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Existence and regularity of time-dependent pullback attractors for the non-autonomous nonclassical diffusion equations

Yuming Qin
Department of Mathematics, Institute for Nonlinear Science, Donghua University, Shanghai 201620, People's Republic of China
yuming_qin@hotmail.com

Bin Yang
College of Information Science and Technology, Donghua University, Shanghai 201620, People's Republic of China binyangdhu@163.com

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In this paper, we prove the existence and regularity of pullback attractors for non-autonomous nonclassical diffusion equations with nonlocal diffusion when the nonlinear term satisfies critical exponential growth and the external force term $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. Under some appropriate assumptions, we establish the existence and uniqueness of the weak solution in the time-dependent space $\mathcal{H}_t(\Omega)$ and the existence and regularity of the pullback attractors.

Keywords: Non-autonomous nonclassical diffusion equations; time-dependent pullback attractors; regularity

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1. Introduction

It is well-known that the study of global, pullback and uniform attractors is of great significance for characterizing the long-time behavior of the solutions of nonlinear evolutionary partial differential equations (see [**7**, **8**, **17**, **27**]). Therefore, in recent decades, as an important class of the nonlinear partial differential equations, the autonomous and non-autonomous diffusion equations have been extensively studied (see [**2**, **3**, **28**, **29**]). However, there are still relatively few results on the existence of time-dependent pullback attractors in the Sobolev space $H^2(\Omega) \cap H^1_0(\Omega)$. To this end, this paper is devoted to studying the existence and regularity of the time-dependent pullback attractors of the non-autonomous diffusion equations with nonlocal diffusion in the time-dependent space $\mathcal{H}_t(\Omega)$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the long-time behavior of the solutions for the following non-autonomous nonclassical diffusion equations:

$$\begin{cases} u_t - \varepsilon(t)\Delta u_t - a(l(u))\Delta u = f(u) + h(t) & \text{in } \Omega \times (\tau, \infty), \\ u = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\tau \in \mathbb{R}$ is the initial time and $\varepsilon(t) \in C^1(\mathbb{R})$ is a decreasing bounded function with respect to the parameter t satisfying

$$\lim_{t \to +\infty} \varepsilon(t) = 1, \tag{1.2}$$

and there exists a constant $L > 0$ such that

$$\sup_{t \in \mathbb{R}} (|\varepsilon(t)| + |\varepsilon'(t)|) \leqslant L. \tag{1.3}$$

For the nonlocal functional $a(l(u))$, we assume that $l(u)$ is a linear functional acting on u that satisfies $l(u) = (u, l)$, whose definition of $(\cdot, \cdot)$ is below, and $a \in C(\mathbb{R}; \mathbb{R}_+)$ is a locally Lipschitz continuous function satisfying

$$\frac{1}{2} < m \leqslant a(s) \leqslant M, \quad \forall s \in \mathbb{R}, \tag{1.4}$$

where m and M are constant. In addition, suppose the nonlinear term $f \in C^1(\mathbb{R})$ and satisfies the following assumptions:

$$\limsup_{|s| \to \infty} \frac{f(s)}{s} < \lambda_1, \tag{1.5}$$

$$f'(s) \leqslant \eta, \quad \forall s \in \mathbb{R}, \tag{1.6}$$

$$|f(s)| \leqslant C\left(1 + |s|^p\right), \quad \forall s \in \mathbb{R}, \tag{1.7}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in Ω with the homogeneous Dirichlet boundary conditions, η and C are arbitrarily positive constants.

Throughout this paper, the inner product of $L^2(\Omega)$ is represented by $(\cdot, \cdot)$, and the corresponding norm is denoted by $\|\cdot\|_2$. For simplicity, $\|\cdot\|_2$ is written as $\|\cdot\|$. The norm of $H^{-1}(\Omega)$ is denoted as $\|\cdot\|_{-1}$, the norm of $H_0^1(\Omega)$ is denoted as $\|\cdot\|_1$, and the dual product between them will be represented by $\langle \cdot, \cdot \rangle$. From [**15**], the chain of dense and continuous embeddings $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ holds. In particular, $H_0^1(\Omega) \subset L^2(\Omega)$ is compact.

When hypothesis (1.7) holds, the Sobolev embedding theorem (see [**1**]) shows that when $n = 1, 2$ and $n \geqslant 3$, the index p satisfies $p > 1$ and $1 < p < \frac{n}{n-2}$, respectively, hence it can be concluded that there is an embedding $H_0^1(\Omega) \subset L^{2p}(\Omega)$, and then by using the Poincaré inequality, we can deduce that

$$|f(u_n)|_2^2 \leqslant C + C \int_\Omega |u_n|^{2p} \, dx \leqslant C + C \|u_n\|^{2p}. \tag{1.8}$$

It is easy to check that if the function u_n is bounded in $L^{2p}(\Omega)$, then $f(u_n)$ is bounded in $L^\infty(\tau, t; L^2(\Omega))$. Furthermore, let the external force term $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$.

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Let us recall some works of problem (1.1) in the literature, which are based on the classic general diffusion equation $u_t - \Delta u = f(u)$. The asymptotic behavior of nonlocal problem $u_t - a(l(u(t)))Au = f$ was studied when there was only one equilibrium point in [**9**]. Later, this result was expanded to investigate the convergence of the solution towards a steady state in [**10**]. In [**11**], the well-posedness of the solutions to

$$u_t - a\left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\Delta u = f$$

was obtained by using the energy method. In order to make the solutions not only exist in finite-time interval, hypothesis (1.4) was introduced and the asymptotic behavior of the solutions was studied in [**12**]. The existence of pullback attractors in $L^2(\Omega)$ for

$$\frac{\mathrm{d}u}{\mathrm{d}t} - a(l(u))\Delta u = f(u) + h(t)$$

was obtained when f is a sublinear function in [**4**]. Besides, some researchers assumed that f satisfies

$$-\kappa - \alpha_1|s|^p \leqslant f(s)s \leqslant \kappa - \alpha_2|s|^p, \quad \forall s \in \mathbb{R},$$

and got similar results (see [**5**, **6**]).

In addition, there are several results in $\mathcal{H}_t(\Omega)$. Note that the definition of time-dependent space $\mathcal{H}_t(\Omega)$ can be checked in § 2. The existence and regularity of the time-dependent global attractors of problem

$$u_t - \varepsilon(t)\Delta u_t - \Delta u + \lambda u = f(u) + g(x)$$

are established by the decomposition method in [**19**]. Besides, the method of contraction function was used to prove the existence of the time-dependent global attractors in [**34**] of problem

$$u_t - \varepsilon(t)\Delta u_t - \Delta u + f(u) = g(x).$$

The Lebesgue-dominated convergence theorem was applied in [**30**] to verify the pullback asymptotic compactness of the global attractors of problem

$$u_t - \varepsilon(t)\Delta u_t - \triangle u + \lambda u + f(u) = g.$$

Furthermore, time-dependent attractors have also been extensively studied in [**14**, **18**, **20**, **21**, **23**] and other papers.

