2, \ldots, k$. Note that

$$h_{S(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t) = \sum_{j=1}^k \frac{\sum_{i=1}^m p_i a_j F_{X_i}^{a_j}(t) r_{X_i}(t)}{1 - \sum_{i=1}^m p_i F_{X_i}^{a_j}(t)}$$

and

$$h_{S(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{Y})}(t) = \sum_{j=1}^k \frac{\sum_{l=1}^m p_l a_j F_{Y_l}^{a_j}(t) r_{Y_l}(t)}{1 - \sum_{l=1}^m p_l F_{Y_l}^{a_j}(t)}.$$

Then, we have $h_{S(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t) \leq h_{S(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{Y})}(t)$, $t \geq 0$, that is, $S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\rm hr} S(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{Y})$.

■

## 4. PARALLEL–SERIES SYSTEM

We consider the parallel–series system consisting of $k$ independent subsystems in parallel, and the $j$th subsystem has $a_j$ components connected in series, $j = 1, 2, \ldots, k$. All components in each subsystem are selected from the $i$th subpopulation with probability $p_i$, $i = 1, 2, \ldots, m$. Figure 5 presents the structural diagram of the parallel–series system. Note that subsystems lifetimes are independent and the $j$th subsystem has reliability function $\sum_{i=1}^m p_i \bar{F}_i^{a_j}(t)$, $j = 1, \ldots, k$, hence the distribution function of $H(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X})$ is given by

$$F_{H(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t) = \prod_{j=1}^k \left[1 - \sum_{i=1}^m p_i \bar{F}_i^{a_j}(t)\right], \quad t \geq 0.$$

The following result shows that if the allocation vector is majorized by another allocation vector, then the parallel–series system corresponding to the second vector is more reliable than that of the other. It is shown that the system reliability may be decreased by balancing the allocation of components. Since proofs of the corresponding results here are extremely similar to those of series–parallel system, for brevity, most results are presented without proof.

Theorem 4.1: Let $(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \in \mathbb{A}$ and $(k, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{X}) \in \mathbb{A}$. If $\boldsymbol{a} \succeq_{\rm m} \boldsymbol{b}$, then $H(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\rm st} H(k, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{X})$.

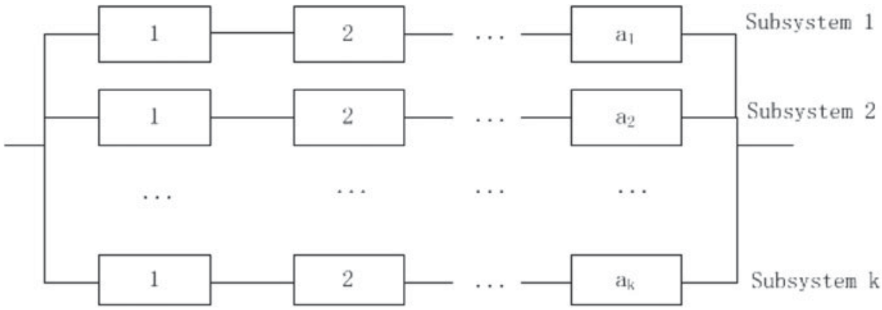


FIGURE 5. Parallel–series system

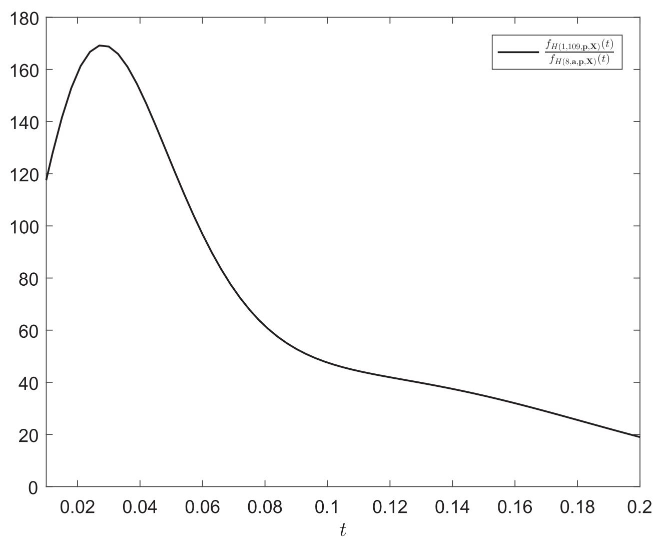


FIGURE 6. The ratio of two density functions

Under certain conditions, the next result shows that if the number of subsystems becomes bigger, then the system should become more reliable.

THEOREM 4.2: Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_{k-1})$, where $b_j \geq a_j$ for all $j = 1, 2, \ldots, k-1$. Then $H(k-1, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{X}) \leq_{\rm st} H(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X})$.

