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Boundary trace of the solutions of the prescribed Gaussian curvature equation

Michèle Grillot
Laboratoire de Mathématiques et Physique Mathématique,
CNRS UMR 6628, Faculté des Sciences, B.P. 6759,
45067 Orléans cedex 2, France

Laurent Véron
Laboratoire de Mathématiques et Physique Théorique,
CNRS UPRES-A 6083, Faculté des Sciences et Techniques,
Parc de Grandmont, 37200 Tours, France

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We study the existence of a boundary trace for minorized solutions of the equation $\Delta u + K(x)e^{2u} = 0$ in the unit open ball B^2 of $\mathbb{R}^2$. We prove that this trace is an outer regular Borel measure on ∂B^2, not necessarily a Radon measure. We give sufficient conditions on Borel measures on ∂B^2 so that they are the boundary trace of a solution of the above equation. We also give boundary removability results in terms of generalized Bessel capacities.

1. Introduction

If one identifies the hyperbolic space $\mathbb{H}^2$ with (B^2, g_H), where $B^2 = \{x \in \mathbb{R}^2 : |x| < 1\}$,

$$g_H = 4g_0/(1 - |x|^2) \quad \text{and} \quad (g_0)_{i,j} = \delta_{i,j} \qquad (1 \leqslant i, j \leqslant 2), \tag{1.1}$$

the expression of the Gaussian curvature K_u of a metric $g_u = e^{2u} g_0$ conformal to the standard one in $\mathbb{H}^2$ is given by

$$K_u(x) = -e^{-2u(x)} \Delta u(x) \qquad (\forall x \in B^2), \tag{1.2}$$

in which formula $\Delta u = \partial^2_{x_1} u + \partial^2_{x_2} u$. Consequently, the problem of finding a metric in $\mathbb{H}^2$ conformal to the standard one, with prescribed Gaussian curvature K is reduced to study the following nonlinear elliptic equation in B^2:

$$\Delta u + K(x)e^{2u} = 0. \tag{1.3}$$

This equation has been studied for a long time and much is known about the existence or non-existence of solutions (see, for example, [1, 9, 10, 15] or [23]). Those questions are deeply related to the sign of K.

In this article we investigate this equation under the completely different point of view of describing all the possible boundary behaviour of the solutions of (1.3).

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This type of problems with different equations has first been introduced by Le Gall [12,13] in the probabilistic framework of the two-dimensional superprocesses. This leads to the following equation

$$\Delta u - u^2 = 0 \tag{1.4}$$

in B^2, and Marcus and Véron [16,17] for the more general equation

$$\Delta u - u^q = 0 \tag{1.5}$$

($q > 1$) in the d-dimensional ball B^d. Concerning related problems much work has also been done by Dynkin [6] and Dynkin and Kuznetzov [7] in the probabilistic framework of the study of branching processes which lead to d-dimensional equations of type (1.5) in which $1 < q \leqslant 2$. The starting point of Marcus and Véron's work [16,17,19,20] is settled upon the existence of a boundary trace for any positive solution of (1.5) in B^d and the fact that this boundary trace is represented by a positive, outer regular Borel measure on ∂B^d, not necessarily a Radon measure. Moreover, they construct positive solutions of (1.5) with given boundary data in the class of positive and (outer) regular Borel measures on ∂B^d, with no condition when $1 < q < (d+1)/(d-1)$, and some compatibility condition when $q \geqslant (d+1)/(d-1)$.

The present work deals with the extension of some of Marcus and Véron's results to the study of boundary trace for solutions of (1.3) in B^2 which are bounded from below by some constant. Let (r, θ) be the polar coordinates in $\mathbb{R}^2\setminus\{0\}$, with S^1 identified with ∂B^2. In the first section no assumption on the sign of the curvature function K is made. We prove that if K is continuous in $\overline{B^2}$, u is a solution in B^2 non-negative near ∂B^2 for the sake of simplicity, and $\omega^+ = \{x \in \partial B^2 : K(x) > 0\}$ (respectively $\omega^- = \{x \in \partial B^2 : K(x) < 0\}$), then the following situation occurs.

I. *For any $\theta \in \omega^-$, either*

(i) *there exists a relatively open neighbourhood U of θ such that, for every $\zeta \in C_0^\infty(U)$*

$$\lim_{r\to 1} \int_U u(r,\sigma)\zeta(\sigma)\,\mathrm{d}\sigma = \ell(\zeta), \tag{1.6}$$

where ℓ is a positive linear functional on $C_0^\infty(U)$, or,

(ii) *for every relatively open neighbourhood U of θ, there holds*

$$\lim_{r\to 1} \int_U u(r,\sigma)\,\mathrm{d}\sigma = \infty. \tag{1.7}$$

II. *For any $\theta \in \omega^+$ there exists a relatively open neighbourhood U of θ such that (1.6) holds for every $\zeta \in C_0^\infty(U)$, where ℓ is a positive linear functional on $C_0^\infty(U)$.*

If $\theta \in \partial B^2$, we shall say that θ is *regular* with respect to u either if statement (I(i)) holds or if $\theta \in \omega^+$. The set of *regular points* is relatively open in ∂B^2 and we denote it by $\mathcal{R}$. As in [12] or [16], there exists a positive Radon measure μ on $\mathcal{R}$ such that

$$\lim_{r\to 1} \int_{\mathcal{R}} u(r,\sigma)\zeta(\sigma)\,\mathrm{d}\sigma = \int_{\mathcal{R}} \zeta(\sigma)\,\mathrm{d}\mu(\sigma) \tag{1.8}$$

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for every $\zeta \in C_0^\infty(\mathcal{R})$. The set of $\theta \in \partial B^2$ such that statement (I(ii)) holds, that we denote $\mathcal{S}$, is called the *singular set* of u and we have

$$\partial B^2 = \mathcal{R} \cup \mathcal{S} \cup \omega^0, \tag{1.9}$$

where $\omega^0 = \{x \in \partial B^2 : K(x) = 0\}$. We denote $\mathrm{tr}_{|\partial B^2}(u) = (\mathcal{S}, \omega^0, \mu)$.

In the next section it is supposed that K is negative in $\overline{B^2}$ (and always continuous) and therefore the boundary trace is described by the set $\mathcal{S}$ and the Radon measure μ on $\mathcal{R} = \partial B^2 \backslash \mathcal{S}$. It is known from Gmira and Véron [8] that isolated boundary points are removable singularities for (1.3), and therefore any too concentrated measures cannot be the regular part of the boundary trace of a bounded from below solution of (1.3) in B^2. Let P be the Poisson kernel in B^2, μ a positive Radon measure on S^1 with Lebesgue decomposition

$$\mu = \mu_{\mathrm{R}}\, \mathrm{d}H^1 + \mu_{\mathrm{S}}, \tag{1.10}$$

where μ_{R} is the regular part with respect to the one-dimensional Hausdorff measure $\mathrm{d}H^1$ and μ_{S} the singular part, and let $P_{\mu_{\mathrm{S}}}$ denote the Poisson potential of μ_{S}; it is defined by

$$P_{\mu_{\mathrm{S}}}(x) = \int_{\partial B^2} P(x, y)\, \mathrm{d}\mu_{\mathrm{S}}(y) \qquad (\forall x \in B^2). \tag{1.11}$$

We say that μ is *admissible* for (1.3) if there exists $p \in (1, \infty]$ such that

$$\left.\begin{array}{ll} \text{(i)} & \exp(2P_{\mu_{\mathrm{S}}}) \in L^{p/(p-1)}(B^2, (1 - |x|)\, \mathrm{d}x), \\ \text{(ii)} & \exp(2\mu_{\mathrm{R}}) \in L^{p-1}(\partial B^2). \end{array}\right\} \tag{1.12}$$

With this condition the existence of a unique solution u of (1.3) in B^2 with boundary data μ is proved. By a solution we mean a function $u \in L^1(B^2)$ such that $K\mathrm{e}^{2u} \in L^1(B^2, (1 - |x|)\, \mathrm{d}x)$, which satisfies

$$\int_{B^2} (-u\Delta\zeta - K(x)\mathrm{e}^{2u}\zeta)\, \mathrm{d}x = -\int_{\partial B^2} \frac{\partial \zeta}{\partial n}\, \mathrm{d}\mu, \tag{1.13}$$

for every ζ in the space $C_0^{1,1}(\overline{B^2})$ of C^1-functions in $\overline{B^2}$ which vanish on ∂B^2 and have uniformly Lipschitz gradient.

If $\mathcal{R}$ is a relatively open subset of ∂B^2 and μ a positive Radon measure on $\mathcal{R}$, we shall say that μ is *locally admissible* if for every compact subset F of $\mathcal{R}$ the restriction $\mu_{|F}$ of μ to F is admissible. If μ is locally admissible, Marcus and Véron's method can be adapted to define a minimal solution u_μ of (1.3) in B^2 whose regular part of the boundary trace is μ. Set $S = \partial B^2 \backslash \mathcal{R}$ and denote by $\partial_\mu \mathcal{S}$ the singular part of the boundary trace of u_μ. For $\varepsilon > 0$ we define $\mathcal{S}_\varepsilon = \{\theta \in S^1 : \mathrm{dist}(\theta, \mathcal{S}) \leqslant \varepsilon\}$ and construct a solution $u_{\mathcal{S}_\varepsilon}$ with boundary trace $(\mathcal{S}_\varepsilon, 0)$; when ε goes to 0, the sequence $\{u_{\mathcal{S}_\varepsilon}\}$ decreases and converges to some solution $u_{\mathcal{S}}^*$ of (1.3) with boundary trace $\mathcal{S}^* \subset \mathcal{S}$. As in [17, 19], we always have

$$\mathcal{S}^* \cup \partial_\mu \mathcal{S} \subset \mathcal{S}, \tag{1.14}$$

and the reverse inclusion gives a necessary and sufficient condition in order $(\mathcal{S}, \mu)$ be the boundary trace of a solution u of (1.3) in B^2 with μ locally admissible.

The meaning of such a result is the following: locally the singular set $\mathcal{S}$ is either intrinsically not removable, or it is created by the unboundedness of the measure μ. In general such a solution u is not uniquely determined from its boundary trace.

In the last section, we study under what conditions a subset of ∂B^2 is removable for the equation (1.3). We introduce an extended notion of Besov capacity that we call the $C_{0,1,\ln}$-capacity, and we prove that if a set $E \subset \partial B^2$ has zero $C_{0,1,\ln}$-capacity, any solution of (1.3) in B^2 which coincides on $\partial B^2 \backslash E$ with a continuous function defined in whole ∂B^2 can be extended as a continuous function in $\bar{B}^2$ (and solution of the equation in B^2 in the weak sense (1.13)). In particular, it is worthwhile noticing that if a set $E \subset \partial B^2$ has Hausdorff dimension $\delta < 1$, it has zero $C_{0,1,\ln}$-capacity.

The present paper is organized as follows: § 2 deals with the boundary trace; § 3 deals with the negative curvature case; § 4 deals with removable singularities.

2. The boundary trace

Throughout this section K is a Hölder continuous function in the closure $\overline{B^2}$ of the unit open ball $B^2 = \{x \in \mathbb{R}^2 : |x| < 1\}$ of the plane. A solution of the equation

$$\Delta u + K\mathrm{e}^{2u} = 0 \tag{2.1}$$

in B^2 is by definition a $C^2(B^2)$-function which satisfies (2.1) in B^2. We denote by $\mathcal{E}$ the class of the solutions of (2.1) which are bounded from below by some negative constant. If u is an element of $\mathcal{E}$, we call m_u such a minorant. Consequently, $u - m_u = v$ is non-negative and satisfies

$$\Delta v + K\mathrm{e}^{2m_u}\mathrm{e}^{2v} = 0 \tag{2.2}$$

in B^2. Let (r, σ) be the polar coordinates in $\mathbb{R}^2\backslash\{0\}$, we set

$$\left.\begin{aligned} \omega^+ &= \{x \in \partial B^2 : K(x) > 0\}, \\ \omega^- &= \{x \in \partial B^2 : K(x) < 0\}, \\ \omega^0 &= \{x \in \partial B^2 : K(x) = 0\}. \end{aligned}\right\} \tag{2.3}$$

If U is any open subset of S^1, we call $\mathcal{T}$ the topology on $C_0(U)$ of inductive limit, the dual space of which being the space of Radon measures on U. The main result is the following.

THEOREM 2.1. *Let u be an element of $\mathcal{E}$, then the following dichotomy occurs.*

I. *For every $\theta \in \omega^-$, either*

(i) *there exists a relatively open neighbourhood U of θ such that for every $\zeta \in C_0^\infty(U)$*

$$\lim_{r\to 1} \int_U u(r,\sigma)\zeta(\sigma)\,\mathrm{d}\sigma = \ell(\zeta), \tag{2.4}$$

where ℓ is a linear functional on $C_0^\infty(U)$ continuous in the $\mathcal{T}$-topology, or

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(ii) *for every relatively open neighbourhood* U *of* θ, *there holds*

$$\lim_{r\to 1}\int_U u(r,\sigma)\,\mathrm{d}\sigma = \infty. \tag{2.5}$$

II. *For every* $\theta\in\omega^+$ *there exists a relatively open neighbourhood* U *of* θ *such that (2.4) holds for every* $\zeta\in C_0^\infty(U)$, *where* ℓ *is a linear functional on* $C_0^\infty(U)$ *continuous in the* $\mathcal{T}$*-topology.*

The following result dealing with the properties of ω^+ is in fact a local version of a classical result due to Doob [5] concerning the boundary trace of positive superharmonic functions (and in fact Doob's theorem involves also an almost everywhere convergence).

LEMMA 2.2. *Suppose* u *is an element of* $\mathcal{E}$ *and let* $\theta\in\omega^+$*; then the assertion II of theorem 2.1 holds.*

Proof. Let m_u be a minorant and $u-m_u=v$. This function v is non-negative and satisfies (2.2). We set $\tilde{v}(t,\sigma)=v(r,\sigma)$ with $t=\ln(1/r)\in(0,\infty)$, then

$$\partial_t^2\tilde{v}+\partial_\sigma^2\tilde{v}=-\tilde{K}\mathrm{e}^{-2t}\mathrm{e}^{2\tilde{v}} \tag{2.6}$$

in $(0,\infty)\times S^1$, with $K_u=K\mathrm{e}^{2\tilde{m}_u}$. Let U_θ be the connected component (an interval) of ω^+ which contains θ and $U\subset\bar{U}\subset U_\theta$ be an open interval containing θ. Since K is continuous we have $\tilde{K}(t,\sigma)>0$ if $(t,\sigma)\in[0,a]\times U$ for some $a>0$. We denote ψ_U the first eigenfunction of $\mathrm{d}^2/\mathrm{d}\sigma^2$ in $W_0^{1,2}(U)$ normalized by

$$0\leqslant\psi_U\leqslant\max_U\psi_U=1 \tag{2.7}$$

and λ_U the corresponding eigenfunction. Integrating (2.6) on U yields

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_U\tilde{v}(t,\sigma)\psi_U\,\mathrm{d}\sigma-\lambda_U\int_U\tilde{v}(t,\sigma)\psi_U\,\mathrm{d}\sigma$$
$$=-\int_U\mathrm{e}^{-2t}\tilde{K}\psi_U\mathrm{e}^{2\tilde{v}(t,\sigma)}\,\mathrm{d}\sigma+\left[\tilde{v}\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right]_{\sigma=\theta^-}^{\sigma=\theta^+} \tag{2.8}$$

on $(0,a]\times U=(0,a]\times(\theta^-,\theta^+)$, where $\theta^-<\theta<\theta^+$ are the two end-points of U; here we identify functions defined on S^1 with 2π-periodic functions. Therefore, the right-hand side of (2.9) is non-positive; if we set $X(t)=\int_U(\tilde{v}\psi_U)(t,\sigma)\,\mathrm{d}\sigma$, then $X\geqslant 0$ and

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2}-\lambda_U X\leqslant 0 \tag{2.9}$$

holds on $(0,a]$.