Since problem (1.1) contains $a(\cdot)$, $\varepsilon(t)$, $f(u)$ and $h(t)$, which results in some difficulties to study the long-time behavior of solutions. To this end, our main solutions are as follows:

(1) When using the condition C, the contraction function or the decomposition method to prove the asymptotic compactness of pullback attractors, we are supposed to select suitable test functions and inequalities to transform $(1.1)_1$ into a formula to ensure it satisfies the Gronwall lemma. But because $a = a(l(u))$ is a compound function, these methods do not work. To overcome

this prominent technical difficulty, we use the diagonal method in functional analysis to obtain the upper and lower bounds of the process of problem (1.1), which is very challenging.

(2) The time-dependent function $\varepsilon(t)$ complicates calculations of energy estimation. Some common techniques, such as multiplying by $(1.1)_1$ with u or u_t as a test function, does not offer any meaningful results. In order to make the Gronwall inequality work, we bring

$$\varepsilon(t)\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon(t)\|\nabla u\|^2\right) - \varepsilon'(t)\|\nabla u\|^2$$

into a priori estimate, then obtain (3.1). Although this seems to complicate the energy equation, it actually helps to discuss the existence of solutions in $\mathcal{H}_t(\Omega)$.

(3) It is worth mentioning that the nonlinear term $f(u)$ and the external force term $h(t)$ makeproblem (1.1) be studied in a more general functional framework. To obtain the dissipative properties of the process, we assume that f satisfies (1.5)–(1.7), which is weaker than the conditions in [**22**].

The structure of our paper is organized as follows. In § 2, some function spaces, abstract definitions and functions to be used later are introduced. As is known, when studying the pullback attractors of an equation, it is often necessary to attain the existence and uniqueness of the solution at first, which will be obtained by the standard Faedo-Galerkin approximations in § 3. The most important parts § 4 and § 5 are arranged in the last two sections. We mention here the energy method, the diagonal method, the decomposition method and multiple inequalities are used to overcome the difficulties in proving the existence and regularity caused by the nonlocal function and nonlinear term.

2. Preliminaries

In this section, we shall introduce the definitions of some spaces and functions involved in the paper, and some abstract concepts related to the time-dependent pullback attractors theory.

For any t, let X_t be a family of normed spaces, where the sphere with radius R is denoted as

$$\bar{B}_{X_t}(R) = \left\{u \in X_t : \|u\|^2_{X_t} \leqslant R\right\}.$$

Besides, for any $t \in \mathbb{R}$, the time-dependent space $\mathcal{H}_t(\Omega)$ is endowed with the norms

$$\|u\|^2_{\mathcal{H}_t} = \|u\|^2_2 + \varepsilon(t)\|\nabla u\|^2_2,$$

and the space $\mathcal{H}^1_t(\Omega)$, more regular than $\mathcal{H}_t(\Omega)$, is endowed with the norms

$$\|u\|^2_{\mathcal{H}^1_t} = \|\nabla u\|^2_2 + \varepsilon(t)\|\Delta u\|^2_2.$$

We define the Hausdorff semidistance of two nonempty sets $A, B \subset \mathcal{H}_t(\Omega)$ by

$$dist_{\mathcal{H}_t}(A, B) = \sup_{x\in A}\inf_{y\in B}\|x - y\|_{\mathcal{H}_t}.$$

LEMMA 2.1 Aubin-Lions Compactness Lemma [**16**]. *Let X_0, X and X_1 be Banach spaces, and satisfy that $X \subset X_0 \subset X_1$ are dense and continuous embeddings and $X \subset X_0$ is a compact embedding. Assuming that $p \geqslant 1$, $1 \leqslant q \leqslant +\infty$ and $T > 0$ is given. Let*

$$\overline{W} = \left\{ u \in L^p([0,T]; X), \frac{du}{dt} \in L^q([0,T]; X) \right\},$$

then

(i) *if $p < +\infty$, $\overline{W} \subset L^p([0,T]; X_0)$ is compact;*

(ii) *if $p = +\infty$, $\overline{W} \subset C([0,T]; X_0)$ is compact.*

REMARK 2.1. Let $p = q = 2$, X_0, X and X_1 are Hilbert spaces, $X \subset X_0 \subset X_1$ are dense and continuous embeddings, if X_0 is the interpolation space between X and X_1 and the coefficient is $\frac{1}{2}$, then $W \subset C([0,T]; X_0)$ is a continuous embedding.

DEFINITION 2.2 [**19**, **34**]. Let $\{\mathcal{H}_t\}_{t\in\mathbb{R}}$ be a family of time-dependent normed spaces. A process or a two-parameter semigroup on $\mathcal{H}_t$ is a family $\{U(t,\tau) \mid t, \tau \in \mathbb{R}, t \geqslant \tau\}$ of continuous mapping $U(t,\tau) : \mathcal{H}_\tau \to \mathcal{H}_t$ satisfies that $U(\tau,\tau)u = u$ for any $u \in \mathcal{H}_\tau$ and $U(t,s)U(s,\tau) = U(t,\tau)$ for all $t \geqslant s \geqslant \tau$.

DEFINITION 2.3 [**6**, **22**]. For any $\delta > 0$, let $\mathcal{D}_{\delta,\mathcal{H}_t}$ be a nonempty class of all families of parameterized sets $\widehat{D}_\delta = \{D_\delta(t) : t \in \mathbb{R}\} \subset \Gamma(\mathcal{H}_t)$ such that

$$\lim_{\tau\to-\infty} \left(e^{\delta\tau} \sup_{u\in D_\delta(\tau)} \|u\|^2_{\mathcal{H}_t} \right) = 0,$$

where $\Gamma(\mathcal{H}_t)$ denotes the family of all nonempty subsets of $\mathcal{H}_t(\Omega)$.

DEFINITION 2.4 [**22**, **33**]. The process $\{U(t,\tau)\}_{t\geqslant\tau}$ is said to be pullback $\mathcal{D}_{\delta,\mathcal{H}_t}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{D}_\delta \in \mathcal{D}_{\delta,\mathcal{H}_t}$, any sequence $\tau_n \to -\infty$, and any sequence $x_n \in D_\delta(\tau_n) \subset \mathcal{H}_t(\Omega)$, the sequence $\{U(t,\tau)x_n\}_{n=1}^\infty$ is relatively compact in $\mathcal{H}_t(\Omega)$.

DEFINITION 2.5 [**22**, **33**]. It is said that $\widehat{D}_0 \in \mathcal{D}_{\delta,\mathcal{H}_t}$ is pullback $\mathcal{D}_{\delta,\mathcal{H}_t}$-absorbing for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ if for any $t \in \mathbb{R}$ and $\widehat{D}_\delta \in \mathcal{D}_{\delta,\mathcal{H}_t}$, there exists a $\tau_0 = \tau_0(t, \widehat{D}_\delta) < t$ such that $U(t,\tau)D_\delta(\tau) \subset D_0(t)$, for all $\tau \leqslant \tau_0(t, \widehat{D}_\delta)$.

