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ALGEBRAIC VALUES OF CERTAIN ANALYTIC FUNCTIONS DEFINED BY A CANONICAL PRODUCT

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Abstract

We give a partial answer to a question attributed to Chris Miller on algebraic values of certain transcendental functions of order less than one. We obtain $C(\log H)^\eta$ bounds for the number of algebraic points of height at most H on certain subsets of the graphs of such functions. The constant C and exponent η depend on data associated with the functions and can be effectively computed from them.

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1. Introduction

We investigate the asymptotic density (in terms of height) of algebraic values of bounded height and degree on graphs of transcendental functions. Given bounds H and d for the height and degree respectively, a trivial upper bound for this density takes the form $C(d)H^{2d}$ and follows immediately from quantitative versions of Northcott's theorem. As such, *polylogarithmic* bounds in H are considered very good and are often nontrivial to prove.

We begin by recalling the definition of the *absolute multiplicative height* of an algebraic number, which is the height notion used throughout the paper. Then, in order to place our main result in context, we briefly discuss some related results.

Let $P(z) \in \mathbb{C}[z]$ be a polynomial with complex coefficients. Writing $P(z)$ as

$$P(z) = a\prod_{j=1}^{n}(z-\alpha_j),$$

the *Mahler measure* $\mathcal{M}(P)$ of the polynomial P is the quantity

$$\mathcal{M}(P) = |a|\prod_{j=1}^{n}\max\{1, |\alpha_j|\}.$$

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If α an algebraic number of degree d, the *logarithmic* height of α, $h(\alpha)$, is defined as

$$h(\alpha) = \frac{\log \mathcal{M}(\alpha)}{d},$$

where $\mathcal{M}(\alpha)$ is the Mahler measure of the minimal polynomial of α over $\mathbb{Z}$.

The *absolute multiplicative height* of α, $H(\alpha)$, is defined as

$$H(\alpha) = \exp\left\{\frac{\log \mathcal{M}(\alpha)}{d}\right\} = \mathcal{M}(\alpha)^{1/d}.$$

If α and β are algebraic numbers, we use the notation $H(\alpha, \beta)$ to represent the quantity

$$\max\{H(\alpha), H(\beta)\}.$$

1.1. Some known results. This area of research can be traced back to the seminal paper of Bombieri and Pila [2], wherein they established the celebrated Bombieri–Pila theorem for counting lattice points on graphs of real analytic functions.

Given a set $\Gamma \subset \mathbb{R}^2$ and a positive number $t \geq 1$, the homothetic dilation of Γ by t, denoted by $t\Gamma$, is the set

$$t\Gamma := \{(tx_1, tx_2) : (x_1, x_2) \in \Gamma\}.$$

In [2], Bombieri and Pila considered, among several other variants, the following question. Let $f : [0, 1] \to \mathbb{R}$ be an analytic function and denote by $X_f \subset \mathbb{R}^2$ the graph of f. Given $t \geq 1$, how does the quantity $|tX_f \cap \mathbb{Z}^2|$ depend on t? For a transcendental function f, they proved the following theorem.

Theorem 1.1 (Bombieri and Pila [2, Theorem 1]). *Let f be a real analytic function on a closed and bounded interval I and suppose that f is not algebraic. Let X_f be the graph of f and let $\epsilon > 0$. Then there is a constant $c(f, \epsilon)$ such that*

$$|tX_f \cap \mathbb{Z}^2| \leq c(f, \epsilon)t^{\epsilon}$$

for all $t \geq 1$.

In [7], Pila extended and refined some of the results from [2]. In particular, he obtained the following refinement (counting rational points) of Theorem 1.1.

