Iterating circum-medial triangles

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1. *Introduction*

When considering 'convergence' many people think of number sequences or contexts arising from calculus. But there are also interesting phenomena of convergence – easy to visualize – arising in elementary geometry. Some of them are so elementary that they can be dealt with at school, for instance an example of iteration that is described in [1, p. 171f] (see also [2, p. 59], [3, p. 222f] and [4, p. 42ff]).

> Given an arbitrary initial triangle $A_0B_0C_0$, the points of tangency with the incircle form the next triangle $A_1B_1C_1$. Continuing this procedure step by step one can observe that the triangles $A_nB_nC_n$ get 'more and more equilateral'.

The convergence of the shape to equilaterality can be shown easily in this case (without the need of matrices and limits of powers of matrices, see [5]). A similar problem is described in [1, p. 173] (also easy to solve, see [5]). Another problem of this type (treated in [1, p. 176ff]; [6]) is much harder to solve even though it may – at the first glance – seem as simple as the ones described above.

In these problems the incircle is involved in the iteration process, and thus the resulting triangles get smaller and smaller at every step (they even become arbitrarily small). When looking at them using Dynamic Geometry Software (DGS) one has to zoom into the sketch deeper and deeper. This disadvantage is avoided if iteration problems are regarded that involve the circumcircle instead of the incircle. We found hardly any references in the literature on this topic. It may well be that there are more; we would highly appreciate hints to corresponding references.

2. *Circum-medial triangles*

Since the circumcircle iterations involving the perpendicular bisectors are really elementary (see [5]) we will turn our attention to the most famous cevians of a triangle, the angle bisectors, the altitudes and the medians. Let us take the medians first.

Example 1: Given an arbitrary triangle $A_0B_0C_0$ with its circumcircle k. We construct the medians of the triangle $A_0B_0C_0$ and intersect them with k. The points of intersection are the points A_1, B_1, C_1, forming the next triangle* $A_1B_1C_1$. Continuing this procedure step by step one can observe that the triangles $A_nB_nC_n$ get 'more and more equilateral'. (A more open formulation: What can be observed looking at the triangles $A_nB_nC_n$?) Is this always the case? Can you give reasons for this phenomenon?

* The triangle $A_1B_1C_1$ is called the 'circumcevian triangle of $A_0B_0C_0$ with respect to the centroid'. Another name for it is 'circum-medial triangle' (see also [7]).

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First we will check the situation with a DGS construction; in Figure 1 three steps are shown.

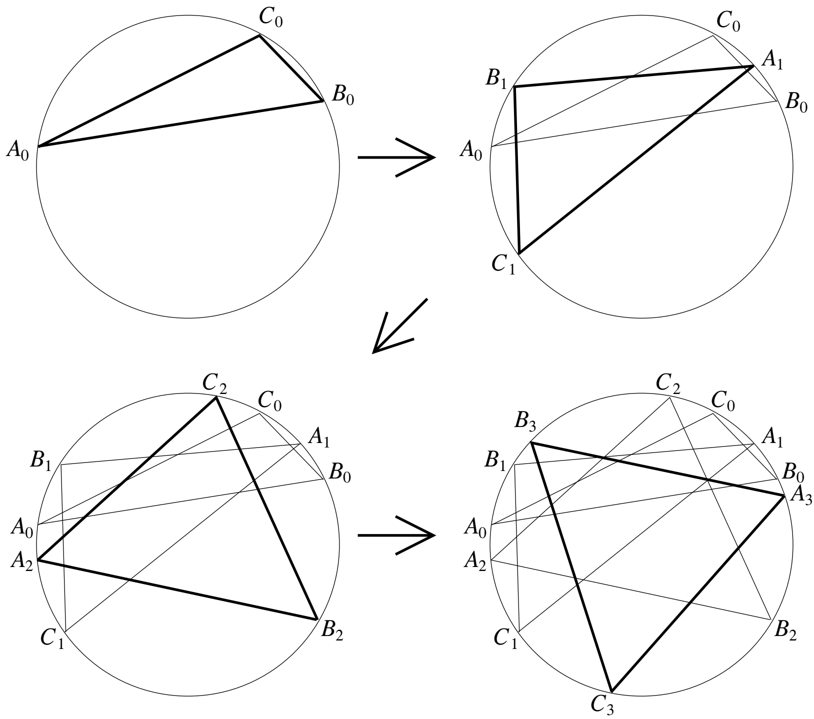

FIGURE 1: Three steps in the iteration process using circum-medial triangles

The shape of the triangles seems to converge to *equilaterality*. But how can we prove that? The method used in [5] (also appropriate for high school – identify the sequence of the 'errors' (deviations of equilaterality) as a geometric sequence with factor $|q| < 1$) is not successful here. Instead, we use the well-known theorem that a monotonic and bounded sequence is convergent. This theorem is commonly met in upper secondary school but not in a sufficiently deep and reflective way; this more advanced treatment will be restricted to students at university.