DEFINITION 2.6 [**22**, **33**]. A family $\widehat{\mathcal{A}}_t = \{\mathcal{A}(t) : t \in \mathbb{R}\} \subset \Gamma(\mathcal{H}_t(\Omega))$ is said to be a time-dependent pullback $\mathcal{D}_{\delta,\mathcal{H}_t}$-attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\mathcal{H}_t(\Omega)$ if

(i) $\mathcal{A}(t)$ is compact in $\mathcal{H}_t(\Omega)$ for any $t \in \mathbb{R}$;

(ii) $\widehat{\mathcal{A}}_t$ is pullback $\mathcal{D}_{\delta,\mathcal{H}_t}$-attracting in $\mathcal{H}_t(\Omega)$, i.e.,

$$\lim_{\tau\to-\infty} \mathrm{dist}_{\mathcal{H}_t} \left(U(t,\tau)D_\delta(\tau), \mathcal{A}(t) \right) = 0,$$

for all $\widehat{D}_\delta \in \mathcal{D}_{\delta,\mathcal{H}_t}$ and $t \in \mathbb{R}$;

(iii) $\widehat{\mathcal{A}}_t$ is invariant, i.e., $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$, for any $-\infty < \tau \leqslant t < +\infty$.

3. Existence and uniqueness of solutions

In order to obtain the existence of the time-dependent pullback attractors of problem (1.1), we need to establish the existence and uniqueness of solutions. First, we give the definition of the weak solution.

DEFINITION 3.1. A weak solution to problem (1.1) is a function $u \in C([\tau, t], \mathcal{H}_t(\Omega))$ for any $t \geqslant \tau$, with $u(\tau) = u_\tau$, and such that for all $\varphi \in H_0^1(\Omega)$, it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[(u(t),\varphi)+\varepsilon(t)(\nabla u(t),\nabla\varphi)]+(2a(l(u))-\varepsilon'(t))(\nabla u(t),\nabla\varphi)\\ =2(f(u(t)),\varphi)+2\langle h(t),\varphi\rangle.\end{aligned} \tag{3.1}$$

REMARK 3.1. The equation (3.1) is supposed to be understood in the sense of the generalized function space $\mathcal{D}'(\tau, +\infty)$.

REMARK 3.2. If $u(x,t)$ is a weak solution of problem (1.1), then it satisfies the following energy equality:

$$\begin{aligned}\|u(t)\|^2+\varepsilon(t)\|\nabla u(t)\|^2+\int_s^t(2a(l(u))-\varepsilon'(r))\|\nabla u(r)\|^2\,\mathrm{d}r\\ =\|u(s)\|^2+\varepsilon(s)\|\nabla u(s)\|^2+2\int_s^t(f(u(r)),u(r))\,\mathrm{d}r+2\int_s^t(h(r),u(r))\,\mathrm{d}r.\end{aligned} \tag{3.2}$$

The following theorems, theorems 3.2–3.3 will clarify the existence and uniqueness of the solution to problem (1.1). In addition, the former theorem is proved by the classic Faedo-Galerkin method, and the proof also involves the energy estimate method.

THEOREM 3.2. *Assume that $a(\cdot)$ is a local Lipschitz continuous function and satisfies (1.4), $f \in C^1(\mathbb{R})$ and satisfies (1.5)–(1.7), $h \in L^2_{loc}(\mathbb{R}; H^{-1})$, $l(\cdot)$ is given, and the initial value $u_\tau \in \mathcal{H}_t(\Omega)$, then for any $\tau \in \mathbb{R}$ and $t \geqslant \tau$, there exists a weak solution to problem (1.1), which satisfies $u \in C([\tau, t]; \mathcal{H}_t(\Omega))$ and $u_t \in L^2(\tau, t; \mathcal{H}_t(\Omega))$. Moreover, the solution u in $\mathcal{H}_t(\Omega)$ depends continuously on the initial values.*

Proof. Using the spectral theory, we conclude that there exists a sequence $\{\omega_j\}_{j=1}^\infty$ of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, which is a Hilbert basis of $L^2(\Omega)$. Fix an integer j, then $\{\omega_1, \omega_2, \cdots, \omega_j\}$ are j linearly independent functions in $H_0^1(\Omega)$, which can be expanded into a j-dimensional linear subspace, denoted as $W_j(\Omega) = \operatorname{span}\{\omega_1, \omega_2, \cdots, \omega_j\}$. The Faedo-Galerkin method needs to find the approximate sequence $u_k(t, \tau; u_\tau) = \sum_{j=1}^{k} r_{k,j}(t)\omega_j(x)$, so that for any $k \geqslant n$, the following approximate system holds:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}[(u_k(t),\omega_j)+\varepsilon(t)(\nabla u_k(t),\nabla\omega_j)]+(2a(l(u_k))-\varepsilon'(t))(\nabla u_k(t),\nabla\omega_j)\\ =2(f(u_k(t)),\omega_j)+2\langle h(t),\omega_j\rangle, \quad \forall u_k\in W_j(\Omega),\\ u_{k,\tau}(x)=u_\tau(x).\end{cases} \tag{3.3}$$

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Step 1: (A priori estimate) In order to ensure that for any $t \in [\tau, +\infty)$, there is a weak solution u_k of the system (3.3), therefore a priori estimate needs to be established. Taking $\gamma_{k,j}(t)$ as the test function of the above approximate system, and then summing j from 1 to k, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|u_k(t)\|^2 + \varepsilon(t)\|\nabla u_k(t)\|^2) + \left(2m - \varepsilon'(t)\|\nabla u_k(t)\|^2\right)\\&\quad\leqslant 2\left(f\left(u_k\right), u_k\right) + 2\left(h(t), u_k\right).\end{aligned} \tag{3.4}$$

From (1.5) and the Poincaré inequality, it follows that there is a constant $w \in (0, m\lambda_1)$ such that

$$\left(f\left(u_k\right), u_k\right) \leqslant (\lambda_1 u_k, u_k) \leqslant \frac{w}{\lambda_1}\|\nabla u_k\|^2 + C. \tag{3.5}$$

By the Hölder, Young and Poincaré inequalities, we have

$$(h(t), u_k) \leqslant \|h(t)\|_{-1}\|u_k\|_1 \leqslant \varepsilon\|\nabla u_k\|^2 + c(\varepsilon)\|h(t)\|_{-1}^2, \tag{3.6}$$

where $\varepsilon = m - \frac{w}{\lambda_1}$, $c(\varepsilon) = (2\varepsilon)^{-1}2^{-1} = \frac{\lambda_1}{4(m\lambda_1 - w)}$. Substituting (3.5) and (3.6) into (3.4), we can derive

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_k(t)\|^2 + \varepsilon(t)\|\nabla u_k(t)\|^2) - \varepsilon'(t)\|\nabla u_k(t)\|^2 \leqslant \frac{\lambda_1}{2\left(m\lambda_1 - w\right)}\|h(t)\|_{-1}^2. \tag{3.7}$$

Integrating (3.7) with respect to t in $[\tau, t]$, we deduce that

$$\begin{aligned}&\|u_k(t)\|^2 + \varepsilon(t)\|\nabla u_k(t)\|^2 - \int_\tau^t \varepsilon'(s)\|\nabla u_k(s)\|^2\,\mathrm{d}s\\&\quad\leqslant \|u_k(\tau)\|^2 + \varepsilon(\tau)\|\nabla u_k(\tau)\|^2 + \frac{\lambda_1}{2\left(m\lambda_1 - w\right)}\int_\tau^t \|h(s)\|_{-1}^2\,\mathrm{d}s.\end{aligned}$$