The next theorem shows that $(n, 1, \boldsymbol{p}, \boldsymbol{X})$ is the best model and $(1, n, \boldsymbol{p}, \boldsymbol{X})$ is the worst model among all admissible models in the sense of the reversed hazard rate order.

THEOREM 4.3: For any $(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \in \mathbb{A}$, $H(n, 1, \boldsymbol{p}, \boldsymbol{X}) \geq_{\rm rh} H(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\rm rh} H(1, n, \boldsymbol{p}, \boldsymbol{X})$.

The following example shows that the reversed hazard rate order of Theorem 4.3 cannot be strengthened to the likelihood ratio order.

*Example 4.4*: Let $n = 109$, $k = 8$, $m = 5$, $\boldsymbol{a} = (13, 13, 13, 14, 14, 14, 14, 14)$, $\boldsymbol{p} = (0.15, 0.25, 0.15, 0.35, 0.1)$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (0.2, 0.5, 1.5, 3, 9)$, and $\bar{F}_i(t) = e^{-t^{\gamma_i}}$, $i = 1, 2, \ldots, 5$. From Figure 6, we can see that $H(1, 109, \boldsymbol{p}, \boldsymbol{X}) \leq_{\rm lr} H(8, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X})$ is invalid.

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Let $P$ and $Q$ be two discrete random variables which have been described in Section 3.

THEOREM 4.5: Let $(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \in \mathbb{A}$ and $(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X}) \in \mathbb{A}$.

(i) If $P \leq_{\mathrm{hr}} Q$ and $\boldsymbol{X} \in D_{\mathrm{lr}}$, then $H(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\mathrm{rh}} H(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$.

(ii) If $P \leq_{\mathrm{rh}} Q$ and $\boldsymbol{X} \in E_{\mathrm{lr}}$, then $H(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \leq_{\mathrm{rh}} H(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$.

(iii) If $P \leq_{\mathrm{st}} Q$ and $\boldsymbol{X} \in E_{\mathrm{st}}(D_{st})$, then $H(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \leq_{\mathrm{st}} (\geq_{\mathrm{st}}) H(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$.

PROOF: (i) Note that

$$1 - \sum_{i=1}^{m} p_i \bar{F}_i^{a_j}(t) = E[1 - \bar{F}_P^{a_j}(t)] \text{ and } 1 - \sum_{i=1}^{m} q_i \bar{F}_i^{a_j}(t) = E[1 - \bar{F}_Q^{a_j}(t)].$$

Let $\alpha(i) = 1 - \bar{F}_i^{a_j}(t_1)$ and $\beta(i) = 1 - \bar{F}_i^{a_j}(t_2)$ for any $t_2 \geq t_1$. Suppose $X_i(a_j)$ is a random variable with distribution function $1 - \bar{F}_i^{a_j}(t)$, $i = 1, 2, \ldots, m$. It is easy to verify that $X_1 \geq_{\mathrm{lr}} X_2 \geq_{\mathrm{lr}} \cdots \geq_{\mathrm{lr}} X_m$ implies $X_1(a_j) \geq_{\mathrm{lr}} X_2(a_j) \geq_{\mathrm{lr}} \cdots \geq_{\mathrm{lr}} X_m(a_j)$. Hence $X_1(a_j) \geq_{\mathrm{rh}} X_2(a_j) \geq_{\mathrm{rh}} \cdots \geq_{\mathrm{rh}} X_m(a_j)$. Then, we obtain that

$$\frac{\alpha(1)}{\beta(1)} \leq \frac{\alpha(2)}{\beta(2)} \leq \cdots \leq \frac{\alpha(m)}{\beta(m)}.$$

That is, $\alpha(i)/\beta(i)$ is increasing in $i$. On the other hand, $X_1(a_j) \geq_{\mathrm{rh}} X_2(a_j) \geq_{\mathrm{rh}} \cdots \geq_{\mathrm{rh}} X_m(a_j)$ implies $X_1(a_j) \geq_{\mathrm{st}} X_2(a_j) \geq_{\mathrm{st}} \cdots \geq_{\mathrm{st}} X_m(a_j)$. Then we have

$$\beta(1) \leq \beta(2) \leq \cdots \leq \beta(m),$$

that is, $\beta(i)$ is increasing in $i$. By Lemma 2.5 (ii), if $P \leq_{\mathrm{hr}} Q$, then

$$\frac{1 - \sum_{i=1}^{m} p_i \bar{F}_i^{a_j}(t_1)}{1 - \sum_{i=1}^{m} q_i \bar{F}_i^{a_j}(t_1)} = \frac{E[\alpha(P)]}{E[\alpha(Q)]} \leq \frac{E[\beta(P)]}{E[\beta(Q)]} = \frac{1 - \sum_{i=1}^{m} p_i \bar{F}_i^{a_j}(t_2)}{1 - \sum_{i=1}^{m} q_i \bar{F}_i^{a_j}(t_2)}$$

