Astin Bulletin (2023), **53**, pp. 84–103
doi:10.1017/asb.2022.27

International Actuarial Association
Association Actuarielle Internationale

RESEARCH ARTICLE

Target benefit pension plan with longevity risk and intergenerational equity[‡]

Ximin Rong[1,2], Cheng Tao[1,*] and Hui Zhao[1]

[1]School of Mathematics, Tianjin University, Tianjin 300350, PR China and [2]Center for Applied Mathematics, Tianjin University, Tianjin 300072, PR China
*Corresponding author. E-mail: taocheng@tju.edu.cn

Received: 16 May 2022; **Revised:** 24 October 2022; **Accepted:** 7 December 2022; **First published online:** 12 January 2023

Keywords: Target benefit plan; longevity risk; intergenerational risk sharing; intergenerational equity; postponing retirement

Abstract
We study a stochastic model for a target benefit pension plan suffering from rising longevity and falling fertility. Policies for postponing retirement are carried out to hedge the payment difficulties caused by the aging population. The plan members' contributions are set in advance while the pension payments reflect intergenerational equity by a target payment level and intergenerational risk sharing by an adjustment. The pension fund is invested in both a risk-free asset and a risky asset. Applying the stochastic optimal control methods, we derive analytic solutions for optimal investment and benefit payment strategies which minimize the benefit risk. Besides, an optimal delayed retirement age which can hedge against the aging phenomenon under certain parameters is given. Therefore, it can provide a basis for quantifying the delay of retirement time.

1. Introduction

The target benefit pension (TBP) plan, as a new type of pension system combining traditional defined benefit (DB) and defined contribution (DC) pension funds, has attracted more and more attention. The target benefit plan is a collective pension scheme featuring fixed contributions and a target benefit level preset by trustees, and actual benefits may exceed or fall short of the target, depending on the return of the fund in the financial market. Its primary advantage is that risks are shared between different generations rather than being borne individually (Wang *et al*., 2018, 2019), which may enhance welfare compared to either traditional defined benefit or defined contribution plans (Gollier, 2008; Cui *et al*., 2011). Nowadays, TBP has developed in Canada and Germany (CIA, 2015; Chen and Rach, 2021). Zhu *et al*., 2021) study fair transitions from defined benefit pension schemes to target benefit pension schemes. Besides, the flexible selection of targets gives TBP a huge space for optimization in design. Chen *et al*. (2022) design a TBP plan possessing a linear risk-sharing structure with cyclical parameters that balances the sustainability and the benefit stability.

Recently, the intergenerational risk-sharing pension design has attracted more and more attention. In this pension design, investment and other risks are distributed among different generations. Many risk-sharing schemes have emerged around the world, including risk-sharing DB plan in Japan (Pugh and Yermo, 2008; Turner, 2014), target benefit plan in Canada (Munnell and Sass, 2013; CIA, 2015), and collective DC plans from the Netherlands (Kortleve, 2013; Bovenberg *et al*., 2016). Some academic discussions on collective pension schemes with risk sharing have been put forward (Baumann and Müller, 2008; Gollier, 2008; Cui *et al*., 2011; Khorasanee, 2012; Chen *et al*., 2017; Wang *et al*., 2018; Wang and Lu, 2019).

[‡]This article has been updated since its original publication. A notice detailing this change can be found here: https://doi.org/10.1017/asb.2023.9

In addition to financial risk, risks that need to be shared between generations also include longevity risk. Generally speaking, longevity trends means people born later live longer than those born before, and longevity risk is a fluctuation that interferes with people's estimation of longevity trends. In recent decades, countries around the world have been enduring demographic transition, characterized by increasing life expectancy at birth and declining lifetime fertility (Oeppen and Vaupel, 2002; Medford, 2017; Pascariu *et al.*, 2018). The proportion of the global population aged 65 years or over is expected to increase from 9.3% in 2020 to about 16% in 2050 (United Nations, 2020). This development is a particular challenge for pension systems.

The negative effects of the increase of life expectancy and the decrease of fertility rate on collective pension systems are mainly reflected in two aspects. First, rising longevity and falling fertility threaten the financial sustainability of pension plans (Heeringa and Bovenberg, 2009; Villegas and Haberman, 2014). Neglecting longevity trends may cause inadequate funding for pension liabilities (Coughlan *et al.*, 2011). Second, the inequality of pension redistribution may increase in the sense of transfers of wealth from the poorer (usually with shorter lifespans) to the richer (with longer lifespans) (Ayuso *et al.*, 2017; Holzmann *et al.*, 2019). Cohorts with a high longevity are compensated at the expense of cohorts with a relatively short longevity (Andersen, 2014), which is unavoidable in collective pension systems, and rising longevity will exacerbate this trend. Therefore, there is an increased need to reform pension systems in many countries.

In order to maintain financial sustainability, postponing retirement has become a seemingly default method in designing pension systems (Galasso, 2008; Alvarez *et al.*, 2021). One way is to increase the retirement age to make the remaining life expectancy fixed. In Denmark, the statutory retirement age will be gradually raised to the target age at which the remaining life expectancy is 14.5 years, see (Danish Ministry of Economic Affairs and Interior, 2017, 2018). Another way is to set the retirement age proportional to cohort-specific longevity (Knell, 2018). However, if an increase in life expectancy does not lead to difficulties in pension payments, it may not be necessary to delay retirement. Therefore, in this paper, we abandon the assumption that retirement ages directly link to life expectancy and explore the demographic conditions for how long to delay retirement in the view of the pension fund's investments and payments.

In the TBP plan, the amount of pension is based on the target and the adjustment of risk sharing, so in addition to establishing a good risk-sharing mechanism, the determination of a reasonable target also needs to be considered. In this paper, the target setting is based on the principle of actuarial equity, which can reflect intergenerational equity. In the multigenerational pension model, intergenerational equity is important, since different generations face different life curves and external environments (e.g., salaries). If their pension annuities are the same, intergenerational equity will be damaged. In addition, people who delay retirement should enjoy higher pension income than those who retire normally, otherwise it will undermine fairness and cause dissatisfaction. Therefore, each generation has its own target annuity, and its amount is determined by the principle of actuarial equity, which means the expected future value of accumulated contributions at the retirement date is equal to the present value at that time of the expected total benefits to be received in retirement. The actual benefit (target plus adjustment) for each generation draws lessons from longevity-linked life annuities (LLLA), where the welfare is adjusted up and down according to the life index calculated by using the reference population (Denuit *et al.*, 2011; Bravo and De Freitas, 2018).

To the best of the authors' knowledge, the proposed study is a maiden attempt at investigating the proper length of postponing retirement under the condition of longevity risk and fertility decline in the target benefit pension plan. In this paper, we study a stochastic model for a TBP suffering from rising longevity and falling fertility. Policies for postponing retirement are carried out to hedge the payment difficulties caused by aging population. The pension fund is invested in both a risk-free asset and a risky asset. Applying the stochastic optimal control methods, we derive analytic solutions for optimal investment and benefit payment strategies which minimize the benefit risk. Compared with Chen *et al.* (2022), where the target benefit level is externally given and the longevity risk is ignored, the target benefit in this paper is endogenous by the principle of actuarial equity and can be affected by the longevity risk and

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press

postponing retirement. Thus, our paper reflects the adaptability of the TBP plan in the face of population structure changes.

This study contributes three primary developments. First, we explore the potential of the TBP in intergenerational equity and intergenerational risk sharing. In this paper, the target benefit reflects intergenerational equity and the adjustment (the difference between the actual benefit and target benefit) reflects intergenerational risk sharing. Second, we consider longevity risk (systematic) and fertility decline in a multigenerational superposition pension model. Through endogenous target benefits, we can dynamically respond to demographic changes and postponing retirement. Third, we obtain an optimal delayed retirement age which is able to maximize the interests of the pension fund and individuals.

This paper is organized as follows. In Section 2, we introduce the basic settings of our model. The optimal stochastic problem is discussed in Section 3. In Section 5, we analyze some numerical examples to demonstrate our results. Finally, Section 6 concludes the paper.

2. Model formulation

2.1. Longevity risk

The aging population is driven by two demographic factors: declining fertility rates and increasing life expectancy. We set $n(h)$ to be the instantaneous density of the newborn cohort at time h, which drops over time to represent declining fertility rates.

The longevity risk can be classified into systematic and nonsystematic (Boon *et al.*, 2022). Here we focus on the systematic longevity risk. To describe the systematic longevity risk, we introduce the assumed survival function $\bar{S}(x, h)$ to be the assumed probability that a member of cohort h survives to age x. We denote the random death time of cohort h as $k(h)$, and then we have $\mathbb{P}(k(h) > x) = \bar{S}(x, h)$. The values of $\bar{S}(x, h)$ come from an initial life table. When signing a pension annuity contract, the insurer determines the value of the annuity according to $\bar{S}(x, h)$.