STEP 1. We claim that $\lim_{t\to 0}X(t)=\Lambda_U$ for some $\Lambda_U>0$. If we set $Y=\mathrm{e}^{-2t\sqrt{\lambda_U}}X$, then

$$\frac{\mathrm{d}^2Y}{\mathrm{d}t^2}+2\sqrt{\lambda_U}\frac{\mathrm{d}Y}{\mathrm{d}t}=\mathrm{e}^{-2t\sqrt{\lambda_U}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2t\sqrt{\lambda_U}}\frac{\mathrm{d}Y}{\mathrm{d}t}\right)\leqslant 0, \tag{2.10}$$

and the function $t \mapsto \mathrm{e}^{2t\sqrt{\lambda_U}}Y'(t)$ is decreasing on $(0,a]$. Consequently, it admits some limit $L > -\infty$ at 0 and this limit is the same as the one of $t \mapsto Y'(t)$. If L is finite, Y satisfies the Cauchy criterion near 0, then Y admits a finite limit Λ_U at 0 and the same holds for X. If $L = \infty$, then

$$\lim_{t\to 0}(X' - 2\sqrt{\lambda_U}X)(t) = \infty \quad \text{and} \quad \lim_{t\to 0} X'(t) = \infty.$$

As a consequence X is increasing near 0 and again it admits a finite limit Λ_U at 0.

STEP 2 (End of the proof). Let $\zeta \in C_0^\infty(U_\theta)$, $\zeta \geqslant 0$, then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{U_\theta} \tilde{v}(t,\sigma)\zeta\,\mathrm{d}\sigma + \int_{U_\theta} \tilde{v}(t,\sigma)\frac{\mathrm{d}^2\zeta}{\mathrm{d}\sigma^2}\,\mathrm{d}\sigma + \int_{U_\theta} \mathrm{e}^{-2t}\tilde{K}(t,\sigma)\mathrm{e}^{2\tilde{v}(t,\sigma)}\zeta\,\mathrm{d}\sigma = 0 \tag{2.11}$$

for $0 < t \leqslant a$. Integrating (2.11) twice yields

$$\begin{aligned}\int_{U_\theta} \tilde{v}(t,\sigma)\zeta\,\mathrm{d}\sigma - \int_{U_\theta} \tilde{v}(a,\sigma)\zeta\,\mathrm{d}\sigma + (a-t)\int_{U_\theta} \frac{\mathrm{d}\tilde{v}}{\mathrm{d}t}(a,\sigma)\zeta\,\mathrm{d}\sigma \\ + \int_t^a (s-t)\int_{U_\theta}\left(\tilde{v}(s,\sigma)\frac{\mathrm{d}^2\zeta}{\mathrm{d}\sigma^2} + \mathrm{e}^{-2s}\tilde{K}(s,\sigma)\mathrm{e}^{2\tilde{v}(s,\sigma)}\zeta\right)\mathrm{d}\sigma\,\mathrm{d}s = 0.\end{aligned} \tag{2.12}$$

If $U = \mathrm{supp}\,.(\zeta)$, one deduces from step 1 that

$$\int_{U_\theta} \tilde{v}(t,\sigma)\zeta(\sigma)\,\mathrm{d}\sigma$$

remains bounded independently of t and it is the same with

$$\int_t^a (s-t)\int_{U_\theta} \tilde{v}(s,\sigma)\frac{\mathrm{d}^2\zeta}{\mathrm{d}\sigma^2}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s.$$

Consequently,

$$S(t) = \int_t^a (s-t)\int_{U_\theta} \mathrm{e}^{-2s}|\tilde{K}(s,\sigma)|\mathrm{e}^{2\tilde{v}}\zeta(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s \leqslant M \tag{2.13}$$

for some positive constant M. Therefore $S(t)$ admits a finite limit when t goes to 0 and

$$\begin{aligned}\lim_{t\to 0}\int_{U_\theta} \tilde{v}(t,\sigma)\zeta\,\mathrm{d}\sigma &= \int_{U_\theta} \tilde{v}(a,\sigma)\zeta\,\mathrm{d}\sigma - a\int_{U_\theta}\frac{\mathrm{d}\tilde{v}}{\mathrm{d}t}(a,\sigma)\zeta\,\mathrm{d}\sigma \\ &\quad - \int_0^a s\int_{U_\theta}\left(\tilde{v}(s,\sigma)\frac{\mathrm{d}^2\zeta}{\mathrm{d}\sigma^2} + \mathrm{e}^{-2s}\tilde{K}(s,\sigma)\mathrm{e}^{2\tilde{v}(s,\sigma)}\zeta\right)\mathrm{d}\sigma\,\mathrm{d}s \\ &= \tilde{\ell}(\zeta),\end{aligned} \tag{2.14}$$

clearly the mapping $\zeta \mapsto \tilde{\ell}(\zeta)$ defines a positive linear functional on $C_0^\infty(U_\theta)$. The claim follows by setting $\ell = \tilde{\ell} + m_u$. □

LEMMA 2.3. *Let U be a connected open subset of S^1 and v a non-negative continuous function on S^1. Then there exists a constant $C = C(U) > 0$ such that for every*

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$\varepsilon > 0$ and $\alpha \geqslant 4$, there exists $M(\varepsilon) > 0$ such that

$$\int_U v\psi_U^{\alpha-2}\left(\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right)^2 \mathrm{d}\sigma \leqslant \varepsilon \int_U \mathrm{e}^{2v}\psi_U^{\alpha}\,\mathrm{d}\sigma + CM(\varepsilon)\int_U \psi_U^{\alpha-8/3}\left|\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right|^2 \mathrm{d}\sigma. \tag{2.15}$$

Proof. Set $\eta \in (0,1)$ and $n \in \mathbb{N}$, $n \geqslant 2$. There holds

$$\begin{aligned}\int_U v\psi_U^{\alpha-2}\left(\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right)^2 \mathrm{d}\sigma &= \int_U v\eta^{1/n}\psi_U^{\alpha/n}\psi_U^{\alpha(1-1/n)}\eta^{-1/n}\left(\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right)^2 \mathrm{d}\sigma\\ &\leqslant \frac{\eta}{n}\int_U v^n\psi_U^{\alpha}\,\mathrm{d}\sigma + \left(1-\frac{1}{n}\right)\eta^{-1/(n-1)}\int_U \psi_U^{\alpha-2n/(n-1)}\left(\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right)^{2n/(n-1)} \mathrm{d}\sigma.\end{aligned} \tag{2.16}$$

Moreover, $(1-1/n)\eta^{-1/(n-1)} \leqslant \eta^{-1}$, $\psi_U^{\alpha-2n/(n-1)} \leqslant \psi_U^{\alpha-4} = \psi_U^{\beta}$ with $\beta = \alpha - 4$ and

$$\left|\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right|^{2n/(n-1)} \leqslant C\left(\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right)^2, \quad \text{with } C = \max\left(1, \left\|\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right\|^2_{L^\infty(U)}\right). \tag{2.17}$$

Consequently,

$$\int_U v\psi_U^{\alpha-2}\left(\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right)^2 \mathrm{d}\sigma \leqslant \frac{\eta}{n}\int_U v^n\psi_U^{\alpha}\,\mathrm{d}\sigma + \frac{C}{\eta}\int_U \psi_U^{\beta}\left(\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right)^2 \mathrm{d}\sigma, \tag{2.18}$$

and

$$\frac{2^n}{(n-1)!}\int_U v\psi_U^{\alpha-2}\left(\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right)^2 \mathrm{d}\sigma \leqslant \frac{2^n\eta}{n!}\int_U v^n\psi_U^{\alpha}\,\mathrm{d}\sigma + \frac{2^nC}{\eta(n-1)!}\int_U \psi_U^{\beta}\left(\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right)^2 \mathrm{d}\sigma. \tag{2.19}$$

By summing those inequalities from $n = 2$ to infinity, one obtains

$$(\mathrm{e}^2-1)\int_U v\psi_U^{\alpha-2}\left(\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right)^2 \mathrm{d}\sigma \leqslant \mathrm{e}^2\eta\int_U \mathrm{e}^{2v}\psi_U^{\alpha}\,\mathrm{d}\sigma + \frac{C(\mathrm{e}^2-1)}{\eta}\int_U \psi_U^{\beta}\left(\frac{\mathrm{d}\psi_U}{\mathrm{d}\sigma}\right)^2 \mathrm{d}\sigma, \tag{2.20}$$

which is the desired result with $\eta = \varepsilon\mathrm{e}^2/(\mathrm{e}^2-1)$ and $M(\varepsilon) = (1-\mathrm{e}^{-2})\varepsilon^{-1}$. □

LEMMA 2.4. *Suppose u is an element of $\mathcal{E}$, $\theta \in \omega^-$ with connected component U_θ in ω^-. If U is any relatively open connected subset of U_θ, with $\theta \in U \subset \bar{U} \subset U_\theta$, $\alpha \geqslant 8/3$ and ψ_U is as in lemma 2.2, the following alternative holds: either*

(i)

$$\int_0^1\int_U |K|\mathrm{e}^{2u}\psi_U^{\alpha}(1-r)\,\mathrm{d}\sigma\,\mathrm{d}r = \infty, \tag{2.21}$$

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and in that case

$$\lim_{r\to 1}\int_U (u\psi_U^{\alpha})(r,\sigma)\,\mathrm{d}\sigma=\infty, \tag{2.22}$$

or

(ii)

$$\int_0^1\int_U |K|\mathrm{e}^{2u}\psi_U^{\alpha}(1-r)\,\mathrm{d}\sigma\,\mathrm{d}r<\infty, \tag{2.23}$$

and in that case, for any function $\zeta\in C^2(U)$ *satisfying*

$$0\leqslant\zeta\leqslant k\psi_U^{\alpha}\quad \text{and}\quad \left|\frac{\mathrm{d}^2\zeta}{\mathrm{d}\sigma^2}\right|\leqslant k\psi_U^{\alpha-2}\quad \text{in } U \tag{2.24}$$

for some $k>0$*, the following limit exists*

$$\lim_{r\to 1}\int_U u(r,\sigma)\zeta(\sigma)\,\mathrm{d}\sigma=\ell(\zeta). \tag{2.25}$$

If m_U *is a negative minorant of* u *the mapping* $\zeta\mapsto\ell(\zeta)-m_U\int_U\zeta(\sigma)\,\mathrm{d}\sigma$ *is a positive linear functional defined on the set of functions* $\zeta\in C^2(U)$ *satisfying (2.24).*

Proof. We define $\tilde v(t,\sigma)=v(r,\sigma)=u(r,\sigma)-m_u$ as in lemma 2.2, with $r=\mathrm{e}^{-t}$, $K_u=K\mathrm{e}^{2\tilde m_u}$ and $a>0$ such that

$$\delta<-\tilde K(t,\theta)\leqslant 1/\delta\qquad (\forall(t,\sigma)\in[0,a]\times\bar U) \tag{2.26}$$

for some $\delta>0$. Then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_U\tilde v(t,\sigma)\psi_U^{\alpha}\,\mathrm{d}\sigma+\int_U\tilde v(t,\sigma)\frac{\mathrm{d}^2\psi_U^{\alpha}}{\mathrm{d}\sigma^2}\,\mathrm{d}\sigma+\int_U\mathrm{e}^{-2t}\tilde K(t,\sigma)\psi_U^{\alpha}\mathrm{e}^{2\tilde v(t,\sigma)}\,\mathrm{d}\sigma=0. \tag{2.27}$$

For simplicity we set $\psi=\psi_U$ and $\lambda=\lambda_U$. Since

$$\int_U\tilde v\frac{\mathrm{d}^2\psi^{\alpha}}{\mathrm{d}\sigma^2}\,\mathrm{d}\sigma=-\alpha\lambda\int_U\tilde v\psi^{\alpha}\,\mathrm{d}\sigma+\alpha(\alpha-1)\int_U\tilde v\psi^{\alpha-2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\sigma}\right)^2\mathrm{d}\sigma,$$

equation (2.27) reads as

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_U\tilde v(t,\sigma)\psi^{\alpha}\,\mathrm{d}\sigma+\alpha(\alpha-1)\int_U\tilde v(t,\sigma)\psi^{\alpha-2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\sigma}\right)^2\mathrm{d}\sigma&\\ -\alpha\lambda\int_U\tilde v(t,\sigma)\psi^{\alpha}\,\mathrm{d}\sigma+\int_U\mathrm{e}^{-2t}\tilde K(t,\sigma)\psi^{\alpha}\mathrm{e}^{2\tilde v(t,\sigma)}\,\mathrm{d}\sigma&=0.\end{aligned} \tag{2.28}$$

Case 1. Let us assume that (2.21) holds, then

$$\int_0^a t\int_U\mathrm{e}^{2\tilde v}\psi^{\alpha}\,\mathrm{d}\sigma\,\mathrm{d}t=\infty. \tag{2.29}$$

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It follows (2.26), (2.28) and lemma 2.3 that, for some $\varepsilon > 0$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_U \tilde{v}(t,\sigma)\psi^{\alpha}\,\mathrm{d}\sigma + \alpha(\alpha-1)\biggl(\varepsilon\int_U \mathrm{e}^{2\tilde{v}(t,\sigma)}\psi^{\alpha}\,\mathrm{d}\sigma + CM(\varepsilon)\int_U \psi^{\alpha-8/3}\biggl(\frac{\mathrm{d}\psi}{\mathrm{d}\sigma}\biggr)^2\mathrm{d}\sigma\biggr) \\ - \alpha\lambda\int_U \tilde{v}(t,\sigma)\psi^{\alpha}\,\mathrm{d}\sigma \geqslant \mathrm{e}^{-2a}\delta\int_U \psi^{\alpha}(\sigma)\mathrm{e}^{2\tilde{v}(t,\sigma)}\,\mathrm{d}\sigma \geqslant 0. \quad (2.30)$$

Choosing $\alpha(\alpha-1)\varepsilon = \mathrm{e}^{-2a}\delta/2$ one deduces

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_U \tilde{v}\psi^{\alpha}(t,\sigma)\,\mathrm{d}\sigma \geqslant A\int_U \mathrm{e}^{2\tilde{v}(t,\sigma)}\psi^{\alpha}\,\mathrm{d}\sigma - B \quad (2.31)$$

for some positive constants A and B, independent of $t \in (0,a]$. Integrating (2.31) twice and using (2.29) yields

$$\lim_{t\to 0}\int_U \tilde{v}\psi^{\alpha}(t,\sigma)\,\mathrm{d}\sigma = \infty, \quad (2.32)$$

which is (2.22).

Case 2. Let us assume that (2.24) holds, then

$$\int_0^a t\int_U \mathrm{e}^{2\tilde{v}}\psi^{\alpha}\,\mathrm{d}\sigma\,\mathrm{d}t < \infty \quad (2.33)$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_U \tilde{v}(t,\sigma)\psi^{\alpha}\,\mathrm{d}\sigma - \alpha\lambda\int_U \tilde{v}(t,\sigma)\psi^{\alpha}\,\mathrm{d}\sigma \leqslant \delta^{-1}\int_U \psi^{\alpha}(\sigma)\mathrm{e}^{2\tilde{v}(t,\sigma)}\,\mathrm{d}\sigma. \quad (2.34)$$

Setting

$$\beta^2 = \alpha\lambda, \quad X(t) = \int_U \tilde{v}(t,\sigma)\psi_U^{\alpha}\,\mathrm{d}\sigma \quad \text{and} \quad F(t) = \delta^{-1}\int_U \mathrm{e}^{2\tilde{v}(t,\sigma)}\psi_U^{\alpha}\,\mathrm{d}\sigma;$$

then $X'' - \beta^2 X \leqslant F$ on $(0,a]$. If $Y(t) = \mathrm{e}^{-\beta t}X(t)$, it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\mathrm{e}^{2\beta t}\frac{\mathrm{d}Y}{\mathrm{d}t}(t)\biggr) \leqslant \mathrm{e}^{\beta t}F(t), \quad (2.35)$$

which yields

$$\frac{\mathrm{d}Y}{\mathrm{d}t}(t) \geqslant \mathrm{e}^{-2\beta(t-a)}\frac{\mathrm{d}Y}{\mathrm{d}t}(a) - \mathrm{e}^{-2\beta t}\int_t^a \mathrm{e}^{\beta s}F(s)\,\mathrm{d}s \quad (2.36)$$

by integration. Consequently, the function $\varPhi$

$$t \mapsto \varPhi(t) = \mathrm{e}^{\beta t}X(t) + \frac{\mathrm{e}^{-2\beta(t-a)}}{2\beta}Y'(a) + \int_t^a \mathrm{e}^{-2\beta s}\int_s^a \mathrm{e}^{\beta\tau}F(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \quad (2.37)$$

is non-decreasing. Since (2.33) implies $\int_0^a \mathrm{e}^{-2\beta s}\int_s^a \mathrm{e}^{\beta\tau}F(\tau)\,\mathrm{d}\tau\,\mathrm{d}s < \infty$, one deduces that

$$\lim_{t\to 0}X(t) = \varLambda_U = \lim_{t\to 0}\int_U \tilde{v}(t,\sigma)\psi_U^{\alpha}\,\mathrm{d}\sigma = \lim_{r\to 1}\int_U v(r,\sigma)\psi_U^{\alpha}\,\mathrm{d}\sigma, \quad (2.38)$$

for some non-negative constant Λ_U. Let $\zeta \in C^2(U)$ be a non-negative function satisfying the relations (2.24). Then, as in lemma 2.2,

$$\int_U \tilde{v}(t,\sigma)\zeta\,\mathrm{d}\sigma - \int_U \tilde{v}(a,\sigma)\zeta\,\mathrm{d}\sigma + (a-t)\int_U \frac{\mathrm{d}\tilde{v}}{\mathrm{d}t}(a,\sigma)\zeta\,\mathrm{d}\sigma \\ = -\int_t^a (s-t)\int_U \left(\tilde{v}(s,\sigma)\frac{\mathrm{d}^2\zeta}{\mathrm{d}\sigma^2} + \mathrm{e}^{-2s}\tilde{K}(s,\sigma)\mathrm{e}^{2\tilde{v}(t,\sigma)}\zeta\right)\mathrm{d}\sigma\,\mathrm{d}s. \quad (2.39)$$

Since

$$\int_0^a t\tilde{v}(s,\sigma)\int_U \left|\frac{\mathrm{d}^2\zeta}{\mathrm{d}\sigma^2}(\sigma)\right|\mathrm{d}\sigma\,\mathrm{d}s < \infty,$$

lemma 2.3, (2.24) and (2.33) implies that the right-hand side of (2.39) admits a limit when t goes to 0; consequently,

$$\lim_{t\to 0}\int_{U_\theta} \tilde{v}(t,\sigma)\zeta(\sigma)\,\mathrm{d}\sigma = \tilde{\ell}(\zeta), \quad (2.40)$$

where $\tilde{\ell}$ is a positive functional on the set of $C^2(U)$-functions satisfying (2.24) and the proof is completed. □

The proof of theorem 2.1 is an immediate consequence of lemmas 2.2 and 2.4, as in [18].