From $\varepsilon'(t) < 0$, we conclude that

$$\|u_k(t)\|^2 + \varepsilon(t)\|\nabla u_k(t)\|^2 \leqslant \|u_k(\tau)\|^2 + \varepsilon(\tau)\|\nabla u_k(\tau)\|^2 + C\int_\tau^t \|h(s)\|_{-1}^2\,\mathrm{d}s. \tag{3.8}$$

It can be seen from (3.8) that for any $t \geqslant \tau$, $\{u_k\}_{k \geqslant n}$ is bounded in $L^\infty(\tau, t; H_t(\Omega)) \cap L^2(\tau, t; H_0^1(\Omega)) \cap L^p(\tau, t; L^p(\Omega))$, hence $\{-a(l(u_n)\Delta u_n\}$ is bounded in $L^2(\tau, t; H^{-1}(\Omega))$. From (1.5), the Hölder inequality and Sobolev

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embedding theorem, we have

$$\int_{\tau}^{t}\int_{\Omega}|f(u_k(s))|^q \,\mathrm{d}x\,\mathrm{d}s \leqslant C\int_{\tau}^{t}\|u_k(s)\|_{L^p(\Omega)}^p\,\mathrm{d}s + C, \tag{3.9}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Therefore, for any $t\geqslant\tau$, we have

$$\{f(u_k)\}_{k\geqslant n} \textit{ is bounded in } L^q(\tau,t;L^q(\Omega)). \tag{3.10}$$

Step 2: (Uniform estimate for the time derivatives) Multiplying the approximate system (3.3) by $\gamma'_{k,j}(t)$ and summing from 1 to k, we obtain that

$$\begin{aligned}\|(u_k(t))_t\|^2 + \varepsilon(t)\|\nabla(u_k(t))_t\|^2 + a(l(u_k))\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_k(t)\| \\ = 2\left(f(u_k)+h(t),(u_k(t))_t\right).\end{aligned} \tag{3.11}$$

Integrating (3.11) in $[\tau,t]$, and then using (1.4), and the boundedness of function f and h, the following equation can be obtained through the similar estimates in step 1:

$$m\|\nabla u_k(t)\|^2 + \int_{\tau}^{t}\|(u_k(s))_s\|^2 + \varepsilon(s)\|\nabla(u_k(s))_s\|^2\,\mathrm{d}s \leqslant m\|\nabla u_k(\tau)\|^2 + C. \tag{3.12}$$

Then it is easy to get that $\{(u_k)_t\}_{k\geqslant n}$ is bounded in $L^\infty(\tau,\tau;\mathcal{H}_t(\Omega))$.

Step 3: (Existence of solutions) From the boundedness of functions $\{u_k\}_{k\geqslant n}$, $\{f(u_k)\}_{k\geqslant n}$, $\{(u_k)_t\}_{k\geqslant n}$ and lemma 2.1, we can deduce that for any $t\geqslant\tau$, there exist functions $u\in L^\infty(\tau,t;H_t(\Omega))\cap L^2(\tau,t;H_0^1(\Omega))\cap L^p(\tau,t;L^p(\Omega))$, $u_t\in L^\infty(\tau,\tau;\mathcal{H}_t(\Omega))$, $\xi\in L^2(\tau,t;H_0^1(\Omega))$ and $\chi\in L^q(\tau,t;L^q(\Omega))$ such that

$$u_k \rightharpoonup u \quad \text{weakly-star in } L^\infty(\tau,t;\mathcal{H}_t(\Omega)); \tag{3.13}$$

$$a(l(u_k))u_k \rightharpoonup \xi \quad \text{weakly in } L^2\left(\tau,t;H_0^1(\Omega)\right); \tag{3.14}$$

$$f(u_k) \rightharpoonup \chi \quad \text{weakly in } L^q(\tau,t;L^q(\Omega)); \tag{3.15}$$

$$(u_k)_t \rightharpoonup u_t \quad \text{weakly in } L^2(\tau,t;\mathcal{H}_t(\Omega)); \tag{3.16}$$

$$u_k \to u \quad \text{strongly in } L^2\left(\tau,t;L^2(\Omega)\right); \tag{3.17}$$

$$u_k \to u \quad \textit{a.e.}\ (x,t)\in\Omega\times[\tau,+\infty). \tag{3.18}$$

From lemma 2.1, it is easy to prove that $a(l(u))u=\xi$ and $f(u)=\chi$. Then we can conclude that

$$f(u_k)\to f(u) \quad \text{a. e. in } \Omega\times[\tau,+\infty).$$

When $W_j(\Omega)$ is dense in $H_0^1(\Omega)$, then from the convergence obtained above, we can see that u is a weak solution of problem (1.1), while when $W_j(\Omega)$ is not dense in $H_0^1(\Omega)$, this more general case will be demonstrated below. Let u_l be a weak solution of problem (1.1). Now estimating the energy of u_k and u_l respectively, and

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then using (1.6) and the Poincaré, Hölder inequalities, and the Sobolev embedding theorem, we can obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|u_k-u_l\|^2+\varepsilon(t)\|u_k-u_l\|_1^2\right]+(2al(u))-\varepsilon'(t)\|u_k-u_l\|_1^2\\
&\quad\leqslant 2\left|\left(f(u_k)-f(u_l),u_k-u_l\right)\right|\\
&\quad= 2\left|\left(f'(\theta u_k+(1-\theta)u_l)(u_k-u_l),u_k-u_l\right)\right|\\
&\quad\leqslant C\|u_k-u_l\|^2\\
&\quad\leqslant C\|u_k-u_l\|_1^2.
\end{aligned}\tag{3.19}$$

Moreover, from $\varepsilon'(t)\leqslant 0$ and noting that $a(\cdot)$ is a positive bounded function, we can obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|u_k-u_l\|^2+\varepsilon(t)\|u_k-u_l\|_1^2]\leqslant C\|u_k-u_l\|_1^2.\tag{3.20}$$

Applying the generalized Gronwall lemma (see [**24**–**26**]) to (3.20), we can conclude

$$\|u_k-u_l\|_{\mathcal{H}_t}\leqslant \mathrm{e}^{C(t-\tau)}\|u_k-u_l\|_{\mathcal{H}_t}^2,\tag{3.21}$$

which implies that $\{u_k\}\to\{u_l\}$ in $u\in C([\tau,t];\mathcal{H}_t(\Omega))$, for any $t\geqslant\tau$.

