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Minimal interval exchange transformations with flips

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Abstract. We consider interval exchange transformations of n intervals with k flips, or (n, k)-IETs for short, for positive integers k, n with $k \leq n$. Our main result establishes the existence of minimal uniquely ergodic (n, k)-IETs when $n \geq 4$; moreover, these IETs are self-induced for $2 \leq k \leq n-1$. This result extends the work on transitivity in Gutierrez *et al* [Transitive circle exchange transformations with flips. *Discrete Contin. Dyn. Syst.* **26**(1) (2010), 251–263]. In order to achieve our objective we make a direct construction; in particular, we use the Rauzy induction to build a periodic Rauzy graph whose associated matrix has a positive power. Then we use a result in the Perron–Frobenius theory [Pullman, A geometric approach to the theory of non-negative matrices. *Linear Algebra Appl.* **4** (1971) 297–312] which allows us to ensure the existence of these minimal self-induced and uniquely ergodic (n, k)-IETs, $2 \leq k \leq n-1$. We then find other permutations in the same Rauzy class generating minimal uniquely ergodic $(n, 1)$- and (n, n)-IETs.

1. *Introduction*

Given $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ we define an *n-interval exchange transformation*, or *n-IET* for short, as an injective map $T : D \subset (0, l) \to (0, l)$ such that:

(i) D is the union of n pairwise disjoint open intervals, $D = \bigcup_{i=1}^{n} I_i$, with $I_i = (a_i, a_{i+1})$, $0 = a_1 < a_2 < a_3 < \cdots < a_{n+1} = l$;

(ii) $T|_{I_i}$ is an affine map of constant slope equal to 1 or -1.

If T *reverses the orientation* of each interval I_f of the interval set $\mathcal{F} = \{I_{f_1}, I_{f_2}, \ldots, I_{f_k}\}$ (the slope is -1 in these intervals) for some $1 \leq f_j \leq n$, then we say that T is *an interval exchange transformation of n intervals with k flips* (for this reason we denote the indices by $f_1, \ldots, f_k$) or simply *an* (n, k)*-IET*; otherwise we say that T is *an interval exchange transformation of n intervals without flips* or simply *an oriented interval exchange transformation of n intervals*. If we replace $[0, l]$ by $\mathbb{S}^1 = [0, l]/\equiv$, $(0 \equiv l)$, then we obtain the notion of *circle exchange transformation of n intervals with k flips* (abbreviated as (n, k)*-CET*) or *circle exchange transformation of n intervals without flips* (abbreviated as n*-CET*). Observe that the right continuous extension of an (n, k)-IET has at most $n - 1$ discontinuity points; when it has exactly $n - 1$ we say that it is a *proper* (n, k)*-IET*.

This type of maps has been intensively studied due to its intrinsic interest and its application in different research areas, for instance surface flows [**5**, **11**], Teichmüller flows [**3**, **4**], continued fraction expansions [**33**] and polygonal billiards [**22**].

Let $x \in (0, l)$. The orbit of this point, generated by T, is the set

$$\mathcal{O}_T(x) = \{T^n(x) : n \text{ is an integer and } T^n(x) \text{ makes sense}\},$$

where $T^0 = \mathrm{Id}$ and $T^n = T \circ T^{n-1}$ for any integer n. Moreover, $\mathcal{O}_T(0) = \{0\} \cup \mathcal{O}_T(\lim_{x\to 0^+} T(x))$ and $\mathcal{O}_T(l) = \{l\} \cup \mathcal{O}_T(\lim_{x\to l^-} T(x))$. T is said to be *minimal* if $\mathcal{O}_T(x)$ is dense in $[0, l]$ for any $x \in [0, l]$. Recall that *transitivity* is a weaker condition: T is said to be *transitive* if there exists some $x \in [0, l]$ such that $\mathcal{O}_T(x)$ is dense in $[0, l]$.

Remark 1. According to [**14**, Corollary 14.5.12], if T has a dense orbit and it has no finite orbits then any orbit is dense in $[0, l]$. Thus the notion of minimality introduced here is equivalent to that used in [**12**], namely, a transitive map without finite orbits. We note that transitivity does not imply minimality; see the IETs T_1 and T_2 after Theorem 4.

In this paper we focus in the topics of minimality and unique ergodicity. Let δ denote a finite measure on $[0, l]$. Then δ is said to be an *invariant measure of T* if, for any measurable set $A \subset [0, l]$, we have $\delta(T^{-1}(A)) = \delta(A)$; T is said to be *ergodic (with respect to* δ*)* if δ is an invariant measure for T and, for any subset $E \subset [0, l]$ satisfying $T(E) = E$, either $\delta(E) = 0$ or $\delta(E) = 1$. In the following, we will denote by μ_L the standard Lebesgue measure on $[0, l]$.

It is easy to see that μ_L (and any of its multiples) is an invariant measure for any interval exchange transformation T. Moreover, T is *uniquely ergodic* if it does not admit another invariant probability measure. It is worth mentioning that, for IETs, unique ergodicity implies ergodicity with respect to Lebesgue measure; cf. [**20**, §II.6, Theorem 6.1].

We will introduce coordinates in the set of IETs. Let $n \in \mathbb{N}$. Then there exists a natural injection between the set of n-IETs and $\mathcal{C}_n = \Lambda^n \times S_n^\sigma$, where $\mathbb{R}_+ = (0, \infty)$, Λ^n is the cone $\mathbb{R}_+^n$ and S_n^σ is the set of *signed permutations*. By a signed permutation we mean an injective map $\pi : N_n = \{1, 2, \ldots, n\} \to N_n^\sigma = \{-n, -(n-1), \ldots, -1, 1, 2, \ldots, n\}$ such that $|\pi| : N_n \to N_n$ is bijective, that is, a *standard permutation*; a *non-standard permutation* will be a signed permutation π such that $\pi(i) < 0$ for some i. As in the case of standard permutations, π will be represented by the vector $(\pi(1), \pi(2), \ldots, \pi(n)) \in (N_n^\sigma)^n$. Let T be an n-IET as in the preceding paragraph. Then its associated coordinates in $\mathcal{C}_n$ are (λ, π) defined as follows.

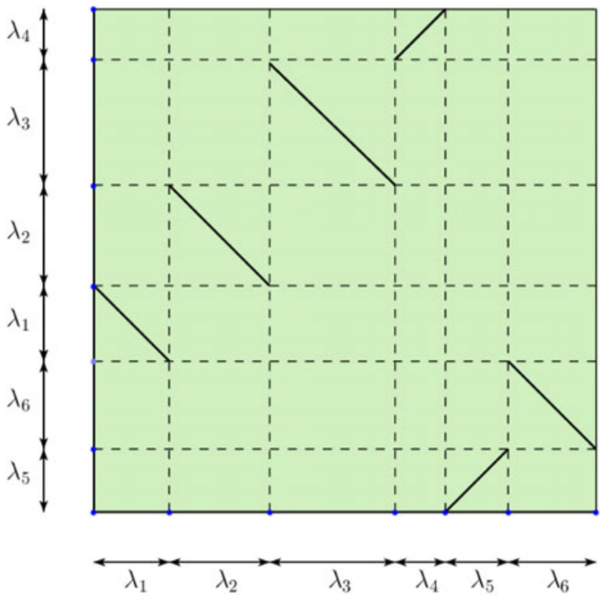

FIGURE 1. Example of proper (6, 4)-IET with associated coordinates (λ, π), where λ is the positive vector $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ and π the signed permutation $(-3, -4, -5, 6, 1, -2)$.

- $\lambda_i = a_{i+1} - a_i$ for all $i \in N_n$.
- $\pi(i)$ is positive (respectively, negative) if $T|_{I_i}$ has slope 1 (respectively, -1). Moreover, $|\pi(i)|$ is the position of the interval $T(I_i)$ in the set $\{T(I_i)\}_{i=1}^n$ taking into account the usual order in $\mathbb{R}$.

Conversely, given a pair (λ, π) we can associate to it a unique n-IET, $T : D \subset [0, l] \to [0, l]$, where:

- $l = |\lambda| := \sum_{i=1}^n \lambda_i$;
- $I_1 = (0, \lambda_1)$;
- $I_i = \big(\sum_{j=1}^{i-1} \lambda_j, \sum_{j=1}^{i} \lambda_j\big)$ for any $1 < i \leq n$;
- $T|_{I_i}(x) = \big(\sum_{j=1}^{|\pi|(i)-((\sigma(\pi(i))+1)/2)} \lambda_{|\pi|^{-1}(j)}\big) + \sigma(\pi(i))\big[x - \big(\sum_{j=1}^{i-1} \lambda_j\big)\big]$, for any $1 \leq i \leq n$, where $\sigma(z)$ denotes the *sign* of $z \in \mathbb{R} \setminus \{0\}$, namely, $\sigma(z) = z/|z|$. Notice that if we define

$$K_{|\pi|^{-1}(j)} := \left(\sum_{s=1}^{j-1} \lambda_{|\pi|^{-1}(s)}, \sum_{s=1}^{j} \lambda_{|\pi|^{-1}(s)}\right),$$

  we have $T(I_i) = K_{|\pi|^{-1}(\pi(i))}$.