Remark 2.5. The result of theorem 2.1 still holds if the two-dimensional ball is replaced by the d-dimensional one ($d > 2$). The only differences are technical and come from lemmas 2.2–2.4, in which the relatively open subset U needs to have a smooth boundary. Moreover, equation (2.1) has to be written in spherical coordinates $(r,\sigma) \in (0,1) \times S^{d-1}$ as

$$\partial_r^2 u + (d-1)r^{-1}\partial_r u + r^{-2}\Delta_{S^{d-1}}u + K(r,\sigma)\mathrm{e}^{2u} = 0 \quad (2.41)$$

and the logarithm change of variable has to be replaced by the following one,

$$t = r^{d-2}, \qquad \tilde{v}(t,\sigma) = r^{d-2}(u(r,\sigma) - m_u), \quad (2.42)$$

for some negative minorant m_u. This leads to

$$t^2\partial_t^2\tilde{v} + (d-2)^{-2}(\Delta_{S^{d-1}}\tilde{v} + t^{(d-2)/d}\tilde{K}(t,\sigma)\mathrm{e}^{2\tilde{v}/t}) = 0 \quad (2.43)$$

in $(0,a] \times S^{d-1}$. Since lemma 2.3 is valid, the analysis of this equation is the same as the one developed in [18].

Remark 2.6. In the statement of lemma 2.2, it is possible to replace the relatively open subset U by the connected component U_θ of $\theta \in \omega^+$ if it is assumed that $K(r,\sigma) > 0$ when $(r,\sigma) \in [0,a] \times U_\theta$ for some $a > 0$. Moreover, if $K(r,\sigma) \geqslant 0$ in some $(r,\sigma) \in [0,a] \times U$ (for some relatively open subset U on ∂B^2), then lemma 2.2 and the part II of theorem 2.1 also hold.

Definition 2.7. If $\theta \in \partial B^2$, we shall say that θ is *regular with respect to u* either if statement I(i) holds or if $\theta \in \omega^+$. The *set of regular points* is relatively open in

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∂B^2 and we denote it by $\mathcal{R}$. As in [12] or [16], there exists a Radon measure μ on $\mathcal{R}$ such that

$$\lim_{r\to 1}\int_{\mathcal{R}} u(r,\sigma)\zeta(\sigma)\,\mathrm{d}\sigma = \int_{\mathcal{R}} \zeta(\sigma)\,\mathrm{d}\mu(\sigma) \tag{2.44}$$

for every $\zeta \in C_0^\infty(\mathcal{R})$, and by density, this limit holds if $\zeta \in C_0(\mathcal{R})$. The set of $\theta \in \partial B^2$ such that statement I(ii), that we denote $\mathcal{S}$, is called the *singular set* of u and we have

$$\partial B^2 = \mathcal{R} \cup \mathcal{S} \cup \omega^0, \tag{2.45}$$

where $\omega^0 = \{x \in \partial B^2 \colon K(x) = 0\}$. The triplet $\mathrm{tr}_{|\partial B^2}(u) = (\mathcal{S}, \omega^0, \mu)$ is by definition the boundary trace of u.

Remark 2.8. As in [18], expression (2.44) is equivalent to

$$\int_{B^2} (-u\Delta\xi + K\mathrm{e}^{2u}\xi)\,\mathrm{d}x = -\int_{\partial B^2} \frac{\partial \xi}{\partial n}\,\mathrm{d}\mu \tag{2.46}$$

for any $\xi \in C_0^{1,1}(B^2 \cup \mathcal{R})$, which is by definition the space of $C^1(\overline{B^2})$-functions, with compact support in $B^2 \cup \mathcal{R}$ and Lipschitz-continuous gradient.

3. The negative curvature case

At the beginning of this section we study the Dirichlet problem in a general regular domain Ω for an equation of type (1.3) with a negative $K = -G$ and boundary data belonging to some Lebesgue spaces. We first recall the following unpublished result due to Brezis (see [8, Appendix] or [29] for a proof).

Lemma 3.1. *Suppose Ω is a bounded regular domain in $\mathbb{R}^d$, $\rho(x) = \mathrm{dist}(x, \partial\Omega)$, g is a continuous, non-decreasing, real-valued function and $f \in L^1(\partial\Omega)$. Then there exists a unique $w \in L^1(\Omega)$ with $g(w) \in L^1(\Omega, \rho\,\mathrm{d}x)$ with the property that*

$$\int_{\Omega} (-w\Delta\zeta + g(w)\zeta)\,\mathrm{d}x = -\int_{\partial\Omega} f\frac{\partial \zeta}{\partial n}\,\mathrm{d}S \tag{3.1}$$

for any $\zeta \in C_0^{1,1}(\bar{\Omega})$. Moreover, if $(\tilde{w}, \tilde{f})$ is another couple, the following estimate holds

$$\|w - \tilde{w}\|_{L^1(\Omega)} + \|\rho(g(w) - g(\tilde{w}))\|_{L^1(\Omega)} \leqslant C\|f - \tilde{f}\|_{L^1(\partial\Omega)} \tag{3.2}$$

and the mapping $f \mapsto w$ is non-decreasing.

In the above monotonicity result the function $g(r)$ can be replaced by $g(x, r)$, provided $g \in C(\bar{\Omega} \times \mathbb{R})$ is non-decreasing with respect to r, for fixed x. We begin with the following estimates.

Proposition 3.2. *Suppose Ω is a bounded and regular domain in $\mathbb{R}^d$, G is a continuous and non-negative function in $\bar{\Omega}$, ψ_1 is the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$ normalized by*

$$0 \leqslant \psi_1 \leqslant \max_{\Omega} \psi_1 = 1, \tag{3.3}$$

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λ_1 the corresponding eigenvalue, $\rho(x)=\operatorname{dist}(x,\partial\Omega)$ and $p\in(1,\infty)$. Then for any $f\in L^1(\partial\Omega)$ such that $\mathrm{e}^{2(p-1)f}\in L^1(\partial\Omega)$, there exists a unique $v\in L^1(\Omega)$ with $\mathrm{e}^{2v}\in L^1(\Omega,\rho\,\mathrm{d}x)$, satisfying

$$\int_\Omega(-v\Delta\zeta+G\mathrm{e}^{2v}\zeta)\,\mathrm{d}x=-\int_{\partial\Omega}f\frac{\partial\zeta}{\partial n}\,\mathrm{d}S \tag{3.4}$$

for any $\zeta\in C_0^{1,1}(\bar{\Omega})$, and the following estimate holds:

$$\frac{\lambda_1}{2(p-1)}\int_\Omega \mathrm{e}^{2(p-1)v}\psi_1\,\mathrm{d}x+2(p-1)\int_\Omega \mathrm{e}^{2(p-1)v}|\nabla v|^2\psi_1\,\mathrm{d}x+\int_\Omega \mathrm{e}^{2pv}G\psi_1\,\mathrm{d}x \leqslant \frac{-1}{2(p-1)}\int_{\partial\Omega}\frac{\partial\psi_1}{\partial n}\mathrm{e}^{2(p-1)f}\,\mathrm{d}S. \tag{3.5}$$

In particular,

$$\|G\mathrm{e}^{2v}\|_{L^p(\Omega,\rho)}\leqslant\left(\frac{C}{p-1}\right)^{1/p}\|\mathrm{e}^{2f}\|^{1-1/p}_{L^{p-1}(\partial\Omega)}, \tag{3.6}$$

where $C=C(\Omega)>0$.

LEMMA 3.3. *Suppose that $f\in L^1(\partial\Omega)$ is such that*

$$\mathrm{e}^{2(p-1)f}\in L^1(\partial\Omega)\quad\textit{and}\quad\inf_{\partial\Omega}\operatorname{ess} f>-\infty,$$

then there exists a sequence $\{f_n\}\subset C^2(\partial\Omega)$ with the following property:

$$\lim_{n\to\infty}f_n=f\quad\textit{in } L^1(\partial\Omega), \tag{3.7}$$

$$\lim_{n\to\infty}\mathrm{e}^{2(p-1)f_n}=\mathrm{e}^{2(p-1)f}\quad\textit{in } L^1(\partial\Omega), \tag{3.8}$$

$$\inf_n\inf_{\partial\Omega}f_n\geqslant\inf_{\partial\Omega}\operatorname{ess} f. \tag{3.9}$$

Proof. Set $k=\inf\operatorname{ess}_{\partial\Omega}f$. There exists a sequence $\{h_n\}\subset C^2(\partial\Omega)$ such that

$$h_n\geqslant 0\quad\text{on }\partial\Omega, \tag{3.10}$$

$$\lim_{n\to\infty}h_n=\mathrm{e}^{2(p-1)f}-\mathrm{e}^{2(p-1)k}\quad\text{in }L^1(\partial\Omega). \tag{3.11}$$

Setting

$$f_n=\frac{1}{2(p-1)}\ln(h_n+\mathrm{e}^{2(p-1)k}),$$

then

$$\mathrm{e}^{2(p-1)f_n}=h_n+\mathrm{e}^{2(p-1)k}\xrightarrow[n\to\infty]{}\mathrm{e}^{2(p-1)f}\quad\text{in }L^1(\partial\Omega).$$

Moreover, $f_n\geqslant k$ for any n and

$$\begin{aligned}|\mathrm{e}^{2(p-1)f_n}-\mathrm{e}^{2(p-1)f}|&=2(p-1)\mathrm{e}^{2(p-1)(\theta f_n+(1-\theta)f)}|f_n-f|,\qquad\theta\in(0,1),\\&\geqslant 2(p-1)\mathrm{e}^{2(p-1)k}|f_n-f|.\end{aligned} \tag{3.12}$$

Therefore

$$\|f - f_n\|_{L^1(\partial\Omega)} \leqslant \frac{e^{2(1-p)k}}{2(p-1)} \|e^{2(p-1)f} - e^{2(p-1)f_n}\|_{L^1(\partial\Omega)}, \tag{3.13}$$

which is (3.7). □

LEMMA 3.4. *Suppose that Ω, G, p, ψ_1 and λ_1 are as in proposition 3.2. Then for any $f \in C^2(\partial\Omega)$ there exists a unique v belonging to $W^{2,q}(\Omega)$ for any $1 < q < \infty$, such that*

$$\left.\begin{aligned} -\Delta v + G e^{2v} &= 0 \quad \text{in } \Omega, \\ v &= f \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.14}$$

Moreover,

$$\frac{\lambda_1}{2(p-1)} \int_\Omega e^{2(p-1)v} \psi_1 \, dx + 2(p-1) \int_\Omega e^{2(p-1)v} |\nabla v|^2 \psi_1 \, dx + \int_\Omega e^{2pv} G \psi_1 \, dx = \frac{-1}{2(p-1)} \int_{\partial\Omega} \frac{\partial \psi_1}{\partial n} e^{2(p-1)f} \, dS. \tag{3.15}$$

Proof. Existence and uniqueness follows from lemma 3.1. From the maximum principle v is bounded from above by the Poisson potential P_f of f, and from below by $P_f - \mathcal{G}_{Ge^{2P_f}}$, where $\mathcal{G}_h$ denotes the Green potential of a function h in Ω. Therefore, it is bounded and regularity follows from the elliptic equations theory. Multiplying the equation by $e^{2(p-1)v}\psi_1$, one obtains

$$\int_\Omega e^{2pv} G \psi_1 \, dx + \int_\Omega \nabla v \cdot \nabla (e^{2(p-1)v} \psi_1) \, dx = 0. \tag{3.16}$$

Since

$$\int_\Omega \nabla v \cdot \nabla (e^{2(p-1)v} \psi_1) \, dx = 2(p-1) \int_\Omega |\nabla v|^2 e^{2(p-1)v} \psi_1 \, dx + \int_\Omega e^{2(p-1)v} \nabla v \cdot \nabla \psi_1 \, dx$$

and

$$\begin{aligned} \int_\Omega e^{2(p-1)v} \nabla v \cdot \nabla \psi_1 \, dx &= \frac{1}{2(p-1)} \int_\Omega \nabla (e^{2(p-1)v}) \cdot \nabla \psi_1 \, dx \\ &= \frac{-1}{2(p-1)} \int_\Omega e^{2(p-1)v} \Delta \psi_1 \, dx + \frac{1}{2(p-1)} \int_{\partial\Omega} e^{2(p-1)v} \frac{\partial \psi_1}{\partial n} \, dS \\ &= \frac{\lambda_1}{2(p-1)} \int_\Omega e^{2(p-1)v} \psi_1 \, dx + \frac{1}{2(p-1)} \int_{\partial\Omega} e^{2(p-1)v} \frac{\partial \psi_1}{\partial n} \, dS, \end{aligned} \tag{3.17}$$

equation (3.15) follows. □

Proof of proposition 3.2. We set $f = f^+ - f^-$ and denote by v_- the solution of

$$\int_\Omega (-v_- \Delta \zeta + G e^{2v_-} \zeta) \, dx = \int_{\partial\Omega} f^- \frac{\partial \zeta}{\partial n} \, dS \qquad (\forall \zeta \in C_0^{1,1}(\bar{\Omega})) \tag{3.18}$$

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and v the one of (3.4), both in the sense of lemma 3.1. Then $v \geqslant v_-$ in Ω; moreover, $V_- \leqslant v_- \leqslant 0$, where V_- is the solution of

$$\left.\begin{aligned} -\Delta V_- &= -\|G\|_{L^\infty} \quad \text{in } \Omega, \\ V_- &= -f_- \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.19}$$

Suppose $k > -\infty$ and set $f^k = \max(f, k)$. Then there exists a sequence $\{f_n^k\} \subset C^2(\partial\Omega)$ such that $f_n^k \geqslant k$ and

$$\lim_{n\to\infty} (\|f_n^k - f^k\|_{L^1(\partial\Omega)} + \|e^{2(p-1)f_n^k} - e^{2(p-1)f^k}\|_{L^1(\partial\Omega)}) = 0. \tag{3.20}$$

If $v_n = v_n^k$ is the solution of the problem

$$\left.\begin{aligned} -\Delta v_n + Ge^{2v_n} &= 0 \quad \text{in } \Omega, \\ v_n &= f_n^k \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.21}$$

then the following identity holds:

$$\frac{\lambda_1}{2(p-1)} \int_\Omega e^{2(p-1)v_n} \psi_1 \, dx + 2(p-1) \int_\Omega e^{2(p-1)v_n} |\nabla v_n|^2 \psi_1 \, dx + \int_\Omega e^{2pv_n} G\psi_1 \, dx = \frac{-1}{2(p-1)} \int_{\partial\Omega} \frac{\partial \psi_1}{\partial n} e^{2(p-1)f_n^k} \, dS. \tag{3.22}$$

Let v^k be the limit of the v_n when n goes to infinity, since

$$\|v_n - v_m\|_{L^1(\Omega)} + \|\rho G(e^{2v_n} - e^{2v_m})\|_{L^1(\Omega)} \leqslant C\|f_n^k - f_m^k\|_{L^1(\partial\Omega)}, \tag{3.23}$$

one gets

$$\frac{\lambda_1}{2(p-1)} \int_\Omega e^{2(p-1)v^k} \psi_1 \, dx + 2(p-1) \int_\Omega e^{2(p-1)v^k} |\nabla v^k|^2 \psi_1 \, dx + \int_\Omega e^{2pv^k} G\psi_1 \, dx \leqslant \frac{-1}{2(p-1)} \int_{\partial\Omega} \frac{\partial \psi_1}{\partial n} e^{2(p-1)f^k} \, dS \tag{3.24}$$

from Fatou's Lemma, and v^k solves (3.4) with f replaced by f^k. When k goes to $-\infty$, $\{v^k\}$ decreases and converges to the solution v of (3.4). Finally, v satisfies (3.5) from Fatou's Lemma. □

If Ω is a bounded regular domain of $\mathbb{R}^d$ we recall that $(x, y) \mapsto P(x, y)$ is the Poisson kernel defined in $\Omega \times \partial\Omega$. If μ any Radon measure on $\partial\Omega$, the function

$$P_\mu(x) = \int_{\partial\Omega} P(x, y) \, d\mu(y) \qquad (\forall x \in \Omega) \tag{3.25}$$

is harmonic in Ω and has μ as boundary trace.