Theorem 1.2 (Pila [7]). *Let f be a transcendental real analytic function on a closed and bounded interval I. Let X_f be the graph of f and let $\epsilon > 0$. Then there is a constant $c(f, \epsilon)$ such that, for any positive integer H, the number of rational points of height at most H on X_f is at most $c(f, \epsilon)H^{\epsilon}$.*

An example in [2] shows that this is the best possible bound in general. However, for certain special cases, such as those arising from additional hypotheses on f, or when f is some concrete function, it is sometimes possible to improve the bound to one of the form $c(\log H)^{\eta}$ for some $c, \eta > 0$. For example, in [6], Masser proved the following result for the number of rational points on the graph of the Riemann ζ-function restricted to the interval $(2, 3)$.

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THEOREM 1.3 (Masser [6]). *Let ζ be the restriction of the Riemann ζ-function to the interval $(2,3)$. There is an effective constant $c > 0$ such that, for all $H \geq e^e$, the number of rational points of height at most H on the graph of ζ is at most*

$$c\Big(\frac{\log H}{\log\log H}\Big)^2.$$

In [1], adapting Masser's method, Besson studied the density of algebraic points of bounded degree and height on the graph of the Γ-function restricted to the interval $[n-1, n]$. He obtained the following result.

THEOREM 1.4 (Besson [1]). *There exists an effective constant $c > 0$ such that, for integers $d \geq 1$, $H \geq 3$ and $n \geq 2$, the number of algebraic points of degree at most d and height at most H on the graph of the Γ-function restricted to the interval $[n-1, n]$ is at most*

$$c(n^2 \log n)\Big(\frac{(d^2 \log H)^2}{\log(d \log H)}\Big).$$

In [9], assuming only that f is complex analytic and transcendental, Surroca achieved the rather exciting bound of $Cd^3(\log H)^2$ for the number of algebraic points of degree at most d and height at most H on the restriction to a compact subset of the graph of f. However, the bound is valid only for infinitely many H.

THEOREM 1.5 (Surroca [9]). *Let $0 < r < R$ and suppose f is a transcendental function complex analytic on a neighbourhood of $B(0, R)$. Then, for any integer $d \geq 1$, there exist a real number $C > 0$ and infinitely many real numbers $H \geq 1$ such that the number of algebraic points of degree at most d and height at most H, with argument belonging to $\overline{B(0, r)}$, is at most $Cd^3(\log H)^2$.*

The constant C effectively depends on r, R and f. It is also shown that the theorem cannot be improved any further. That is, one cannot replace the 'infinitely many real $H \geq 1$' in the conclusion of the theorem with 'for all sufficiently large H'.

Recall that the order and lower order of an entire function f are respectively

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

REMARK 1.6. If ρ is finite, then ρ is the infimum of the set of all α such that $M(r, f) \leq e^{r^\alpha}$ for sufficiently large r and λ is the supremum of the set of all β such that $e^{r^\beta} \leq M(r, f)$ for sufficiently large r.

In [4], motivated by the earlier work of Masser in [6], Boxall and Jones studied the density of algebraic points of bounded height and degree on graphs of entire functions of finite order ρ and positive lower order λ restricted to compact subsets of $\mathbb{C}$. They gave a bound of the form $C(\log H)^\eta$, where the constant C and the exponent η are effective and η depends only on ρ and λ.

THEOREM 1.7 (Boxall and Jones [4]). *Let f be a nonconstant entire function of order ρ and lower order λ. Suppose $0 < \lambda \le \rho < \infty$ and let $d \ge 1$ and $r > 0$. There is a constant $C > 0$ such that, for all $H > e$, there are at most $C(\log H)^{\eta(\lambda,\rho)}$ complex numbers z such that $|z| \le r$, $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \le d$ and $H(z, f(z)) \le H$.*

The Boxall–Jones theorem immediately prompts two questions towards possible generalisations or improvements. On the one hand, one can ask if the same type of bound holds for meromorphic functions. Using Nevanlinna theory, we explore this theme in a paper currently in preparation. On the other hand, one can ask when the region to which f was initially restricted can be enlarged. In fact, more generally, for which functions can one drop the restriction to compact sets and count (possibly) *all* points of bounded height and degree on the graph of f? In this paper, we explore the second theme for a specific class of entire functions of order less than one, following a question asked by Chris Miller and brought to our attention by Jones.