These coordinates allow us to make the identification $T = (\lambda, \pi)$; see Figure 1 to clarify the idea. For a fixed permutation π, we can consider the Lebesgue measure of the cone Λ^n on the set of n-IETs having associated permutation π.

Remark 2. $T = (\lambda, \pi)$ is a proper (n, k)-IET provided that π satisfies $\pi(j+1) - \pi(j) \neq 1$ for any $j \in \{1, 2, \ldots, n-1\}$. Notice that π is a signed permutation.

Remark 3. IETs and CETs are closely related: a proper (n, k)-IET $T = (\lambda, \pi) : D \subset [0, l] \to [0, l]$ generates an $(\tilde{n}, \tilde{k})$-CET $\hat{T} : \hat{D} \subset \mathbb{S}^1 \to \mathbb{S}^1$ by identifying $0 \equiv l$. Even more, if T is a proper (n, k)-IET then $\hat{T}$ is a proper (n, k)-CET if either the signs of $\pi(j)$ and $\pi(j+1)$ are different or they coincide and then $\pi(j+1) - \pi(j) \not\equiv 1 \bmod (n)$ for any $j \in \{1, 2, \ldots, n\}$ (note that we use arithmetic modulo n and also that for integers a, b, n, $a \equiv b \bmod (n)$ means that n divides $a - b$). We stress that, in any case, the minimality and unique ergodicity of T imply those of $\hat{T}$.

Our aim is to construct minimal interval exchange transformations with flips. This goal is important in itself but also because it will allow us to construct minimal flows on open non-orientable surfaces of finite genus $g \geq 4$ by means of the standard procedure of suspensions of IETs; see [**2**, **9**, **19**, **29**]. We recall that the first steps in this direction were made by Gutierrez [**10**], who constructed a minimal proper $(5, 2)$-IET, $T = (\lambda, \pi)$, with $\pi = (3, -4, 5, 1, -2)$ which is *self-induced*, which means that the return map induced by T on a suitable subinterval, $\widetilde{T} = (\widetilde{\lambda}, \widetilde{\pi})$, satisfies $\widetilde{\pi} = \pi$ and $\widetilde{\lambda} = \rho\lambda$ for some $\rho \in (0, 1)$.

Nogueira [**23**] generalized Gutierrez's construction to obtain, for any $n \geq 2$, self-induced minimal proper $(2n + 1, n)$-IETs, (λ, τ), with $\tau(i) = (-1)^{i+1}(i + 2)$ for any $1 \leq i \leq 2n - 1$, $\tau(2n) = 1$ and $\tau(2n + 1) = -2$. It is worth mentioning that Nogueira and Gutierrez IETs can be used to obtain minimal proper $(2n, n)$-CETs and a minimal proper $(4, 2)$-CET, respectively.

Both authors used the above mentioned CETs to build transitive flows on compact and connected surfaces. Moreover, the constructed flows are minimal on some open surfaces; in particular, Gutierrez obtained a minimal flow on N_4^{**} (the resulting surface after removing two points from the non-orientable compact surface of genus 4, N_4). The suspensions of Nogueira $(2n, n)$-CETs induce minimal flows on the non-orientable compact surface of genus $2 + n$ where n points were removed. Notice that in order to obtain minimal flows on any non-orientable surface of genus greater than 4 it would be interesting to suspend other interval exchange transformations that generate minimal flows on non-orientable compact surfaces with a single hole. The IETs which will be constructed in the proof of main theorem have this property; see [**9**].

Recent works about exchange transformations with flips are [**12**, **13**, **25**]. In particular, [**12**] is due to Gutierrez *et al*, and it states its main result as follows.

THEOREM 4. *Given $n \geq k \geq 1$, there exists a transitive proper (n, k)-CET if and only if $n + k \geq 5$.*

The 'if' part of the proof of Theorem 4 is obtained by introducing some minimal self-induced IETs. In particular they build $(4, 2)$, $(4, 3)$, $(4, 4)$, $(5, 3)$ and $(5, 5)$ self-induced minimal IETs and two operators in the set of IETs. Given a transitive (n, f)-IET $T : D \subset [0, 1] \to [0, 1]$, they define a transitive $(n + 1, f)$-IET, $T_1 : D_1 \subset [0, 2] \to [0, 2]$, and a transitive $(n + 2, f + 2)$-IET, $T_2 : D_2 \subset [0, 3] \to [0, 3]$, in the following way:

$$T_1(x) = \begin{cases} T(x) + 1 & \text{if } x \in D \cap [0, 1], \\ x - 1 & \text{if } x \in (1, 2), \end{cases} \qquad T_2(x) = \begin{cases} T(x) + 1 & \text{if } x \in D \cap [0, 1], \\ -x + 4 & \text{if } x \in (1, 2), \\ -x + 3 & \text{if } x \in (2, 3). \end{cases}$$

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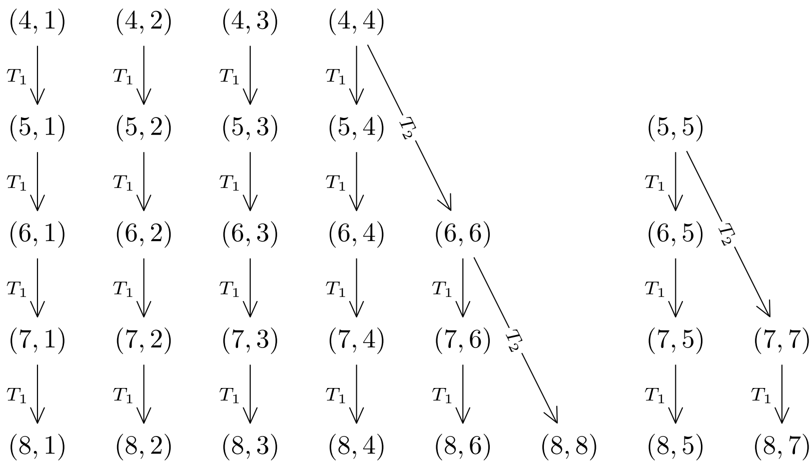

FIGURE 2. Tree of transitive IETs generated by means of T_1 and T_2.

With the (4, 2), (4, 4), (4, 3), (5, 3) and (5, 5) self-induced minimal proper IETs and the way of generating new transitive IETs by means of T_1 and T_2, Theorem 4 is proved. We remark that T_1 and T_2 are transitive but not minimal since $\mathcal{O}_{T_2}(2) = \{2\}$ and $\mathcal{O}_{T_1}(1) = \{1\}$. Figure 2 gives the idea of the proof.

Neither the problem of finding minimal uniquely ergodic proper (n, k)-IETs nor the problem of finding minimal non-uniquely ergodic (n, k)-IETs is completed for $k \geq 1$; see [**12**]. We solve the first problem in the following way.

MAIN THEOREM. *Given $n, k \in \mathbb{N}$ with $n \geq 4$ and $1 \leq k \leq n$, there exist minimal, uniquely ergodic, proper (n, k)-IETs.*

We claim that the (n, k)-IETs constructed in the proof of the previous theorem are self-induced when $2 \leq k \leq n - 1$. The idea behind our construction is to build a periodic Rauzy graph (see §3) whose associated matrix has a positive power and to use some results on the Rauzy–Veech theory that also apply to the non-oriented case as we show. In this scheme, a key point is the use of Perron–Frobenius theory and, in particular, the use of the nature of the core of a matrix analysed in [**26**]. It is important to stress the existence of signed permutations which do not generate self-induced IETs; see Remark 26.

As a consequence of our main result we obtain the following generalization of Theorem 4.

PROPOSITION A. *Given $n \geq k \geq 1$, there exists a minimal proper (n, k)-CET if and only if $n + k \geq 5$.*

The rest of this paper is organized as follows. In §2 we introduce some folklore results on the theory of IETs. Then, in §3, we give the induction procedure of Rauzy and adapt the proofs of some results on oriented IETs to the flip case. Among them, we stress the relevance of Theorems 23–25 which guide the construction of our minimal IETs. The following sections are devoted to the proof of our main result, distinguishing separately

(n, k)-IETs with even $n \geq 8$ and with odd $n \geq 9$. In §6 we analyse the particular cases $n = 4, 5, 6, 7$, and in the final section we present the proof of Proposition A.

2. *Folklore results*

We emphasize that oriented interval exchange transformations are usually defined in the literature on the whole of $[0, l)$ using the right continuous extension in the discontinuity points; see [**15**]. However, when working with non-oriented interval exchange transformations we cannot use the extension mentioned if we want the IETs to remain one-to-one. For example, the right continuous extension of a $(4, 3)$-IET with associated permutation $\pi = (4, -2, 3, -1)$ is never injective. This is the reason for working with IETs which are not defined in the discontinuity points.

Interval exchange transformations without flips have been widely studied† and there is a characterization (in terms of the 'orbits' corresponding to discontinuity points) of those being minimal due to the pioneering work by Keane [**15**].