Here we will work primarily with angles (as in [8]), whereas in [9] we focused on the sides lengths in solving this problem.

With $\alpha_n, \beta_n, \gamma_n$ $(n \geqslant 0)$ we will denote the angles of the triangles after n iteration steps.

We will prove that with $\alpha_0 \leqslant \beta_0 \leqslant \gamma_0$ all three new angles $\alpha_1, \beta_1, \gamma_1$ are not smaller than α_0. Let $m_n = \min\{\alpha_n, \beta_n, \gamma_n\}$ $(n = 0, 1, 2, \dots)$ then the following must hold: $m_0 \leqslant m_1$ and in general $m_{n-1} \leqslant m_n$, i.e. the sequence (m_n) is monotonically increasing and bounded above (by 60°).

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Therefore, (m_n) must converge*. Since in case of an equilateral triangle† the angles do not change from one step to the next, it is clear that the shape of the triangles converges to equilaterality (in other words the angles α_n, β_n, γ_n converge to 60°).

Our goal is to prove: $\alpha_0 \leqslant \beta_0 \leqslant \gamma_0 \Rightarrow \alpha_1, \beta_1, \gamma_1 \geqslant \alpha_0$.

We assume $\alpha_0 \leqslant \beta_0 \leqslant \gamma_0$ or equivalently $a_0 \leqslant b_0 \leqslant c_0$ (initial side lengths). The angles $\mu_{1,2}$, $\nu_{1,2}$, $\rho_{1,2}$ are the parts of α_0, β_0, γ_0 that result from the medians (see Figure 2).

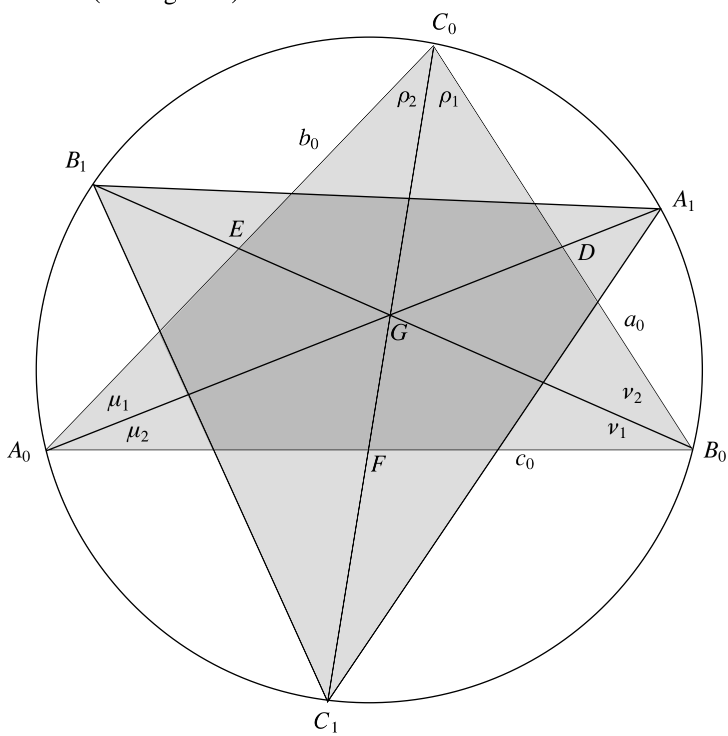

FIGURE 2: The first step $A_0B_0C_0 \rightarrow A_1B_1C_1$

We denote the midpoints of a_0, b_0, c_0 by D, E, F, and the centroid of the initial triangle by G.

From the equality of angles in the same segment we get immediately:

$$\begin{aligned} \alpha_1 &= \nu_1 + \rho_2, \\ \beta_1 &= \rho_1 + \mu_2, \\ \gamma_1 &= \mu_1 + \nu_2. \end{aligned} \tag{1}$$

* We could also work with M_n : max $\{\alpha_n, \beta_n, \gamma_n\}$; then the sequence (M_n) would be monotonically decreasing and bounded below.

† And *only* in this case (see the end of the proof below).