In this connection and at the expense of gaining an extra $\log H$ factor, Boxall and Jones [3] unified and extended the results from [6] and [1] and obtained a $C(\log H)^3(\log\log H)^3$ bound for the number of *all* rational points $(x, f(x))$ of height at most H with $x > 0$ on the graphs of entire functions satisfying a general growth condition and a decay condition along the positive ray. The functions to which their result applies include the Riemann ζ-function (derived by counting points on $f(z) := (z-1)(\zeta(z)-1)$) and the Γ-function (by counting points on $f(z) := 1/\Gamma(z)$).

1.2. A proposition of Masser. A crucial part of our proof strategy involves 'converting' the question of counting algebraic points on the graph of the function f to that of counting (or finding an upper bound for) the number of zeros of a related function g, say, which is considerably easier to handle via analytic methods. This requires the construction (or existence) of a certain nonzero auxiliary polynomial $P(X, Y) \in \mathbb{Z}[X, Y]$ such that $P(z, f(z)) = 0$ whenever

$$(z, f(z)) \in \overline{\mathbb{Q}}^2, \quad \deg(z, f(z)) \le d \quad \text{and} \quad H(z, f(z)) \le H.$$

We use the auxiliary polynomial constructed by Masser in [6, Proposition 2], which we give below. The 'moreover' part in the conclusion of the lemma does not appear in the original proposition as given by Masser, but it can be deduced easily from his proof of the proposition, as observed by Boxall and Jones in [4].

LEMMA 1.8 (Masser [6, Proposition 2]). *Let $d \ge 1$ and $T \ge \sqrt{8d}$ be positive integers and A, Z, M and H positive real numbers with $H \ge 1$. Let f_1, f_2 be functions analytic on an open neighbourhood of $\overline{B(0, 2Z)}$, with $\max\{|f_1(z)|, |f_2(z)|\} \le M$ on this set. Suppose $\mathcal{Z} \subset \mathbb{C}$ is finite and the following conditions are satisfied for all $z, w \in \mathcal{Z}$:*

- $|z| \le Z$,
- $|w - z| \le 1/A$,
- $[\mathbb{Q}(f_1(z), f_2(z)) : \mathbb{Q}] \le d$,
- $H(f_1(z), f_2(z)) \le H$.

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Then there is a nonzero polynomial $P(X, Y)$ of total degree at most T such that $P(f_1(z), f_2(z)) = 0$ for all $z \in \mathcal{Z}$ provided

$$(AZ)^T > (4T)^{96d^2/T}(M + 1)^{16d}H^{48d^2}.$$

Moreover, if $|\mathcal{Z}| \geq T^2/8d$, then $P(X, Y)$ can be chosen such that all the coefficients are integers each with absolute value at most

$$2^{1/d}(T + 1)^2H^T.$$

Remark 1.9. When using this lemma, we will take $f_1(z) = z$ and $f_2(z) = f(z)$.

2. Preliminaries and auxiliary lemmas

2.1. The function *f* and a brief discussion of the strategy. Let $1 \leq z_1 \leq z_2 \leq \cdots$ be an increasing and unbounded sequence of positive real numbers with $\sum_{n=1}^{\infty} 1/z_n < \infty$. Then the infinite product

$$f(z) := \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right) \tag{2.1}$$

necessarily defines an entire function of order ρ where $0 \leq \rho < 1$.

Chris Miller asked for the density of algebraic points of height at most H and degree at most d on graphs of functions defined in this way. For these functions, the lower order λ coincides with the order ρ. Hence when the order of f is positive, the Boxall–Jones theorem applies for restrictions of f to sets of the form $\overline{B(0, r)}$ for $r > 0$. The bound one gets is of the form $C(\log H)^{\eta}$, where $C = C(r, f, d, \rho)$ and $\eta = \eta(\rho)$.