Definition 5. (Generalized Keane condition) Let T be an n-IET with domain $D = \bigcup_{i=1}^{n}(a_i, a_{i+1})$. We define $T(a_i^{\oplus}) := \lim_{x \to a_i^+} T(x)$ for $1 \leq i \leq n$ and $T(a_i^{\ominus}) := \lim_{x \to a_i^-} T(x)$ for $2 \leq i \leq n+1$. We also write $T(a_1^{\ominus}) = T(a_1^{\oplus})$ and $T(a_{n+1}^{\oplus}) = T(a_{n+1}^{\ominus})$. We say that T satisfies the *Keane condition* if and only if

$$T^m(a) \neq a_j, \quad \text{for all } m \geq 1,\ 2 \leq j \leq n \text{ and } a \in \bigcup_{i=1}^{n+1}\{a_i^{\oplus}, a_i^{\ominus}\}. \tag{1}$$

THEOREM 6. (Keane [**15**]) *Let T be an oriented n-IET that satisfies the Keane condition‡. Then T is minimal.*

Remark 7. The notion of minimality introduced here is slightly different than that of the papers on oriented IETs. Let $T : \bigcup_{i=1}^{n}(a_i, a_{i+1}) \subset [0, l] \to [0, l]$ be an IET and let $\overline{T} : \bigcup_{i=1}^{n}[a_i, a_{i+1}) = [0, l) \to [0, l]$ be the right continuous extension of T. Then it could happen that $\overline{T}$ is minimal (it has all orbits dense) while T is not, because the points a_i, $1 \leq i \leq n$, have forward orbit by $\overline{T}$ but they do not have this forward orbit by T. However, if T satisfies the Keane condition then the points a_i have infinite backward orbit and the minimality of $\overline{T}$ implies that of T; cf. [**14**, Corollary 14.5.12].

Remark 8. In [**14**] the authors use a notion related to the Keane condition, namely, the notion of saddle connection. A *saddle connection* for T is a set

$$\mathcal{S} = \{a_i, T^1(a_i^{\otimes}), \ldots, T^k(a_i^{\otimes}) = a_j\}$$

with $k \geq 1$, $\otimes \in \{\oplus, \ominus\}$, $\mathcal{S} \cap \{a_r\}_{r=1}^{n+1} = \{a_i, a_j\}$ (the case $i = j$ is not excluded). Observe that any IET has saddle connections, with $a_j \in \{0, 1\}$: these are called *trivial saddle*

† See [**32**] for an exhaustive review with unified notation.

‡ Although Definition 5 for oriented IETs is equivalent to the classical Keane condition for the right continuous extension, this is not the case for IETs with flips. To see this, consider minimal IETs, U and V, defined on dense open subsets of $[0, 1]$, whose associated permutations are $\pi_U = (-3, -4, 5, 1, -2)$ and $\pi_V = (-2, -3, -5, -1, -4)$; see §6. Then define $T : D \subset [0, 2] \to [0, 2]$ by $T(x) = U(x)$ if $x \in [0, 1]$ and $T(x) = V(x - 1)$ if $x \in [1, 2]$. It is easy to prove that T does not satisfies Definition 5 but its right continuous extension satisfies the classical Keane condition.

connections. It is a simple task to realize that the absence of non-trivial saddle connection is equivalent to the Keane condition introduced in Definition 5. Also, it is important to stress that in [**14**, Corollary 14.5.12] the hypothesis on the absence of saddle connection refers to the absence of non-trivial saddle connection.

A permutation $\pi : N_n \to N_n^\sigma$ is said to be irreducible if $|\pi(\{1, 2, \ldots t\})| \neq \{1, 2, \ldots, t\}$ for any $1 \leq t < n$. The set of *irreducible permutations* is denoted by $S_n^{\sigma,*}$. We will write $S_n^{\sigma,+}$ to denote the set of permutations, $\pi \in S_n^\sigma$, satisfying $|\pi|(n) \neq n$. Observe that $S_n^{\sigma,*} \subset S_n^{\sigma,+} \subset S_n^\sigma$. It is easily seen that if (λ, π) is a minimal n-IET (not necessarily oriented) then π is irreducible. Given an oriented n-IET, $T = (\lambda, \pi)$, with π irreducible, if the components of λ are rationally independent then T satisfies the Keane condition and is minimal; however, the Keane condition does not imply that the components of λ are rationally independent (see, for example, [**15**, §6.3]). The last condition on λ allows us to easily construct minimal n-IETs; in fact it gives relevant information, expressed in the following theorem.

THEOREM 9. (Keane [**15**]) *Let $\pi : N_n \to N_n$ be a fixed irreducible standard permutation. Then almost all (with respect to the Lebesgue measure induced on Λ^n) n-IETs of the form (λ, π) are minimal.*

Contrary to what was conjectured since the first work of Keane [**15**], the minimality of an oriented n-IET does not guarantee its unique ergodicity. Counterexamples to this conjecture were first provided by Sataev; see [**28**] and [**6**, Theorem 2, p. 134]. Also Keynes and Newton [**17**] and Keane [**16**] constructed minimal non-uniquely ergodic oriented 5- and 4-IETs, respectively. In answer to another conjecture by Keane, Veech and Masur independently proved† the following theorem.

THEOREM 10. (Masur [**21**, Theorem 1]; Veech [**31**, Theorem 13.10]) *Let $\pi : N_n \to N_n$ be an irreducible standard permutation. Then almost all (with respect to the Lebesgue measure) n-IETs of the form (λ, π) are uniquely ergodic.*

Masur's proof derives from the study of measured foliations on oriented surfaces, while Veech's approach is based on the powerful *Rauzy induction*. Although the latter technique was developed for oriented IETs, it was adapted for non-oriented ones by Nogueira [**24**]. Before stating it, we introduce one relevant result concerning IETs with flips which shows that the behaviour in the non-oriented case is rather different from the oriented case, in particular the previous theorems by Keane, Veech and Masur are no longer true.

THEOREM 11. (Nogueira, [**24**]) *Let π be an irreducible non-standard permutation. Then almost all n-IETs of the form $T = (\pi, \lambda)$ admit periodic points and, therefore, they are not minimal.*

We finish this section by emphasizing that Nogueira and Danthony generalized the notion of IET by introducing *linear involutions*; see [**7**, **8**]. While the first return map of a flow to a transversal segment is closely related to an IET; see [**2**, **11**, **19**], the first return map of a (non-orientable) measured foliation is linked to a linear involution.

† Previously, Veech gave in [**30**] a criterion to obtain the unique ergodicity of (λ, π) in terms of irreducible matrices obtained from the Rauzy induction process; see §3.

3. *Rauzy induction and invariant measures*

Roughly speaking, the *generalized Rauzy induction* is an operator in the set of IETs which sends any $T : D \subset [0, l] \to [0, l]$ to its first return map on some subinterval $[0, l'] \subsetneq [0, l]$. The aim of this section is to give a formalization of this operator, by means of the maps a and b defined on S_n^σ. In the final part of the section we investigate the relationship between the Rauzy induction and the existence of minimal uniquely ergodic IETs with flips.

Let $x \in \mathbb{R}\backslash\{0\}$. Recall that the sign of x is denoted by $\sigma(x)$. The *generalized Rauzy maps* were introduced by Nogueira in [**24**] (cf. also [**27**]). Map a is given by

$$\begin{aligned} a : S_n^{\sigma,+} &\longrightarrow S_n^\sigma \\ \pi &\to a(\pi) \end{aligned}$$

where $a(\pi)$ is the permutation defined depending on the sign of $\pi(n)$ by

$$a(\pi)(i) = \begin{cases} \pi(i) & \text{if } |\pi(i)| \le |\pi(n)| - \dfrac{1-\sigma(\pi(n))}{2}, \\ \sigma(\pi(n))\sigma(\pi(i)) & \\ \quad \times \left(|\pi(n)| + \dfrac{1+\sigma(\pi(n))}{2}\right) & \text{if } |\pi(i)| = n, \\ \sigma(\pi(i))(|\pi(i)|+1) & \text{otherwise.} \end{cases} \tag{2}$$

Map b is given by

$$\begin{aligned} b : S_n^{\sigma,+} &\longrightarrow S_n^\sigma \\ \pi &\to b(\pi) \end{aligned}$$

where $b(\pi)$ is the permutation defined depending on the sign of $\pi(|\pi|^{-1}(n))$ by

$$b(\pi)(i) = \begin{cases} \pi(i) & \text{if } i \le |\pi|^{-1}(n) + \dfrac{\sigma(\pi(|\pi|^{-1}(n))) - 1}{2}, \\ \sigma(\pi(|\pi|^{-1}(n)))\pi(n) & \text{if } i = |\pi|^{-1}(n) + \dfrac{\sigma(\pi(|\pi|^{-1}(n))) - 1}{2} + 1, \\ \pi(i-1) & \text{otherwise.} \end{cases} \tag{3}$$

Together with these maps, we also define the *generalized Rauzy matrices* associated to a permutation $\pi \in S_n^{\sigma,+}$, $M_a(\pi)$ and $M_b(\pi)$. Given $1 \le i, j \le n$, $E_{i,j}$ denotes the $n \times n$ matrix having zeros in all the positions except for the position (i, j) which is equal to 1, and I_n denotes the $n \times n$ identity matrix. The definitions of $M_a(\pi)$ and $M_b(\pi)$ are

$$\begin{aligned} M_a(\pi) &= I_n + E_{n,|\pi|^{-1}(n)}, \\ M_b(\pi) &= \left(\sum_{i=1}^{|\pi|^{-1}(n)} E_{i,i}\right) + E_{n,|\pi|^{-1}(n)+(1+\sigma(\pi(|\pi|^{-1}(n))))/2} + \left(\sum_{i=|\pi|^{-1}(n)}^{n-1} E_{i,i+1}\right). \end{aligned} \tag{4}$$

Now, as a trivial consequence of this definition, we obtain the following claim.