Obviously, u is a weak solution of problem (1.1). □

THEOREM 3.3. *Under the assumptions of theorem 3.2, if the weak solution of problem (1.1) exists, then it is a unique solution.*

Proof. Assuming that the solutions corresponding to the initial values $u_{\tau,1}$ and $u_{\tau,2}$ are u_1 and u_2, respectively, they satisfy the following equations, respectively:

$$\begin{cases}(u_1)_t-\varepsilon(t)\Delta(u_1)_t-a(l(u_1))\Delta u_1=f(u_1)+h(t) & \text{in } \Omega\times(\tau,\infty),\\ u_1=0 & \text{on } \partial\Omega\times(\tau,\infty),\\ u_1(x,\tau)=u_{\tau,1}(x), & x\in\Omega,\end{cases}\tag{3.22}$$

and

$$\begin{cases}(u_2)_t-\varepsilon(t)\Delta(u_2)_t-a(l(u_2))\Delta u_2=f(u_2)+h(t) & \text{in } \Omega\times(\tau,\infty),\\ u_2=0 & \text{on } \partial\Omega\times(\tau,\infty),\\ u_2(x,\tau)=u_{\tau,2}(x), & x\in\Omega.\end{cases}\tag{3.23}$$

Subtracting (3.22) from (3.23), and setting $u=u_1-u_2$, then u satisfies

$$\begin{cases}u_t-\varepsilon(t)\Delta u_t-a\left(l\left(u_1\right)\right)\Delta u_1+a\left(l\left(u_2\right)\right)\Delta u_2 & \\ \quad=f\left(u_1\right)-f\left(u_2\right) & \text{in } \Omega\times(\tau,\infty)\\ u(x,t)=0 & \text{on } \partial\Omega\times(\tau,\infty)\\ u(x,\tau)=u_{\tau,1}-u_{\tau,2}, & x\in\Omega.\end{cases}\tag{3.24}$$

Taking $u=u_1-u_2$ as a test function of (3.24), then we can obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|u\|^2+\varepsilon(t)\|\nabla u\|^2\right]+\left(2a\left(l\left(u_1\right)\right)-\varepsilon'(t)\right)\|\nabla u\|^2\\
&\quad=2\left[a\left(l\left(u_2\right)\right)-a\left(l\left(u_1\right)\right)\right]\left(\nabla u_2,\nabla u\right)+2\left(f\left(u_1\right)-f\left(u_2\right),u\right).
\end{aligned}\tag{3.25}$$

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Since $l \in L^2(\Omega)$ is a bounded linear functional and by the Hölder inequality, it is easy to get

$$l(u_1) - l(u_2) = l(u_1 - u_2) = (l, u_1 - u_2) \leqslant \|l\|_2 \|u\|_2. \tag{3.26}$$

Besides, since $a(\cdot)$ is a locally Lipschitz continuous function, when $l(u_1), l(u_2) \in [0, C_1]$, we have

$$|a(l(u_2)) - a(l(u_1))| \leqslant \mathrm{Lip}(C_1) \|l\|_2 \|u\|_2, \tag{3.27}$$

where $Lip(C_1)$ is the Lipschitz constant of $a(\cdot)$. By (3.26) and (3.27) and the Young inequality, we obtain

$$\begin{aligned} |(a(l(u_2)) - a(l(u_1)))(\nabla u_2, \nabla u)| \leqslant d\|\nabla u\|^2 + \frac{1}{4d}(\mathrm{Lip}(C_1))^2 \\ \times \|l\|^2 \|u\|^2 \|\nabla u_2\|^2. \end{aligned} \tag{3.28}$$

From the proof of theorem 3.2, we deduce that

$$|(f(u_1) - f(u_2), u)| \leqslant c\|\nabla u\|^2. \tag{3.29}$$

In addition, substituting (3.28)–(3.29) into (3.25), then we can obtain the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|u\|^2 + \varepsilon(t)\|\nabla u\|^2\right] \leqslant C\left(\|u\|^2 + \varepsilon(t)\|\nabla u\|^2\right). \tag{3.30}$$

Applying the Gronwall lemma to (3.30), we conclude

$$\|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 \leqslant \mathrm{e}^{C(t-\tau)}\left(\|u_\tau\|^2 + \varepsilon(\tau)\|\nabla u_\tau\|^2\right). \tag{3.31}$$

Consequently, the uniqueness of the solution follows readily. □

4. The time-dependent pullback attractors

In this section, we will verify the existence of the time-dependent pullback attractors in $\mathcal{H}_t$. In order to prove the existence of time-dependent pullback attractors for the process $\{U(t,\tau)\}_{t\geqslant\tau}$, we need to check that the process U is pullback $D_{s,\mathcal{H}_t}$-asymptotically compact, thus we first need to give the following lemma.

Lemma 4.1. *Under the assumptions of theorem 3.2, then for any $t \geqslant \tau$, the solution of problem (1.1) satisfies*

$$\begin{aligned} \|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 \leqslant e^{-\delta(t-\tau)}\|u_\tau\|^2_{\mathcal{H}_t} + \frac{C}{\delta} \\ + \frac{e^{-\delta t}}{2(m - w\lambda_1^{-1}) - \delta}\int_\tau^t e^{\delta s}\|h(s)\|^2_{-1}\,ds, \end{aligned} \tag{4.1}$$

where $0 < \delta < \min\{2(m - w\lambda_1^{-1}), \frac{-\varepsilon'(t)\lambda_1}{1+\lambda_1\varepsilon(t)-\lambda_1}\}$.

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Proof. From the energy equality (3.2) of problem (1.1), it can be seen that the weak solution u satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|u(t)\|^2+\varepsilon(t)\|\nabla u(t)\|^2\right]+(2a(l(u(t)))-\varepsilon'(t))\,\|\nabla u(t)\|^2$$
$$=2(f(u(t)),u(t))+2(h(t),u(t)). \tag{4.2}$$

Using the Young, Cauchy-Schwartz and Poincaré inequalities, from (1.2), (4.2) and $w\in(0,m\lambda_1)$, we can derive

$$\|u(t)\|^2+\varepsilon(t)\|\nabla u(t)\|^2+\delta\left(\|u(t)\|^2+\varepsilon(t)\|\nabla u(t)\|^2\right)$$
$$\leqslant C+\frac{1}{2\left(m-w\lambda_1^{-1}\right)-\delta}\|h(s)\|_{-1}^2. \tag{4.3}$$

Then by the Gronwall lemma, (4.1) follows directly. □

REMARK 4.1. The choice of δ in lemma 4.1 is not unique. For example, if for any $t\geqslant\tau$, $0<\delta<\min\{\frac{-\varepsilon'(t)}{\varepsilon(t)},2(m-w\lambda_1^{-1})\lambda_1\}$ is selected, the solution u of problem (1.1) can be obtained through similar calculations, which satisfies

$$\|u(t)\|^2+\varepsilon(t)\|\nabla u(t)\|^2\leqslant \mathrm{e}^{-\delta(t-\tau)}\,\|u_\tau\|_{\mathcal{H}t}^2+\frac{C}{\delta}$$
$$+\frac{\mathrm{e}^{-\delta t}}{2\left(m-w\lambda^{-1}\right)-\delta\lambda_1^{-1}}\int_\tau^t \mathrm{e}^{\delta s}\|h(s)\|_{-1}^2\,\mathrm{d}s. \tag{4.4}$$

As a consequence, the existence of the time-dependent pullback attractors can be obtained by both (4.3) and (4.4).