The real survival function is denoted by $S(x, h)$ and is nonnegative. The real survival function often exceeds the insurer's expectation due to the systematic longevity risk, which will bring losses to the insurer. The difference between the real survival function $S(x, h)$ and the hypothetical survival function $\bar{S}(x, h)$ constitutes the systematic longevity risk. It holds that $S(0, h) = 1$ and $S(\omega, h) = 0$, where ω is a sufficient large constant to be an upper bound lifespan for all cohorts h considered in this model. For fixed h, $S(x, h)$ is decreasing in x, and for fixed x, $S(x, h)$ is increasing in h. It follows that the life expectancy of cohort h is nondecreasing which can be calculated by integrating the survival function with respect to age x from 0 to ω.

Assume all the members join the TBP at age a. Let the retirees' minimum age (also the active members' maximum age) at time t be $r(t)$. Then the density of those who reach age x at time t is $n(t - x)S(x, t - x)$. Then the expected total number of active members at time t is given by

$$A(t) = \int_a^{r(t)} n(t - x)S(x, t - x)\mathrm{d}x.$$

Similarly, the expected total number of retired members at time t is

$$R(t) = \int_{r(t)}^{\omega} n(t - x)S(x, t - x)\mathrm{d}x.$$

The old-age dependency ratio which describes the average number of elderly people to support per working population is thus

$$D(t) = \frac{R(t)}{A(t)}.$$

Generally speaking, the aging population is accompanied by the increase of the dependency ratio.

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press

2.2. Postponing the retirement age

By assumption, the TBP fund manager decides to implement a postponing retirement policy. Before time $t = 0$, active members retire at the age of r_0. Then at time $t = 0$, a postponing retirement policy is carried out that the statutory retirement age is set to r_1, which means active members need to work until they are r_1 years old. Those who have retired before $t = 0$ can continue to enjoy their retirement life, even if part of them do not reach the new retirement age r_1. After $t = 0$, active members are supposed to work until they reach the new retirement age r_1. For example, if we set the initial retirement age r_0 to 55 and the new retirement age r_1 to 60, then at time 0, the minimum age of retirees is still 55, while at time $t = 1$, the retirees' minimum age becomes 56. More specifically, at time $t = 1$, except those who died, the retired members are the same as a year ago, because the increase of the statutory retirement age causes that there are no new retirees this year. However, when $t \geq 5$, new retirees begin to appear, then the retirees' minimum age will be fixed at age 60. To conclude, when $t \leq r_1 - r_0$, the retirees' minimum age is $r_0 + t$ and the active members' ages range from a to $r_0 + t$. For $t > r_1 - r_0$, the retirees' minimum age (also the active members' maximum age) will always be r_1. We now introduce $r(t)$ to be the retirees' minimum age at time t, which is subtly different from the statutory retirement age. It holds that

$$r(t) = \begin{cases} r_0, & t \leq 0, \\ r_0 + t, & 0 < t \leq r_1 - r_0, \\ r_1, & t > r_1 - r_0. \end{cases}$$

2.3. Plan provisions

Consider a plan consisting of active and retirement members. While the active members make contributions to the pension fund, the retired members receive benefits paid from the pension fund.

Denote by $Y(x, h)$ the annual salary rate of an active member aged x of cohort h. Then each active member aged x contributes $cY(x, t - x)$ to the pension fund at time t, where c is the contribution ratio related to the salary. Thus, the aggregate instantaneous contribution rate of all active members at time t is given by

$$C(t) = \int_a^{r(t)} n(t - x)S(r(t), t - x)cY(x, t - x)\mathrm{d}x.$$

The reason why we use $S(r(t), t - x)$ rather than $S(x, t - x)$ here is that for those who pass away before retirement, their contributions are refunded. When a member joins the pension fund, she will continue to make contributions to her account. At the moment, her account is part of the pension fund, and the manager will invest it. If she dies before retirement, the pension fund will return all the accumulated contributions in her account. We do not care how the manager invests the funds in her account, because no matter what the investment is, when she dies before retirement, the funds in her account will be withdrawn from the pension fund. Therefore, we only consider the contribution of people who live to retirement age.

When the retirees reach their retirement age r_0 or r_1, they begin to receive pension benefits. Let $b(t, h)$ be the actual benefit at time t for retirees of cohort h, which is given by

$$b(t, h) = \bar{b}(h) + \theta(t), \tag{2.1}$$

where $\bar{b}(h)$ is the target annuity for the cohort h, and $\theta(t)$ reflects the risk sharing at time t, including investment risks and longevity risks. Their expressions are discussed in the next two subsections.

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press

2.4. Financial market

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, where $\mathbb{P}$ is the real-world probability measure on Ω and $\mathscr{F} = \{\mathscr{F}_t\}_{t\geq 0}$ is a complete and right continuous filtration generated by a standard Brownian motion.

We assume that there are two underlying assets available to the pension trustees: one risk-free asset (a bank account) and one risky asset (a stock). The evolution of the value of the risk-free asset $S_1(t)$ is given by

$$\mathrm{d}S_1(t) = m(t)S_1(t)\mathrm{d}t, \qquad t \geq 0, \tag{2.2}$$

where $m(t)$ represents the risk-free interest rate.

The price process of the risky asset is denoted by $S_2(t)$, and $S_2(t)$ is supposed to follow a stochastic differential equation of the form

$$\mathrm{d}S_2(t) = S_2(t)(\mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t)), \qquad t \geq 0, \tag{2.3}$$

where $\mu(t)$ is the appreciation rate of the stock and $\sigma(t)$ is the volatility rate, and $W(t)$ is a standard Brownian motion. To exclude arbitrage opportunities, we assume that $\mu(t) > m(t)$.

2.5. Target benefit: intergenerational equity

The pension plan provides members with a life annuity payable continuously commencing at retirement age r_0 or r_1 and has a pre-set retirement benefit target $\bar{b}(h)$ for retired members of cohort h (which means they are born at time h). The target benefit is assumed as a fixed annuity, which means retirees of the cohort h get a fixed annuity $\bar{b}(h)$ until they die. The value of $\bar{b}(h)$ is based on the principle of actuarial equity, that is, the expected future value of accumulated contributions at the retirement date is equal to the present value at that time of the expected total benefits to be received in retirement. Since we only consider people who can live to retirement age, we introduce the conditional probability that an individual aged z at time $h + z$ can live to x years old here, which is denoted by

$$\bar{S}(x, z, h) = \mathbb{P}(k(h) > x | k(h) > z) = \frac{\bar{S}(x, h)}{\bar{S}(z, h)}.$$

When the pension contract is generated, the basis of the insurer is the assumed conditional survival function $\bar{S}(x, z, h)$ rather than the real survival function $S(x, h)$.

For the cohort h with $h \geq -r_0$ (thus their retirement age is deferred), they encounter a delay in retirement and the value of $\bar{b}(h)$ should conform to actuarial equity, that is,

$$\int_a^{r_1} e^{\int_x^{r_1} m(s)\mathrm{d}s} cY(x, h)\mathrm{d}x = \int_{r_1}^{\omega} e^{\int_{r_1}^{x} -m(s)\mathrm{d}s} \bar{b}(h)\bar{S}(x, r_1, h)\mathrm{d}x.$$

The left of the equal sign is the value of accumulated contributions at retirement r_1, and the right is the discounted present value of the expected overall annuity income at r_1.

So the target fixed annuity is

$$\bar{b}(h) = \frac{\int_a^{r_1} e^{\int_x^{r_1} m(s)\mathrm{d}s} cY(x, h)\mathrm{d}x}{\int_{r_1}^{\omega} e^{\int_{r_1}^{x} -m(s)\mathrm{d}s} \bar{S}(x, r_1, h)\mathrm{d}x}, \tag{2.4}$$

when $h \geq -r_0$.

Similarly, for $h < -r_0$, we have

$$\bar{b}(h) = \frac{\int_a^{r_0} e^{\int_x^{r_0} m(s)\mathrm{d}s} cY(x, h)\mathrm{d}x}{\int_{r_0}^{\omega} e^{\int_{r_0}^{x} -m(s)\mathrm{d}s} \bar{S}(x, r_0, h)\mathrm{d}x}.$$

In this way, even if the retirement is delayed, the interests of the policyholders can be guaranteed. In addition, this form of $\bar{b}(h)$ can also correspond well to the change of longevity trends $(\bar{S}(x, r_0, h))$ and the external environment $(Y(x, h))$.

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press

From the perspective of the insurer, the target aggregate payment rate $\bar{B}(t)$ for all retirees at time t is given by

$$\bar{B}(t) = \int_{r(t)}^{\omega} n(t-x)S(x, t-x)\bar{b}(t-x)\mathrm{d}x.$$

Since the target benefit $\bar{b}(h)$ of each cohort h is based on actuarial equity, we say that this target aggregate payment $\bar{B}(t)$ reflects intergenerational equity.