Claim 12. Let $\lambda \in \Lambda^n$ and $\pi \in S_n^{\sigma,+}$. Then $M_v(\pi)\lambda \in \Lambda^n$, where $v \in \{a, b\}$.

Given a matrix $A \in M_{n\times n}(\mathbb{R})$, if $\{\mathrm{a}_i\}_{i=1}^n$ are the columns of A we will write $A = (\mathrm{a}_1; \mathrm{a}_2; \mathrm{a}_3; \ldots; \mathrm{a}_{n-1}; \mathrm{a}_n)$, so $\mathrm{a}_j = (a_{1j}, a_{2j}, \ldots, a_{nj})^t$, $1 \le j \le n$, where t denotes the transpose of a matrix. Let $2 \le j \le n$ and let $\{i_1, i_2, \dots, i_j\} \subset \{1, 2, \dots, n\}$. Then we will denote by

$$\mathrm{a}_{i_1,i_2,\dots,i_j} := \sum_{l=1}^{j} \mathrm{a}_{i_l}$$

the column consisting of the sum of certain columns of A. For $P, Q \in M_{n\times n}(\mathbb{R})$, $P \ge Q$ will mean that the non-zero entries of Q are also non-zero entries of P; the values of these entries may of course not coincide.

Positive matrices will play a relevant role in our study of minimality of IETs. A *non-negative* matrix $A \in M_{n\times n}(\mathbb{R})$, that is, with $a_{i,j} \ge 0$ for any $i, j \in \{1, 2, \dots, n\}$, is said to be *positive* if these inequalities are strict. In the following, the diagonal, a row or a column of a matrix is said to be positive if all the entries in the corresponding diagonal, row or column are positive.

Now it is a simple task to verify the following lemma. It suffices to apply the corresponding definitions and to consider the summation equal to 0 whenever the upper bound of the summation is less than the lower one (we leave the proof to the reader).

LEMMA 13. *Let $n \in \mathbb{N}$, $A = (\mathrm{a}_1; \mathrm{a}_2; \mathrm{a}_3; \dots; \mathrm{a}_{n-1}; \mathrm{a}_n) \in M_{n\times n}(\mathbb{R})$, $B \in M_{n\times n}(\mathbb{R})$, and let $\pi \in S_n^{\sigma,+}$, $M_a(\pi)$ and $M_b(\pi)$ be as defined in equation (4). Then:*

(1) $A \cdot E_{i,j} = (0; 0; \dots; 0; \underbrace{\mathrm{a}_i}_{j\text{th column}}; 0; \dots; 0)$ *for all $i, j \in \{1, \dots, n\}$;*

(2) $A \cdot M_a(\pi) = (\mathrm{a}_1; \mathrm{a}_2; \dots; \mathrm{a}_{|\pi|^{-1}(n)-1}; \mathrm{a}_{|\pi|^{-1}(n),n}; \mathrm{a}_{|\pi|^{-1}(n)+1}; \dots; \mathrm{a}_n)$

whenever $|\pi|^{-1}(n) > 1$, and $A \cdot M_a(\pi) = (\mathrm{a}_{|\pi|^{-1}(n),n}; \mathrm{a}_2; \mathrm{a}_3; \dots; \mathrm{a}_n)$ if $|\pi|^{-1}(n) = 1$;

(3) *if $\sigma(\pi(|\pi|^{-1}(n))) = 1$,*

$$A \cdot M_b(\pi) = (\mathrm{a}_1; \mathrm{a}_2; \dots; \mathrm{a}_{|\pi|^{-1}(n)}; \mathrm{a}_{|\pi|^{-1}(n),n}; \mathrm{a}_{|\pi|^{-1}(n)+1}; \mathrm{a}_{|\pi|^{-1}(n)+2}; \dots; \mathrm{a}_{n-1})$$

whenever $|\pi|^{-1}(n) < n-1$, and $A \cdot M_b(\pi) = (\mathrm{a}_1; \mathrm{a}_2; \dots; \mathrm{a}_{n-1}; \mathrm{a}_{n-1,n})$ if $|\pi|^{-1}(n) = n-1$;

(4) *if $\sigma(\pi(|\pi|^{-1}(n))) = -1$,*

$$A \cdot M_b(\pi) = (\mathrm{a}_1; \mathrm{a}_2; \dots; \mathrm{a}_{|\pi|^{-1}(n)-1}; \mathrm{a}_{|\pi|^{-1}(n),n}; \mathrm{a}_{|\pi|^{-1}(n)}; \mathrm{a}_{|\pi|^{-1}(n)+1}; \mathrm{a}_{|\pi|^{-1}(n)+2}; \dots; \mathrm{a}_{n-1})$$

whenever $|\pi|^{-1}(n) > 1$, and $A \cdot M_b(\pi) = (\mathrm{a}_{1,n}; \mathrm{a}_1; \mathrm{a}_2; \dots; \mathrm{a}_{n-1})$ if $|\pi|^{-1}(n) = 1$;

(5) *if A and B are non-negative and B has positive diagonal, $AB \ge A$;*

(6) $M_a(\pi)^{-1} = I_n - E_{n,|\pi|^{-1}(n)}$*;*

(7) *if $\sigma(\pi(|\pi|^{-1}(n))) = 1$,*

$$M_b(\pi)^{-1} = \left(\sum_{i=1}^{|\pi|^{-1}(n)-1} E_{i,i}\right) + E_{|\pi|^{-1}(n),|\pi|^{-1}(n)} - E_{|\pi|^{-1}(n),n} + E_{|\pi|^{-1}(n)+1,n} + \left(\sum_{i=|\pi|^{-1}(n)+2}^{n} E_{i,i-1}\right);$$

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(8) *if* $\sigma(\pi(|\pi|^{-1}(n))) = -1$,

$$M_b(\pi)^{-1} = \left(\sum_{i=1}^{|\pi|^{-1}(n)-1} E_{i,i}\right) + E_{|\pi|^{-1}(n),n} - E_{|\pi|^{-1}(n)+1,n} + E_{|\pi|^{-1}(n)+1,|\pi|^{-1}(n)} + \left(\sum_{i=|\pi|^{-1}(n)+2}^{n} E_{i,i-1}\right);$$

COROLLARY 14. *Let* $n \in \mathbb{N}$, $\pi \in S_n^{\sigma,+}$, $A = (\mathrm{a}_1; \mathrm{a}_2; \mathrm{a}_3; \ldots; \mathrm{a}_{n-1}; \mathrm{a}_n) \in M_{n\times n}(\mathbb{R})$ *and* $B \in M_{n\times n}(\mathbb{R})$, *both non-negative. Then:*

(1) $A \cdot M_a(\pi) \geq A$;

(2) *if* $\sigma(\pi(|\pi|^{-1}(n))) = 1$,

$$A \cdot M_b(\pi) \geq (\mathrm{a}_1; \mathrm{a}_2; \ldots; \mathrm{a}_{|\pi|^{-1}(n)}; \mathrm{a}_n; \mathrm{a}_{|\pi|^{-1}(n)+1}; \mathrm{a}_{|\pi|^{-1}(n)+2}; \ldots; \mathrm{a}_{n-1})$$

whenever $|\pi|^{-1}(n) < n-1$, *and* $A \cdot M_b(\pi) \geq A$ *and* $A \cdot M_b(\pi) \geq (\mathrm{a}_1; \mathrm{a}_2; \ldots; \mathrm{a}_{n-1}; \mathrm{a}_{n-1})$ *if* $|\pi|^{-1}(n) = n-1$;

(3) *if* $\sigma(\pi(|\pi|^{-1}(n))) = -1$,

$$A \cdot M_b(\pi) \geq (\mathrm{a}_1; \mathrm{a}_2; \ldots; \mathrm{a}_{|\pi|^{-1}(n)-1}; \mathrm{a}_n; \mathrm{a}_{|\pi|^{-1}(n)}; \mathrm{a}_{|\pi|^{-1}(n)+1}; \mathrm{a}_{|\pi|^{-1}(n)+2}; \ldots; \mathrm{a}_{n-1}) \quad \text{if } |\pi|^{-1}(n) > 1$$

and

$$A \cdot M_b(\pi) \geq (\mathrm{a}_n; \mathrm{a}_1; \mathrm{a}_2 \ldots; \mathrm{a}_{n-1}) \quad \text{whenever } |\pi|^{-1}(n) = 1;$$

(4) *if* $A \geq B$, $A \cdot M_a(\pi) \geq B \cdot M_a(\pi)$ *and* $A \cdot M_b(\pi) \geq B \cdot M_b(\pi)$.