Definition 3.5. Let μ be a Radon measure on $\partial\Omega$, with Lebesgue decomposition

$$\mu = \mu_R \, dH^{d-1} + \mu_S, \tag{3.26}$$

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where μ_{R} is the regular part with respect to the $(d-1)$-dimensional Hausdorff measure $\mathrm{d}H^{d-1}$ and μ_{S} the singular part. We say that μ is *admissible* if there exists $p \in (1,\infty]$ such that

$$\left.\begin{aligned} &\text{(i)} \qquad \exp(2P_{\mu_{\mathrm{S}}}) \in L^{p/(p-1)}(\Omega, \rho\,\mathrm{d}x), \\ &\text{(ii)} \qquad \exp(2\mu_{\mathrm{R}}) \in L^{p-1}(\partial\Omega). \end{aligned}\right\} \tag{3.27}$$

We say that μ is bounded from below if $\mu + m\,\mathrm{d}H^{d-1}$ is positive for some real number m.

THEOREM 3.6. *Let Ω be as below and G be a continuous positive function defined in $\bar{\Omega}$. Then for any admissible, bounded from below Radon measure μ on $\partial\Omega$, there exists a unique $u \in L^1(\Omega)$ such that $\mathrm{e}^{2u} \in L^1(\Omega, \rho\,\mathrm{d}x)$ satisfying*

$$\int_\Omega (-u\Delta\zeta + G\mathrm{e}^{2u}\zeta)\,\mathrm{d}x = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\,\mathrm{d}\mu \qquad (\forall\zeta \in C_0^{1,1}(\bar{\Omega})). \tag{3.28}$$

Moreover, the mapping $\mu \mapsto u$ is non-decreasing.

Proof. Uniqueness follows from monotonicity and [8, theorem 2.1, p. 282]. Without any loss of generality it can be assumed that μ is positive since we can always replace μ by $\mu + m\,\mathrm{d}H^{d-1}$, and it is the same with μ_{R} and μ_{S}. For existence we shall distinguish according to whether p is finite or not.

Case 1. $p < \infty$. From lemma 3.3 there exists a sequence of functions $\{f_n\} \subset C^2(\partial\Omega)$, $f_n \geqslant 0$ such that

$$\lim_{n\to\infty} (\|\mathrm{e}^{2(p-1)f_n} - \mathrm{e}^{2(p-1)\mu_{\mathrm{R}}}\|_{L^1(\partial\Omega)} + \|f_n - \mu_{\mathrm{R}}\|_{L^1(\partial\Omega)}) = 0. \tag{3.29}$$

For $k \in \mathbb{N}^*$ we define the non-decreasing function e_k by $e_k(r) = \min(\mathrm{e}^{2r}, \mathrm{e}^{2k})$. Let $u_{n,k}$ be the solution of

$$\left.\begin{aligned} -\Delta u_{n,k} + Ge_k(u_{n,k}) = 0 \quad &\text{in } \Omega, \\ u_{n,k} = f_n + \mu_{\mathrm{S}} \quad &\text{on } \partial\Omega, \end{aligned}\right\} \tag{3.30}$$

in the weak sense, which means

$$\int_\Omega (-u_{n,k}\Delta\zeta + Ge_k(u_{n,k})\zeta)\,\mathrm{d}x = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\,\mathrm{d}\mu_{\mathrm{S}} - \int_{\partial\Omega} \frac{\partial\zeta}{\partial n} f_n\,\mathrm{d}S \qquad (\forall\zeta \in C_0^{1,1}(\bar{\Omega})); \tag{3.31}$$

the existence of $u_{n,k}$ is a consequence of [8, theorem 2.1]. Denoting by $v_{n,k}$ the solution of

$$\left.\begin{aligned} -\Delta v_{n,k} + Ge_k(v_{n,k}) = 0 \quad &\text{in } \Omega, \\ v_{n,k} = f_n \quad &\text{on } \partial\Omega, \end{aligned}\right\} \tag{3.32}$$

and setting $w_{n,k} = v_{n,k} + P_{\mu_{\mathrm{S}}}$, one has

$$-\Delta w_{n,k} + Ge_k(w_{n,k}) \geqslant 0 \tag{3.33}$$

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in the weak sense, since $P_{\mu_{\rm S}} \geqslant 0$. But $u_{n,k}$ and $w_{n,k}$ coincide in the sense of measures on $\partial\Omega$. We deduce again from [8, lemma 2.2] that $u_{n,k} \leqslant w_{n,k}$. Moreover, $\Psi \leqslant u_{n,k}$, where Ψ is the solution of

$$\left.\begin{aligned} -\Delta\Psi + G\mathrm{e}^{2\Psi} = 0 \quad \text{in } \Omega, \\ \Psi = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.34}$$

Since $e_k(r)$ increases with k the two sequences $\{u_{n,k}\}_k$ and $\{v_{n,k}\}_k$ decrease and converge respectively to u_n and v_n. Clearly, v_n is the classical solution of

$$\left.\begin{aligned} -\Delta v_n + G\mathrm{e}^{2v_n} = 0 \quad \text{in } \Omega, \\ v_n = f_n \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.35}$$

In fact, for $k \geqslant k_0$ large enough, $e_k(v_{n,k}) = \mathrm{e}^{2v_{n,k}}$ and $v_{n,k} = v_n$, since

$$v_{n,k}(x) \leqslant \int_{\partial\Omega} P(x,y) f_n(y)\,\mathrm{d}y \quad \text{in } \Omega. \tag{3.36}$$

Moreover, $e_k(u_{n,k}) \to \mathrm{e}^{2u_n}$ a.e. in Ω. We also have the following estimate:

$$0 \leqslant e_k(u_{n,k}) \leqslant e_k(v_{n,k} + P_{\mu_{\rm S}}) \leqslant \mathrm{e}^{2(v_{n,k}+P_{\mu_{\rm S}})} = \mathrm{e}^{2v_{n,k}}\mathrm{e}^{2P_{\mu_{\rm S}}} = \mathrm{e}^{2v_n}\mathrm{e}^{2P_{\mu_{\rm S}}}. \tag{3.37}$$

But $\mathrm{e}^{2v_n} \in L^p(\Omega, \rho\,\mathrm{d}x)$ from lemma 3.4 and $\mathrm{e}^{2P_{\mu_{\rm S}}} \in L^{p/(p-1)}(\Omega, \rho\,\mathrm{d}x)$; therefore, it follows from Lebesgue's theorem that $\lim_{k\to\infty} Ge_k(u_{n,k}) = G\mathrm{e}^{2u_n}$ in $L^1(\Omega, \rho\,\mathrm{d}x)$. Going to the limit in (3.31) yields

$$\int_\Omega (-u_n\Delta\zeta + G\mathrm{e}^{2u_n}\zeta)\,\mathrm{d}x = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\,\mathrm{d}\mu_{\rm S} - \int_{\partial\Omega} \frac{\partial\zeta}{\partial n} f_n\,\mathrm{d}S. \tag{3.38}$$

Set $k > 0$ such that $G(x) \geqslant k > 0$ in $\bar{\Omega}$. From the Keller–Osserman-type estimate of [27] and the maximum principle, it follows

$$\Psi(x) \leqslant v_n(x) \leqslant \ln(1/\rho(x)) + \ln(4/k) \qquad (\forall x \in \Omega) \tag{3.39}$$

(Ψ is given by (3.34)). Moreover,

$$\Psi(x) \leqslant u_n(x) \leqslant P_{\mu_{\rm S}}(x) + v_n(x) \quad \text{and} \quad \mathrm{e}^{2u_n(x)} \leqslant \mathrm{e}^{2P_{\mu_{\rm S}}(x)}\mathrm{e}^{2v_n(x)} \tag{3.40}$$

in Ω. Again from lemma 3.4, $\{\mathrm{e}^{2v_n}\}$ remains bounded in $L^p(\Omega, \rho\,\mathrm{d}x)$ independently of n. From (3.39) and the elliptic equations local regularity theory there exist a subsequence $\{v_{n_t}\} \subset \{v_n\}$ and a function $v \in \bigcap_{1\leqslant q<\infty} W^{2,q}_{\rm loc}(\Omega)$, with $q \in (1,\infty)$ as large as needed, such that

$$v_{n_t} \xrightarrow[t\to\infty]{} v \quad \text{in the } C^1_{\rm loc}(\Omega)\text{-topology}, \tag{3.41}$$

$$\mathrm{e}^{2v_{n_t}} \xrightarrow[t\to\infty]{} \mathrm{e}^{2v} \quad \text{in the weak } (L^p(\Omega, \rho\,\mathrm{d}x), L^{p/(p-1)}(\Omega, \rho\,\mathrm{d}x))\text{-topology}. \tag{3.42}$$

Moreover, for every Borel subset $\omega \subset \Omega$ there holds

$$\int_\omega G\mathrm{e}^{2P_{\mu_{\rm S}}}\mathrm{e}^{2v_{n_t}}\rho\,\mathrm{d}x \leqslant \biggl(\int_\omega \mathrm{e}^{2pv_{n_t}}\rho\,\mathrm{d}x\biggr)^{1/p}\biggl(\int_\omega \mathrm{e}^{2pP_{\mu_{\rm S}}/(p-1)}\rho\,\mathrm{d}x\biggr)^{1-1/p} \max_\Omega G. \tag{3.43}$$

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Therefore, the sequence $\{e^{2P_{\mu_S}} e^{2v_{n_t}}\}$ is equi-integrable for the measure $\rho\,dx$, in the sense that

$$\forall\varepsilon>0,\quad \exists\delta>0,\quad \forall\omega\subset\Omega,\quad \omega \text{ measurable},$$
$$\int_\omega \rho\,dx<\delta \quad\Rightarrow\quad \int_\omega e^{2P_{\mu_S}} e^{2v_{n_t}}\rho\,dx<\delta. \tag{3.44}$$

Estimate (3.39) implies that $\{v_{n_t}\}$ is bounded in any $L^q(\Omega)$ $(1<q<\infty)$ and therefore

$$v_{n_t} \xrightarrow[t\to\infty]{} v$$

weakly in $L^q(\Omega)$. Going to the limit in the weak formulation of equation (3.35) one gets

$$\int_\Omega(-v\Delta\zeta+Ge^{2v}\zeta)\,dx=-\int_{\partial\Omega}\frac{\partial\zeta}{\partial n}\,d\mu_S \qquad (\forall\zeta\in C_0^{1,1}(\bar\Omega)). \tag{3.45}$$

Because of the uniqueness of the solution v of (3.45) one can replace $\{v_{n_t}\}$ by the full sequence $\{v_n\}$. From (3.39), (3.41) and the exponential estimate coming from the admissibility assumption the sequence $\{u_n\}$ is bounded in $L^q(\Omega)$, $1<q<\infty$. From (3.41) and (3.44) the sequence $\{e^{2u_n}\}$ is equi-integrable for the measure $\rho\,dx$ and therefore it is relatively compact in the weak $(L^1(\Omega,\rho\,dx),L^\infty(\Omega))$-topology (from the Dunford–Pettis weak compactness theorem). Consequently, there exist a function $u\in\bigcap_{1\leqslant q<\infty}W^{2,q}_{\mathrm{loc}}(\Omega)$ and a subsequence $\{u_{n_t}\}\subset\{u_n\}$ such that

$$\left.\begin{aligned} u_{n_t}&\xrightarrow[t\to\infty]{} u \quad \text{in the } C^1_{\mathrm{loc}}(\Omega)\text{-topology and weakly in } L^q(\Omega),\\ e^{2u_{n_t}}&\xrightarrow[t\to\infty]{} e^{2u} \quad \text{in weakly in } L^1(\Omega,\rho\,dx). \end{aligned}\right\} \tag{3.46}$$

Because of the relation

$$\lim_{t\to\infty}\left(\int_{\partial\Omega}\frac{\partial\zeta}{\partial n}\,d\mu_S+\int_{\partial\Omega}\frac{\partial\zeta}{\partial n}f_{n_t}\,dS\right)=\int_{\partial\Omega}\frac{\partial\zeta}{\partial n}\,d\mu \qquad (\forall\zeta\in C_0^{1,1}(\bar\Omega)) \tag{3.47}$$

one deduces from (3.45) that (3.28) holds. Since u is unique $\{u_{n_t}\}$ can be replaced by $\{u_n\}$. Finally, the fact that the mapping $\mu\mapsto u$ is increasing follows from the construction and the uniqueness.