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First the following inequalities hold:

$$\begin{aligned} \mu_1 &\geqslant \mu_2, \\ \nu_1 &\leqslant \nu_2, \\ \rho_1 &\geqslant \rho_2. \end{aligned} \tag{2}$$

One possible explanation for $\rho_1 \geqslant \rho_2$ (the others are analogous): because of $a_0 \leqslant b_0$ the point C_0 lies somewhere in the 'right' part when drawing the perpendicular bisector of A_0B_0, and therefore C_1 is in the 'left' part. Thus we can conclude for the arclengths $\left|\widehat{C_1B_0}\right| \geqslant \left|\widehat{C_1A_0}\right|$ and $\rho_1 \geqslant \rho_2$.

Now we use the well-known formulas for the lengths of the medians (which can be derived by applying the law of cosines twice):

$$|A_0D| = \frac{\sqrt{2(b_0^2 + c_0^2) - a_0^2}}{2}, \quad |B_0E| = \frac{\sqrt{2(a_0^2 + c_0^2) - b_0^2}}{2}, \quad |C_0F| = \frac{\sqrt{2(a_0^2 + b_0^2) - c_0^2}}{2}. \tag{3}$$

From (3) and $a_0 \leqslant b_0 \leqslant c_0$ we can deduce $|C_0F| \leqslant |B_0E| \leqslant |A_0D|$ and by multiplying by the factor $\frac{2}{3}$ we get $|C_0G| \leqslant |B_0G| \leqslant |A_0G|$. This inequality implies (because in a triangle we know $a \leqslant b \Leftrightarrow \alpha \leqslant \beta$):

$$\begin{aligned} \mu_1 &\leqslant \rho_2, \\ \nu_1 &\geqslant \mu_2, \\ \rho_1 &\geqslant \nu_2. \end{aligned} \tag{4}$$

Using (2) and (4) we have:

$$\alpha_1 = \nu_1 + \rho_2 \geqslant \mu_2 + \mu_1 = \alpha_0,$$

$$\beta_1 = \mu_2 + \rho_1 \geqslant \mu_2 + \rho_2 \geqslant \mu_2 + \mu_1 = \alpha_0,$$

$$\gamma_1 = \mu_1 + \nu_2 \geqslant \mu_1 + \nu_1 \geqslant \mu_1 + \mu_2 = \alpha_0.$$

Only in the case of an equilateral triangle does the set of the angles not change from one step to the next (i.e. only the equilateral triangle is a possible 'limiting shape'). We assume $\alpha_0 \leqslant \beta_0 \leqslant \gamma_0$. In the cases $\alpha_0 < \beta_0 = \gamma_0$, $\alpha_0 = \beta_0 < \gamma_0$ and $\alpha_0 < \beta_0 < \gamma_0$ with the same arguments as above we get $\alpha_1, \beta_1, \gamma_1 > \alpha_0$; thus the triangle $A_1B_1C_1$ cannot be congruent to $A_0B_0C_0$, and this completes the proof.

Teaching situations with this topic

As said above, this topic is restricted to students who know that a monotonic and bounded sequence is convergent. This is mostly used in situations that arise from calculus; in geometry there are not so many situations using it, but this is an accessible one. No matter whether students work on the problem in groups or alone, the teacher will need to give several hints because the problem is really not easy. We may first ask students to prove that (m_n) is monotonically increasing, in other words: $\alpha_0 \leqslant \beta_0 \leqslant \gamma_0 \Rightarrow \alpha_1, \beta_1, \gamma_1 \geqslant \alpha_0$. The way of proving this geometrically should be

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guided by leading questions like:

- What can you observe for α_1, β_1, γ_1 as functions of $\mu_{1,2}$, $\nu_{1,2}$, $\rho_{1,2}$? Give arguments for it. This should lead to (1).
- Let us assume $\alpha_0 \leqslant \beta_0 \leqslant \gamma_0$. Can you observe relations between μ_1 and μ_2 (analogously for $\nu_{1,2}$ and $\rho_{1,2}$)? Give your reasons. This should lead to (2).
- What do you observe concerning the angles adjacent to a_0, namely ν_2 and ρ_1 (analogously for the other sides b_0 and c_0)? In order to prove this, look for formulas for the medians and use the fact that the centroid divides the medians in the ratio 2 : 1. This should lead to (3) and (4) and is probably the most difficult step.

Having questions like those leaves enough work to students and gives them a chance to come successfully to a proof. For all three questions students can use the measurement features of a DGS and come to conjectures which should finally be proved.