LEMMA 4.2. *Under the assumptions of theorem 3.2, moreover if h satisfies that*

$$\int_{-\infty}^t e^{\delta s}\|h(s)\|_{-1}^2 d_s<+\infty, \tag{4.5}$$

for some $0<\delta<\min\{2(m-w\lambda_1^{-1}),\frac{-\varepsilon'(t)\lambda_1}{1+\lambda_1\varepsilon(t)-\lambda_1}\}$. Then the family $\widehat{D}_\delta=\{D_\delta(t):t\in\mathbb{R}\}$ with $D_\delta(t)=\bar{B}_{\mathcal{H}_t}(0,\rho(t))$, the closed ball in $\mathcal{H}_t(\Omega)$ of centre zero and radius $\rho(t)$, where

$$\rho^2(t)=1+\frac{C}{\delta}+\frac{e^{-\delta t}}{2\left(m-w\lambda^{-1}\right)-\delta}\int_\tau^t e^{\delta s}\|h(s)\|_{-1}^2\,ds \tag{4.6}$$

is pullback $\mathcal{D}_{\delta,\mathcal{H}_t}$-absorbing for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ of the solution of the equation (1.1). Moreover, we have $\widehat{D}_\delta\in\mathcal{D}_{\delta,\mathcal{H}_t}$.

Proof. It is clear that $\widehat{D}_\delta$ is pullback $\mathcal{D}_{\delta,\mathcal{H}_t}$-absorbing for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ is an immediate consequence of lemma 4.1. Thanks to (4.5), for any $t\geqslant\tau$, we have $\mathrm{e}^{\delta t}\rho^2(t)\to 0$, as $t\to\infty$. Then, $\widehat{D}_\delta$ belongs to $\mathcal{D}_{\delta,\mathcal{H}_t}$. □

In order to prove the existence of time-dependent pullback attractors for the process $\{U(t,\tau)\}_{t\geqslant\tau}$, we need to check that the process U is pullback $D_{s,\mathcal{H}_t}$-asymptotically compact, which is stated in the next lemma.

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LEMMA 4.3. *Under the assumptions of theorem 3.2, the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *is pullback* $D_{s,\mathcal{H}_t}$*-asymptotically compact in* $\mathcal{H}_t(\Omega)$.

Proof. To prove the lemma, obviously we ought to estimate that for any $u_{\tau_n}\in D_\delta(\tau_n)$, $\tau_n\in(-\infty,t)$, $n\in\mathbb{N}^+$ and $t\in\mathbb{R}$, the sequence $\{U(t,\tau_n)u_{\tau_n}\}_{n=1}^\infty$ is relatively compact in $\mathcal{H}_t(\Omega)$.

To simplify the notation, let $U(t,\tau_n)u_{\tau_n}=u_n(t)$. In the following, we will use the diagonal method to get the compactness of the process $\{U(t,\tau_n)u_{\tau_n}\}_{n=1}^\infty$.

From lemma 4.3, we can conclude that for any $r\geqslant 0$, there is $\tau_r(\widehat{D}_\delta,t)\leqslant t-r$ such that $U(t-r)\widehat{D}_\delta(\tau)\subset D_\delta(t-r)$ for any $\tau\leqslant\tau_r(\widehat{D}_\delta,t)$. Besides, we can also obtain that $\widehat{D}_\delta(t)$ is bounded in $\mathcal{H}_t(\Omega)$.

According to the diagonal parameter method, there is a subsequence $\{u_{\tau_m}\}\subset\{u_{\tau_n}\}$ for any $r\geqslant 0$ such that

$$U\left(t-r,\tau_m\right)u_{\tau_m}\rightharpoonup u_k \quad in\ \mathcal{H}_t(\Omega), \tag{4.7}$$

where $u_k\in D_\delta(t-r)$.

From theorems 3.2–3.3, (1.4) and (1.7), we can conclude that for a fixed interval $[t-r,t]$, the sequence $\{u_n\}$ is bounded in $L^\infty(t-r,t;\mathcal{H}_t(\Omega))$, $\{-a(l(u))\Delta u_n\}$ is bounded in $L^2(t-r,t;H^{-1}(\Omega))$, and $f(u_n)$ is bounded in $L^q(t-r,t;L^q(\Omega))$. Therefore, there is a subsequence $\widetilde{u}_n$ of $\{u_n\}_{n=1}^\infty$ that belongs to $L^\infty(t-r,t;\mathcal{H}_t(\Omega))$ and satisfies

$$\begin{aligned} &u_m\rightharpoonup\tilde{u}_n \quad \text{weakly-star in } L^\infty\left(t-r,t;\mathcal{H}_t(\Omega)\right);\\ &u_m\rightharpoonup\tilde{u}_n \quad \text{weakly in } L^2\left(t-r,t;\mathcal{H}_t(\Omega)\right);\\ &u_m\rightarrow\tilde{u}_n \quad \text{strongly in } L^2\left(t-r,t;L^2(\Omega)\right);\\ &f\left(u_m\right)\rightharpoonup f\left(\tilde{u}_n\right) \quad \text{weakly in } L^q\left(t-r,t;L^q(\Omega)\right). \end{aligned} \tag{4.8}$$

Furthermore, from (4.7) and (4.8), we can conclude that

$$\tilde{u}_n=U(t,t-r)u_r, \tag{4.9}$$

$$U(t,t-r)U\left(t-r,\tau_m\right)u_{\tau_m}\rightharpoonup U(t,t-r)u_r \quad \text{in } L^2\left(t-r,t;\mathcal{H}_t(\Omega)\right), \tag{4.10}$$

and

$$U(t,t-r)U\left(t-r,\tau_m\right)u_{\tau_m}\rightarrow U(t,t-r)u_r \quad \text{in } L^2\left(t-r,t;L^2(\Omega)\right). \tag{4.11}$$

Thanks to (4.8), we have

$$\|\tilde{u}_n\|\leqslant\lim_{m\to\infty} inf\left\|U\left(t,\tau_m\right)u_{\tau_m}\right\|. \tag{4.12}$$

In order to prove the lemma, we only need to check that

$$\lim_{m\to\infty} sup\left\|U\left(t,\tau_m\right)u_{\tau_m}\right\|\leqslant\|\tilde{u}_n\|. \tag{4.13}$$

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Multiplying both sides of the energy equation (3.2) by $\mathrm{e}^{\delta t}$, and then integrating it on $[t-r, t]$, we have

$$\begin{aligned} \|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 = {} & \mathrm{e}^{-\delta r}\left(\|u(t-r)\|^2 + \varepsilon(t)\|\nabla u(t-r)\|^2\right) \\ & + \delta \int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}\|u(s)\|^2 \,\mathrm{d}s \\ & + \delta \int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}\varepsilon(s)\|\nabla u(s)\|^2 \,\mathrm{d}s \\ & - \int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}\left(2a(l(u(s))) - \varepsilon'(s)\right)\|\nabla u(s)\|^2 \,\mathrm{d}s \\ & + 2\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}(f(u(s)), u(s)\,\mathrm{d}s \\ & + 2\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}(h(s), u(s))\,\mathrm{d}s. \end{aligned} \tag{4.14}$$