Case 2. $p=\infty$. We first construct a solution u_k of

$$\left.\begin{aligned} -\Delta u_k+Ge_k(u_k)&=0 \quad \text{in } \Omega,\\ u_k&=\mu_R+\mu_S \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.48}$$

where e_k is defined as in case 1. If we set $L=\|\mu_R\|_{L^\infty}$, then

$$0\leqslant u_k(x)\leqslant L+P_{\mu_S}(x) \quad\text{and}\quad e_k(u_k(x))\leqslant e^{2L}e^{2P_{\mu_S}}(x) \quad \text{in } \Omega. \tag{3.49}$$

Similarly to case 1, there exists a function $u\in\bigcap_{1\leqslant q<\infty}W^{2,q}_{\mathrm{loc}}(\Omega)$ and a subsequence $\{u_{k_t}\}\subset\{u_k\}$ such that

$$\left.\begin{aligned} u_{k_t}&\xrightarrow[t\to\infty]{} u \quad \text{in the } C^1_{\mathrm{loc}}(\Omega)\text{-topology and weakly in } L^q(\Omega),\\ e_k(u_{k_t})&\xrightarrow[t\to\infty]{} e^{2u} \quad \text{weakly in } L^1(\Omega,\rho\,dx). \end{aligned}\right\} \tag{3.50}$$

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Letting t go to infinity in

$$\int_{\Omega}(-u_{k_t}\Delta\zeta + Ge_k(u_{k_t})\zeta)\,\mathrm{d}x = -\int_{\partial\Omega}\frac{\partial\zeta}{\partial n}\,\mathrm{d}\mu, \tag{3.51}$$

where $\zeta \in C_0^{1,1}(\bar{\Omega})$, one obtains (3.28). □

In the remaining part of this section we assume that $\Omega = B^2 \subset \mathbb{R}^2$, we denote by P the Poisson kernel in B^2 and suppose that K is continuous and negative in $\overline{B^2}$, where it satisfies $0 < k \leqslant -K(x) \leqslant k^{-1}$. From theorem 2.1 the boundary trace of a minorized solution u of (1.3) in B^2 (i.e. an element of the class $\mathcal{E}$) is characterized by the singular set $\mathcal{S} \subset \partial B^2$, which is a closed subset, and a bounded from below Radon measure on the relatively open set $\mathcal{R} = \partial B^2\backslash\mathcal{S}$, and we denote $\mathrm{tr}_{|\partial B^2}(u) = (\mathcal{S}, \mu)$. In the same way as in [18], one can define the trace in terms of outer regular Borel measure on ∂B^2: for every minorized regular Borel measure $\tilde{\mu}$ on ∂B^2 we define the set of *regular points* $\mathcal{R}_{\tilde{\mu}}$ and the set of *blow-up points* $\mathcal{S}_{\tilde{\mu}}$ as follows,

$$\mathcal{R}_{\tilde{\mu}} = \{\sigma \in \partial B^2 : \exists \text{ a relatively open neighbourhood } U \text{ of } \sigma, \text{ s.t. } \tilde{\mu}(U) < \infty\}, \tag{3.52}$$

$$\mathcal{S}_{\tilde{\mu}} = \partial B^2\backslash\mathcal{R}_{\tilde{\mu}}, \tag{3.53}$$

and for any relatively open neighbourhood U of $\sigma \in \mathcal{S}_{\tilde{\mu}}$, $\tilde{\mu}(U) = \infty$. Therefore, $\mu = \tilde{\mu}_{|\mathcal{R}_{\tilde{\mu}}}$ is a minorized Radon measure on $\mathcal{R}_{\tilde{\mu}}$. Conversely, to each couple $(\mathcal{S}, \mu)$ where $\mathcal{S}$ is a closed subset of ∂B^2 and μ minorized Radon measure on $\mathcal{R} = \partial B^2\backslash\mathcal{S}$, we associate a regular and minorized Borel measure $\tilde{\mu}$ by

$$\tilde{\mu}(A) = \begin{cases} \mu(A) & \text{if } A \subset \mathcal{R}, \\ \infty & \text{if } A \cap \mathcal{S} \neq \emptyset, \end{cases} \tag{3.54}$$

for every Borel subset A of ∂B^2. It is proved in [18] that $\mathcal{R}_{\tilde{\mu}} = \mathcal{R}$, $\mathcal{S}_{\tilde{\mu}} = \mathcal{S}$ and that the correspondence $(\mathcal{S}, \mu) \leftrightarrow \tilde{\mu}$ is one to one. With this result we denote

$$\mathrm{tr}_{|\partial B^2}(u) = (\mathcal{S}, \mu) \Leftrightarrow \mathrm{Tr}_{|\partial B^2}(u) = \tilde{\mu}. \tag{3.55}$$

Definition 3.7. Let μ be a Radon measure on a relatively open subset $\mathcal{R} \subset \partial B^2$. We say that μ is locally admissible if for any compact subset $F \subset \mathcal{R}$ the restriction $\mu_{|F}$ of μ to F defined by

$$\mu_{|F}(A) = \mu(A \cap F), \quad \text{for every Borel subset } A \subset \partial B^2 \tag{3.56}$$

is admissible in the sense of definition 3.5.

If $\mathcal{S}$ is a closed subset of ∂B^2 and $\varepsilon > 0$, we set

$$\mathcal{S}_\varepsilon = \{x \in \partial B^2 : \mathrm{dist}(x, \mathcal{S})\} < \varepsilon = \bigcup_{\omega\in S} D_\varepsilon(\omega), \qquad \bar{\mathcal{S}}_\varepsilon = \bigcup_{\omega\in S} \bar{D}_\varepsilon(\omega), \tag{3.57}$$

where $D_\varepsilon(\omega)$ is the open geodesic disc on ∂B^2 with centre ω and radius ε (in fact, it is just an interval on the circle ∂B^2). We also set $\mathcal{R} = \partial B^2\backslash\mathcal{S}$ and $\mathcal{R}_\varepsilon = \partial B^2\backslash\bar{\mathcal{S}}_\varepsilon$. The following result follows from [23, theorem 7.2] (see [2,28] for related uniqueness results).

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PROPOSITION 3.8. *Suppose K is continuous and negative in $\overline{B^2}$. Then there exists one and only one function Φ which satisfies*

$$\left.\begin{aligned} \Delta\Phi + K\mathrm{e}^{2\Phi} &= 0 \quad \text{in } B^2, \\ \lim_{|x|\to 1} \Phi(x) &= \infty. \end{aligned}\right\} \tag{3.58}$$

This Φ is the maximal solution of (1.3) in the sense that it dominates any other solution. When K is constant and equal to $-k$, then

$$\Phi_k(x) = \ln\biggl(\frac{4}{k(1-|x|^2)}\biggr). \tag{3.59}$$

For $\varepsilon > 0$ and $n \in \mathbb{N}^*$, we denote $u_{n,\mathcal{S}_\varepsilon}$ the solution of

$$\left.\begin{aligned} \Delta u_{n,\mathcal{S}_\varepsilon} + K\mathrm{e}^{2u_{n,\mathcal{S}_\varepsilon}} &= 0 \quad \text{in } B^2, \\ u_{n,\mathcal{S}_\varepsilon} &= n\chi_{\mathcal{S}_\varepsilon} \quad \text{on } \partial B^2. \end{aligned}\right\} \tag{3.60}$$

LEMMA 3.9. *The sequence $\{u_{n,\mathcal{S}_\varepsilon}\}$ is increasing and converges to a solution $u_{\mathcal{S}_\varepsilon}$ of (1.3) the boundary trace of which is $(\bar{\mathcal{S}}_\varepsilon, 0)$.*

Proof. The monotonicity of the sequence follows from the maximum principle. In order to prove that the zero boundary condition is maintained on $\mathcal{R}_\varepsilon$, we pick an x_0 on $\mathcal{R}_\varepsilon$ and a positive ρ_0 such that $\operatorname{dist}(x_0, \mathcal{S}_\varepsilon) > \rho_0$. Clearly, $u_{n,\mathcal{S}_\varepsilon}$ is dominated in $B^2 \cap B_{\rho_0}(x_0)$ by the maximum solution of the equation (1.3), where $K(x)$ is replaced by $k = \min_{B^2}(-K)$ (k is positive by assumption), and this maximal solution is

$$x \mapsto \ln\biggl(\frac{2\rho_0}{\sqrt{k}(\rho_0^2 - |x-x_0|^2)}\biggr). \tag{3.61}$$

Consequently, $u_{n,\mathcal{S}_\varepsilon}$ remains locally uniformly bounded near $\mathcal{R}_\varepsilon$. By the elliptic equations regularity theory, the boundary conditions on $\mathcal{R}_\varepsilon$ remain and $u_{n,\mathcal{S}_\varepsilon}$ converges to a solution $u_{\mathcal{S}_\varepsilon}$ with boundary trace $(\bar{\mathcal{S}}_\varepsilon, 0)$. In fact, $u_{\mathcal{S}_\varepsilon}$ satisfies

$$\lim_{r\to 1} \frac{u_{\mathcal{S}_\varepsilon}(r,\sigma)}{\ln(1/(1-r^2))} = 1 \tag{3.62}$$

uniformly on compact subsets of $\mathcal{S}_\varepsilon$. This precise estimate follows by scaling techniques as in [23, theorem 7.2] or local comparison techniques as in [3]. When ε goes to 0, $\{u_{\mathcal{S}_\varepsilon}\}$ decreases to a solution $u^*_{\mathcal{S}}$ of (1.3) with boundary trace $(\mathcal{S}^*, 0)$. □

PROPOSITION 3.10. *There always holds $\mathcal{S}^* \subseteq \mathcal{S}$, and $u^*_{\mathcal{S}}$ is the maximal solution of (1.3) among all the solutions in the class $\mathcal{E}$ with boundary trace $(\mathcal{S}^*, 0)$.*

Proof. The fact that $\mathcal{S}^* \subseteq \mathcal{S}$ is proved as in [18, theorem 3.5], but for the sake of self-containedness, we give an outline of a direct argument. If there exists some $\theta \in \mathcal{S}^*\backslash\mathcal{S}$, then $\operatorname{dist}(\theta, \mathcal{S}) = \tau > 0$ and $\theta \notin \bar{\mathcal{S}}_\varepsilon$ for $0 < \varepsilon < \tau$; therefore, $\lim_{r\to 1} u_{\mathcal{S}_\varepsilon} = 0$, uniformly when $\operatorname{dist}(\sigma, \theta) \leqslant (\tau - \varepsilon)/3$. Since $\{u_{\mathcal{S}_\varepsilon}\}_\varepsilon$ decreases with ε, θ is not a singular point of the boundary trace of $u^*_{\mathcal{S}}$.

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If $\tilde{u} \in \mathcal{E}$ has boundary trace $(\mathcal{S}^*, 0)$, then

$$\lim_{r\to 1} \int_{\mathcal{R}^*} u(r,\sigma)\zeta(\sigma)\,\mathrm{d}\sigma = 0 \tag{3.63}$$

for any $\zeta \in C_0^2(\mathcal{R}^*)$, where $\mathcal{R}^* = \partial B^2 \backslash \mathcal{S}^*$.

Step 1. We claim that $\tilde{u}^+(r,\sigma)$ converges to 0 when r goes to 1, locally uniformly in $\mathcal{R}^*$. For any closed subset F or $\mathcal{R}^*$, there holds

$$\int_0^1 \int_F \mathrm{e}^{2\tilde{u}}(1-r)\,\mathrm{d}\sigma\,\mathrm{d}r < \infty. \tag{3.64}$$

Consequently, for any $\gamma > 0$ there exists a connected open subset G of ∂B^2 such that $G \subset \bar{G} \subset F \subset \mathcal{R}^*$, with $\mathrm{dist}(G, \mathcal{S}^*) < \gamma$ and

$$\int_{1/2}^1 |\tilde{u}|(r,\sigma_j)\,\mathrm{d}r < \infty \qquad (j = 1,2), \tag{3.65}$$

since $\partial G = \{\sigma_1, \sigma_2\}$. Therefore, $\tilde{u} \leqslant U$ in the truncated cone $C_G = \{(r,\sigma) \in (1/2,1) \times G\}$, where U is the solution of

$$\left.\begin{aligned} -\Delta U &= 0 && \text{in } C_G, \\ U &= \tilde{u}^+ && \text{on } \partial_\ell C_G \cup \partial_b C_G, \\ U &= 0 && \text{on } \partial_v C_G, \end{aligned}\right\} \tag{3.66}$$

with $\partial_\ell C_G = [1/2,1] \times \partial G$, $\partial_b C_G = \{1/2\} \times G$ and $\partial_v C_G = \{1\} \times G$ expressed in the (r,σ)-variables. Since $U(x)$ goes to 0 locally uniformly in G when $|x|$ goes to 1, the same property holds for the positive part of $\tilde{u}$, and this is also true on any compact subset of $\mathcal{R}^*$.

Step 2. Let Ψ be the solution of

$$\left.\begin{aligned} \Delta\Psi + K\mathrm{e}^{2\Psi} &= 0 \quad \text{in } B^2, \\ \Psi &= 0 \quad \text{on } \partial B^2. \end{aligned}\right\} \tag{3.67}$$

Since Ψ is negative in B^2, it minorizes any solution of (1.3) with non-negative measure boundary data. Let v_1 and v_2 be two such solutions, then we claim that $\varphi = v_1 + v_2 - 2\Psi$ is a supersolution of (1.3). Actually,

$$\Delta\varphi = -K(\mathrm{e}^{2v_1} + \mathrm{e}^{2v_2} - 2\mathrm{e}^{2\Psi}). \tag{3.68}$$

If we define $\beta(x,y) = \mathrm{e}^{2x+2y-4\Psi} - \mathrm{e}^{2x} - \mathrm{e}^{2y} + 2\mathrm{e}^{2\Psi}$ for $(x,y) \in [\Psi,\infty) \times [\Psi,\infty)$, then

$$\partial_x\beta(x,y) = 2\mathrm{e}^{2x+2y-4\Psi} - 2\mathrm{e}^{2x} = 2\mathrm{e}^{2x}(\mathrm{e}^{2y-4\Psi} - 1). \tag{3.69}$$

But $2y - 4\Psi \geqslant 0$ since $y \geqslant \Psi$ and $\Psi \leqslant 0$. Therefore,

$$\beta(x,y) \geqslant \beta(\Psi,y) = \mathrm{e}^{2y-2\Psi} - \mathrm{e}^{2y} - \mathrm{e}^{2\Psi}. \tag{3.70}$$

Since $\partial_y\beta(\Psi,y) = 2(\mathrm{e}^{2y-2\Psi} - \mathrm{e}^{2y}) = 2\mathrm{e}^{2y}(\mathrm{e}^{-2\Psi} - 1) \geqslant 0$, one deduces also that

$$\beta(\Psi,y) \geqslant \beta(\Psi,\Psi) = 1. \tag{3.71}$$

Consequently, $\mathrm{e}^{2\varphi} \geqslant \mathrm{e}^{2x} + \mathrm{e}^{2y} - 2\mathrm{e}^{2\Psi}$ and φ is a supersolution.

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STEP 3. Construction of a dominating solution. Let $\varepsilon > 0$, $\delta > 0$, and let $\eta \in (0,1)$ be such that

$$\tilde{u}^{+}(r,\sigma) \leqslant \delta \qquad (\forall (r,\sigma) \in [1-\eta, 1) \times \mathcal{R}_{\varepsilon}) \tag{3.72}$$

(see step 1). We denote $v_{\eta,\varepsilon} = v^{\delta}_{\eta,\varepsilon}$ the solution of

$$\left.\begin{aligned} \Delta v_{\eta,\varepsilon} + K\mathrm{e}^{2v_{\eta,\varepsilon}} &= 0 && \text{in } B^2_{1-\eta} = \{x : |x| < 1-\eta\}, \\ v_{\eta,\varepsilon}(1-\eta,\sigma) &= \delta && \text{if } \sigma \in \mathcal{R}_{\varepsilon}, \\ v_{\eta,\varepsilon}(1-\eta,\sigma) &= \infty && \text{if } \sigma \in \bar{\mathcal{S}}_{\varepsilon}, \end{aligned}\right\} \tag{3.73}$$

such a solution being obtained by an increasing scheme of approximate solutions as in lemma 3.9. Clearly, $v_{\eta,\varepsilon} \geqslant \tilde{u}$ in $B^2_{1-\eta}$. If we denote $v^1_{\eta,\varepsilon}$ (respectively, $v^2_{\eta,\varepsilon}$) the solution of (1.3) in $B^2_{1-\eta}$ with boundary data $\delta\chi_{\mathcal{R}_{\varepsilon}}$ (respectively, ∞ on $\bar{\mathcal{S}}_{\varepsilon}$ and 0 on $\mathcal{R}_{\varepsilon}$), then $v^1_{\eta,\varepsilon} + v^2_{\eta,\varepsilon} - 2\Psi$ is a supersolution of (1.3) which dominates $v_{\eta,\varepsilon}$. As for the function $v^2_{\eta,\varepsilon}$, it is bounded from above by the function,

$$x \mapsto V_{\eta,\varepsilon}(x) = V_{\varepsilon}(x/(1-\eta)) - \ln(1-\eta), \tag{3.74}$$

where V_{ε} is obtained from lemma 3.4 with K replaced by $-k = \max_{B^2} K$. As for $V_{\eta,\varepsilon}$, it satisfies the same equation as V_{ε}, with infinite boundary value if $\sigma \in \mathcal{S}_{\varepsilon}$ and some positive one if $\sigma \in \mathcal{R}_{\varepsilon}$. When η goes to 0, $v_{\eta,\varepsilon}$ converges to a solution v^{δ}_{ε} of (1.3) which blows-up on $\mathcal{S}_{\varepsilon}$ and takes the value δ on $\mathcal{R}_{\varepsilon}$. Letting δ go to 0 implies that $\delta \mapsto v^{\delta}_{\varepsilon}$ decreases to some v_{ε} which dominates $\tilde{u}$ and has boundary trace $(\bar{\mathcal{S}}_{\varepsilon}, 0)$ (clearly, v_{ε} is bounded from below by Ψ).