Notation 15. If $A = (\mathrm{a}_1; \mathrm{a}_2; \mathrm{a}_3; \ldots; \mathrm{a}_{n-1}; \mathrm{a}_n) \in M_{n\times n}(\mathbb{R})$ and we are going to multiply it, immediately, by one of the two matrices $M_a(\pi)$ or $M_b(\pi)$, then we will also write:

- $A = (\mathrm{a}_1; \mathrm{a}_2; \ldots; \mathbf{a}^{+}_{|\pi|^{-1}(n)}; \ldots; \mathrm{a}_n)$ if $\sigma(\pi(|\pi|^{-1}(n))) = +1$;
- $A = (\mathrm{a}_1; \mathrm{a}_2; \ldots; \mathbf{a}^{-}_{|\pi|^{-1}(n)}; \ldots; \mathrm{a}_n)$ otherwise.

The objective is to simplify the reading so that we know which columns will be modified after the product in accordance with Lemma 13.

We are now ready to present formally the *generalized Rauzy operator* R. Let

$$\mathcal{D} = \{(\lambda, \pi) \in \Lambda^n \times S_n^\sigma : \lambda_n \neq \lambda_{|\pi|^{-1}(n)}\}.$$

Then

$$\begin{aligned} R : \mathcal{D} \subset \Lambda^n \times S_n^\sigma &\longrightarrow \Lambda^n \times S_n^\sigma \\ T = (\lambda, \pi) &\rightarrow T' = (\lambda', \pi') \end{aligned}$$

is defined by

$$T' = (\lambda', \pi') = \begin{cases} (M_a(\pi)^{-1}\lambda, a(\pi)) & \text{if } \lambda_{|\pi|^{-1}(n)} < \lambda_n, \\ (M_b(\pi)^{-1}\lambda, b(\pi)) & \text{if } \lambda_{|\pi|^{-1}(n)} > \lambda_n. \end{cases}$$

If T' is obtained from T by means of the operator a, T is said to be of *type a*, otherwise T is of *type b*. In any case, T' is the *Poincaré first return map* induced by T on $[0, l']$, with $l' = l - \min\{\lambda_n, \lambda_{|\pi|^{-1}(n)}\}$; see [**1**, Proposition 5]. Figure 3 shows the (6, 4)-IET $T = (\lambda, \pi)$ and the induced IETs $T' = (\lambda', \pi')$ and $T'' = (\lambda'', \pi'')$. Observe that, in this

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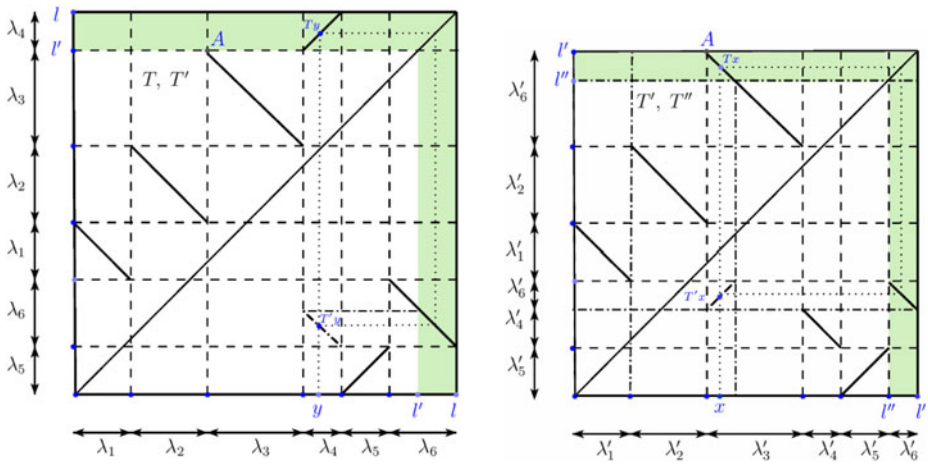

FIGURE 3. Example of an IET T and the induced IETs T' and T''.

example, $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$, $\pi = (-3, -4, -5, 6, 1, -2)$, $\lambda' = (\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4, \lambda'_5, \lambda'_6) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 - \lambda_4)$, $\pi' = a(\pi) = (-4, -5, -6, -2, 1, -3)$, $\lambda'' = (\lambda''_1, \lambda''_2, \lambda''_3, \lambda''_4, \lambda''_5, \lambda''_6) = (\lambda'_1, \lambda'_2, \lambda'_6, \lambda'_3 - \lambda'_6, \lambda'_4, \lambda'_5)$ and $\pi'' = b(\pi') = (-4, -5, 3, -6, -2, 1)$. Notice that $M_a(\pi)^{-1}\lambda = \lambda'$ and $M_b(\pi)^{-1}\lambda' = \lambda''$. In general, given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n_+$ and $\pi \in S^\sigma_n$, it can easily be checked that:

$$M_a(\pi)^{-1}\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n - \lambda_{|\pi|^{-1}(n)}); \tag{5}$$

$$M_b(\pi)^{-1}\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{|\pi|^{-1}(n)-1}, \lambda_{|\pi|^{-1}(n)} - \lambda_n, \lambda_n, \lambda_{|\pi|^{-1}(n)+1}, \ldots, \lambda_{n-1}),$$
$$\text{if } \sigma(\pi(|\pi|^{-1}(n))) = +1 \text{ and } |\pi|^{-1}(n) > 1, \dagger \tag{6}$$

$$M_b(\pi)^{-1}\lambda = (\lambda_1 - \lambda_n, \lambda_n, \lambda_2, \ldots, \lambda_{n-1}),$$
$$\text{if } \sigma(\pi(|\pi|^{-1}(n))) = +1 \text{ and } |\pi|^{-1}(n) = 1, \tag{7}$$

$$M_b(\pi)^{-1}\lambda = (\lambda_1, \ldots, \lambda_{|\pi|^{-1}(n)-1}, \lambda_n, \lambda_{|\pi|^{-1}(n)} - \lambda_n, \lambda_{|\pi|^{-1}(n)+1}, \ldots, \lambda_{n-1}),$$
$$\text{if } \sigma(\pi(|\pi|^{-1}(n))) = -1 \text{ and } |\pi|^{-1}(n) > 1, \ddagger \tag{8}$$

$$M_b(\pi)^{-1}\lambda = (\lambda_n, \lambda_1 - \lambda_n, \lambda_2, \ldots, \lambda_{n-1}),$$
$$\text{if } \sigma(\pi(|\pi|^{-1}(n))) = -1 \text{ and } |\pi|^{-1}(n) = 1. \tag{9}$$

Remark 16. It is worth claiming that if $T = (\lambda, \pi)$ is a proper (n, k)-IET then $R(T)$ may not be a proper (n, k')-IET with $k' \in \{k-1, k, k+1\}$ (notice that if we apply a or b with $\sigma(\pi(n)) = 1$ or $\sigma(\pi(|\pi|^{-1}(n))) = 1$, respectively, then the induced IET keeps the same number of flips, but if $\sigma(\pi(n)) = -1$ or $\sigma(\pi(|\pi|^{-1}(n))) = -1$, then the induced IET can have $k-1$ or $k+1$ flips). For example, take $T_i = (\lambda_i, \pi_i)$, $i \in \{1, 2\}$, with $\pi_1 = (3, -2, -5, 4, 1)$, $\pi_2 = (4, 1, 5, 3, 2)$ and λ_i chosen in such a way that T_i are both of type a. Then $R(T_i) = (\lambda'_i, a(\pi_i))$, with $a(\pi_1) = (4, -3, -2, 5, 1)$ and $a(\pi_2) = (5, 1, 3, 4, 2)$. In this case, T_1 is a proper $(5, 2)$-IET and T_2 is a proper $(5, 0)$-IET, both with four discontinuity points. However, both $R(T_1)$ and $R(T_2)$ have three discontinuity points.

† If $\pi(|\pi|^{-1}(n)) = n-1$ we will understand $M_b(\pi)^{-1}\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-2}, \lambda_{n-1} - \lambda_n, \lambda_n)$.
‡ If $\pi(|\pi|^{-1}(n)) = n-1$, $M_b(\pi)^{-1}\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-2}, \lambda_n, \lambda_{n-1} - \lambda_n)$.

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Claim 17. If $T = (\lambda, \pi)$ is minimal then $T \in \mathcal{D}$, otherwise $\lambda_n = \lambda_{|\pi|^{-1}(n)}$ and there exist $j \in \{1, 2, \ldots, n\}$ and $\otimes \in \{\oplus, \ominus\}$ such that $T(a_j^{\otimes}) = a_n$, which would imply that the orbit of a_n is not dense.

The operators a and b induce in the set $S_n^{\sigma,*}$ a directed graph structure whose vertices are all the points from $S_n^{\sigma,*}$ and the directed edges are arrows labelled by a and b. Given $\pi, \pi' \in S_n^{\sigma,*}$, there exists an arrow labelled by a (respectively, b) from π to π' if and only if $a(\pi) = \pi'$ (respectively, $b(\pi) = \pi'$). Any connected subgraph of this graph, $\mathcal{G}_n$, is called a *Rauzy class* (the Rauzy classes for standard permutations are studied in [**18**]). We remark that we only take into account irreducible permutations because they are the only ones for which the associated IETs can be minimal. Moreover, it is worth noticing that if π is an irreducible standard permutation then $a(\pi)$ and $b(\pi)$ are irreducible, while this is not always the case for non-standard irreducible permutations; observe that $a(-4, 3, 2, -1) = (1, 4, 3, -2)$.