However, as we will see shortly, functions of this form enjoy certain asymptotic approximations that give a more explicit and finer measure of growth than the one provided by just having positive lower order and finite order. Unfortunately, these approximations only hold outside certain subsets of the graphs. In any case, taking advantage of such explicit growth characterisations, for appropriate subsets of the graphs, one can find the density of *all* the algebraic points of bounded height and degree.

The strategy to do this utilises a rather simple but crucial observation. Given an algebraic number z of height at most H and degree at most d, the modulus $|z|$ is bounded above by a function of H and d. Therefore, to count the algebraic points of bounded height and degree on a function f, we can restrict our attention to those (algebraic) arguments z for which $|f(z)|$ is not too large to have height at most H or degree at most d. This is where an explicit lower approximation of f becomes crucial because it gives a handle on the growth of $|f|$.

The remainder of this section is devoted to making the contents of the previous two paragraphs explicit.

2.2. Lemmas. For $0 < \phi < \pi/2$, define the sector $S_\phi = \{z \in \mathbb{C} : -\phi \le \arg z \le \phi\}$. Let the sequence $\{z_k\}_{k=1}^{\infty} \subset S_\phi$ be such that $1 \le z_1 \le z_2 \le \cdots$ and $\sum_{k=1}^{\infty} 1/|z_k|^p < \infty$, where p is a nonnegative integer. In [5, Example 1, pages 66–69], Goldberg and Ostrowski approximate the function

$$g(z) := \prod_{k=1}^{\infty} E\Big(\frac{z}{z_k}, p\Big),$$

where p is a nonnegative integer and

$$E(z, p) := \begin{cases} (1 - z), & \text{if } p = 0, \\ (1 - z)\exp(z + z^2/2 + \cdots + z^p/p), & \text{otherwise,} \end{cases}$$

is the pth *Weierstrass elementary factor*. They obtain an asymptotic inequality approximating $\log g(z)$ in terms of the function $|z|^\rho$ and certain explicit coefficients, where $z \in \mathbb{C} \setminus S_\phi$ and $p \le \rho \le p + 1$. The asymptotic inequality we need is a specialisation of this result to the case where $p = 0$. We give the specific details in the next lemma.

Let $\{z_n\}_{n=1}^{\infty}$ be the sequence of zeros of f defined in (2.1) and denote by $n(r)$ the number of z_n with modulus less than r. Let

$$\mu := \lim_{r\to\infty} \frac{n(r)}{r^\rho},$$

where $\rho \in (0, 1)$ is the order of f.

Lemma 2.1 [5, Corollary of Example 1, page 66]. *Let $0 < \epsilon < 1$ and suppose f, μ, ρ and ϕ are as defined previously. Assume $0 < \mu < \infty$. Then there exists $r_1(\epsilon)$ such that, for all $z \in \mathbb{C}$ with $|z| > r_1(\epsilon)$ and $\phi < \arg z < 2\pi - \phi$,*

$$\Big|\log f(z) - \frac{\mu\pi}{\sin \pi\rho} e^{-i\pi\rho} z^\rho\Big| \le \epsilon D|z|^\rho \csc \frac{\phi}{2}, \tag{2.2}$$

where $D = 6 + 3\mu\pi \csc(\pi\rho)$.

From Lemma 2.1, it follows that

$$\Big|\Re\Big(\log f(z) - \frac{\mu\pi}{\sin \pi\rho} e^{-i\pi\rho} z^\rho\Big)\Big| \le \epsilon D|z|^\rho \csc \frac{\phi}{2}.$$

More explicitly, writing $z = re^{i\theta}$, with $r > r_1(\epsilon)$ and $\phi < \theta < 2\pi - \phi$, we deduce from (2.2) that

$$\Big|\log |f(re^{i\theta})| - \frac{\mu\pi}{\sin \pi\rho} \cos \rho(\theta - \pi) r^\rho\Big| \le \epsilon D r^\rho \csc \frac{\phi}{2}. \tag{2.3}$$

Remark 2.2. By Remark 1.6 and the above inequality, $\lambda = \rho$, where λ is the lower order of f.