STEP 4. End of the proof. We claim that

$$u_{\mathcal{S}_{\varepsilon'}} \leqslant v_{\varepsilon} \leqslant u_{\mathcal{S}_{\varepsilon''}} \qquad (0 < \varepsilon' < \varepsilon < \varepsilon''). \tag{3.75}$$

The above three functions satisfy the blow-up estimate (3.62) locally uniformly in the interior of their respective singular boundary set. For $\tau > 0$ the function

$$x \mapsto V^{\tau}(x) = (1+\tau)v_{\varepsilon}(x) + \tau\|\Psi^{-}\|_{L^{\infty}} \tag{3.76}$$

is a supersolution of (1.3) in B^2. Since $x \mapsto (u_{\mathcal{S}_{\varepsilon'}}(x) - V^{\tau}(x))^{+}$ has compact support one gets $u_{\mathcal{S}_{\varepsilon'}} \leqslant V^{\tau}$. The inequality $u_{\mathcal{S}_{\varepsilon'}} \leqslant v_{\varepsilon}$ is derived by letting τ go to 0. The left-hand side of (3.75) is proved in the same way. If we take $\varepsilon' = \varepsilon/2$, $\varepsilon'' = 2\varepsilon$ and let ε go to 0, then $v_{\varepsilon} \to u_{S^*}$ and $u_{S^*} \geqslant \tilde{u}$ in B^2. □

In the following, we extend to the exponential case the result [19, lemma 3.3].

PROPOSITION 3.11. *Let $\mathcal{R}$ be a relatively open subset of ∂B^2 and μ a non-negative locally admissible Radon measure on $\mathcal{R}$. Then there exists a minimal solution u_{μ} of (1.3) in B^2 with trace μ on $\mathcal{R}$ among the solutions u which satisfy $\lim_{r\to 1} u^{-}(r,\sigma) = 0$ uniformly for $\sigma \in S^1$.*

Proof. STEP 1. Construction of u_{μ}. For $\varepsilon > 0$ we denote $u_{\mu_{\varepsilon}}$ the solution of (1.3) with $\mu_{\varepsilon} = \mu_{|\mathcal{R}_{\varepsilon}}$, the existence of which follows from theorem 3.6 as well as the monotonicity of the correspondence $\varepsilon \mapsto u_{\mu_{\varepsilon}}$. Since

$$\Psi(x) \leqslant u_{\mu_{\varepsilon}}(x) \leqslant \ln(1/(1-|x|)) + \ln(4/k), \tag{3.77}$$

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where Ψ is given by (3.67), $\lim_{r\to 1} u^-_{\mu_\varepsilon}(r,\sigma)=0$ holds uniformly for $\sigma\in S^1$. But

$$\int_{B^2}(-u_{\mu_\varepsilon}\Delta\zeta - K\mathrm{e}^{2u_{\mu_\varepsilon}}\zeta)\,\mathrm{d}x = -\int_{\partial B^2}\frac{\partial\zeta}{\partial n}\,\mathrm{d}\mu_\varepsilon \qquad (\zeta\in C_0^{1,1}(\bar{B}^2)). \tag{3.78}$$

Moreover, u_{μ_ε} converges monotonically to some ω. Therefore, restricting ζ to be non-negative and have compact support in $B^2\cup\mathcal{R}$ and letting ε go to 0, one deduces from the monotone convergence theorem that

$$\int_{B^2}(-\omega\Delta\zeta - K\mathrm{e}^{2\omega}\zeta)\,\mathrm{d}x = -\int_{\partial B^2}\frac{\partial\zeta}{\partial n}\,\mathrm{d}\mu \tag{3.79}$$

holds. It follows from remark 2.8 that ω has boundary trace μ on $\mathcal{R}$; we shall write $\omega = u_\mu$.

STEP 2. u_μ is minimal. Let u be another solution of (1.3) with boundary trace μ on $\mathcal{R}$ and such that $\lim_{r\to1} u^-(r,\sigma)=0$ uniformly on S^1, then $u\geqslant\Psi$. For $\varepsilon>0$, $w=u-u_{\mu_\varepsilon}$ has zero boundary trace on $\mathcal{R}_\varepsilon$ in the sense of measures and

$$\int_0^1\int_{\mathcal{R}_\varepsilon}(\mathrm{e}^{2u}+\mathrm{e}^{2u_{\mu_\varepsilon}})(1-r)r\,\mathrm{d}\sigma\,\mathrm{d}r<\infty. \tag{3.80}$$

Consequently, w solves

$$-\Delta w = f \tag{3.81}$$

in $C_{\mathcal{R}_\varepsilon}=\{x=(r,\sigma): 1/2<r<1, \sigma\in\mathcal{R}_\varepsilon\}$, for some $f\in L^1(C_{\mathcal{R}_\varepsilon},(1-|x|)\,\mathrm{d}x)$. By the same analysis as the one of step 1 of proposition 3.10 and the regularity theory of elliptic equations in L^1, one obtains

$$\lim_{r\to1}\int_{\mathcal{R}_\varepsilon}|u-u_{\mu_\varepsilon}|(r,\sigma)\,\mathrm{d}\sigma = 0 \tag{3.82}$$

(it can always be assumed that u is integrable on $\partial\mathcal{R}_\varepsilon\times(1/2,1)$ by using Fubini's theorem and theorem 2.1). Consequently,

$$\lim_{r\to1}\int_{S^1}(u_{\mu_\varepsilon}-u)^+(r,\sigma)\,\mathrm{d}\sigma = 0.$$

Since

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\int_{S^1}(u_{\mu_\varepsilon}-u)^+(r,\sigma)\,\mathrm{d}\sigma + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_{S^1}(u_{\mu_\varepsilon}-u)^+(r,\sigma)\,\mathrm{d}\sigma \geqslant 0 \tag{3.83}$$

holds in the sense of distributions on $(0,1)$, it follows from the maximum principle that $\int_{S^1}(u_{\mu_\varepsilon}-u)^+(\cdot,\sigma)\,\mathrm{d}\sigma\equiv 0$, and $u_{\mu_\varepsilon}\leqslant u$. Using step 1, one ends the proof by letting ε go to 0. □

Let $\partial_\mu\mathcal{S}$ denote the singular part of the boundary trace on u_μ, then $\partial_\mu\mathcal{S}\subset\mathcal{S}$. The following result which expresses under what condition a couple $(\mathcal{S},\mu)$ is a trace is the analogous of a previous result of Marcus and Véron [19, theorem 3.5] dealing with the d-dimensional equation (1.5) in the case $q\geqslant(d+1)/(d-1)$.

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THEOREM 3.12. *Let $\mathcal{S}$ be a closed subset of ∂B^2 and μ a locally admissible positive Radon measure on $\mathcal{R} = \partial B^2 \backslash \mathcal{S}$. Then there exists a solution u of (1.3) in B^2 with boundary trace $(\mathcal{S}, \mu)$ if and only if*

$$\mathcal{S} = \partial_\mu \mathcal{S} \cup \mathcal{S}^*. \tag{3.84}$$

Proof. STEP 1. The condition is sufficient. Assume that $\mathcal{S} = \partial_\mu \mathcal{S} \cup \mathcal{S}^*$; for $\varepsilon > 0$ and $n \in \mathbb{N}^*$ let u_{μ_ε} and $u_{n,\mathcal{S}_\varepsilon}$ be the functions constructed in (3.60) and proposition 3.11, and $u_{n,\varepsilon}$ the solution of (1.3) with boundary trace $(\emptyset, n\chi_{\mathcal{S}_\varepsilon} + \mu_\varepsilon)$, where $\mu_\varepsilon = \mu_{|\mathcal{R}_\varepsilon}$. From theorem 3.6 and step 2 of proposition 3.10, there holds

$$\max(u_{\mu_\varepsilon}, u_{n,\mathcal{S}_\varepsilon}) \leqslant u_{n,\varepsilon} \leqslant u_{\mu_\varepsilon} + u_{n,\mathcal{S}_\varepsilon} - 2\Psi. \tag{3.85}$$

When n goes to infinity $u_{n,\varepsilon}$ increases and converges to a solution u_ε of (1.3) and

$$\max(u_{\mu_\varepsilon}, u_{\mathcal{S}_\varepsilon}) \leqslant u_\varepsilon \leqslant u_{\mu_\varepsilon} + u_{\mathcal{S}_\varepsilon} - 2\Psi. \tag{3.86}$$

It follows that u_ε has boundary trace $(\mathcal{S}_\varepsilon, \mu_\varepsilon)$. When ε decreases to 0, the two terms $\max(u_{\mu_\varepsilon}, u_{\mathcal{S}_\varepsilon})$ and $u_{\mu_\varepsilon} + u_{\mathcal{S}_\varepsilon} - 2\Psi$ converge respectively to $\max(u_\mu, u^*_{\mathcal{S}})$ and $u_\mu + u^*_{\mathcal{S}} - 2\Psi$. Up to some subsequence $\{\varepsilon_n\}$ with limit 0, $\{u_{\mu_{\varepsilon_n}}\}$ converges in the $C^1_{\text{loc}}(B^2)$-topology to some solution u of (1.3) which satisfies

$$\max(u_\mu, u^*_{\mathcal{S}}) \leqslant u \leqslant u_\mu + u^*_{\mathcal{S}} - 2\Psi. \tag{3.87}$$

But this relation implies that the singular set of the boundary trace of u is $\partial_\mu \mathcal{S} \cup \mathcal{S}^*$. It follows from (3.87) and theorem 2.1 that $\text{tr}_{|\partial B^2}(u) = (\mathcal{S}, \mu)$.

STEP 2. The condition is necessary. Let $\tilde{u}$ be a solution of (1.3) such that $\text{tr}_{|\partial B^2}(\tilde{u}) = (\mathcal{S}, \mu)$, in the class $\mathcal{E}_0$ of functions u which satisfy $\lim_{r \to 1} \|u^-(r, \cdot)\|_{L^\infty(S^1)} = 0$. Then we construct a solution U of (1.3) such that

$$\tilde{u} \leqslant U \leqslant u_\mu + u^*_{\mathcal{S}} - 2\Psi \tag{3.88}$$

in the following way: for $\varepsilon, \delta > 0$, set

$$\varphi_\varepsilon(\sigma) = \begin{cases} u_{\mu_\varepsilon}(1-\delta, \sigma) & \text{if } \sigma \in \mathcal{R}_\varepsilon, \\ \Phi(1-\delta, \sigma) & \text{if } \sigma \in \mathcal{S}_\varepsilon, \end{cases} \tag{3.89}$$

where Φ is the maximal solution defined in proposition 3.2, and denote $U_{\delta,\varepsilon}$ and $W_{\delta,\varepsilon}$ the solutions of (1.3) in $B^2_{1-\delta}$ with respective boundary data φ_ε and $(\tilde{u}(1-\delta, \cdot) - \varphi_\varepsilon(\cdot))^+$. From step 2 of proposition 3.10, $U_{\delta,\varepsilon} + W_{\delta,\varepsilon} - 2\Psi$ is a supersolution of (1.3) and it dominates $\tilde{u}$ on $\partial B^2_{1-\delta}$ provided δ is chosen small enough. Therefore

$$\tilde{u} \leqslant U_{\delta,\varepsilon} + W_{\delta,\varepsilon} - 2\Psi \tag{3.90}$$

in $B^2_{1-\delta}$. Moreover, $W_{\delta,\varepsilon} \leqslant Y_{\delta,\varepsilon}$, which is the solution of (1.3) in $B^2_{1-\delta}$ with boundary data

$$\omega_\varepsilon(\sigma) = \begin{cases} \tilde{u}(1-\delta, \sigma) & \text{if } \sigma \in \mathcal{R}_\varepsilon, \\ 0 & \text{if } \sigma \in \mathcal{S}_\varepsilon. \end{cases} \tag{3.91}$$

As for the function $U_{\delta,\varepsilon}$ it is dominated by $u_{\mu_\varepsilon} + u_{\mathcal{S}_\varepsilon} - 2\Psi$. When δ goes to 0, $U_{\delta,\varepsilon}$ converges to a solution U_ε of (1.3) in B^2 with boundary trace $(\mathcal{S}_\varepsilon, \mu_\varepsilon)$. The function

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$(\tilde u(1-\delta,\cdot)-\varphi_\varepsilon(\cdot))^+$, which is the boundary data of $W_{\delta,\varepsilon}$, vanishes on $\mathcal{S}_\varepsilon$ and is equal to $\tilde u(1-\delta,\cdot)-\varphi_\varepsilon(\cdot)$ on $\mathcal{R}_\varepsilon$ since $u_{\mu_\varepsilon}\leqslant \tilde u$; therefore, it converges to 0 in the weak sense of measures on S^1 as $\delta\to 0$. If we set $W_{\delta,\varepsilon}(x)=W^\delta_{\delta,\varepsilon}(x/(1-\delta))-\ln(1-\delta)$, then

$$\left.\begin{aligned}\Delta W^\delta_{\delta,\varepsilon}+K((1-\delta)x)\mathrm{e}^{2W^\delta_{\delta,\varepsilon}}=0\quad\text{in } B^2,\\ W^\delta_{\delta,\varepsilon}=(\tilde u(1-\delta,\cdot)-\varphi_\varepsilon(\cdot))^+-\ln(1-\delta)\quad\text{on } \partial B^2.\end{aligned}\right\}\tag{3.92}$$

Recalling that Ψ_1 and λ_1 are defined in proposition 3.2, one gets

$$\int_{B^2}(\lambda_1 W^\delta_{\delta,\varepsilon}+K((1-\delta)x)\mathrm{e}^{2W^\delta_{\delta,\varepsilon}})\Psi_1\,\mathrm{d}x=-\int_{\partial B^2}\frac{\partial\Psi_1}{\partial n}W^\delta_{\delta,\varepsilon}\,\mathrm{d}\sigma.\tag{3.93}$$

But $W^\delta_{\delta,\varepsilon|\partial B^2}$ converges to 0 with δ and similarly does the right-hand side of (3.93); consequently, the same holds with $W^\delta_{\delta,\varepsilon}$ and $W_{\delta,\varepsilon}$, and (3.90) yields

$$\tilde u\leqslant U_\varepsilon-2\Psi\leqslant u_{\mathcal{S}_\varepsilon}+u_{\mu_\varepsilon}-4\Psi,\tag{3.94}$$

and finally $\tilde u\leqslant u^*_{\mathcal{S}}+u_\mu-4\Psi$. This implies that $\mathcal{S}\subseteq\mathcal{S}^*\cup\partial_\mu\mathcal{S}$. At the end, if u only belongs to the class $\mathcal{E}$ with $\mathrm{tr}_{|\partial B^2}(u)=(\mathcal{S},\mu)$, then $u^\ell=u+\ell$ has boundary trace $(\mathcal{S},\mu+\ell\,\mathrm{d}\sigma)$ and belongs to $\mathcal{E}_0$ for some ℓ. Obviously, $\partial_{\mu+\ell\,\mathrm{d}H^1}\mathcal{S}=\partial_\mu\mathcal{S}$, and (3.84) follows from the previous case. □

As in [19] it is not true that a given admissible (in the sense of theorem 3.6) boundary trace $(\mathcal{S},\mu)$ characterizes in a unique way a solution u of (1.3) such that $\mathrm{tr}_{|\partial B^2}(u)=(\mathcal{S},\mu)$. Actually, we have the following result.