Furthermore, from (4.14) and definition 2.2, for any solution $U(t,\tau_m)u_{\tau_m} \leqslant t-r$ with $\tau_m \leqslant t-r$, it is easy to check that

$$\begin{aligned} & \left\|U(t,\tau_m)u_{\tau_m}\right\|^2 + \varepsilon(t)\left\|\nabla U(t,\tau_m)u_{\tau_m}\right\|^2 \\ & \quad = \mathrm{e}^{-\delta t}\left(\left\|U(t-r,\tau_m)u_{\tau_m}\right\|^2 + \varepsilon(t)\left\|\nabla U(t-r,\tau_m)u_{\tau_m}\right\|^2\right) \\ & \qquad + \delta\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}\left\|U(s,t-r)U(t-r,\tau_m)u_{\tau_m}\right\|^2 \,\mathrm{d}s \\ & \qquad + \delta\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}\varepsilon(s)\left\|\nabla U(s,t-r)U(t-r,\tau_m)u_{\tau_m}\right\|^2 \,\mathrm{d}s \\ & \qquad - \int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}\left(2a\left(l\left(U(s,\tau_m)u_{\tau_m}\right)\right) - \varepsilon'(s)\right)\left\|\nabla U(s,\tau_m)u_{\tau_m}\right\|^2 \,\mathrm{d}s \\ & \qquad + 2\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}\left(f\left(U(s,\tau_m)u_{\tau_m}\right), U(s,\tau_m)u_{\tau_m}\right)\,\mathrm{d}s \\ & \qquad + 2\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}(h(s), U(s,t-r)U(t-r,\tau_m)u_{\tau_m})\,\mathrm{d}s. \end{aligned} \tag{4.15}$$

By (4.11), we have

$$\begin{aligned} & \lim_{m\to\infty} \delta\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}\left\|U(s,t-r)U(t-r,\tau_m)u_{\tau_m}\right\|^2 \,\mathrm{d}s \\ & \quad = \delta\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}\left\|U(s,t-r)u_r\right\|^2 \,\mathrm{d}s, \end{aligned} \tag{4.16}$$

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and

$$\begin{aligned}
&\lim_{m\to\infty} 2\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)} \left(f\left(U\left(s,\tau_m\right)u_{\tau_m}\right), U\left(s,\tau_m\right)u_{\tau_m}\right) \mathrm{d}s \\
&\quad = 2\lim_{m\to\infty}\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}(f\left(U(s,t-r)U\left(t-r,\tau_m\right)u_{\tau_m}\right), \\
&\quad\quad \times U(s,t-r)U\left(t-r,\tau_m\right)u_{\tau_m}) \,\mathrm{d}s \\
&\quad = 2\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}(f(U(s,t-r)u_r), U(s,t-r)u_r)\,\mathrm{d}s
\end{aligned} \tag{4.17}$$

According to (1.4) and (4.10), we obtain that

$$\begin{aligned}
&\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)} \left(2a\left(l\left(U\left(s,\tau_m\right)u_{\tau_m}\right)\right) - \varepsilon'(s) - \delta\varepsilon(s)\right) \left\|\nabla U\left(s,\tau_m\right)u_{\tau_m}\right\|^2 \,\mathrm{d}s \\
&\quad \leqslant \lim_{m\to\infty} inf \int_{t-r}^{t} \mathrm{e}^{\delta(s-t)} \left(2a\left(l\left(U\left(s,\tau_m\right)u_{\tau_m}\right)\right) - \varepsilon'(s) - \delta\varepsilon(s)\right) \\
&\quad\quad \times \left\|\nabla U(s,t-r)U\left(t-r,\tau_m\right)u_{\tau_m}\right\|^2 \,\mathrm{d}s \\
&\quad = \int_{t-r}^{t} \mathrm{e}^{\delta(s-t)} \left(2a\left(l\left(U(s,t-r)u_r\right)\right) - \varepsilon'(s) - \delta\varepsilon(s)\right) \left\|\nabla U(s,t-r)u_r\right\|^2 \,\mathrm{d}s.
\end{aligned} \tag{4.18}$$

By $\mathrm{e}^{\delta(s-t)}h(s) \in L^2(t-r,t;H^{-1}(\Omega))$ and (4.10), we have

$$\begin{aligned}
&\lim_{m\to\infty} 2\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)} \left(h(s), U(s,t-r)U\left(t-r,\tau_m\right)u_{\tau_m}\right) \mathrm{d}s \\
&\quad = 2\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)} \left(h(s), U(s,t-r)u_r\right) \,\mathrm{d}s.
\end{aligned} \tag{4.19}$$

Using lemmas 4.1 and 4.2, we deduce that

$$\mathrm{e}^{-\delta r}\left(\left\|U\left(t-r,\tau_m\right)u_{\tau_m}\right\|^2 + \varepsilon(t)\left\|\nabla U\left(t-r,\tau_m\right)u_{\tau_m}\right\|^2\right) \leqslant \mathrm{e}^{-\delta r}\rho^2(t-r) \tag{4.20}$$

Taking (4.16)–(4.20) into (4.15), we have

$$\begin{aligned}
&\|\tilde{u}_n\|^2 + \varepsilon(t)\lim_{m\to\infty} sup\left\|\nabla U\left(t,\tau_m\right)u_{\tau_m}\right\|^2 \\
&\quad \leqslant \mathrm{e}^{-\delta r}\rho^2(t-r) + \delta\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)} \left\|U(s,t-r)u_r\right\|^2 \,\mathrm{d}s \\
&\quad\quad - \int_{t-r}^{t} \mathrm{e}^{\delta(s-t)} \left(2a\left(l\left(U(s,t-r)u_r\right)\right) - \varepsilon'(s) - \delta\varepsilon(s)\right) \left\|\nabla U(s,t-r)u_r\right\|^2 \,\mathrm{d}s \\
&\quad\quad + 2\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)}(f(U(s,t-r)u_r, U(s,t-r)u_r)\,\mathrm{d}s \\
&\quad\quad + 2\int_{t-r}^{t} \mathrm{e}^{\delta(s-t)} \left(h(s), U(s,t-r)u_r\right) \,\mathrm{d}s.
\end{aligned} \tag{4.21}$$

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In addition, substituting (4.9) into the equation (1.1) and performing a calculation similar to (4.14), we can deduce that

$$\begin{aligned}\|\tilde{u}_n\|^2+\varepsilon(t)\|\nabla\tilde{u}_n\|^2 \leqslant\ & \mathrm{e}^{-\delta r}\left(\|u_r\|^2+\varepsilon(t)\|\nabla u_r\|^2\right)\\ &+\delta\int_{t-r}^{t}\mathrm{e}^{\delta(s-t)}\|U(s,t-r)u_r\|^2\,\mathrm{d}s\\ &-\int_{t-r}^{t}\mathrm{e}^{\delta(s-t)}\left(2a\left(l\left(U(s,t-r)u_r\right)\right)-\varepsilon'(s)-\delta\varepsilon(s)\right)\\ &\times\|\nabla U(s,t-r)u_r\|^2\,\mathrm{d}s\\ &+2\int_{t-r}^{t}\mathrm{e}^{\delta(s-t)}\left(f\left(U(s,t-r)u_r\right),U(s,t-r)u_r\right)\mathrm{d}s\\ &+2\int_{t-r}^{t}\mathrm{e}^{\delta(s-t)}\left(h(s),U(s,t-r)u_r\right)\mathrm{d}s. \end{aligned} \tag{4.22}$$

Furthermore, by (4.21), (4.22) and the Poincaré inequality, we have

$$\lim_{m\to\infty} sup\left\|U\left(t,\tau_m\right)u_{\tau_m}\right\|^2 \leqslant \mathrm{e}^{-\delta r}\lambda_1^{-1}\rho^2(t-r)+\|\tilde{u}_n\|^2.$$

Obviously, $\lim_{r\to\infty}\mathrm{e}^{-\delta r}\lambda_1^{-1}\rho^2(t-r)=0$ can be obtained from lemma 4.2, therefore (4.13) holds. From (4.12) and (4.13), we have

$$\lim_{m\to\infty} inf\left\|U\left(t,\tau_m\right)u_{\tau_m}\right\| = \lim_{m\to\infty} sup\left\|U\left(t,\tau_m\right)u_{\tau_m}\right\| = \|\tilde{u}_n\|,$$

which implies the process U converges to $\tilde{u}_n$, thus we can obtain that process U is relatively compact. Hence, the proof is complete. □

THEOREM 4.4. *Under the assumptions of theorem 3.2, then from the lemmas in this section, we can conclude that the process of problem (1.1) exists time-dependent pullback attractors $\{\widehat{\mathcal{A}}_t\}_{t\in\mathbb{R}}$, which satisfy nonempty, compact, invariant and pullback attracting.*

5. Regularity of the attractors

In this section, we shall establish the regularity of time-dependent pullback attractors for non-autonomous system (1.1). The methods used in the proof process can also be seen in [**13**, **31**, **32**].