Remarks

The iteration with the medians is in general not uniquely reversible. Given a first step triangle $A_1B_1C_1$ (not equilateral) there are always *two different* initial triangles $A_0B_0C_0$ (with the same circumcircle as $A_1B_1C_1$) which lead to $A_1B_1C_1$ when constructing the circum-medial triangle. Their centroids are the foci of the 'Steiner inellipse' of the triangle $A_1B_1C_1$. The corresponding considerations need other means (e.g. 'isogonally conjugated points') which are beyond the scope of this article.

We found only two references to that problem, [8] and [10]. In [10] pure algebraic methods (Gröbner bases and the like) are used and the paper deals primarily with the generalisation to dimension 3.

Further unsolved (?) *mathematical questions*

- Are there other cevians (except angle bisectors and medians) that lead to equilaterality in the iteration process involving the circumcircle (convergence in shape)?
- Are there other interesting phenomena (not necessarily convergence in shape to equilaterality) using cevians? We give just one example (without proof): If you take the so-called 'symmedians' then $A_2B_2C_2 = A_0B_0C_0$ i.e. the sequence of the triangles is a 2-cycle (see [11, p. 77]).

What about the angle bisectors instead of the medians?

This problem turns out to be an easier one, very similar to the problems in [5].

Example 2: Given an arbitrary triangle $A_0B_0C_0$ with its circumcircle k. We construct the angle bisectors of the triangle $A_0B_0C_0$ and intersect them with k. The points of intersection are the points A_1, B_1, C_1 and they form the next triangle* $A_1B_1C_1$ (see Figure 3). Continuing this procedure step by step one can observe that the triangles $A_nB_nC_n$ get 'more and more equilateral'. (A more open formulation: What can be observed looking at the shapes of the triangles $A_nB_nC_n$?) Is this always the case? Can you give reasons for this phenomenon?

The triangle shapes seem to converge to equilaterality as one can see doing DGS experiments (Figure 3).

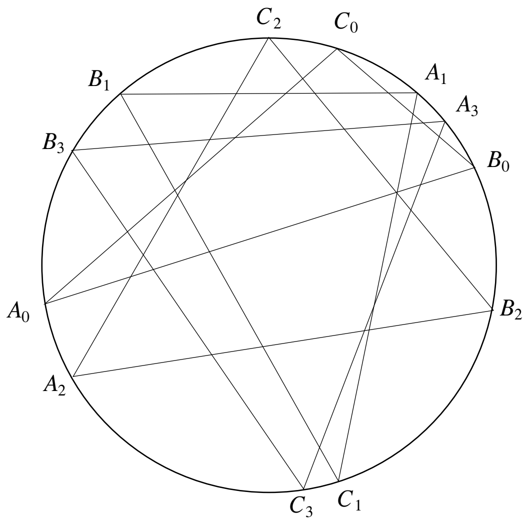

FIGURE 3: Three steps in the iteration process

Idea of the proof: Using the equality of angles in the same segment we get: $\alpha_1 = \frac{1}{2}(\beta_0 + \gamma_0)$, $\beta_1 = \frac{1}{2}(\gamma_0 + \alpha_0)$, $\gamma_1 = \frac{1}{2}(\alpha_0 + \beta_0)$ (see Figure 4). Of course, analogous relations hold for all transitions $n \to n + 1$:

$$\alpha_{n+1} = \frac{\beta_n + \gamma_n}{2}, \qquad \beta_{n+1} = \frac{\gamma_n + \alpha_n}{2}, \qquad \gamma_{n+1} = \frac{a_n + \beta_n}{2}.$$

Here one immediately recognises that the new values are the (pairwise) arithmetic means of the old ones; this fact can be used for an easy proof of the convergence (either on an intuitive level or a more formal one, see [5], where the equivalent case of perpendicular bisectors is treated).

* The triangle $A_1B_1C_1$ is called the 'circumcevian triangle of $A_0B_0C_0$ with respect to the incentre'.