Proposition 3.13. *There exist infinitely many solutions of equation (1.3) in B^2 with boundary trace $(\partial B^2,0)=(S^1,0)$.*

Proof. In the proof we borrow some of the ideas of [19, § 5]. Let $\{a_n\}_{n\in\mathbb{N}}$ be a dense subset in ∂B^2 and $\{\varepsilon_n\}_{n\in\mathbb{N}}$ a sequence of positive numbers to be specified later on. We denote $S_n=\{x\in\partial B^2:|x-a_n|\leqslant\varepsilon_n\}$ for $n\in\mathbb{N}$ and v_n the minimal solution (obtained by an increasing scheme) of

$$\Delta v_n+K(\mathrm{e}^{2v_n}-1)=0\tag{3.95}$$

in B^2 which satisfies

$$\left.\begin{aligned}&\text{(i)}\quad \lim_{r\to1}v_n(r,\sigma)/\ln(1/(1-r^2))=1\quad\text{if }\sigma\in S_n,\\ &\text{(ii)}\quad \lim_{r\to1}v_n(r,\sigma)=0\quad\text{if }\sigma\in S^1\backslash S_n.\end{aligned}\right\}\tag{3.96}$$

The construction and the asymptotics of v_n are obtained as in [3] and [23, § 7]. Moreover, $v_n\geqslant 0$. Since isolated points on ∂B^2 are removable singularity for equation of type (1.3)–(3.95) (see [8]), $\lim_{\varepsilon_n\to0}v_n=0$, uniformly on compact subset of B^2 (because $-K(x)\geqslant k>0$ this convergence does not depend on the position of the point a_n). Since $\mathrm{e}^{a+b}-1\geqslant\mathrm{e}^a-1+\mathrm{e}^b-1$ for any non-negative real numbers a and b, $V_n=\sum_{j=0}^n v_j$ is a supersolution of (3.95). We choose the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that

$$\sum_{j=0}^\infty v_j(0)<\infty.\tag{3.97}$$

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From Harnack inequality $\sum_{j=0}^{\infty} v_j(x) < \infty$ holds uniformly when x remains in a compact subset of B^2. If we denote by U_n the minimal solution of (3.95) which satisfies

$$\left.\begin{aligned} &\text{(i)} && \lim_{r\to 1} U_n(r,\sigma)/\ln(1/(1-r^2)) = 1 \quad \text{if } \sigma \in \Sigma_n = \bigcup_{j=0}^{n} S_j, \\ &\text{(ii)} && \lim_{r\to 1} U_n(r,\sigma) = 0 \quad \text{if } \sigma \in S^1 \backslash \Sigma_n, \end{aligned}\right\} \tag{3.98}$$

then

$$\max_{0\leqslant j\leqslant n} v_j \leqslant U_n \leqslant V_n. \tag{3.99}$$

When n goes to infinity the sequence $\{U_n\}$ increases and converges to a solution U of (3.95) with the property that

$$\max_{j\geqslant 0} v_j \leqslant U \leqslant \sum_{j=0}^{\infty} v_j. \tag{3.100}$$

Clearly, U is a non-negative supersolution of (1.3). Let H be the solution of

$$\left.\begin{aligned} -\Delta H &= -K \quad \text{in } \Omega, \\ H &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.101}$$

The function H is positive and, if $W = U - H$, then

$$-\Delta W = -\Delta U + \Delta H = K(\mathrm{e}^U - 1) + K \geqslant K\mathrm{e}^{U-H} = K\mathrm{e}^W. \tag{3.102}$$

Moreover, U and W satisfy the same boundary conditions; henceforth it is classical that there exists a solution u of (1.3) such that $U - H \leqslant u \leqslant U$. This implies that $\mathrm{tr}_{|\partial B^2}(u) = (\partial B^2, 0)$. Since $\sum_{j=0}^{\infty} v_j(0)$ can be made as small as needed one can construct infinitely many such solutions u. □

REMARK 3.14. By adapting the construction of [19, proposition 5.2], it is possible to prove that for any $\varepsilon > 0$ there exists a solution u_ε of (1.3) with $\mathrm{tr}_{|\partial B^2}(u_\varepsilon) = (\partial B^2, 0)$ and a Borel subset $E_\varepsilon \subset \partial B^2$ with $\mathrm{meas}(E_\varepsilon) < \varepsilon$ such that

$$\lim_{r\to 1} u_\varepsilon(r,\sigma) = 0 \quad \text{for almost all } \sigma \in \partial B^2 \backslash E_\varepsilon.$$

REMARK 3.15. Since uniqueness of a solution of (1.3) with a given trace is not true, we believe that a finer notion should be appropriate to describe the trace. A particular interesting problem would be to prove uniqueness under the mere assumption that the metric $g_u = \mathrm{e}^{2u} g_0$ is complete, which actually reads

$$\mathrm{Length}(\gamma) = \int_0^1 \sqrt{\sum_{i,j=1}^{2} (g_u(\gamma))_{ij}(\dot\gamma_i, \dot\gamma_j)}\,\mathrm{d}t = \int_0^1 \mathrm{e}^{u(\gamma(t))}\,\mathrm{d}t = \infty \tag{3.103}$$

for any geodesic $\gamma \in C^\infty([0,1], \bar B^2)$ with $\gamma([0,1)) \subset B^2$ and $\gamma(1) \in \partial B^2$. Some results in this direction can be found in [23, § 7].

4. Removable singularities

In this section it is still assumed that the curvature function K is continuous and negative in $\overline{B^2}$, and set $G = -K$, with $0 < k \leqslant G(x) < k^{-1}$. In order to describe the removability results, we introduce an extended Bessel capacity framework [4, 21]. We first recall some basic facts about fractional Sobolev–Besov spaces on S^1.

DEFINITION 4.1. Let $0 < \alpha \leqslant 1$ and $1 \leqslant p \leqslant \infty$. Then

(i) if $0 < \alpha < 1$, f belongs to $B_p^{\alpha,p}(S^1) = W^{\alpha,p}(S^1)$ if $f \in L^p(S^1)$ and if the norm below is finite

$$\|f\|_{B_p^{\alpha,p}(S^1)} = \|f\|_{L^p(S^1)} + \left(\int_0^{2\pi} t^{-(1+\alpha p)}\|f(t+.) - f(\cdot)\|_{L^p(S^1)}^p \,\mathrm{d}t\right)^{1/p}; \tag{4.1}$$

(ii) if $\alpha = 1$, f belongs to $B_p^{1,p}(S^1)$ if $f \in L^p(S^1)$ and if the norm below is finite

$$\|f\|_{B_p^{\alpha,p}(S^1)} = \|f\|_{L^p(S^1)} + \left(\int_0^{2\pi} t^{-(1+p)}\|f(t+\cdot) + f(\cdot - t) - 2f(\cdot)\|_{L^p(S^1)}^p \,\mathrm{d}t\right)^{1/p}. \tag{4.2}$$

In both expressions we make the usual modification when $p = \infty$.

The space $B_p^{k+\alpha}(S^1)$, $k \in \mathbb{N}^*$, $0 < \alpha \leqslant 1$ is defined as the space of functions f belonging to $W^{k,p}(S^1)$ such that $f^{(k)} \in B_p^{\alpha,p}(S^1)$. If f belongs to $L^1(S^1)$, we recall that P_f is the Poisson potential of f in B^2. If one writes

$$P_f(r,\sigma) = \mathcal{P}_t(f)(\sigma) \qquad (t = \ln(1/r), \quad \sigma \in S^1), \tag{4.3}$$

the problem

$$\left.\begin{aligned} \partial_t^2\mathcal{P}_t(f) + \partial_\sigma^2\mathcal{P}_t(f) = 0 \quad \text{in } (0,\infty)\times S^1, \\ \mathcal{P}_0(f)(\cdot) = f(\cdot), \quad \mathcal{P}(f) \in L^\infty(0,\infty,L^1(S^1)) \end{aligned}\right\} \tag{4.4}$$

is equivalent to

$$\partial_t\mathcal{P}_t(f) + (-\partial_\sigma^2)^{1/2}\mathcal{P}_t(f) = 0 \quad \text{in } (0,\infty)\times S^1, \quad \mathcal{P}_0(f)(\cdot) = f(\cdot). \tag{4.5}$$

If α and k are as above, it is known that the space $B_p^{\alpha+k,p}(S^1)$ can be expressed by

$$B_p^{\alpha+k,p}(S^1) = \left\{ f \in L^p(S^1) : \int_0^1 (t^{k+1-\alpha}\|\partial_t^k\mathcal{P}_t(f)\|_{L^p})^p \frac{\mathrm{d}t}{t}\right\} < \infty, \tag{4.6}$$

with equivalent norms [25, 26]

$$\|f\|_{B_p^{\alpha+k,p}(S^1)} \approx \|f\|_{L^p(S^1)} + \left(\int_0^1 (t^{k+1-\alpha}\|\partial_t^k\mathcal{P}_t(f)\|_{L^p})^p \frac{\mathrm{d}t}{t}\right)^{1/p}, \tag{4.7}$$

and

$$\|f\|_{B_p^{\alpha+k,p}(S^1)} \approx \|f\|_{L^p(S^1)} + \left(\int_0^1 (t^{k+1-\alpha}\|\partial_\sigma^k\mathcal{P}_t(f)\|_{L^p})^p \frac{\mathrm{d}t}{t}\right)^{1/p}. \tag{4.8}$$

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We turn now to the definition of the Sobolev–Besov capacity $C_{\delta,p}$.

DEFINITION 4.2. Suppose $\delta > 0$ and $1 \leqslant p \leqslant \infty$.

(i) If $K \subset S^1$ is compact,

$$C_{\delta,p}(K) = \inf\{\|f\|_{B_p^{\delta,p}(S^1)} : f \in C_0^\infty(S^1),\ f \geqslant 0,\ f \geqslant 1 \text{ in a neighbourhood of } K\}. \tag{4.9}$$

(ii) If $G \subset S^1$ is open,

$$C_{\delta,p}(G) = \sup\{C_{\delta,p}(K) : K \subset G, K \text{ compact}\}. \tag{4.10}$$

(iii) If $E \subset S^1$,

$$C_{\delta,p}(E) = \inf\{C_{\delta,p}(G) : G \supset E, G \text{ open}\}. \tag{4.11}$$

Accordingly, we define the class $B_{\ln}^{\alpha,p}$ $(0 \leqslant \alpha \leqslant 1,\ 1 \leqslant p \leqslant \infty)$ on S^1 by

$$B_{\ln}^{\alpha,p}(S^1) = \left\{ f \in L^p(S^1) : \int_0^1 (t^{1-\alpha}\|\partial_t \mathcal{P}_t(f)\|_{L^p(S^1)} \ln(1+t^{-1}))^p \frac{\mathrm{d}t}{t} \right\} < \infty, \tag{4.12}$$

with the norm

$$\|f\|_{B_{\ln}^{\alpha,p}(S^1)} = \|f\|_{L^p(S^1)} + \left(\int_0^1 (t^{1-\alpha}\|\partial_t \mathcal{P}_t(f)\|_{L^p(S^1)} \ln(1+t^{-1}))^p \frac{\mathrm{d}t}{t} \right)^{1/p} \tag{4.13}$$

and the corresponding extended capacity $C_{\alpha,p,\ln}$.

DEFINITION 4.3. Suppose $0 \leqslant \alpha \leqslant 1$ and $1 \leqslant p \leqslant \infty$.

(i) If $K \subset S^1$ is compact,

$$C_{\alpha,p,\ln}(K) = \inf\{\|f\|_{B_{\ln}^{\alpha,p}(S^1)} : f \in C_0^\infty(S^1),\ f \geqslant 0,\ f \geqslant 1 \text{ in a neighbourhood of } K\}. \tag{4.14}$$

(ii) If $G \subset S^1$ is open,

$$C_{\alpha,p,\ln}(G) = \sup\{C_{\alpha,p,\ln}(K) : K \subset G, K \text{ compact}\}. \tag{4.15}$$

(iii) If $E \subset S^1$,

$$C_{\alpha,p,\ln}(E) = \inf\{C_{\alpha,p,\ln}(G) : G \supset E, G \text{ open}\}. \tag{4.16}$$

The following properties of this capacities are easy to verify with the help of Hölder and Sobolev inequalities.

PROPOSITION 4.4. *Let $E \subset S^1 = \partial B^2$. Then for any $1 \leqslant p \leqslant \infty$, and $0 \leqslant \alpha \leqslant 1$*

(i) $C_{\alpha,p}(E) \leqslant M(\alpha,p)C_{\alpha,p,\ln}(E)$ $(\alpha > 0)$,

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(ii) $C_{\alpha,p,\ln}(E) \leqslant M(\alpha,\beta,p)C_{\beta,p}(E)$ ($\forall 0 \leqslant \alpha < \beta \leqslant 1$).

(iii) $C_{\beta,p/(1-p(\alpha-\beta))}(E) \leqslant M(\alpha,\beta,p)C_{\alpha,p}(E)$ ($\forall(\alpha,\beta) : (\alpha - p^{-1})_+ < \beta \leqslant \alpha < 1$).

LEMMA 4.5. *Suppose $f \in L^\infty(S^1)$, $f \geqslant 0$, then the following estimate holds:*

$$|\partial_r P_f(r,\sigma)| \leqslant \frac{4}{1-r}\|f(r,\cdot)\|_{L^\infty(S^1)} \qquad (\forall(r,\sigma) \in (0,1) \times S^1). \tag{4.17}$$

Proof. From Poisson's representation formula

$$P_f(r,\sigma) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\sigma)} f(\theta)\,\mathrm{d}\theta, \tag{4.18}$$

one gets

$$\partial_r P_f(r,\sigma) = -\frac{1}{\pi}\int_0^{2\pi} \frac{r}{1+r^2-2r\cos(\theta-\sigma)} f(\theta)\,\mathrm{d}\theta \\ - \frac{1}{\pi}\int_0^{2\pi} \frac{(1-r^2)(r-\cos(\theta-\sigma))}{(1+r^2-2r\cos(\theta-\sigma))^2} f(\theta)\,\mathrm{d}\theta. \tag{4.19}$$

By a straightforward computation,

$$(1-r)|r-\cos\varphi| \leqslant 1+r^2-2r\cos\varphi \qquad (\forall(r,\varphi) \in [0,1]\times[0,2\pi]), \tag{4.20}$$

from which it is derived that

$$-4P_f(r,\sigma) \leqslant (1-r)\partial_r P_f(r,\sigma) \leqslant 2P_f(r,\sigma). \tag{4.21}$$

This implies the claim since $\max_{x\in\mathbb{R}}|P_f(t,\cdot)| \leqslant \max_{x\in\mathbb{R}}|f(\cdot)|$. □

We turn now to the key estimate for solutions of (1.3) with a possible singularity on the boundary with zero $C_{0,1,\ln}$-capacity.

PROPOSITION 4.6. *Let $E \subset S^1 = \partial B^2$ be a compact subset such that $C_{0,1,\ln}(E) = 0$ and u an element of $\mathcal{E}$ such that $u = 0$ on $\partial B^2 \backslash E$. Then*

$$\int_{B^2} \mathrm{e}^{2u(x)}(1-|x|)\,\mathrm{d}x < M \tag{4.22}$$

for some positive constant M depending on G and the lower bound of u in B^2.

Proof. We recall that ψ_1 denotes the first eigenfunction of $-\Delta$ in $W_0^{1,2}(B^2)$ normalized by $\max_{B^2}\psi_1 = 1$ and λ_1 is the corresponding eigenvalue. Let $\eta \in C_0^\infty(\partial B^2)$ with $0 \leqslant \eta \leqslant 1$ such that $\eta(\sigma) = 1$ in a neighbourhood E_η of E; we set $\zeta(x) = \zeta(r,\sigma) = ((1-P_\eta)\psi_1)(r,\sigma)$, take $x \mapsto \zeta_j(x) = \zeta(x/j)$ $(0 < j < 1)$ as a test function and get

$$\int_{|x|<j} (-u\Delta\zeta_j + G\mathrm{e}^{2u}\zeta_j)\,\mathrm{d}x = -\int_{|x|=j} u\frac{\partial\zeta_j}{\partial n}\,\mathrm{d}\sigma. \tag{4.23}$$

STEP 1. We claim that

$$\int_{B^2} (-u\Delta\zeta + G\mathrm{e}^{2u}\zeta)\,\mathrm{d}x = 0. \tag{4.24}$$

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From assumption $u(x) \geqslant \ell$ for some $\ell \in \mathbb{R}$; moreover,

$$\frac{\partial \zeta_j}{\partial r}(x) = \frac{1}{j}\frac{\partial \zeta}{\partial r}\left(\frac{x}{j}\right) = \frac{1}{j}\left(\frac{\partial \psi_1}{\partial r}(1-P_\eta) - \psi_1 \frac{\partial P_\eta}{\partial r}\right)\left(\frac{x}{j}\right),$$

with $x = (r, \sigma)$ and $x/j = (r/j, \sigma)$. But ψ_1 vanishes on ∂B^2, therefore

$$\left|\frac{\partial \zeta_j}{\partial n}(x)\right| \leqslant \frac{C_1}{j}(1-P_\eta)(j,\sigma) \quad \text{for } |x| = j. \tag{4.25}$$

On the other hand

$$\begin{aligned}\ell\chi_{E_\eta}(\sigma) - C_{E_\eta}(1-j)(1-\chi_{E_\eta}(\sigma)) \\ &\leqslant u(r,\sigma) \\ &\leqslant C_0(\ln(1/(1-j)) + \ln(4/k))\chi_{E_\eta}(\sigma) + C_{E_\eta}(1-j)(1-\chi_{E_\eta}(\sigma))\end{aligned}$$

for $|x| = j$ because u vanishes on $\partial B^2 \backslash E$, which contains $\partial B^2 \backslash E_\eta$ (and the positive constant C_{E_η} depends on η). This, together with (4.25) implies

$$\lim_{j \to 1} \int_{|x|=j} u \frac{\partial \zeta_j}{\partial n}\, \mathrm{d}\sigma = 0. \tag{4.26}$$

Moreover,

$$\begin{aligned}\Delta\zeta_j(x) &= j^{-2}\Delta\zeta(x/j) \\ &= j^{-2}((1-P_\eta)\Delta\psi_1 + \psi_1\Delta(1-P_\eta) + 2\nabla\psi_1 \cdot \nabla(1-P_\eta))(x/j), \\ &= j^{-2}(-\lambda_1\psi_1(1-P_\eta) - 2\nabla\psi_1 \cdot \nabla P_\eta)(x/j),\end{aligned} \tag{4.27}$$

and $|\Delta\zeta_j(x)| \leqslant C_3$. Since u is integrable in B^2, it is the same with $u\Delta\zeta$. Consequently, $\mathrm{e}^{2u}\zeta \in L^1(B^2)$ and (4.24) follows by letting j go to 1 in (4.23).