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For $\rho \in (0, \frac{1}{2}]$, we have $\rho(\theta - \pi) \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, so $\sin \frac{1}{2}\theta = \cos \frac{1}{2}(\theta - \pi) \leq \cos \rho(\theta - \pi)$. Therefore, from (2.3),

$$|f(re^{i\theta})| \geq e^{C(\phi,\rho)r^\rho} \quad \text{where } C(\phi, \rho) = \frac{\mu\pi \sin \frac{1}{2}\phi}{\sin \pi\rho} - \epsilon D \csc \frac{\phi}{2}.$$

Given ϕ, we can choose ϵ, say,

$$\epsilon = \min\left\{\frac{\mu\pi \sin^2 \frac{1}{2}\phi}{4D \sin \pi\rho}, \frac{1}{2}\right\}.$$

In this case,

$$C(\phi, \rho) > \frac{\mu\pi \sin \frac{1}{2}\phi}{2 \sin \pi\rho} > 0.$$

The next lemma gives bounds in terms of H and d of the modulus of an algebraic number of height at most H and degree at most d. It is essentially a version of Liouville's inequality for the absolute multiplicative height (see [10, page 82]).

Lemma 2.3. *Let α be a nonzero algebraic number of degree at most d and height at most H. Then*

$$\frac{1}{(2H)^d} \leq |\alpha| \leq (2H)^d.$$

We use the above lemma, assuming that z and $f(z)$ are algebraic, to prove the following result.

Lemma 2.4. *Let $d \geq 1$ and $H \geq e^e$. Let $z = re^{i\theta} \in \mathbb{C}$ be algebraic such that $\deg(z) \leq d$ and $H(z) \leq H$. Define the constant $K(\phi, \rho, d)$ by*

$$K(\phi, \rho, d) = \left(\frac{2(d + 1)}{C(\phi, \rho)}\right)^{1/\rho}.$$

If $r \geq K(\phi, \rho, d)(\log H)^{1/\rho} = R_H$, then $e^{C(\phi,\rho)r^\rho} \geq (2H)^{d+1}$. For $r \geq \max\{r_1(\epsilon), R_H\}$, we therefore have the chain of inequalities

$$|f(re^{i\theta})| \geq e^{C(\phi,\rho)r^\rho} \geq (2H)^{d+1}.$$

By Lemma 2.3, if $f(z)$ is algebraic, then either $H(f(z)) > H$ or $\deg(f(z)) > d$.

Proof. We note that $e^{C(\phi,\rho)r^\rho} \geq (2H)^{d+1}$ if

$$C(\phi, \rho)r^\rho \geq (d + 1) \log(2H).$$

The above inequality follows if

$$C(\phi, \rho)r^\rho \geq 2(d + 1) \log H$$

and this is true if

$$r \geq K(\phi, \rho, d)(\log H)^{1/\rho}.$$

Recalling that $|f(re^{i\theta})| \geq e^{C(\phi,\rho)r^\rho}$ when $r \geq r(\epsilon)$, we obtain the desired chain of inequalities when $r \geq \max\{r_1(\epsilon), R_H\}$, □

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The next lemma gives a quantitative way of covering the zeros of a polynomial $P(z)$ with a collection of disks outside of which $|P(z)| > 1$.

Lemma 2.5 (Boutroux–Cartan [8, Theorem 12.5.7]). *Let $P(z) \in \mathbb{C}[z]$ be a monic polynomial with degree $n \geq 1$. Then $|P(z)| > 1$ for all complex z outside a collection of at most n disks the sum of whose radii is $2e$.*

In the following lemma, the function $n(r, 1/f)$ represents the number of zeros of f in $\overline{B(0, r)}$. This is a standard Nevanlinna-theoretic notation.