2.6. Actual benefit: intergenerational risk sharing

The actual aggregate payment rate is $B(t)$ for all retirees at time t, which is a strategy determined by the plan trustees and thus is affected by market fluctuations. Since $B(t)$ is affected by the return from the financial market, it contains investment risks. The difference between $B(t)$ and $\bar{B}(t)$ is $B(t) - \bar{B}(t)$ which actually reflects risks (including longevity risks and investment risks). The adjustment is denoted by $\theta(t)$, which shares the risks among all retirees. We have

$$\theta(t) = \frac{B(t) - \bar{B}(t)}{R(t)}. \tag{2.5}$$

The value of $\bar{B}(t)$ and $R(t)$ is affected by longevity risks. We aim to achieve this effect: when $B(t) - \bar{B}(t) < 0$, the trend of longevity makes more people share the loss; when $B(t) - \bar{B}(t) > 0$, the trend of longevity makes more people share the benefits. Therefore, θ shares the longevity risk and the investment risk.

Since $\theta(t)$ is independent of cohort h, the value of risk sharing is the same for all cohorts, and it reflects that the risk is shared among all generations.

2.7. Terminal wealth: future security

Let $\{F(t), t \in [0, T]\}$ be the wealth process of the pension fund, where $T > 0$ is the terminal time point. The target terminal wealth is a value that $F(T)$ is supposed to get close and is denoted as M, which contains two components: $M = M_1 + M_2$. To keep the fund size from shrinking, M_1 is set to $F(0)e^{\int_0^T m(s)\mathrm{d}s}$, which is the future value at time T of the initial wealth $F(0)$. If $F(T)$ is close to M_1, it may imply that the pension fund operates stably and is prepared for the next cycle.

Let M_2 be the capital reserve which can ensure that the pension is supported for at least τ years. It is the accumulation of the present discounted value of the difference between income and target expenditure in the next τ years at time T:

$$M_2 = \int_T^{T+\tau} e^{-\int_T^s m(u)\mathrm{d}u}(\bar{B}(s) - C(s))\mathrm{d}s.$$

If the terminal wealth $F(T)$ reaches M_2, the pension fund will last for at least τ years with risk-free investments only. This provides a guarantee for the pension payment in the next τ years.

Therefore M is used to achieve the following two goals: to maintain the fund pool size and to leave enough funds to run for τ years.

3. The stochastic control problem

In this subsection, we present the wealth process of the pension fund including the dynamics of the fund investment, the contributions from active members, and the benefits for all retirees. We then establish a continuous-time stochastic optimal control problem based on the target benefit plan to realize intergenerational risk sharing and equity.

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press

3.1. Objective function

Assume that the plan trustees can invest in both the risk-free and risky assets and use the fund to pay retirement benefits. Let f_0 denote the initial wealth of this fund, $\pi(t)$ denote the amount that the plan manager invests in the risky asset at time t, and $F(t)$ denote the wealth of the pension fund at time t after adopting the investment strategy $\pi(t)$. Then the fund's value follows the dynamics:

$$\begin{cases} dF(t) = \pi(t)\dfrac{dS_2(t)}{S_2(t)} + (F(t) - \pi(t))\dfrac{dS_1(t)}{S_1(t)} + [C(t) - B(t)]dt, \\ F(0) = f_0. \end{cases} \tag{3.1}$$

Using (2.2) and (2.3), we can easily rewrite (3.1) as

$$\begin{cases} dF(t) = \pi(t)(\mu(t)dt + \sigma(t)dW(t)) + (F(t) - \pi(t))(m(t)dt) + [C(t) - B(t)]dt, \\ F(0) = f_0. \end{cases} \tag{3.2}$$

We now define $\eta = \{(\pi(s), B(s))\}_{s\in[t,T]}$ to be a strategy adopted by the plan trustees between time t and T, where $T > 0$ is the terminal time point. This strategy includes the investment strategy $\pi(s)$ which is the amount to be invested in the risky asset at time s and the pension benefit $B(s)$ which is the amount of annuity for the retirees at time s, for $s \in [t, T]$. The definition for a strategy being an *Admissible Strategy* with respect to SDE (3.2) is provided below.

Definition 2.1 (Admissible Strategy). *For any fixed $t \in [0, T]$, a portfolio and benefit payment processes pair $\eta = (\pi(s), B(s))_{s\in[t,T]}$ is said to be admissible if*

(i) η is $\mathscr{F}_t$-adapted;
(ii) for all $s \in [t, T]$, $B(t) \geq 0$ and $\mathrm{E}\left[\int_t^T [\pi(s)^2 + B(s)^2]ds\right] < +\infty$;
(iii) (F^η, η) is the unique solution to SDE (3.2).

Remark 3.1. It will be more realistic if we assume that the fund is prohibited from borrowing ($\pi(t) > F(t)$) and short selling ($\pi(t) < 0$). However, such restrictions may make it impossible to derive an analytic solution. So it is a weakness of our analysis.

We now turn to the objective function of the problem. The composition of the objective function mainly considers the following three aspects. First, in order to reduce the general deviation between the actual benefit and the target benefit, we integrate the squared deviation between $B(t)$ and $\bar{B}(t)$ with respect to time t, and if this term is smaller, $B(t)$ gets closer to $\bar{B}(t)$. Next, under the precondition that the actual and target benefit are close, the deviations are expected to be positive, that is, it will be better if the actual benefits are higher than target benefits. Finally, the actual terminal value of the fund $F(T)$ is supposed to approach M to ensure the security of the pension fund. Combined with the three factors above, then the objective function $J(t, f)$ at time t with the fund value f is defined as

$$J(t,f) = \mathrm{E}_{\pi,B}\left\{ \int_t^T [(B(s) - \bar{B}(s))^2 - \lambda_1(B(s) - \bar{B}(s))]ds \right. \tag{3.3}$$
$$\left. + \lambda_2(F(T) - M)^2 | F(t) = f \right\},$$

where both λ_1 and λ_2 are nonnegative constants, interpreted as penalty weights given to the negative integration of deviation between $B(s)$ and $\bar{B}(s)$ and the achievement of the terminal fund target, respectively. The expectations in (3.3) are conditional on the state of f at time t.

The value function is defined as

$$V(t,f) := \min_{\eta\in\Pi} J(t,f), \qquad t > 0, \tag{3.4}$$

where Π is a set of all the admissible strategies of η.

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press

3.2. Optimal retirement age

Under the certain condition of the aging population, we expect to obtain different values of $V(0, f_0)$ with different deferred retirement ages r_1. In Section 3.1, we find that the smaller value of $V(0, f_0)$ means the benefit $B(t)$ and the terminal wealth $F(T)$ are more close to their targets $\bar{B}(t)$ and M, and the overpayment $B(t) - \bar{B}(t)$ is larger. Thus, our purpose is to determine an optimal deferred retirement age to minimize the value function $V(0, f_0)$.

If the new retirement age r_1 can minimize $V(0, f_0)$, we denote it as the optimal retirement age r^*:

$$r^* = \inf\{\arg \inf_{r_1 \in [r_0, \omega]} V(0, f_0)\}. \tag{3.5}$$

The deferred retirement ages r_1 which make $V(0, f_0)$ reach its minimum value may not be unique, so we select the smallest possible r_1 to be r^*.

4. Main results

In this section, we use standard methods to solve the optimal control problem (3.4) and derive closed-form expressions of the optimal policy, denoted by $\eta^*(t) = (\pi^*(t), B^*(t))$.

First, we derive an HJB equation associated with the stochastic control problem (3.4). We refer to Fleming and Soner (2006) for more details. Using variational methods and Itô's formula, we get the following HJB equation satisfied by the value function $V(t, f)$:

$$V_t + \inf_{\eta} \left\{ \frac{1}{2} V_{ff} \sigma(t)^2 \pi^2 + V_f[\mu(t)\pi + m(t)f - m(t)\pi + C(t) - B] \right. \tag{4.1}$$

$$\left. + (B - \bar{B}(t))^2 - \lambda_1 (B - \bar{B}(t)) \right\} = 0$$

with boundary condition

$$V(T, f) = P(T)f^2 + Q(T)f + K(T) = \lambda_2 f^2 - 2\lambda_2 M f + \lambda_2 M^2, \tag{4.2}$$

where V_t, V_f are partial derivatives of $V(t, f)$.

We state our findings on the optimal asset allocation and benefit payment policies for optimal control problem (3.4) in the following theorem.

Theorem 4.1. *For the optimal control problem* (3.4), *the optimal asset allocation policy and benefit payment policy are given, respectively, by*

$$\pi^*(t) = \pi^*(t, F(t))$$

and

$$B^*(t) = B^*(t, F(t)),$$

where

$$\pi^*(t, f) = \frac{m(t) - \mu(t)}{\sigma(t)^2} \left(f + \frac{Q(t)}{2P(t)} \right), \tag{4.3}$$

$$B^*(t, f) = \bar{B}(t) + \frac{\lambda_1}{2} + P(t)f + \frac{Q(t)}{2}, \tag{4.4}$$

and the corresponding value function is given by

$$V(t, f) = P(t)f^2 + Q(t)f + K(t), \tag{4.5}$$

where

$$P(t) = \left[\frac{1}{\lambda_2} e^{\left(\frac{(\mu(t) - m(t))^2}{\sigma(t)^2} - 2m(t) \right)(T-t)} + e^{\left(\frac{(\mu(t) - m(t))^2}{\sigma(t)^2} - 2m(t) \right)(T-t)} \int_t^T e^{\left(2m(x) - \frac{(\mu(x) - m(x))^2}{\sigma(x)^2} \right)(T-x)} \mathrm{d}x \right]^{-1}, \tag{4.6}$$

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press

$$J(t) = C(t) - \bar{B}(t) - \frac{\lambda_1}{2}, \tag{4.7}$$

$$H(t) = -\frac{(\mu(t) - m(t))^2}{\sigma(t)^2} + m(t) - P(t), \tag{4.8}$$

$$Q(t) = -2\lambda_2 M e^{\int_t^T H(x)\mathrm{d}x} + e^{\int_t^T H(x)\mathrm{d}x} \int_t^T 2P(x)J(x)e^{-\int_x^T H(y)dy}\mathrm{d}x, \tag{4.9}$$

$$K(t) = \lambda_2 M^2 - \int_t^T \left[\frac{(\mu(x) - m(x))^2}{4\sigma(x)^2} \frac{Q(x)^2}{P(x)} - Q(x)J(x) + \frac{1}{4}Q(x)^2 + \frac{\lambda_1^2}{4} \right] \mathrm{d}x. \tag{4.10}$$

The proof can be found in Appendix A.