Step 2. We claim that estimate (4.22) holds. Since $C_{0,1,\ln}(E) = 0$, there exists a sequence $\{\eta_n\} \subset B^{0,1}_{\ln}(S^1)$ such that $0 \leqslant \eta_n \leqslant 1$, $\eta_n = 1$ in a neighbourhood of E and

$$\lim_{n\to\infty} \|\eta_n\|_{B^{0,1}_{\ln}} = 0. \tag{4.28}$$

Since $u \geqslant \ell$, it can be assumed be that u is non-negative by replacing G by $G\mathrm{e}^{2\ell}$. Taking $\eta = \eta_n$ in the construction of ζ one gets from (4.24) and Young's inequality

$$\begin{aligned}\int_{B^2} G\mathrm{e}^{2u}\zeta\,\mathrm{d}x &\leqslant \int_{B^2} u|\Delta\zeta|\,\mathrm{d}x \\ &\leqslant \tfrac{1}{2}k\int_{B^2}\mathrm{e}^{2u}\zeta\,\mathrm{d}x + \frac{1}{2}\int_{B^2}|\Delta\zeta|(\ln|\Delta\zeta| - \ln\zeta - \ln k)\,\mathrm{d}x.\end{aligned} \tag{4.29}$$

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Therefore,

$$k\int_{B^2} \mathrm{e}^{2u}\zeta\,\mathrm{d}x \leqslant \int_{B^2} |\Delta\zeta|(\ln|\Delta\zeta| - \ln\zeta - \ln k)\,\mathrm{d}x$$
$$\leqslant L\int_{B^2} |\Delta\zeta| \ln\biggl(\biggl(\frac{|\Delta\zeta|}{\zeta}+1\biggr)\zeta\biggr)\,\mathrm{d}x$$
$$\leqslant L\int_{B^2} |\Delta\zeta| \ln\biggl(\frac{|\Delta\zeta|}{\zeta}+1\biggr)\,\mathrm{d}x \tag{4.30}$$

for some $L = L(k) > 1$. Since $\zeta = \psi_1(1-P_\eta)$,

$$|\Delta\zeta| = |-\lambda_1\psi_1(1-P_\eta) - 2\nabla\psi_1\cdot\nabla P_\eta| \leqslant \lambda_1 + 2|\nabla\psi_1\cdot\nabla P_\eta|, \tag{4.31}$$

$$\biggl|\frac{\Delta\zeta}{\zeta}\biggr| = |-\lambda_1 - 2\psi_1^{-1}(1-P_\eta)^{-1}\nabla\psi_1\cdot\nabla P_\eta| \leqslant \lambda_1 + 2\psi_1^{-1}(1-P_\eta)^{-1}|\nabla\psi_1\cdot\nabla P_\eta|, \tag{4.32}$$

and one has to estimate

$$A = \int_{B^2} (\lambda_1 + 2|\nabla\psi_1\cdot\nabla P_\eta|)\ln(1+\lambda_1+2\psi_1^{-1}(1-P_\eta)^{-1}|\nabla\psi_1\cdot\nabla P_\eta|)\,\mathrm{d}x. \tag{4.33}$$

But $c(1-r) \leqslant (1-P_\eta) \leqslant 1$, $c(1-r) \leqslant \psi_1 \leqslant 1$, $|\partial_r\psi_1| \leqslant c$ and $|\nabla\psi_1\cdot\nabla P_\eta| = |\partial_r\psi_1\cdot\partial_r P_\eta| \leqslant c|\partial_r P_\eta|$ for some $c > 0$ independent of η. Then by using (4.3),

$$A \leqslant C\int_0^1\int_{S^1} (|\partial_r P_\eta|+1)\ln(1+c^{-1}(1-r)^{-2}|\partial_r P_\eta|)\,\mathrm{d}\sigma\,\mathrm{d}r$$
$$\leqslant C\int_0^\infty\int_{S^1} (|\partial_t\mathcal{P}_t(\eta)|+1)\ln(1+c^{-1}(1-\mathrm{e}^{-t})^{-2}\mathrm{e}^t|\partial_t\mathcal{P}_t(\eta)|)\mathrm{e}^{-t}\,\mathrm{d}\sigma\,\mathrm{d}t$$
$$= A' + A'', \tag{4.34}$$

where

$$A' = C\int_0^1\int_{S^1} (|\partial_t\mathcal{P}_t(\eta)|+1)\ln(1+c^{-1}(1-\mathrm{e}^{-t})^{-2}\mathrm{e}^t|\partial_t\mathcal{P}_t(\eta)|)\mathrm{e}^{-t}\,\mathrm{d}\sigma\,\mathrm{d}t$$
$$\leqslant C\int_0^1\int_{S^1} (|\partial_t\mathcal{P}_t(\eta)|+1)\ln(1+\mathrm{e}c^{-1}t^{-2}|\partial_t\mathcal{P}_t(\eta)|)\,\mathrm{d}\sigma\,\mathrm{d}t = \tilde{A}', \tag{4.35}$$

and

$$A'' = C\int_1^\infty\int_{S^1} (|\partial_t\mathcal{P}_t(\eta)|+1)\ln(1+c^{-1}(1-\mathrm{e}^{-t})^{-2}\mathrm{e}^t|\partial_t\mathcal{P}_t(\eta)|)\mathrm{e}^{-t}\,\mathrm{d}\sigma\,\mathrm{d}t$$
$$\leqslant C\int_1^\infty\int_{S^1} (|\partial_t\mathcal{P}_t(\eta)|+1)\ln(1+c^{-1}\mathrm{e}^t|\partial_t\mathcal{P}_t(\eta)|)\mathrm{e}^{-t}\,\mathrm{d}\sigma\,\mathrm{d}t = \tilde{A}''. \tag{4.36}$$

From lemma 4.5

$$|\partial_t\mathcal{P}_t(\eta)(t,\sigma)| \leqslant C't^{-1}\|\eta\|_{L^\infty(S^1)} = C't^{-1}; \tag{4.37}$$

therefore,

$$\tilde{A}'' \leqslant C'C\int_1^\infty\int_{S^1} (t^{-1}+1)\ln(1+c^{-1}C\mathrm{e}^t t^{-1})\mathrm{e}^{-t}\,\mathrm{d}\sigma\,\mathrm{d}t = C_3. \tag{4.38}$$

Using again lemma 4.5

$$
\begin{aligned}
\tilde{A}' &\leqslant C\int_0^1\int_{S^1}(|\partial_t\mathcal{P}_t(\eta)|+1)\ln(1+C'c^{-1}\mathrm{e}t^{-3})\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\leqslant C\int_0^1 \ln(1+C'c^{-1}\mathrm{e}t^{-3})\,\mathrm{d}t + C\int_0^1\int_{S^1}|\partial_t\mathcal{P}_t(\eta)|\ln(1+t^{-1})\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\leqslant C\int_0^1 \ln(1+C'c^{-1}\mathrm{e}t^{-3})\,\mathrm{d}t + C\|\eta\|_{B^{0,1}_{\ln}(S^1)}. \qquad (4.39)
\end{aligned}
$$

Replacing η by η_n and letting n go to infinity implies that $\lim_{n\to\infty}\zeta_n=\psi_1$. One concludes that (4.22) holds from (4.28), (4.38) and (4.39). □

THEOREM 4.7. *Let $E\subset S^1=\partial B^2$ be a closed subset such that $C_{0,1,\ln}(E)=0$ and u a solution of (1.3) in B^2, which is bounded from below and coincides on $\partial B^2\backslash E$ with $\varphi\in C(\partial B^2)$. Then u can be extended to $\overline{B^2}$ as a continuous function.*

Proof. Let $v\in C(\overline{B^2})$ be the solution of

$$
\left.\begin{aligned}
\Delta v = 0 &\quad \text{in } B^2,\\
v=\varphi &\quad \text{on } \partial B^2,
\end{aligned}\right\} \qquad (4.40)
$$

and $\tilde{u}=u-v$. Then

$$
\Delta\tilde{u}=\tilde{G}\mathrm{e}^{2\tilde{u}} \qquad (4.41)
$$

with $\tilde{G}=G\mathrm{e}^{2v}$, and $\tilde{u}$ vanishes on $\partial B^2\backslash E$. From proposition 4.6,

$$
\int_{B^2}\mathrm{e}^{2\tilde{u}}(1-|x|)\,\mathrm{d}x\leqslant\tilde{M},
$$

where $\tilde{M}$ depends on φ but not on $\tilde{u}$. Moreover, $\mathrm{tr}_{|\partial B^2}(\tilde{u})=(\emptyset,\mu)$, where $\mu\in\mathfrak{M}^+(\partial B^2)$ and $\operatorname{supp}.(\mu)\subset E$. We take $\xi=\xi_n=(1-P_\eta)\psi_1$ for test function where $\eta=\eta_n$ satisfies (4.28), and get as above, in the same way as in proposition 4.6,

$$
\int_{B^2}(-\tilde{u}\Delta\xi+\tilde{G}\mathrm{e}^{2\tilde{u}}\xi)\,\mathrm{d}x=\int_{\partial B^2}\left((1-P_\eta)\frac{\partial\psi_1}{\partial n}-\psi_1\frac{\partial P_\eta}{\partial n}\right)\mathrm{d}\mu=0 \qquad (4.42)
$$

since $1-P_\eta=0$ in a neighbourhood of the support of μ. Therefore,

$$
\begin{aligned}
\int_{B^2}(-\tilde{u}(1-P_\eta)\Delta\psi_1+\tilde{G}\mathrm{e}^{2\tilde{u}}\xi)\,\mathrm{d}x &= \int_{B^2}(\lambda_1\tilde{u}(1-P_\eta)\psi_1+\tilde{G}\mathrm{e}^{2\tilde{u}}\xi)\,\mathrm{d}x\\
&=2\int_{B^2}\tilde{u}\nabla P_\eta\cdot\nabla\psi_1\,\mathrm{d}x \qquad (4.43)
\end{aligned}
$$

and, as in (4.29),

$$
\begin{aligned}
\left|\int_{B^2}\tilde{u}\nabla P_\eta\cdot\nabla\psi_1\,\mathrm{d}x\right| &\leqslant C\int_{B^2}\tilde{u}|\partial_r P_\eta|\,\mathrm{d}x\\
&\leqslant \tfrac{1}{2}C\varepsilon\int_{B^2}\mathrm{e}^{2\tilde{u}}(1-|x|)\\
&\quad+\tfrac{1}{2}C\int_{B^2}|\partial_r P_\eta|(\ln|\partial_r P_\eta|-\ln(1-|x|)-\ln\varepsilon)\,\mathrm{d}x \qquad (4.44)
\end{aligned}
$$

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($\varepsilon > 0$). From step 2 of proposition 4.6

$$\lim_{n\to\infty}\int_{B^2}|\partial_r P_{\eta_n}|(\ln|\partial_r P_{\eta_n}| - \ln(1-|x|) - \ln\varepsilon)\,\mathrm{d}x = 0, \tag{4.45}$$

therefore,

$$\limsup_{n\to\infty}\left|\int_{B^2}\tilde{u}\nabla P_{\eta_n}\cdot\nabla\xi\,\mathrm{d}x\right| \leqslant \tfrac{1}{2}\varepsilon\int_{B^2}\mathrm{e}^{2\tilde{u}}(1-|x|)\,\mathrm{d}x, \tag{4.46}$$

and finally

$$\int_{B^2}(\lambda_1\tilde{u}\psi_1 + \tilde{G}\mathrm{e}^{2\tilde{u}}\psi_1)\,\mathrm{d}x = 0. \tag{4.47}$$

Let $\tilde{\Psi}$ be the solution of

$$\left.\begin{aligned}\Delta\tilde{\Psi} &= \tilde{G}\mathrm{e}^{2\tilde{\Psi}} \quad \text{in } B^2,\\ \tilde{\Psi} &= 0 \quad \text{on } \partial B^2.\end{aligned}\right\} \tag{4.48}$$

Clearly, $\tilde{\Psi}$ is continuous in $\overline{B^2}$ and

$$\int_{B^2}(\lambda_1\tilde{\Psi}\psi_1 + \tilde{G}\mathrm{e}^{2\tilde{\Psi}}\psi_1)\,\mathrm{d}x = 0. \tag{4.49}$$

Because the mapping $\mu \mapsto \tilde{u}$ (with $\mathrm{tr}_{|\partial B^2}(\tilde{u}) = (\emptyset, \mu)$) is monotone and μ is non-negative, we have $\tilde{\Psi} \leqslant \tilde{u}$ in B^2. This together with (4.47) and (4.49) and the fact, that for any x in B^2, the mapping $r \mapsto \lambda_1 r + \tilde{G}(x)\mathrm{e}^{2r}$ is increasing implies that $\tilde{\Psi} = \tilde{u}$. □

Remark 4.8. The relation between capacities and Hausdorff dimension asserts that if a set $E \subset \partial B^2$ is such that

$$H-\dim(E) = \delta = \sup\left\{s > 0 \Big/ \lim_{\varepsilon\to 0}\left(\inf_{E\subset\cup_{i\in I}B(x_i,r_i)}\sum_{i\in I} r_i^s : 0 < r_i \leqslant \varepsilon\right) > 0\right\} < 1,$$

then $C_{\varepsilon,1}(E) = 0$ for $\varepsilon \in (0, 1-\delta)$, and consequently $C_{0,1,\ln}(E) = 0$ from proposition 4.4.

Definition 4.9. We shall say that a Borel subset $E \subset \partial B^2$ is *removable* if any solution $u \in \mathcal{E}$ of (1.3) in B^2 which is locally uniformly continuous on $\bar{B}^2\backslash E$ can be extended to $\bar{B}^2$ as a continuous function.

From theorem 4.7 any closed subset E of ∂B^2 with $C_{0,1,\ln}(E) = 0$ is removable. With this result, we obtain the following consequence.

Corollary 4.10. *Let $\mathcal{S} \subset \partial B^2$ be a closed removable set and μ a locally admissible non-negative Radon measure on $\mathcal{R} = \partial B^2\backslash\mathcal{S}$. Then there exists a unique solution u of (1.3) in B^2 such that $\lim_{r\to 1}\|u^-(r,\cdot)\|_{L^\infty(S^1)} = 0$ and $\mathrm{tr}_{|\partial B^2}(u) = (\mathcal{S}, \mu)$.*

Proof. We recall that Ψ is the solution of

$$\left.\begin{aligned}\Delta\Psi + K\mathrm{e}^{2\Psi} &= 0 \quad \text{in } B^2,\\ \Psi &= 0 \quad \text{on } \partial B^2.\end{aligned}\right\} \tag{4.50}$$

It follows from proposition 3.11, the proof of step 2 of theorem 3.6 and the fact that $u^*_{\mathcal{S}} = \Psi$ (since $\mathcal{S}^* = \emptyset$) that any solution u of (1.3) with $\lim_{r\to1} \|u^-(r,\cdot)\|_{L^\infty(S^1)} = 0$ with $\operatorname{tr}_{|\partial B^2}(u) = (\mathcal{S}, \mu)$ satisfies

$$\Psi \leqslant u_\mu \leqslant u \leqslant u_\mu - 4\Psi, \tag{4.51}$$

where u_μ is the minimal solution constructed in proposition 3.11 (we recall that Ψ is negative in B^2). Therefore, for any $\varepsilon > 0$, $u_\mu + \varepsilon$ is a supersolution which dominates u near ∂B^2. Letting ε go to zero yields $u = u_\mu$. □

REMARK 4.11. Most of the results which are presented here are extendible to the d-dimensional case of equation (1.3) in B^d, with the help of some of the techniques introduced in [19] (see also remark 2.5).

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