**Linear algebra** by Elizabeth S. Meckes and Mark W. Meckes, pp. 427, £49.99 (hard), ISBN 978-1-10717-790-1, Cambridge University Press (2018)

This is an introductory course on linear algebra. Unlike many others, these authors take the 'introductory' seriously by not taking for granted any experience of abstractness on the part of their readers. It is directed at an audience at the transition 'from calculus to rigorous mathematics', keeping in mind that some will go on to have mathematics as their main interest, while some will be heading for engineering and other sciences with this as their only exposure to linear algebra.

So the computational aspects (Gaussian elimination, differential and integral operators, transformation geometry, etc) figure largely in the early parts of the book. But in the end this is a pure mathematics text, with a gentle climb and sympathetic guides to get readers to this point.

The great number of exercises illustrating the variety of behaviours of linear systems (independence, redundancy, consistency, etc) and their relation to the reduced row echelon form of the corresponding matrices certainly help to make the learning curve gentle.

This leisurely and thorough style continues through the rest of the first chapter (of 62 pages) with vectors defined initially as columns of real numbers, writing linear systems in vector form, then relating the usual vector algebra concepts back to linear equations and to the geometric picture of vectors.

Still in Chapter 1, there is an elementary but thorough treatment of fields (the scalars have been taken to be in $\mathbb{R}$ up to this point) and all the operations on linear systems and vectors are shown to work in general fields. Although the definition of a field is rigorous, I rather like the authors' 'bottom line: a field consists of a bunch of things that you can add, subtract, multiply and divide with, and these operations behave basically as you would expect.' The fields used for illustration are $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $F_2$.

The chapter ends by putting all this together in the basic algebra of vector spaces. Here too there is a nice balance between operating in general vector spaces and in familiar 'concrete' ones. Another nice touch is the inclusion of comments on why, in seeking a proof of a result, one choice of initial step may be more productive than an apparently equally attractive one. The teaching message from these authors becomes clear: mathematical maturity cannot be left to happen by accident.

But in spite of the authors' care for their readers, chapter 2 on linear maps and their matrices inevitably takes a step up the abstraction slope with linear maps whose domains are themselves linear maps. All results are proved in detail, and there are many of them: eigenvalues and eigenspaces, vector space isomorphism, elementary matrices, matrix inverses and transposes, LU and LUP decomposition, most of which are related back to the solving of linear systems in Chapter 1, and the connection with row echelon and pivot elements. The final section of Chapter 2 brings much of the previous work together with an account of linear error-detecting and error-correcting codes.

As I read on I began to fall out of love with the treatment, probably because of the relentless density of the results proved – a new one every $1\frac{1}{2}$ pages after chapter 1. I suspect that readers will simply get tired and become content to settle for just a superficial grasp of the later chapters. The only exceptions to this are likely to be the very committed students heading along a very pure mathematical path. I started to wonder what 'elementary' means, since the preface identifies this book with the 'elementary' label. So don't expect elementary to mean easy, brief, or covering a small selection of topics. It is hard going, very thorough and detailed, and every conceivable topic of undergraduate linear algebra is included. The remaining list is linear independence, bases, dimension, rank and nullity, coordinates, change of

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basis, triangularisation, inner products, orthonormal bases, orthogonal projection, optimisation, normed spaces, isometries, singular value decomposition, adjoint maps, the spectral theorem, determinants and characteristic polynomials.

So my 'falling out of love with the treatment' could be unfair. Perhaps I just have to acknowledge that linear algebra is tough and there is a lot of it. Failing to see the wood for the trees is probably the main problem, and these authors try very hard to avoid that by various methods: they provide plenty of interconnections between matrices, linear maps, geometry and some applications; they give a helping hand over the more difficult steps; extensive end-of-section exercises are provided (solutions to half of them), and quick exercises scattered throughout the text, having an 'are you with me so far?' function, with answers at the foot of the page.

It is a book well worth considering both for learning and teaching this important area of mathematics.

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**Introduction to approximate groups** by Matthew C. H. Tointon, pp. 205, £26.99 (paper), ISBN 978-1-10845-644-9, Cambridge University Press (2019)

The theory of approximate groups is a rapidly growing one; it is the focus of considerable current research and has surprising applications to many areas of mathematics, including graph theory, number theory, geometric group theory, and differential geometry. The book now under reviews offers an excellent introduction to this field, written at the level of a first or second year graduate student who has taken a good prior course on group theory and also has some acquaintance with basic notions of analysis such as open sets and measurable functions.

Approximate groups are not hard to define. If $A$ is a subset of a group $G$ and $K$ is a positive integer, we say that $A$ is a $K$-approximate group if $A$ contains the identity element 1 of $G$, is symmetric (i.e. closed under the taking of group inverses), and satisfies an approximate closure condition. To define this condition more precisely, recall that, for subsets $A$ and $B$ of $G$, the set $AB$ denotes the set of all elements $ab$ ($a \in A$, $b \in B$); using this terminology, the requirement that $A$ is closed under the group operation can be rephrased as $A^2 (= AA) \subseteq A$. In this notation, the approximate closure condition referred to earlier is that there exists a set $X$ of size at most $K$ such that $A^2 \subseteq XA$. An actual subgroup is obviously an approximate group with $K = 1$ and $X = \{1\}$. Of course, there are lots of approximate groups that fail to be actual groups: any finite subset of $G$ of size $K$ is a $K$-approximate group.

A slightly weaker notion is that of a *small doubling set*, namely a finite subset $A$ of a group $G$ with the property that the quotient of the cardinalities of $A^2$ and $A$ is bounded by a constant $K \geqslant 1$. (Unfortunately there is a typo in the definition of 'small doubling' found in Definition 1.1.2, but it is fairly obvious and easily corrected.) While there are small doubling sets that are not approximate groups, the two notions are very closely related, and both objects are discussed in the book.

Chapter 1 of the text is introductory: it introduces the notion of an approximate group, gives a few examples, provides a useful two-page survey of the history of the subject, and discusses some background material and notation. Chapters 2 and 3 discuss basic results and examples. Chapters 4–10 discuss small doubling sets and approximate groups in various kinds of groups, including abelian and nilpotent groups and the general linear group. Chapter 11 is devoted to applications; as the author points out, there are too many examples to discuss in detail, but 'since the applicability of approximate groups is