A *vector of operators* is an element of $\{a, b\}^L$, where $L \in \mathbb{N}$ or $L = \infty$ (when $L = \infty$, $\{a, b\}^L = \{a, b\}^{\mathbb{N}}$). An easy way of constructing Rauzy subgraphs is to take a vertex $\pi \in S_n^{\sigma,*}$ and to construct other vertices recursively by applying a vector of operators. The *Rauzy subgraph associated to* $\pi_1 \in S_n^{\sigma,*}$ *and* $v \in \{a, b\}^L$, $\mathcal{G}^{\pi_1, v}$, is the graph of vertices $\{\pi_i\}_{i=1}^L$ satisfying $v_i(\pi_i) = \pi_{i+1}$, $1 \le i \le L - 1$, the edges of this graph being arrows labelled by v_i from π_i to π_{i+1}. Observe that any n-IET, $T = (\lambda, \pi) \in \mathcal{D}$, defines a Rauzy subgraph in a natural way, the one associated to π and the vector of operators v defined by Rauzy induction, that is, v_i is a (respectively, b) if $R^{i-1}(T)$ is of type a (respectively, b); we denote this subgraph by $\mathcal{G}^T$. We will say that T is *infinitely inducible* if v has infinite dimension, that is, $v \in \{a, b\}^{\mathbb{N}}$.

LEMMA 18. *Let* $\gamma \in \Lambda^n$, $\pi \in S_n^{\sigma,*}$ *and* $v \in \{a, b\}$. *Then the IET* $S = (M_v(\pi)\gamma, \pi)$ *is of type* v.

Proof. Write $\lambda = M_v(\pi)\gamma$. Assume first that $v = a$. Then $\lambda_n = \gamma_n + \gamma_{|\pi|^{-1}(n)}$, $\lambda_{|\pi|^{-1}(n)} = \gamma_{|\pi|^{-1}(n)}$ and S is of type a. Second, if $v = b$ we obtain $\lambda_{|\pi|^{-1}(n)} = \gamma_{|\pi|^{-1}(n)} + \gamma_{|\pi|^{-1}(n)+1}$, while $\lambda_n \in \{\gamma_{|\pi|^{-1}(n)}, \gamma_{|\pi|^{-1}(n)+1}\}$. Thus S is of type b in this case. □

COROLLARY 19. *Let* $v \in \{a, b\}^L$ *for some* $L \in \mathbb{N}$ *and let* $\pi_1 \in S_n^{\sigma,*}$ *be such that* $\mathcal{G}^{\pi_1, v}$ *has all its vertices in* $S_n^{\sigma,*}$. *Then there exists an IET* $T = (\lambda, \pi_1)$ *such that* $\mathcal{G}^T = \mathcal{G}^{\pi_1, v}$. *Moreover, given* $\gamma \in \Lambda^n$, *there exists an IET* $T = (\lambda, \pi_1)$ *such that* $R^L(T) = (\gamma, \pi_{L+1})$.

Proof. It suffices to take $\lambda = M_{v_1}(\pi_1) M_{v_2}(\pi_2) \cdots M_{v_L}(\pi_L)\gamma$. □

THEOREM 20. *Let* $T = (\lambda^1, \pi_1)$ *be an* n*-IET such that* $\pi_1 \in S_n^{\sigma,*}$, T *infinitely inducible. For any* $i \ge 1$, *let* $R^i(\lambda^1, \pi_1) = (\lambda^{i+1}, \pi_{i+1})$. *The Rauzy graph of* T, $\mathcal{G}^T$, *is the one associated to* π_1 *and* $v \in \{a, b\}^{\mathbb{N}}$. *Put*

$$\mathcal{C}(\mathcal{G}^T) := \bigcap_{i=1}^{\infty} M_{v_1}(\pi_1) \cdot M_{v_2}(\pi_2) \cdot \cdots \cdot M_{v_i}(\pi_i)\Lambda^n. \tag{10}$$

Assume also that, for any $i \in \mathbb{N}$, π_i *is irreducible. Then:*

(1) $\lambda^1 \in \mathcal{C}(\mathcal{G}^T)$;

(2) *if* $\gamma \in \mathcal{C}(\mathcal{G}^T)$ *and* $S = (\gamma, \pi_1)$, $\mathcal{G}^T = \mathcal{G}^S$.

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Proof. Let us prove the first item. Observe that $v_i(\pi_i) = \pi_{i+1}$ and $\lambda^i = M_{v_i}(\pi_i)\lambda^{i+1}$ for any $i \geq 1$. Thus $\lambda^1 = M_{v_1}(\pi_1)M_{v_2}(\pi_2)\ldots M_{v_i}(\pi_i)\lambda^{i+1}$ for any $i \in \mathbb{N}$ and then $\lambda^1 \in \bigcap_{i=1}^{\infty} M_{v_1}(\pi_1) \cdot M_{v_2}(\pi_2) \cdot \cdots \cdot M_{v_i}(\pi_i)\Lambda^n$.

We now prove the second item. Since $\gamma \in \mathcal{C}(\mathcal{G}^T)$, for any $i \in \mathbb{N}$ there exists $\gamma^i \in \Lambda^n$ such that $\gamma = M_{v_1}(\pi_1)M_{v_2}(\pi_2)\ldots M_{v_i}(\pi_i)\gamma^{i+1}$ and $\gamma^i = M_{v_i}(\pi_i)\gamma^{i+1}$. Now, by Lemma 18, $S = (\gamma, \pi_1) = (M_{v_1}(\pi_1)\gamma^2, \pi_1)$ is of type v_1 and $R(S) = (\gamma^2, v_1(\pi_1))$. Assume now that $R^i(\gamma, \pi_1) = (\gamma^{i+1}, \pi_{i+1}) = (M_{v_{i+1}}(\pi_{i+1})\gamma^{i+2}, \pi_{i+1})$, for any $1 \leq i \leq j$. Then $R^j(\gamma, \pi_1)$ is, by Lemma 18, of type v_{j+1}. Therefore $R^{j+1}(\gamma, \pi_1) = (\gamma^{j+2}, \pi_{j+2}) = (M_{v_{j+2}}(\pi_{j+2})\gamma^{j+3}, \pi_{j+2})$. By recurrence we obtain the result. □

The next result relates the *cone* introduced in (10) to two IETs, one being induced from the other.

PROPOSITION 21. *Let $T = (\lambda^1, \pi_1)$ be an infinitely inducible n-IET such that $\pi_i \in S_n^{\sigma,*}$ and $R^i(\lambda^1, \pi_1) = (\lambda^{i+1}, \pi_{i+1})$ for any $i \geq 1$. Assume that the Rauzy graph of T, $\mathcal{G}^T$, is the one associated to π_1 and $v \in \{a, b\}^{\mathbb{N}}$. Let $S = (\lambda^0, \pi_0)$ be an IET such that $\pi_0 \in S_n^{\sigma,*}$, $R(S) = T$, and $\pi_1 = v_0(\pi_0)$ for some $v_0 \in \{a, b\}$. Then*

$$M_{v_0}(\pi_0)\mathcal{C}(\mathcal{G}^T) = \mathcal{C}(\mathcal{G}^S).$$

Proof. In order to prove the first inclusion, take $\lambda \in \mathcal{C}(\mathcal{G}^T)$. Then there exists $\lambda^i \in \Lambda^n$, $i \in \mathbb{N}$, such that $\lambda = (\prod_{j=1}^{i} M_{v_j}(\pi_j))\lambda^i$ and $M_{v_0}(\pi_0)\lambda = (\prod_{j=0}^{i} M_{v_j}(\pi_j))\lambda^i$ for any $i \in \mathbb{N}$. Therefore $M_{v_0}(\pi_0)\mathcal{C}(\mathcal{G}^T) \subseteq \mathcal{C}(\mathcal{G}^S)$.

Let us proceed with the second inclusion. To this end take $\gamma \in \mathcal{C}(\mathcal{G}^S)$, so there exist $\gamma^i \in \Lambda^n$, $i \in \mathbb{N} \cup \{0\}$, such that $\gamma = (\prod_{j=0}^{i} M_{v_j}(\pi_j))\gamma^i$ for any $i \geq 0$. Thus $M_{v_0}(\pi_0)^{-1}\gamma = (\prod_{j=1}^{i} M_{v_j}(\pi_j))\gamma^i$ for any $i \geq 1$. Then $M_{v_0}(\pi_0)^{-1}\gamma \in \mathcal{C}(\mathcal{G}^T)$, $\gamma \in M_{v_0}(\pi_0)\mathcal{C}(\mathcal{G}^T)$ and finally $M_{v_0}(\pi_0)\mathcal{C}(\mathcal{G}^T) \supseteq \mathcal{C}(\mathcal{G}^S)$. □

In the following, given two reals a and b, we put $\langle a, b\rangle := (a, b)$ if $b \geq a$, otherwise $\langle a, b\rangle := (b, a)$. Assume now that μ is a non-negative non-zero finite invariant measure for the minimal IET $T = (\lambda, \pi)$. Since T is minimal, μ has no atoms and if O is a non-empty open set then $\mu(O) > 0$. Moreover, $\varphi_\mu(x) := \mu(0, x)$ is a homeomorphism between I and $I_\mu := (0, \mu(I))$. We next define T_μ as the map that makes the following diagram commutative.

$$\begin{array}{ccc} I & \xrightarrow{T} & I \\ \varphi_\mu \downarrow & \circlearrowleft & \downarrow \varphi_\mu \\ I_\mu & \xrightarrow{T_\mu} & I_\mu \end{array}$$

Thus $T_\mu \circ \varphi_\mu(x) = \varphi_\mu \circ T(x)$ for any $x \in I$.