for any $t_2 \geq t_1$, which means that

$$\frac{1 - \sum_{i=1}^{m} p_i \bar{F}_i^{a_j}(t)}{1 - \sum_{i=1}^{m} q_i \bar{F}_i^{a_j}(t)}$$

is increasing in $t$. Therefore,

$$\frac{F_{H(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t)}{F_{H(k,\boldsymbol{a},\boldsymbol{q},\boldsymbol{X})}(t)} = \frac{\Pi_{j=1}^{k} \left[1 - \sum_{i=1}^{m} p_i \bar{F}_i^{a_j}(t)\right]}{\Pi_{j=1}^{k} \left[1 - \sum_{i=1}^{m} q_i \bar{F}_i^{a_j}(t)\right]}$$

is increasing in $t$, that is, $H(k, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{X}) \geq_{\mathrm{rh}} H(k, \boldsymbol{a}, \boldsymbol{q}, \boldsymbol{X})$.

(ii) Note that $X_1 \leq_{\mathrm{lr}} X_2 \leq_{\mathrm{lr}} \cdots \leq_{\mathrm{lr}} X_m$ implies $X_1(a_j) \leq_{\mathrm{rh}} X_2(a_j) \leq_{\mathrm{rh}} \cdots \leq_{\mathrm{rh}} X_m(a_j)$, $j = 1, 2, \ldots, k$. By Theorem 1.B.52 of Shaked and Shanthikumar [18], $P \leq_{\mathrm{rh}} Q$ implies that

$$\frac{1 - \sum_{i=1}^{m} q_i \bar{F}_i^{a_j}(t)}{1 - \sum_{i=1}^{m} p_i \bar{F}_i^{a_j}(t)}$$

is increasing in $t$. Hence

$$\frac{F_{H(k,\boldsymbol{a},\boldsymbol{q},\boldsymbol{X})}(t)}{F_{H(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X})}(t)}$$

is increasing in $t$, that is, $H(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X}) \leq_{\text{rh}} H(k,\boldsymbol{a},\boldsymbol{q},\boldsymbol{X})$.

(iii) It may be proved in a very similar manner to Theorem 3.6 (iii).

■

By Theorems 4.1 and 4.5 (iii), we have the following result.

THEOREM 4.6: Let $\boldsymbol{p} \in D_{[0,1]}, \boldsymbol{q} \in D_{[0,1]}, \boldsymbol{X} \in D_{\text{st}}(\in E_{\text{st}}), (k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X}) \in \mathbb{A}$, and $(k,\boldsymbol{b},\boldsymbol{q},\boldsymbol{X}) \in \mathbb{A}$. If $\boldsymbol{a} \succeq_{\text{m}} (\preceq_{\text{m}})\boldsymbol{b}$ and $\boldsymbol{p} \succeq_{\text{m}} \boldsymbol{q}$, then $H(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X}) \geq_{\text{st}} (\leq_{\text{st}})H(k,\boldsymbol{b},\boldsymbol{q},\boldsymbol{X})$.

The next result shows that if components of a system are drawn from the subpopulations with more reliable components, then the corresponding system is more reliable as well.

THEOREM 4.7: Let $(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X}) \in \mathbb{A}$ and $(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{Y}) \in \mathbb{A}$.

(i) If $X_i \geq_{\text{st}} Y_i$ for all $i = 1,2,\ldots,m$, then $H(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X}) \geq_{\text{st}} H(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{Y})$.

(ii) If $X_i \geq_{\text{lr}} Y_l$ for all $i = 1,2,\ldots,m$ and $l = 1,2,\ldots,m$, then $H(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{X}) \geq_{\text{rh}} H(k,\boldsymbol{a},\boldsymbol{p},\boldsymbol{Y})$.

## 5. CONCLUDING REMARKS

This paper compares two different components grouping policies in a series–parallel (parallel–series) system composed of several subsystems. Components of each subsystem are drawn from a heterogeneous population composed of $m$ different subpopulations. Components grouping policies in this paper include the decision of the number of components in each subsystem and the number of subsystems such that the series–parallel (parallel–series) system is more reliable. We also study the effect of the selection probabilities of subpopulations on the reliability of the series–parallel (parallel–series) system. We can mathematically prove that the majorization order of Theorem 3.1 (Theorem 4.1) can be weakened to the supermajorization order. This paper considers the series–parallel system and the parallel–series system, respectively. It would be of interest to further study the corresponding components grouping policy in the $k$-out-of-$n$ system.

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