THEOREM 5.1. *Under the assumptions of theorem 3.2, then $\{\widehat{\mathcal{A}}_t\}_{t\in\mathbb{R}}$ is bounded in $\mathcal{H}_t^1(\Omega)$.*

Proof. Since $L^2(\Omega)\subset H^{-1}(\Omega)$ is dense, for any h, there exists a function $h^\xi\in L^2(\Omega)$ such that

$$\|h-h^\xi\|<\xi, \tag{5.1}$$

where $\xi\geqslant 0$ is a constant.

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Fix $\tau\in\mathbb{R}$, suppose $u_\tau\in\{\widehat{\mathcal{A}}_t\}_{t\in\mathbb{R}}$, decompose $U(t,\tau)u_\tau=u(t)$ into $U(t,\tau)u_\tau=U_1(t,\tau)u_\tau+U_2(t,\tau)u_\tau$, where $U_1(t,\tau)u_\tau=v(t)$ and $U_2(t,\tau)u_\tau=g(t)$ satisfy the following two equations respectively,

$$\begin{cases} v_t-\varepsilon(t)\Delta v_t-a\left(l(u)\right)\Delta v=h(t)-h^{\xi}(t) & \text{in } \Omega\times(\tau,\infty),\\ v=0 & \text{on } \partial\Omega\times(\tau,\infty),\\ v(x,\tau)=u_\tau(x), & x\in\Omega,\end{cases} \tag{5.2}$$

and

$$\begin{cases} g_t-\varepsilon(t)\Delta g_t-a\left(l(u)\right)\Delta g=f(u)+h^{\xi}(t) & \text{in } \Omega\times(\tau,\infty),\\ g=0 & \text{on } \partial\Omega\times(\tau,\infty),\\ g(x,\tau)=0, & x\in\Omega.\end{cases} \tag{5.3}$$

Multiplying (5.2)$_1$ by $-\Delta v$ and integrating it in Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|^2+\varepsilon(t)\|\Delta v\|^2\right)+\left(2a(l(u))-\varepsilon'(t)\right)\|\Delta v\|^2=2\left(h(t)-h^{\xi}(t),-\Delta v\right).$$

By (1.4), (5.1), the Cauchy and Poincaré inequalities, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)+\delta_1E_1(t)\leqslant\xi^2, \tag{5.4}$$

where $E_1(t)=\|\nabla v\|^2+\varepsilon(t)\|\,\Delta v\|^2$ and $0<\delta_1\leqslant\frac{2m-\varepsilon'(t)-1}{\lambda_1^{-1}+\varepsilon(t)}$.

Using the Gronwall lemma, we obtain

$$\|v(t)\|^2_{\mathcal{H}^1_t}=\|U_1(t,\tau)u_\tau\|^2_{\mathcal{H}^1_t}\leqslant \mathrm{e}^{-\delta_1(t-\tau)}\|u_\tau\|^2_{\mathcal{H}^1_t}+\frac{\xi^2}{\delta_1}. \tag{5.5}$$

Then, multiplying (5.3)$_1$ by $-\Delta g$ and integrating it in Ω, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(\|\nabla g\|^2+\varepsilon(t)\|\Delta g\|^2\right)+\left(2a(l(u))-\varepsilon'(t)\right)\|\Delta g\|^2\\ &=2(f(u),-\Delta g)+2\left(h^{\xi}(t),-\Delta g\right).\end{aligned} \tag{5.6}$$

Besides, from (1.5) and the Young inequality, we have

$$2(f(u),-\Delta g)\leqslant 2\lambda_1^2\|u\|^2+\frac{1}{2}\|Ag\|^2, \tag{5.7}$$

and

$$2(h^{\xi}(t),-\Delta g)\leqslant 2\|h^{\xi}(t)\|^2+\frac{1}{2}\|Ag\|^2, \tag{5.8}$$

where $A=-\Delta$.

In addition, taking (5.7), (5.8) into (5.6), and from (1.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t)+\delta_1E_2(t)\leqslant 2\lambda_1^2\|u\|^2+2\|h^{\xi}(t)\|^2, \tag{5.9}$$

where $E_2(t)=\|\nabla g\|^2+\varepsilon(t)\|\,\Delta g\|^2$.

It follows from (4.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) + \delta_1 E_2(t) \leqslant 2\lambda_1^2 R_1 + 2\|h^{\xi}(t)\|^2, \tag{5.10}$$

where $R_1 = \mathrm{e}^{-\delta(t-\tau)}\,\|u_\tau\|^2_{\mathcal{H}_t} + \frac{C}{\delta} + \frac{\mathrm{e}^{-\delta t}}{2(m-w\lambda_1^{-1})-\delta}\int_\tau^t \mathrm{e}^{\delta s}\|h(s)\|^2_{-1}\,\mathrm{d}s$.

By the Gronwall lemma, we conclude that

$$\|g(t)\|^2_{\mathcal{H}_t^1} = \|U_2(t,\tau)u_\tau\|^2_{\mathcal{H}_t^1} \leqslant R_2, \tag{5.11}$$

where $R_2 = \mathrm{e}^{-\delta_1 t}\int_\tau^t \mathrm{e}^{\delta_1 s}(2\lambda_1^2 R_1 + 2\left\|h^{\xi}(s)\right\|^2_{-1})\,\mathrm{d}s$.

Thanks to (5.5) and (5.11), for any $t \in \mathbb{R}$, we can deduce that

$$\operatorname{dist}\left(\mathcal{A}_t, \bar{B}_{\mathcal{H}_t^1}(R_2)\right) = \operatorname{dist}\left(U(t,\tau)\mathcal{A}_\tau, \bar{B}_{\mathcal{H}_t^1}(R_2)\right) \leqslant C\,\mathrm{e}^{-\delta_2(t-\tau)} \to 0, \quad \tau \to -\infty, \tag{5.12}$$

where $\delta_2 > 0$.

As a result, we have $\mathcal{A}_t \subseteq \bar{B}_{\mathcal{H}_t^1}$, which represents the time-dependent pullback attractor $\{\widehat{\mathcal{A}}_t\}_{t\in\mathbb{R}}$ is bounded in $\mathcal{H}_t^1(\Omega)$. □

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