Corollary 4.2. *If $m(t)$, $\sigma(t)$, and $\mu(t)$ are constants (m, σ, and μ, respectively), then $P(t)$ is a positive and monotonically increasing function.*

Proof. Denoting $\frac{(\mu-m)^2}{\sigma^2} - 2m$ as γ, we can simplify (4.6) as

$$P(t) = \gamma e^{\gamma(t-T)} \left(e^{\gamma(T-t)} - 1 + \frac{\gamma}{\lambda_2} \right)^{-1},$$

for $\gamma \neq 0$.

If $\gamma = 0$, then we have

$$P(t) = \left(\frac{1}{\lambda_2} + T - t \right)^{-1},$$

which is positive and monotonically increasing function.

Note that $\lambda_2 > 0$. If $\gamma > 0$, then $\gamma e^{\gamma(t-T)}$ and $(e^{\gamma(T-t)} - 1 + \frac{\gamma}{\lambda_2})^{-1}$ are both positive and monotonically increasing functions. Therefore, their product $P(t)$ is a positive, monotonically increasing function.

Conversely, if $\gamma < 0$, then $\gamma e^{\gamma(t-T)}$ and $(e^{\gamma(T-t)} - 1 + \frac{\gamma}{\lambda_2})^{-1}$ are both negative and monotonically decreasing function. Therefore, their product $P(t)$ is a positive, monotonically increasing function. □

Remark 4.3. If we allow $m(t)$, $\sigma(t)$, and $\mu(t)$ to be deterministic functions, not just constants, then $P(t)$ is still positive (not necessarily monotonic increasing). It is an important item of the adjustment $\theta(t)$, which can be seen in the following corollary.

Corollary 4.4. *The adjustment $\theta(t)$ is*

$$\theta(t) = \frac{\lambda_1 + 2P(t)F(t) + Q(t)}{2R(t)} \tag{4.11}$$

or

$$\theta(t) = \frac{\lambda_1(m(t) - \mu(t)) + 2\sigma(t)^2\pi^*(t)P(t)}{2R(t)(m(t) - \mu(t))}. \tag{4.12}$$

Proof. According to (2.5) and (4.4), the adjustment $\theta(t)$ is given by

$$\theta(t) = \frac{B^*(t) - \bar{B}(t)}{R(t)} = \frac{B^*(t, F(t)) - \bar{B}(t)}{R(t)}$$
$$= \frac{\lambda_1 + 2P(t)F(t) + Q(t)}{2R(t)}.$$

Rearranging (4.3), we have

$$F(t) = \frac{\sigma(t)^2}{m(t) - \mu(t)}\pi^*(t) - \frac{Q(t)}{2P(t)}. \tag{4.13}$$

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press

Substituting (4.13) into (4.4), we have

$$B^{*}(t) = \bar{B}(t) + \frac{\lambda_1}{2} + \frac{\sigma(t)^2}{m(t) - \mu(t)}\pi^{*}(t)P(t). \tag{4.14}$$

Therefore, the adjustment $\theta(t)$ can be written as

$$\theta(t) = \frac{\lambda_1(m(t) - \mu(t)) + 2\sigma(t)^2\pi^{*}(t)P(t)}{2R(t)(m(t) - \mu(t))}.$$

□

Remark 4.5. Reviewing (3.3), we find that λ_1 is the weight of the amount exceeding the target payment $\bar{B}(t)$. Note that the expressions of $P(t)$ and $Q(t)$ do not contain λ_1. According to both (4.11) and (4.12), the effect of λ_1 on $\theta(t)$ is obvious: under the same condition, the larger λ_1 is, the larger $\theta(t)$ is.

We regard the difference between the real payment $B^{*}(t)$ and the target payment $\bar{B}(t)$ as the risks which contains both financial risks and longevity risks, since $B^{*}(t)$ is affected by the financial market and $\bar{B}(t)$ is affected by the number of retirees. Then $\theta(t)$ is the risk divided by the number of retirees $R(t)$. Assuming that the numerator in (4.11) is fixed, the large $R(t)$ caused by the longevity trend will lead to $\theta(t)$ close to 0. For example, if the fixed numerator is positive, then θ is positive and decreases to 0 with a large enough $R(t)$. If the fixed numerator is negative, then θ is negative and increases to 0 with a large enough $R(t)$. Through this mechanism, the longevity trend increases the number of people taking financial risks, which realizes the effect of reducing risks.

Proposition 4.6. *If we regard r_1 as an independent variable of $\bar{b}(h)$, the monotonicity of $\bar{b}(h)$ with respect to r_1 is ambiguous.*

Proof. If we assume $Y(x, h)$ to be a constant $y > 0$ and $m(s)$ to be zero, then we can simplify (2.4) as

$$\bar{b}(h) = \frac{cyr_1}{\int_{r_1}^{\omega}\bar{S}(x, r_1, h)\mathrm{d}x}, \quad h \geq -r_0. \tag{4.15}$$

Since $\bar{S}(x, r_1, h)$ is monotonically decreasing with respect to r_1, we can introduce a special case where

$$\bar{S}(x, r_1, h) = \frac{1}{(\omega - r_1)^2}, \quad 0 \leq r_1 < \omega.$$

Substituting it into (4.15) we have

$$\bar{b}(h) = cyr_1(\omega - r_1), \quad h \geq -r_0,$$

which is quadratic function with respect to r_1 and is monotonically increasing for $r_1 \in [r_0, \frac{\omega}{2}]$ (if $r_0 < \frac{\omega}{2}$) and monotonically decreasing for $r_1 \in [\frac{\omega}{2}, \omega)$.

Through this special case, we find that the monotonicity of $\bar{b}(h)$ with respect to r_1 is ambiguous. □

Remark 4.7. Unlike intuitive thinking, delaying retirement does not necessarily lead to a higher target annuity $\bar{b}$. This is because we adopt the assumption that the contribution of members who pass away before retirement is refunded and thus the target annuity is related to form of the conditional probability $\bar{S}(x, r_1, h)$, whose monotonicity is not obvious. Considering extreme cases, if $r_1 = a$, which means the individual skips the contribution process, naturally $\bar{b}(h)$ should be 0. If $r_1 = \omega$, which means no one can live to pensionable age and all their contributions are refunded, then $\bar{b}(h)$ should be set to 0 too.

If we replace $\bar{S}(x, h)$ with $\bar{S}(x, r_1, h)$, which means the pension does not refund the contribution of those who pass away early, then $\bar{b}(h)$ will be monotonically increasing.

Proposition 4.8. *Let $U(s)$ denote $V(0, f_0)|_{r_1=s}$, where $s \in [r_0, \omega]$, then $U(s)$ is continuous w.r.t. s.*

Proof. Let $G(s, t) = \int_a^{\omega} Z(s, t, x)\mathrm{d}x$, where $Z(s, t, x)$ is continuous on $[r_0, \omega] \times [0, T] \times [a, \omega]$. Then $Z(s, t, x)$ is bounded and by Lebesgue-dominated convergence theorem, $G(s, t)$ is continuous w.r.t. s. Here $G(s, t)$ can be $C(t)|_{r_1=s}$ and then $Z(s, t, x) = \chi_{[a, r(t)|_{r_1=s}]}(x)n(t - x)S(r(t)|_{r_1=s}, t - x)cY(x, t - x)$ is

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press

continuous if we assume that $n(\cdot)$, $S(\cdot, \cdot)$ and $Y(\cdot, \cdot)$ are continuous. Thus, $C(t)$ is continuous w.r.t. r_1. Similarly, we have $\bar{B}(t)$, $J(t)$, $Q(t)$, and $K(t)$ are continuous w.r.t. r_1. Then by (4.5), $V(t, f)$ is continuous w.r.t. r_1. □

Remark 4.9. Since $U(s)$ is continuous w.r.t. s in $[r_0, \omega]$, we obtain the existence of r^*. The explicit expression of r^* is difficult to get. Therefore, we obtain r^* by substituting some possible discrete r_1 into $U(s)$ and comparing them in the numerical analysis.