Following [**32**, §28] (see also [**30**, §1]), we can adapt the proofs to the case of IETs with flips. We implicitly use Claim 17.

THEOREM 22. *Let $T = (\lambda, \pi)$ be a minimal IET and let μ, μ^* be non-negative non-zero invariant measures. Denote by $\lambda(\mu)$ the positive vector having as ith component $\lambda(\mu)_i = \mu(I_i)$, $i \in \{1, 2, \ldots, n\}$. Then:*

(1) *$T_\mu = (\lambda(\mu), \pi)$;*

(2) *the types of T and T_μ coincide;*

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(3) *the subgraphs* $\mathcal{G}^T$ *and* $\mathcal{G}^{T_\mu}$ *coincide;*
(4) *if* $\mu \neq \mu^*$, *we have* $\lambda(\mu) \neq \lambda(\mu^*)$, *where* $\lambda(\mu^*)$ *is defined in the same way as* $\lambda(\mu)$.

Proof. Let $T : D = \bigcup_{i=1}^n I_i \to [0, l]$ and let $x \in I_i = (a_i, a_{i+1})$. Then (recall the notation used on page 3103):

$$\begin{aligned}\varphi_\mu(T(x)) &= \mu\left(\left(0, \left(\sum_{j=1}^{|\pi|(i)-(\sigma(\pi(i))+1)/2} \lambda_{|\pi|^{-1}(j)}\right) + \sigma(\pi(i))\left[x - \sum_{j=1}^{i-1}\lambda_j\right]\right)\right)\\ &= \mu\left(\left(0, \sum_{j=1}^{|\pi|(i)-(\sigma(\pi(i))+1)/2} \lambda_{|\pi|^{-1}(j)}\right)\right)\\ &\quad + \sigma(\pi(i))\mu\left(\left\langle \sum_{j=1}^{|\pi|(i)-(\sigma(\pi(i))+1)/2} \lambda_{|\pi|^{-1}(j)}, \left(\sum_{j=1}^{|\pi|(i)-(\sigma(\pi(i))+1)/2} \lambda_{|\pi|^{-1}(j)}\right)\right.\right.\\ &\quad \left.\left. + \sigma(\pi(i))\left[x - \sum_{j=1}^{i-1}\lambda_j\right]\right\rangle\right).\end{aligned}$$

Notice that the invariance of μ by T implies

$$\begin{aligned}\mu\left(\left(0, \sum_{j=1}^{|\pi|(i)-(\sigma(\pi(i))+1)/2} \lambda_{|\pi|^{-1}(j)}\right)\right) &= \mu\left(\bigcup_{j=1}^{|\pi|(i)-(\sigma(\pi(i))+1)/2} I_{|\pi|^{-1}(j)}\right)\\ &= \sum_{j=1}^{|\pi|(i)-(\sigma(\pi(i))+1)/2} \mu(I_{|\pi|^{-1}(j)})\\ &= \sum_{j=1}^{k} \lambda(\mu)_{|\pi|^{-1}(j)} \qquad (11)\end{aligned}$$

and

$$\begin{aligned}&\mu\left(\left\langle \sum_{j=1}^{|\pi|(i)-(\sigma(\pi(i))+1)/2} \lambda_{|\pi|^{-1}(j)}, \left(\sum_{j=1}^{|\pi|(i)-(\sigma(\pi(i))+1)/2} \lambda_{|\pi|^{-1}(j)}\right)\right.\right.\\ &\quad \left.\left. + \sigma(\pi(i))\left[x - \sum_{j=1}^{i-1}\lambda_j\right]\right\rangle\right) = \mu\left(T\left(\left(\sum_{j=1}^{i-1}\lambda_j, x\right)\right)\right). \qquad (12)\end{aligned}$$

Equations (11)–(12) give

$$\begin{aligned}T_\mu(\varphi_\mu(x)) = \varphi_\mu(T(x)) &= \sum_{j=1}^{|\pi|(i)-(\sigma(\pi(i))+1)/2} \lambda(\mu)_{|\pi|^{-1}(j)} + \sigma(\pi(i))\mu\left(\left(\sum_{j=1}^{i-1}\lambda_j, x\right)\right)\\ &= \sum_{j=1}^{|\pi|(i)-(\sigma(\pi(i))+1)/2} \lambda(\mu)_{|\pi|^{-1}(j)}\\ &\quad + \sigma(\pi(i))\left[\varphi_\mu(x) - \sum_{j=1}^{i-1}\lambda(\mu)_j\right].\end{aligned}$$

This proves part (1) of the theorem (cf. the definition of an IET introduced on page 3103).

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Turning to part (2), we begin by noticing that the Rauzy type of T and T_μ coincide. Indeed, if T is of type a then $T(I_{|\pi|^{-1}(n)}) \subsetneq I_n$ and $\mu(I_{|\pi|^{-1}(n)}) = \lambda(\mu)_{|\pi|^{-1}(n)} < \lambda(\mu)_n = \mu(I_n)$, and T_μ is also of type a. If, in exchange, T is of type b then $T(I_{|\pi|^{-1}(n)}) \supsetneq I_n$ and $\mu(I_{|\pi|^{-1}(n)}) = \lambda(\mu)_{|\pi|^{-1}(n)} > \lambda(\mu)_n = \mu(I_n)$. Thus T_μ is also of type b.

We now prove part (3). Let $T' : D' = \bigcup_{i=1}^n I'_i \to [0, l']$ be the map induced by $T = (\lambda, \pi)$ by means of the Rauzy procedure, where $l' = l - \min\{\lambda_n, \lambda_{|\pi|^{-1}(n)}\}$, and observe that if we continue writing μ to denote the measure $\mu|_{[0,l']}$, then μ is an invariant measure for T' (notice also that $T'|_{I'_j}$ is either $T|_{I'_j}$ or $T^2|_{I'_j}$). We will now show that $(T')_\mu = (T_\mu)'$. We will use that $T_\mu = (\lambda(\mu), \pi)$, $T' = (\lambda', c(\pi))$, $(T_\mu)' = (\lambda(\mu)', c(\pi))$, $(T')_\mu = (\lambda'(\mu), c(\pi))$, where c is the type of T and T_μ; see part (2). First, we assume that $c = a$. Then (see equation (5))

$$\lambda(\mu)'_i = \lambda(\mu)_i = \mu(I_i) = \mu(I'_i) = \lambda'(\mu)_i, \quad 1 \le i \le n-1,$$
$$\lambda(\mu)'_n = \lambda(\mu)_n - \lambda(\mu)_{|\pi|^{-1}(n)} = \mu(I_n \backslash T(I_{|\pi|^{-1}(n)})) = \mu(I'_n) = \lambda'(\mu)_n.$$

Now, if $c = b$ (see equations (6)–(9)) then

$$\lambda(\mu)'_i = \lambda(\mu)_i = \mu(I_i) = \mu(I'_i) = \lambda'(\mu)_i, \qquad i < |\pi|^{-1}(n),$$
$$\lambda(\mu)'_i = \lambda(\mu)_{i-1} = \mu(I_{i-1}) = \mu(I'_i) = \lambda'(\mu)_i, \quad i > |\pi|^{-1}(n) + 1.$$

Moreover, if $\sigma(\pi(|\pi|^{-1}(n))) = -1$ then

$$\begin{aligned}
\lambda(\mu)'_{|\pi|^{-1}(n)} &= \lambda(\mu)_n = \mu(I_n) = \mu(I'_{|\pi|^{-1}(n)}) = \lambda'(\mu)_{|\pi|^{-1}(n)},\\
\lambda(\mu)'_{|\pi|^{-1}(n)+1} &= \lambda(\mu)_{|\pi|^{-1}(n)} - \lambda(\mu)_n = \mu(I_{|\pi|^{-1}(n)}) - \mu(I_n)\\
&= \mu(I_{|\pi|^{-1}(n)}) - \mu(T^{-1}(I_n))\\
&= \mu(I_{|\pi|^{-1}(n)} \backslash T^{-1}(I_n)) = \mu(I'_{|\pi|^{-1}(n)+1}) = \lambda'(\mu)_{|\pi|^{-1}(n)+1}.
\end{aligned}$$

However, if $\sigma(\pi(|\pi|^{-1}(n))) = +1$ then

$$\begin{aligned}
\lambda(\mu)'_{|\pi|^{-1}(n)} &= \lambda(\mu)_{|\pi|^{-1}(n)} - \lambda(\mu)_n = \mu(I_{|\pi|^{-1}(n)}) - \mu(I_n)\\
&= \mu(I_{|\pi|^{-1}(n)}) - \mu(T^{-1}(I_n))\\
&= \mu(I_{|\pi|^{-1}(n)} \backslash T^{-1}(I_n)) = \mu(I'_{|\pi|^{-1}(n)}) = \lambda'(\mu)_{|\pi|^{-1}(n)},\\
\lambda(\mu)'_{|\pi|^{-1}(n)+1} &= \lambda(\mu)_n = \mu(I_n) = \mu(I'_{|\pi|^{-1}(n)+1}) = \lambda'(\mu)_{|\pi|^{-1}(n)+1}.
\end{aligned}$$

In any case, we have proved that $\lambda'(\mu) = \lambda(\mu)'$, hence $(T')_\mu = (T_\mu)'$, and reasoning by recurrence we obtain that $\mathcal{G}^T$ and $\mathcal{G}^{T_\mu}$ coincide.