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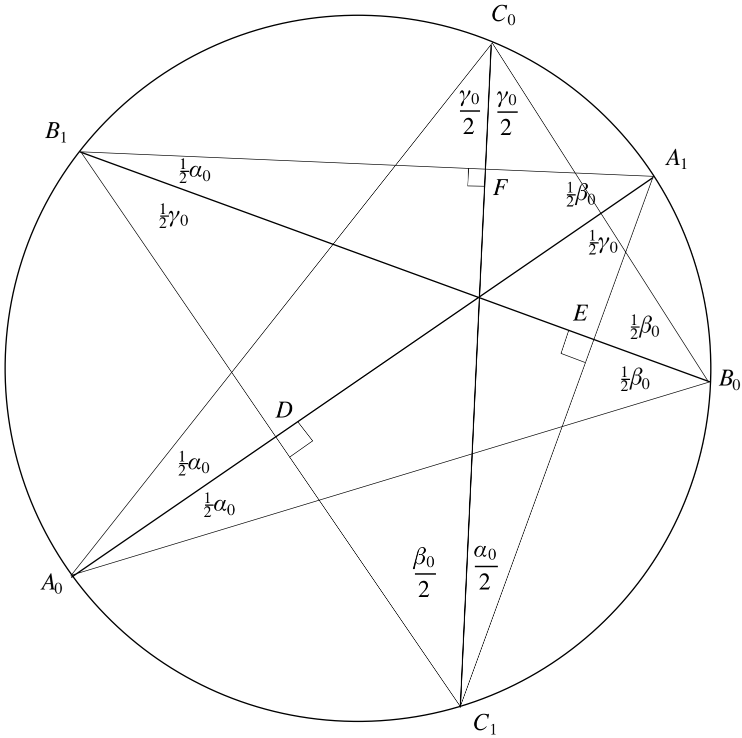

FIGURE 4: The first iteration step using angle bisectors

Remarks:

- It turns out that this iteration with the angle bisectors is actually the same as using the perpendicular bisectors (because the angle bisectors of a triangle intersect the circumcircle in the same point as the perpendicular bisectors of the opposite side). Therefore, we have convergence of the triangle shapes in all three cases: medians, angle bisectors, perpendicular bisectors.
  The use of perpendicular bisectors or angle bisectors in case of a triangle iteration is a special version of a more general result (cf. [12]).
- We cannot expect convergence of the shapes using the altitudes for the iteration process because this iteration turns out to be the same as the iteration using the angle bisectors but reversed. Since we had convergence of the shape to equilaterality in the case of the angle bisectors it is obvious that we will not have convergence in the general case using the altitudes (except for some special initial triangles).

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References

1. D. Ismailescu, J. Jacobs, On sequences of nested triangles, *Periodica Mathematica Hungarica*, **53** (1–2) (2006) pp. 169-184.
2. Stephen Jones, Two iteration examples, *Math. Gaz.* **74** (March 1990) pp. 58-62.
3. S. Abbott, Averaging sequences and triangles, *Math. Gaz.* **80** (March 1996) pp. 222-224.
4. M. De Villiers, Over and over again: two geometric iterations with triangles, *Learning & Teaching Mathematics* **16** (July 2014) pp. 40-45, also available at http://dynamicmathematicslearning.com/geometric-iteration-examples.pdf
5. H. Humenberger, F. Embacher, Convergence with respect to triangle shapes – elementary geometric iterations, *Learning & Teaching Mathematics* **25** (December 2018) pp. 15-19.
6. S. Y. Trimble, The limiting case of triangles formed by angle bisectors, *Math. Gaz.* **80** (November 1996) pp. 554-556.
7. G. Leversha, *The geometry of the triangle*, United Kingdom Mathematics Trust (2013).
8. G. Bourgeois, J.-P. Lechêne, Etude d'une itération en géométrie du triangle, *Bulletin de l'APMEP* **1**(409) (1997) pp. 147-154 also available at https://www.apmep.fr/IMG/pdf/AAA97013.pdf
9. H. Humenberger, F. Embacher, Konvergenz bei Dreiecken – interessante geometrische Iterationen, *Mathematische Semesterberichte* 65 (2) (2018) pp. 253-275 also available at http://link.springer.com/article/10.1007/s00591-017-0206-3
10. G. Bourgeois, S. Orange, Dynamical systems of simplices in dimension two or three. In: T. Sturm, Ch. Zengler, (Ed., 2011): *Automated deduction in geometry*. 7th International Workshop, Shanghai, China, September 22-24, 2008. Springer Berlin-Heidelberg, 1-21 also available at https://arxiv.org/pdf/0906.0352.pdf
11. R. Honsberger, *Episodes in nineteenth and twentieth century Euclidean geometry*, The Mathematical Association of America (1995).
12. M. F. Gomez, P. Taslakian, G. T. Toussaint, Convergence of the shadow sequence of inscribed polygons, *18th Fall Workshop on Computational Geometry*, New York, EEUU (2008). http://ac.aua.am/ptaslakian/web/publications/shadowConvergence.pdf

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