5. Illustration

We have obtained in the previous section explicit expressions of the optimal investment allocation and benefit strategy as well as the value function for the TBP model. To demonstrate our results regarding optimal strategies, we present numerical illustrations obtained through simulation in this section under different scenarios.

5.1. Assumptions

Now we make two assumptions that allow for numerical solutions with respect to the changing demographic structure. First, we assume that the number of people entering the plan per unit time evolves in an exponential manner, that is:

$$n(h) = N \exp(-w(h - t_0)\chi_{\{h \geq t_0\}}),$$

where w is the decreasing rate, $t_0 = -80$ is the time point when the fertility decline begins, and N is the initial pension entry rate at time $t = t_0$. This setting is similar to Knell (2018). Note that the fertility rate begins to decrease at time t_0 rather than 0. We begin to observe the behavior of the pension fund at $t = 0$, while the demographic structure may have changed for a long time.

The second assumption is that the survival function $S(x, h)$ satisfies the Gompertz-Makeham law, which goes back to Gompertz (1825). Milevsky (2020) summarizes a Gompertz-Makeham calculation model with the compensation law from the life data of various countries. This enlightens us that we can use the mortality rate of low life span countries evolving to that of high life span countries with generation to describe the longevity trend. Now we introduce the mortality rate $q(x, h)$:

$$q(x, h) = \begin{cases} \rho + \dfrac{1}{\beta(h)} e^{\frac{x-\alpha(h)}{\beta(h)}}, & x \in [0, x^*], \\ \rho + e^L, & x \in (x^*, \omega], \end{cases} \tag{5.1}$$

where ρ is a nonage-dependent accidental death rate (Makeham constant), and x^*, L, α, and β are Gompertz parameters. In Milevsky (2020), it is observed that $\rho + e^L$ is an upper limit of the mortality rate, which is reached by all countries near the age of x^*, and x^* and L are suggested to be set to 100 and -1, respectively. This setting is followed in this paper. Note that $q(x, h)$ has an argument h, which means the mortality rate changes with generation. We assume α and β involves with the cohort h to simulate the longevity trend, thus they are denoted as $\alpha(h)$ and $\beta(h)$. If we adopt an equivalent form of $q(x)$, that is, $q(x) = \rho + \kappa e^{\varsigma x}$, $x \in [0, x^*]$, where $\kappa = (1/\beta)e^{-\frac{\alpha}{\beta}}$ and $\varsigma = 1/\beta$, then the compensation law means a linear relationship between κ and ς, that is, $ln[\kappa] = L - x^*\varsigma$. We can understand the compensation law as countries with a relatively low initial natural mortality rate κ tend to have a higher mortality growth rate ς and vice versa. This consequently forces a relationship between $\alpha(h)$ and $\beta(h)$:

$$\alpha(h) = x^* - \beta(h)(L + ln(\beta(h))).$$

Next, we assume that

$$\beta(h) = \beta_0 - d(h - t_0)\chi_{\{h \geq t_0\}},$$

where d is the parameter that controls the speed of the longevity trend, and $t_0 = -80$ is the time point when the longevity trend begins. Milevsky (2020) provides a table of Gompertz-Makeham parameters around the world, which implies the smaller the value of β, the longer the lifespan of people. The initial value β_0 of cohort t_0 is set to 14, which is similar with that of Belarus ($\beta = 14.5$). If $d = 0.05$ (the maximum value of d in the following numerical analysis), after 80 years from $h = t_0$, we have $\beta(0) = 10$ (the value of β is 9.73 for Japan), which means the lifespan increases with the cohort h. Thus, we establish a way to reflect the change of the mortality rate with cohorts.

The survival function $S(x, h)$ has the following form:

$$S(x,h) = \exp\left(-\int_0^x q(s,h)\mathrm{d}s \right).$$

And the equation can be calculated as follows by (5.1):

$$S(x,h) = \begin{cases} \rho x + e^{-\frac{\alpha(h)}{\beta(h)}}(e^{\frac{x}{\beta(h)}} - 1), & x \in [0, x^*], \\ \rho x^* + e^{-\frac{\alpha(h)}{\beta(h)}}(e^{\frac{x^*}{\beta(h)}} - 1) + (\rho + e^L)(x - x^*), & x \in (x^*, \omega]. \end{cases}$$

The assumed survival function $\bar{S}(x, h)$ has the same expression with $S(x, h)$ and the longevity speed is relatively $\bar{d}$.

We also make the following assumptions for our numerical illustration.

- The time horizon is 20 years, that is, $T = 20$.
- Members enter the TBP at age 25 and retire at age 55 initially, that is, $a = 25$, $r_0 = 55$.
- The maximum age for all cohorts considered in this model is 130, that is, $\omega = 130$.
- The initial pension entry rate N at time $t = t_0$ is 10.
- The Makeham constant ρ is set to 2.66×10^{-4}, which is an average constant from Milevsky (2020).
- The parameters in the two asset models (2.2) and (2.3) are $m(t) = m = 0.01$, $\mu(t) = \mu = 0.05$, $\sigma(t) = \sigma = 0.15$ (thus the Sharpe ratio of the risky asset is about 0.27).
- The salary rate $Y(x, h)$ increases in accordance with the risk-free interest in order to account for inflation, that is, $Y(x, h) = e^{0.01(x+h)}$.
- The penalty weight λ_1 given to the benefit shortfall risk is chosen to be 8, and the weight λ_2 given to the risk at the terminal time is chosen to be 0.1. The estimation is given in Section 5.6.
- The initial fund wealth $F(0)$ is set to 100, and the contribution rate c of each active member is set to 0.1.
- The reserved preparation time τ is 5 years.

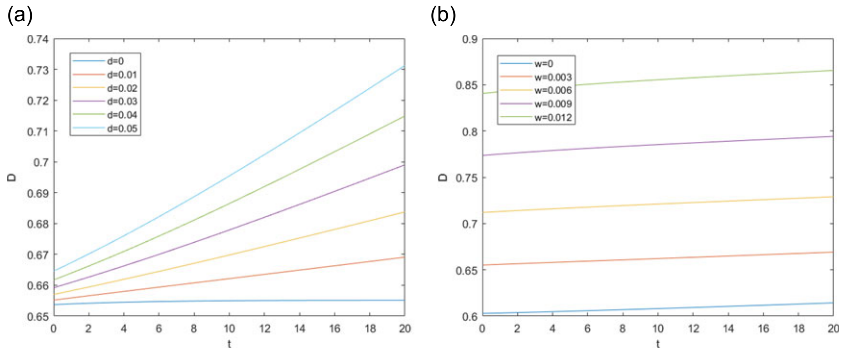
5.2. Demographic transition

The dependency ratio $D(t)$ is the quotient of the number of retirees $R(t)$ and the number of workers $A(t)$, which can reflect the degree of the social pension burden. Figure 1(a) shows that the longevity trend continuously increases $D(t)$ with time t and $D(t)$ grows faster with a larger longevity speed d. Figure 1(b) shows that the fertility decline also increases $D(t)$ with time t and when the decline speed w gets larger, $D(t)$ will rise.

5.3. Intergenerational equity

Figure 2 shows the impact of the assumed longevity speed $\bar{d}$ and the deferred retirement age r_1 on the individual target annuity $\bar{b}(h)$. In Figure 2(a), before $t_0 = -80$ (the time point when the longevity trend begins), $\bar{b}(h)$ increases with $Y(x, h)$ and stays the same for all $\bar{d}$ because the longevity trend has not started yet. After $t_0 = -80$, people's life expectancy has increased and thus $\bar{b}(h)$ decreases with h according to actuarial equity. Besides, if we delay the retirement age, then people will work longer years and their

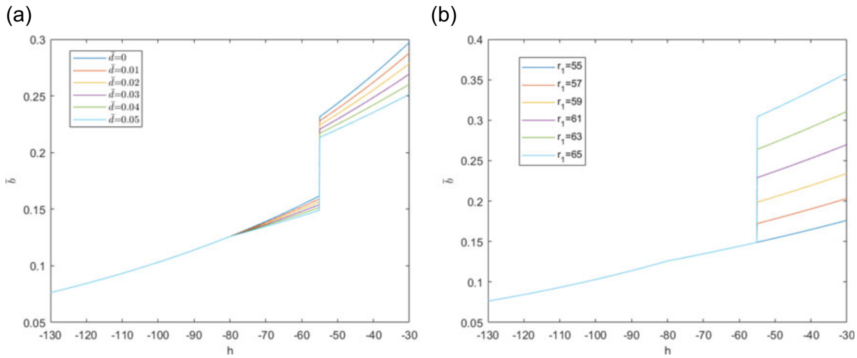
https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press



The dependency ratio $D(t)$ with different longevity speeds d ($w = 0.003$)

The dependency ratio $D(t)$ with different fertility decline speeds w ($d = 0.01$)

Figure 1. *Factors affecting D(t).*



Effects of the assumed longevity speed $\bar{d}$ ($r_1 = 60$)

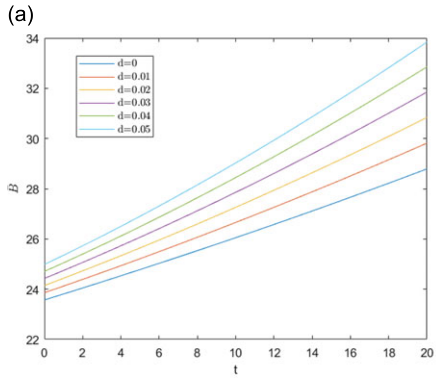
Effects of the deferred retirement age r_1 ($d = 0.01$)

Figure 2. *Factors affecting $\bar{b}(h)$.*

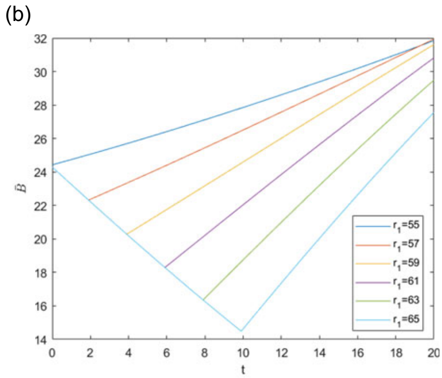
remaining life expectancy will also be reduced, so according to (2.4), their annuity $\bar{b}(h)$ will increase. In Figure 2(b), it can be found that if we delay retirement for a longer time, the target annuity $\bar{b}(h)$ will also increase.