Lemma 2.6 (A corollary of Jensen's formula). *Let G be a nonconstant entire function such that $G(0) \neq 0$. Let $0 < r < R < \infty$. Then*

$$n\Big(r, \frac{1}{G}\Big) \leq \frac{1}{\log R/r} \log\Big(\frac{M(R, G)}{|G(0)|}\Big).$$

3. Main result

We can now state and prove our main result. Since f is an entire function of positive lower order and finite order, our argument is an adaptation of that of Boxall and Jones in [4].

Theorem 3.1. *Let $f(z) = \prod_{n=1}^{\infty}(1 - z/z_n)$, where $1 \leq z_1 \leq z_2 \leq \cdots$ and $\sum_{n=1}^{\infty} 1/z_n < \infty$. Suppose the order ρ of f is such that $0 < \rho \leq \frac{1}{2}$. Let $0 < \phi < \frac{1}{2}\pi$. Let d, α, β, γ satisfy $d \geq 1$, $\alpha = 1 + \rho$, $\beta = \rho/2$ and $\gamma = (2\alpha + \rho)/\beta\rho$. Then there is a constant $C > 0$ such that, for all $H > e$, there are at most $C(\log H)^{2\alpha(\gamma+1)/\rho}$ numbers $z \in \mathbb{C} \setminus S_\phi$ such that $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$ and $H(z, f(z)) \leq H$.*

Proof. Let $H > e^e$. Throughout our proof, the height bound H is assumed to be sufficiently large. We shall denote by C a positive constant independent of H. The constant C may not be the same at each occurrence. Let $|P|$ denote the modulus of the coefficient of the polynomial P with largest absolute value.

We would first like to obtain a nonzero polynomial $P(X, Y) \in \mathbb{Z}[X, Y]$ of degree at most $T = C(\log H)^{2\alpha/\rho}$ such that $|P| \leq 2^{1/d}(T + 1)^2 H^T$ and $P(z, f(z)) = 0$ whenever $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$ and $H(z, f(z)) \leq H$ and $z \notin S_\phi$. To this end, let

$$A = \frac{1}{2R_H}, \quad Z = C(\log H)^{1/\rho}, \quad T = C(\log H)^{2\alpha/\rho} \quad \text{and} \quad M = e^{(2Z)^\alpha}.$$

Then $\max\{|z|, |f(z)|\} \leq M$ for all $z \in \overline{B(0, 2Z)}$. Furthermore,

$$\log(AZ)^T = C(\log H)^{2\alpha/\rho} > C\Big(\frac{\log\log H}{(\log H)^{2\alpha/\rho}}\Big) + C(\log H)^{\alpha/\rho} + C\log H.$$

Therefore

$$(AZ)^T > (4T)^{96d^2/T}(M + 1)^{16d} H^{48d^2}.$$

We note that the bound we are trying to prove is worse than $C(\log H)^{4\alpha/\rho}$. We can thus assume that there are at least $T^2/8d$ complex numbers such that $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$

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and $H(z, f(z)) \le H$. The hypotheses of Lemma 1.8 are thus satisfied and therefore there is a polynomial $P(X, Y)$ satisfying all our requirements.

In light of Lemma 2.4, our choice of A and Z further ensures that the algebraic arguments of height at most H and degree at most d to which we restrict our attention are *all* the 'admissible' arguments—that is, those whose image (if it is algebraic) will also be of height at most H and degree at most d.

Let $G(z) = P(z, f(z))$. We would like to bound the number of zeros of G in $\overline{B(0, R_H)}$. To do this, first let k be the highest power of Y in $P(X, Y)$. We can assume $k \ge 1$. Let $\tilde{P}(X, Y) = Y^k P(X, 1/Y)$, $R(X) = \tilde{P}(X, 0)$ and $Q(X, Y) = \tilde{P}(X, Y) - R(X)$. We note that $R(X)$ is not identically zero. Let $\tilde{Q}(X, Y) = Q(X, Y)/Y$. The highest power of X in $\tilde{Q}$ is at most T and $|\tilde{Q}| \le |P| \le 2^{1/d}(T + 1)^2 H^T$. Finally, $\tilde{Q}$ has at most $(T + 1)^2$ terms.