Figure 3 shows the impact of the longevity speed d, the deferred retirement age r_1, and the fertility decline speed w on aggregate target benefit $\bar{B}(t)$. In Figure 3(a), we assume $\bar{d} = 0$, $r_1 = r_0$, and $w = 0$ and find that as the longevity trend becomes serious, $\bar{B}(t)$ increases, which is consistent with common sense. Figure 3(b) focuses on the change of new retirement age r_1 with $d = 0.03$ and $w = 0$. We find that delaying retirement can significantly reduce $\bar{B}(t)$ within $r_1 - r_0$ years. This is because if we set the new retirement age be r_1 at time $t = 0$, then there will be no new retirees in the next $r_1 - r_0$ years. Meanwhile the number of original retirees will decrease with age. So the number of retirees $R(t)$ decreases which lowers $\bar{B}(t)$. In addition, we also find that the later the delayed retirement, the faster the growth rate of $\bar{B}(t)$ after $r_1 - r_0$. This is because the delay in retirement increases the personal annuity $\bar{b}(h)$ (for $h \geq -r_0$), and the cohorts affected by delaying retirement will begin to receive the pension at time $r_1 - r_0$. Figure 3(c) shows that the decline in fertility will reduce the target payment $\bar{B}(t)$ because fewer people enter the pension plan.

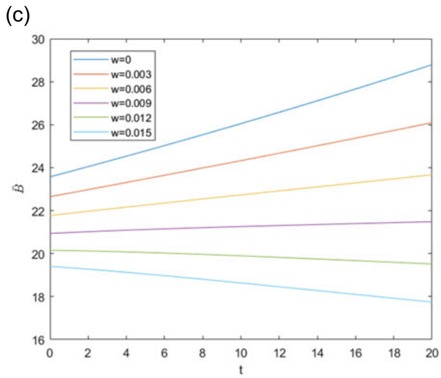
https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press



Effects of longevity speed d



Effects of the deferred retirement age r_1



Effects of the fertility decline speed w

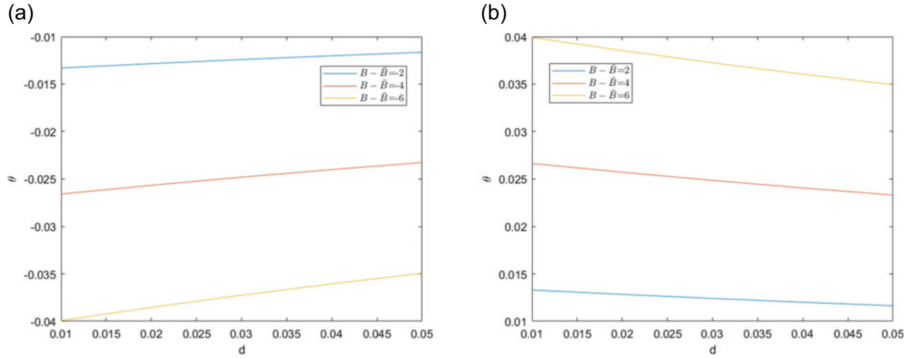
Figure 3. *Factors affecting* $\bar{B}(t)$*.*

5.4. Intergenerational risk sharing

In Figure 4, we study the value of risk sharing $\theta(t)$ at time T with different levels of overpayment $B(T) - \bar{B}(T)$. We assume that $\bar{d} = d$, $w = 0$, $r_1 = r_0$. When $B - \bar{B} < 0$, which means the actual payment does not reach the target payment, we have $\theta < 0$ according to (2.5) and the loss will be borne by all retirees. As d increases, θ increases, which implies the loss shared by everyone decreases with the trend of longevity. When $B - \bar{B} > 0$, which means the adjustment θ is positive and in addition to the target annuity $\bar{b}$, each retiree also receives the income from the insurer's investment. Under this condition, θ decreases as d increases, implying that the longevity trend also thins everyone's extra benefit. The phenomena reflected in these two figures verify the effectiveness of the risk-sharing mechanism.

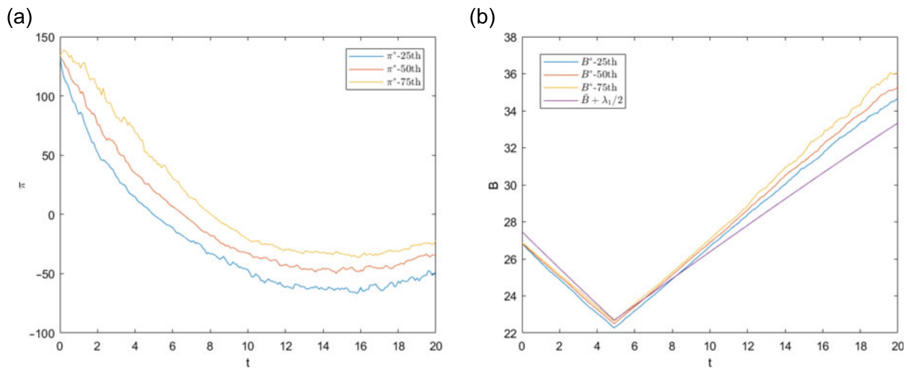
We next study the behaviors of the optimal investment and benefit strategies $\pi^*(t)$ and $B^*(t)$ by Monte Carlo methods. Figure 5 shows three investment paths $\pi^*(t)$ and three benefit paths $B^*(t)$ with $r_1 = 60$, $w = 0.003$, $\bar{d} = 0.02$, and $d = 0.05$. The interval between adjacent time points is 0.1. The paths are the 25th, 50th, and 75th percentiles of the investment and the benefit payment at each date in time (calculated from the 10,000 simulations). Note that $\pi^*(t)$ shows a downward trend on the whole. This is because in the early investment, the discounted future income is much higher than the current pension fund which implies a high leverage strategy. Besides, it can be found that the actual payment $B^*(t)$ fluctuates up and down around $\bar{B}(t) + \frac{\lambda_1}{2}$, and it can be explained by (4.14).

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press



$B - \bar{B} < 0$ $B - \bar{B} > 0$

Figure 4. *The value of risk sharing θ for different longevity speeds d.*



$\pi^*(t)$ $B^*(t)$

Figure 5. *The 25th, 50th, and 75th percentiles of investment and benefit strategies.*

5.5. Effects of postponing retirement

In Figure 6, we compare $V(0, f_0)$ under different r_1 and d with $\bar{d} = 0$ and $w = 0.006$. By definition, a smaller V implies that the pension fund gets closer to the targets and can afford more benefit payments. We find that all functions of V experience a decline and an increase w.r.t. r_1 and the minimum point is the optimal deferred retirement age r^*. The decline is because the pension burden is reduced for at least $r_1 - r_0$ years, and this period gives an opportunity to try more risky and profitable investments. The increase is because the loss of the pension fund in the trend of longevity is caused by the underestimation of the survival function, while due to actuarial fairness, deferred retirement does not really reduce the amount of benefits, but delays the time of benefits. Therefore, appropriately delaying the retirement age is good for the pension fund, but too much delay will bring losses. Note that the value of V is smaller with a larger d. This does not mean that a larger d will be better, because when changing d, the scale of the pension fund has changed, and more benefits $B^*(t)$ are brought by more retirees, which will not be better for individuals. (Note that the adjustment $\theta(t)$ will be diluted by the number of retirees $R(t)$.) For example, even if $\theta(t)$ is the same, the total value V of the pension scheme with more retirees will be larger than that with fewer retirees, which does not mean that the former is better than the latter. Besides, we find that the distribution of the optimal retirement age r^* with respect to the speed of longevity d is relatively concentrated ($r^* = 61$ or 62).

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press

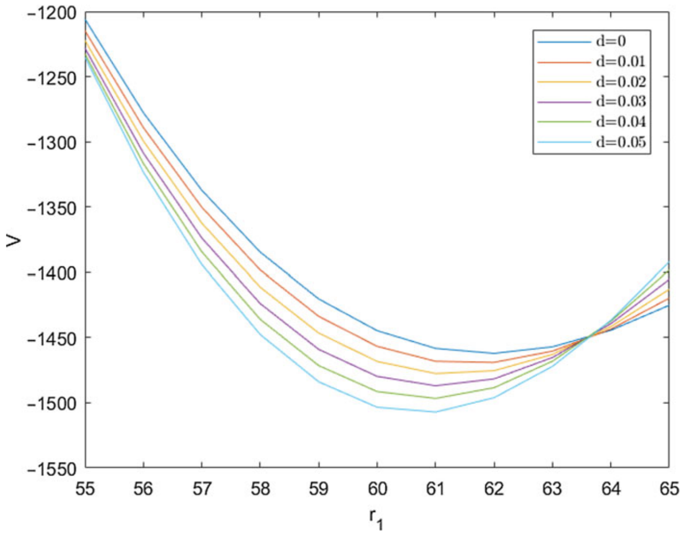


Figure 6. *The total value V for different retirement ages r_1 and longevity speeds d.*

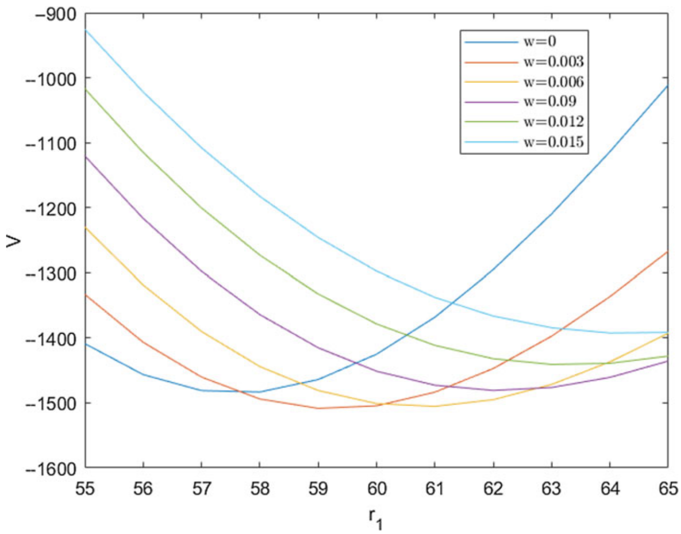


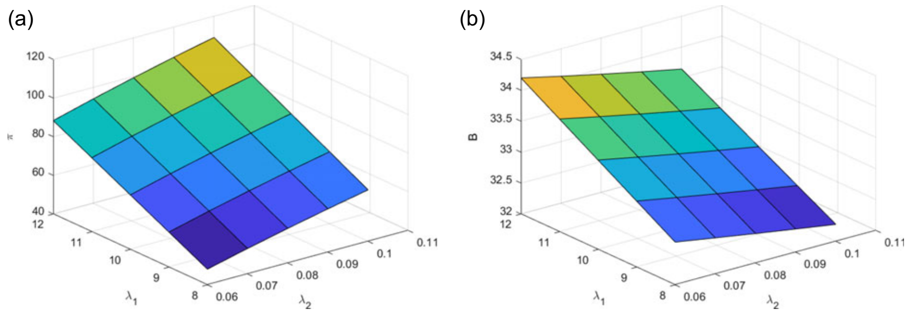
Figure 7. *The total value V for different retirement ages r_1 and fertility decline rates w.*

Figure 7 shows the values of $V(0, f_0)$ under different r_1 and w with $\bar{d} = 0$ and $d = 0.05$. We find that the optimal deferred retirement age r^* (the minimum point of each curve) increases with the fertility decline rate w. Note that fertility is essentially the rate of expansion of the pension plan. This may explain the importance of scale for the pension plan. A pension plan that expands faster will have less demand for delaying retirement.

5.6. The sensitivity of λ_1 and λ_2

We first estimate the values of λ_1 and λ_2. Basically, we have $\lambda_1 > 0$ and $\lambda_2 > 0$ by definition. Besides, we assume that the overpayment $B^*(t) - \bar{B}(t)$ is bounded, that is, $B^*(t) - \bar{B}(t) \in [\mathscr{I}_{\min}, \mathscr{I}_{\max}]$, where $\mathscr{I}_{\min}$

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press



$\pi^*(10)$ $B^*(10)$

Figure 8. *Strategies response to λ_1 and λ_2 at $t = 10$.*

and $\mathscr{I}_{\max}$ are lower and upper bounds determined by the plan trustees. Note that $P(t) \in [0, \lambda_2]$. To avoid borrowing and short selling, the risky investment $\pi^*(t)$ should be larger than 0 and smaller than $F(t)$, although this is not considered in the solution (see Remark 3.1). Since we set the terminal wealth target $F(T)$ to M (abbreviated as M), $\pi^*(t)$ can be assumed to be smaller than M. By (4.14) and assuming that $\pi^*(t) \in [0, M]$, we can roughly estimate the range of $B^*(t) - \bar{B}(t)$ as $[\frac{\lambda_1}{2} + \frac{\sigma^2}{m-\mu}M\lambda_2, \frac{\lambda_1}{2}]$. Therefore, we rearrange the restrictions of λ_1 and λ_2 as the following linear programming problem:

$$\begin{cases} \lambda_1 > 0, \\ \lambda_1 < 2\mathscr{I}_{\max} \\ \lambda_2 > 0, \\ \dfrac{\lambda_1}{2} + \dfrac{\sigma^2}{m-\mu}M\lambda_2 > \mathscr{I}_{\min}. \end{cases}$$

The values of $\mathscr{I}_{\min}$ and $\mathscr{I}_{\max}$ are set to -6 and 6, respectively. Then we take $(\lambda_1, \lambda_2) \in [8, 12] \times [0.06, 0.1]$ satisfying above restrictions for the numerical analysis.

Figure 8 shows the values of the optimal investment $\pi^*(t)$ and the optimal benefit $B^*(t)$ at time $t = 10$ with different λ_1 and λ_2, where $F(10) = 100$, $w = 0.003$, $\bar{d} = 0.02$, and $d = 0.05$. We find that when λ_1 and λ_2 increase, the optimal investment $\pi^*(10)$ also increases. This is because the rising λ_1 and λ_2 bring more pressure on the pension fund to pay and to accumulate terminal wealth, and the investment strategy tends to be more risky. It can be found in Figure 8(b) that the optimal benefit $B^*(10)$ increases with λ_1 and decreases with λ_2. This is because the pension fund tends to overpay with a large λ_1, and a large λ_2 means the pension fund reduces benefits to ensure sufficient terminal wealth.

6. Conclusion

We have studied a model of optimal investment and payment strategies in the target benefit pension plans under the condition of longevity risks and fertility decline. Through this model, we establish a target benefit pension plan that can take into account intergenerational equity and intergenerational risk sharing. Equity is reflected in three aspects: actuarial fairness of contributions and benefits, the change of terminal target M with the population structure, and the refund of the contributions of members who die before retirement. An intergenerational risk sharing mechanism shares longevity risks and investment risks among retirees. In addition, we find an optimal new retirement age to maximize the interests of the pension fund and individuals.

https://doi.org/10.1017/asb.2022.27 Published online by Cambridge University Press

Acknowledgments. This research was supported by National Natural Science Foundation of China (Grant Nos. 11871052, 12171360, 11771329), Key Project of the National Social Science Foundation of China (Grant No. 21AZD071), and Natural Science Foundation of Tianjin City (Grant No. 20JCYBJC01160).

References

Alvarez, J.-A., Kallestrup-Lamb, M. and Kjærgaard, S. (2021) Linking retirement age to life expectancy does not lessen the demographic implications of unequal lifespans. *Insurance: Mathematics and Economics*, **99**, 363–375.

Andersen, T.M. (2014) Intergenerational redistribution and risk sharing with changing longevity. *Journal of Economics*, **111**(1), 1–27.

Ayuso, M., Bravo, J.M. and Holzmann, R. (2017) Addressing longevity heterogeneity in pension scheme design. *Journal of Finance and Economics*, **6**(1), 1–21.

Baumann, R.T. and Müller, H.H. (2008) Pension funds as institutions for intertemporal risk transfer. *Insurance: Mathematics and Economics*, **42**, 1000–1012.

Boon, L.-N., Brière, M. and Werker, B.J.M. (2022) Systematic longevity risk: To bear or insure? *Journal of Pension Economics and Finance*, **19**, 409–441.

Bovenberg, A.L., Mehlkopf, R. and Nijman, T. (2016) The promise of defined ambition plans: Lessons for the united states. In *Reimagining Pensions: The Next 40 Years* (eds. O. Mitchell and R. Shea), pp. 215–246. Oxford: Oxford University Press.

Bravo, J.M. and De Freitas, N.E.M. (2018) Valuation of longevity-linked life annuities. *Insurance: Mathematics and Economics*, **78**, 212–229.