Now we would like to find some $w_i \in \mathbb{C}$ such that $|G(w_i)| = |P(w_i, f(w_i))| \ge 1$. To this end, we first find some sufficiently large radius r such that if $|z| \ge r$ then $|Q(z, 1/f(z))| \le \frac{1}{2}$. Let $z = re^{i\theta} \in \mathbb{C}$ be such that $|f(z)| = M(r, f) \ge 1$. Then

$$\left|\tilde{Q}\left(z, \frac{1}{f(z)}\right)\right| \le 2^{1/d}(T + 1)^4 H^T r^T.$$

Therefore

$$\left|Q\left(z, \frac{1}{f(z)}\right)\right| \le \frac{1}{2}$$

provided

$$2^{1/d}(T + 1)^4 H^T r^T \le \tfrac{1}{2} M(r, f).$$

We note that (for a large enough C), if

$$r \ge C(\log H)^{(2\alpha+\rho)/\beta\rho},$$

then

$$2^{1/d}(T + 1)^4 H^T r^T \le \tfrac{1}{2} e^{r^\beta}$$

and (since $\beta < \rho = \lambda$) by Remark 1.6,

$$e^{r^\beta} \le M(r, f).$$

Thus

$$\left|Q\left(z, \frac{1}{f(z)}\right)\right| \le \frac{1}{2} \quad \text{when } r \ge C(\log H)^{(2\alpha+\rho)/\beta\rho}.$$

Note that the degree of $R(X)$ is also at most T. For $i = 1, \ldots, [T] + 14$, say, let r_i be the ith integer after $C(\log H)^\gamma$, where $\gamma := (2\alpha + \rho)/\beta\rho$. Let w_i be such that $|w_i| = r_i$ and $|f(w_i)| = M(r_i, f)$. By Lemma 2.5, there will be at least one i such that $|R(w_i)| > 1$. For such i,

$$\left|\tilde{P}\left(w_i, \frac{1}{f(w_i)}\right)\right| \ge \frac{1}{2}.$$

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Again by Remark 1.6,

$$|G(w_i)| = |P(w_i, f(w_i))| = \left| f(w_i)^k \tilde{P}\Big(w_i, \frac{1}{f(w_i)}\Big)\right| \geq \frac{1}{2} e^{kr_i^\beta},$$

and therefore

$$|G(w_i)| \geq 1.$$

Recall that R_H is of the form $C(\log H)^{1/\rho}$, while on the other hand,

$$r_i \leq C(\log H)^\gamma + T + 14.$$

So, $\overline{B(0, R_H)} \subset \overline{B(w_i, s)}$, where $s = C(\log H)^\gamma$.

By the maximum modulus principle and Lemma 1.8,

$$n\Big(R_H, \frac{1}{G}\Big) \leq \frac{1}{\log 2} \log\Big(\frac{M(3s, G)}{|G(w_i)|}\Big) \leq \frac{\log M(3s, G)}{\log 2}.$$

By Remark 1.6,

$$M(3s, G) \leq |P|(T + 1)^2 (3s)^T e^{T(3s)^\alpha}.$$

Since $s = C(\log H)^\gamma$ and $T = C(\log H)^{2\alpha/\rho}$, we deduce that

$$\log M(3s, G) \leq C(\log H)^{2\alpha(\gamma+1)/\rho}.$$

Therefore

$$n\Big(R_H, \frac{1}{G}\Big) \leq C(\log H)^{2\alpha(\gamma+1)/\rho}$$

as required. The constant C effectively depends on μ, ρ, ϕ and d. □

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