Bruce, N. and Turnovsky, S.J. (2013) Demography and growth: A unified treatment of overlapping generations. *Macroeconomic Dynamics*, **17**, 1605–1637.

Chen, A. and Rach, M. (2021) Current developments in German pension schemes: What are the benefits of the new target pension? *European Actuarial Journal*, **11**(1), 21–47.

Chen, D.H., Beetsma, R.M., Broeders, D.W. and Pelsser, A.A. (2017) Sustainability of participation in collective pension schemes: An option pricing approach. *Insurance: Mathematics and Economics*, **74**, 182–196.

Chen, L., Li, D., Wang, Y. and Zhu, X. (2022) The optimal cyclical design for a target benefit pension plan. *Journal of Pension Economics and Finance*, 1–20. https://doi.org/10.1017/S1474747222000099

Cia (2015) Report of the task force on target benefit plans. Accessible via: https: www.cia-ica.ca/docs/default-source/2015/215043e.pdf.

Coughlan, G.D., Khalaf-Allah, M., Ye, Y., Kumar, S., Cairns, A.J., Blake, D. and Dowd, K. (2011) Longevity hedging 101: A framework for longevity basis risk analysis and hedge effectiveness. *North American Actuarial Journal*, **15**(2), 150–176.

Cui, J., De Jong, F. and Ponds, E. (2011) Intergenerational risk sharing within funded pension schemes. *Journal of Pension Economics and Finance*, **10**(01), 1–29.

Danish Ministry of Economic Affairs and Interior (2017) Denmark's Convergence Programme 2017. Technical report, Danish Government, Copenhagen.

Danish Ministry of Economic Affairs and Interior (2018) Denmark's Convergence Programme 2018. Technical report, Danish Government, Copenhagen.

Denuit, M., Haberman, S. and Renshaw, A. (2011) Longevity-indexed life annuities. *North American Actuarial Journal*, **15**, 97–111.

Dybvig, P.H. (1995) Dusenberry's racheting of consumption: Optimal dynamic consumption and investment given intolerance for any decline in standard of living. *Review of Economic Studies*, **62**(2), 287–313.

Fleming, W.H. and Soner, H.M. (2006) *Controlled Markov Processes and Viscosity Solutions*, Vol. 25. New York: Springer Science & Business Media.

Galasso, V. (2008) Postponing retirement: The political effect of aging. *Journal of Public Economics*, **92**, 2157–2169.

Gollier, C. (2008) Intergenerational risk-sharing and risk-taking of a pension fund. *Journal of Public Economics*, **92**(5), 1463–1485.

Gompertz, B. (1825) On the nature of the function expressive of the law of human mortality and on a new mode of determining the value of life contingencies. *Philosophical Transactions of the Royal Society of London*, **115**, 513–583.

Heeringa, W. and Bovenberg, A.L. (2009) Stabilizing Pay-as-You-Go pension schemes in the face of rising longevity and falling fertility: An application to the Netherlands, Netspar Discussion Paper 08/2009-025.

Holzmann, R., Alonso-García, J., Labit-Hardy, H. and Villegas, A.M. (2019) *NDC schemes and heterogeneity in longevity: Proposals for redesign*. World Bank.

Khorasanee, Z.M. (2012) Risk-sharing and benefit smoothing in a hybrid pension plan. *North American Actuarial Journal*, **16**, 449–461.

Knell, M. (2018) Increasing life expectancy and NDC pension systems. *Journal of Pension Economics and Finance*, **17**(2), 170–199.

Kortleve, N. (2013) The 'defined ambition' pension plan: A dutch interpretation. *Rotman International Journal of Pension Management*, **6**, 6–11.

Medford, A. (2017) Best-practice life expectancy: An extreme value approach. *Demographic Research*, **36**, 989–1014.

Milevsky, M.A. (2020). Calibrating Gompertz in reverse: What is your longevity-risk-adjusted global age? *Insurance: Mathematics and Economics*, **92**, 147–161.

Munnell, A.H. and Sass, S.A. (2013) New Brunswick's new shared risk pension plan. Center for Retirement Research at Boston College, Boston, (33).
Oeppen, J. and Vaupel, J.W. (2002) Broken limits to life expectancy. *Science*, **296**(5570), 1029–1031.
Pascariu, M.D., Canudas-Romo, V. and Vaupel, J.W. (2018) The double-gap life expectancy forecasting model. *Insurance: Mathematics and Economics*, **78**, 339–350.
Pugh, C. and Yermo, J. (2008) Funding regulations and risk sharing. OECD working papers on insurance and private pensions, No. 17, OECD Publishing. https://doi.org/10.1787/241841441002.
Turner, J.A. (2014) *Hybrid pensions: Risk sharing arrangements for pension plan sponsors and participants*. Society of Actuaries.
United Nations, Department of Economic and Social Affairs (2020) Government policies to address population ageing. Population facts No. 2020/1.
Villegas, A.M. and Haberman, S. (2014) On the modeling and forecasting of socioeconomic mortality differentials: An application to deprivation and mortality in England. *North American Actuarial Journal*, **18**(1), 168–193.
Wang, S. and Lu, Y. (2019) Optimal investment strategies and risk-sharing arrangements for a hybrid pension plan. *Insurance: Mathematics and Economics*, **89**, 46–62.
Wang, S., Lu, Y. and Sanders, B. (2018) Optimal investment strategies and intergenerational risk sharing for target benefit pension plans. *Insurance: Mathematics and Economics*, **80**, 1–14.
Wang, S., Rong, X. and Zhao, H. (2019) Optimal investment and benefit payment strategy under loss aversion for target benefit pension plans. *Applied Mathematics and Computation*, **346**, 205–218.
Zhu, X., Hardy, M. and Saunders, D. (2021) Fair transition from defined benefit to target benefit. *ASTIN Bulletin: The Journal of the IAA*, **51**(3), 873–904.

Appendix A. Theorem 3.1

Proof. First, differentiating expression in bracket of (4.1) w.r.t. π and B, setting to zero and solving them give immediately

$$\pi^*(t,f) = \frac{V_f(m(t) - \mu(t))}{V_{ff}\sigma(t)^2}, \tag{A1}$$

$$B^*(t,f) = \bar{B}(t) + \frac{\lambda_1}{2} + \frac{V_f}{2}. \tag{A2}$$

Obviously, there is one sufficient condition for π to be minimum, which is

$$V_{ff} > 0. \tag{A3}$$

We verify this condition once we obtain the expression of V.

Next we find an explicit expression for $V(t, f)$. By the terminal condition (4.2), we postulate that $V(t,f)$ is of the form

$$V(t,f) = P(t)f^2 + Q(t)f + K(t), \tag{A4}$$

where $P(t)$, $Q(t)$, $K(t)$ are functions of t to be determined. The boundary condition (4.2) implies that

$$P(T) = \lambda_2,\ \ Q(T) = -2\lambda_2 M,\ \ K(T) = \lambda_2 M^2. \tag{A5}$$

From (A4), we obtain

$$\begin{aligned} V_t &= P_t f^2 + Q_t f + K_t, \\ V_f &= 2P(t)f + Q(t),\ \ V_{ff} = 2P(t). \end{aligned} \tag{A6}$$

Substituting (A6) into (A1) and (A2), the optimal policies (control variables) can be expressed in terms of the functions $P(t)$ and $Q(t)$.

Substituting (A6), (A1), and (A2) into the HJB Equation (4.1) and grouping terms with f^2, f, and 1, we get

$$[P_t - P^2 + (2m(t) - \frac{(\mu(t)-m(t))^2}{\sigma(t)^2})P]f^2$$

$$+ [Q_t + [-\frac{(\mu(t)-m(t))^2}{\sigma(t)^2} + m(t) - P(t)]Q + 2P(t)J(t)]f$$

$$+ K_t - \frac{(\mu(t)-m(t))^2}{4\sigma(t)^2}\frac{Q(t)^2}{P(t)} + Q(t)J(t) - \frac{1}{4}Q(t)^2 - \frac{\lambda_1^2}{4} = 0,$$

where

$$J(t) = C(t) - \bar{B}(t) - \frac{\lambda_1}{2}.$$

The coefficients of f^2, f, and 1 are all zeros, which leads to the following system of differential equations:

$$P_t - P^2 + (2m(t) - \frac{(\mu(t)-m(t))^2}{\sigma(t)^2})P = 0, \tag{A7}$$

$$Q_t + [-\frac{(\mu(t)-m(t))^2}{\sigma(t)^2} + m(t) - P(t)]Q + 2P(t)J(t) = 0, \tag{A8}$$

$$K_t - \frac{(\mu(t)-m(t))^2}{4\sigma(t)^2}\frac{Q(t)^2}{P(t)} + Q(t)J(t) - \frac{1}{4}Q(t)^2 - \frac{\lambda_1^2}{4} = 0, \tag{A9}$$

with boundary conditions (A5).

The three ordinary differential equations are easy to solve, the solutions of which are given by (4.6)–(4.10).

We now verify the constraint condition (A3), which is

$$V_{ff} = 2P(t) > 0,$$

where $P(t)$ is given by (4.6), which is always positive by Corollary 4.2 and Remark 4.3. ■