For part (4) one realizes that the proof for orientable IETs applies directly to IETs with flips; see [**32**, Lemma 28.4]. □

The next result has been proved for orientable IETs only by Viana [**32**], but it also holds for IETs with flips as mentioned in [**12**]. To fill the gap, in [**1**], Angosto and the second author have related the generalized Keane condition, minimal IETs and infinitely inducible IETs.

THEOREM 23. *Let $T = (\lambda^1, \pi_1)$ be an n-IET such that $\pi_1 \in S_n^{\sigma,*}$, T infinitely inducible. For any $i \ge 1$, let $R^i(\lambda^1, \pi_1) = (\lambda^{i+1}, \pi_{i+1})$. The Rauzy graph of T, $\mathcal{G}^T$, is the one associated to π_1 and $v \in \{a, b\}^{\mathbb{N}}$. Assume also that, for any $i \in \mathbb{N}$, π_i is irreducible. Then:*

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(1) *T satisfies the generalized Keane's condition;*
(2) *$R^j(T)$ is minimal for any $j \in \mathbb{N} \cup \{0\}$;*
(3) *if $\mathcal{C}(\mathcal{G}^T)$ is a half line, T is uniquely ergodic.*

Proof. Parts (1) and (2) follow from [**1**]. We prove (3): let μ and μ^* be two Borel, non-negative, non-zero, finite, invariant measures and observe that by Theorem 22(3), $\mathcal{G}^{T_\mu} = \mathcal{G}^{T_{\mu^*}}$, and by Theorem 20(1), $\{\lambda(\mu), \lambda(\mu^*)\} \subset C(\mathcal{G}^T)$. Since $C(\mathcal{G}^T)$ is one-dimensional, there exists $\kappa \in \mathbb{R}_+$ such that $\lambda(\mu^*) = \kappa\lambda(\mu) = \lambda(\kappa\mu)$, thus $\mu^* = \kappa\mu$, by Theorem 22(4), and the claim follows. □

The *core* of a non-negative $n \times n$ matrix M is the set $\bigcap_{j\in\mathbb{N}} M^j \Lambda^n$ with $n \in \mathbb{N}$. We take the next result from [**26**, Theorem 4.1]; the reader can also consult [**30**, Proposition 3.30 and Lemma 3.28] for a more general setting.

THEOREM 24. *Let M be a positive $n \times n$ matrix. Then $\bigcap_{j\in\mathbb{N}} M^j \Lambda^n = \{\lambda v : \lambda \in \mathbb{R}_+\}$ for some (positive vector) $v \in \Lambda^n$.*

The next result gives a method for constructing minimal IETs by means of Rauzy graphs. Let $\mathcal{G}^{\pi_1,\mathbf{v}}$ be the graph of vertices $\{\pi_i\}_{i\in\mathbb{N}}$ associated to $\pi_1 \in S_n^{\sigma,*}$ and $\mathbf{v} \in \{a, b\}^{\mathbb{N}}$. We say that $\mathcal{G}^{\pi_1,\mathbf{v}}$ is *periodic* if there exists a minimal $p \in \mathbb{N}$ such that $\pi_{j+p} = \pi_j$ and $v_{j+p} = v_j$ for any positive integer j. The *period* of $\mathcal{G}^{\pi_1,\mathbf{v}}$ is p. The matrix associated to the periodic Rauzy graph, $\mathcal{G}^{\pi_1,\mathbf{v}}$, of period p is $M^{\mathcal{G}}_{\pi_1,\mathbf{v}} = M_{v_1}(\pi_1) \cdot M_{v_2}(\pi_2) \cdot \cdots \cdot M_{v_p}(\pi_p)$.

THEOREM 25. *Let $\mathcal{G}^{\pi_1,\mathbf{v}}$ be a periodic graph of period p associated to $\pi_1 \in S_n^{\sigma,*}$ and $\mathbf{v} \in \{a, b\}^{\mathbb{N}}$. Assume that the sth power of $M^{\mathcal{G}}_{\pi_1,\mathbf{v}}$ is positive for some $s \in \mathbb{N}$. Then:*
(1) *there exists $\lambda^1 \in \Lambda^n$ such that $|\lambda^1| = 1$ and $T = (\lambda^1, \pi_1)$ is minimal and uniquely ergodic, and, in particular, $\mathcal{C}(\mathcal{G}^T)$ is one-dimensional;*
(2) *the associated graph to T is $\mathcal{G}^{\pi_1,\mathbf{v}}$;*
(3) *$R^j(T)$ is minimal, uniquely ergodic and self-induced for any $j \in \mathbb{N} \cup \{0\}$.*

Proof. (1) First, we claim that

$$\bigcap_{i=1}^{\infty} M_{v_1}(\pi_1) \cdot M_{v_2}(\pi_2) \cdot \cdots \cdot M_{v_i}(\pi_i)\Lambda^n = \bigcap_{j\in\mathbb{N}} (M^{\mathcal{G}}_{\pi_1,\mathbf{v}})^{sj} \Lambda^n$$

(notice that, by Theorem 24, $\bigcap_{j\in\mathbb{N}}(M^{\mathcal{G}}_{\pi_1,\mathbf{v}})^{sj}\Lambda^n \neq \emptyset$). The inclusion '$\subseteq$' is trivial. For the other inclusion, take $\lambda \in \bigcap_{j\in\mathbb{N}}(M^{\mathcal{G}}_{\pi_1,\mathbf{v}})^{sj}\Lambda^n$. Then for any $j \in \mathbb{N}$ there exists $\lambda^j \in \Lambda^n$ such that $\lambda = (M^{\mathcal{G}}_{\pi_1,v})^{sj}\lambda^j$. Let us show that $\lambda \in \bigcap_{i=1}^{\infty} M_{v_1}(\pi_1) \cdot M_{v_2}(\pi_2) \cdot \cdots \cdot M_{v_i}(\pi_i)\Lambda^n$. For $i \in \mathbb{N}$, write $i = spc + r$, with $0 \le r < sp$. Since $(M^{\mathcal{G}}_{\pi_1,v})^{s(c+1)}\lambda^{c+1} = \lambda$, we deduce

$$\begin{aligned}
&[M_{v_1}(\pi_1) \cdot M_{v_2}(\pi_2) \cdots M_{v_p}(\pi_p)]^{sc} \cdot M_{v_1}(\pi_1) \cdot M_{v_2}(\pi_2) \cdots M_{v_r}(\pi_r) \\
&\quad \cdot [M_{v_{r+1}}(\pi_{r+1}) \cdots M_{v_{sp}}(\pi_{sp})] \cdot \lambda^{c+1} \\
&= M_{v_1}(\pi_1) \cdot M_{v_2}(\pi_2) \cdots M_{v_i}(\pi_i) \cdot \gamma = \lambda,
\end{aligned}$$

with $\gamma = (M_{v_{r+1}}(\pi_{r+1}) \cdots M_{v_{sp}}(\pi_{sp})) \cdot \lambda^{c+1} \in \Lambda^n$ (see Claim 12). Then

$$\lambda \in \bigcap_{i=1}^{\infty} M_{v_1}(\pi_1) \cdot M_{v_2}(\pi_2) \cdot \cdots \cdot M_{v_i}(\pi_i)\Lambda^n,$$

which proves the claim.

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Next, it suffices to apply Theorem 24 to obtain that $\bigcap_{j\in\mathbb{N}}(M^{\mathcal{G}}_{\pi_1,\mathbf{v}})^{sj}\Lambda^n$ is one-dimensional. Therefore, by Theorem 23†, there exists a unique λ^1, $|\lambda^1| = 1$, such that $T = (\lambda^1, \pi)$ is minimal and uniquely ergodic, hence (1) follows.

Part (2) is a direct consequence of repeatedly applying Lemma 18.

Finally, we show part (3). Observe that for any $j \in \mathbb{N}$, $R^j(T)$ is minimal by Theorem 23(2). Moreover, Proposition 21 implies that $\mathcal{C}(\mathcal{G}^{T^j})$, $j \in \mathbb{N}$, is one-dimensional since $\mathcal{C}(\mathcal{G}^T)$ is, and then Theorem 23(3) guarantees the unique ergodicity of T^j. $R^j(T)$ is also self-induced since $\mathcal{G}^{\pi_1,\mathbf{v}}$ is periodic too. □

Remark 26. Although in this paper we construct minimal self-induced IETs we now introduce some permutations which do not generate self-induced IETs. Consider $\pi \in S_n^{\sigma,*}$ such that $\pi(1) = -n$ and $\pi(n) = -1$. Then $a(\pi)$ and $b(\pi)$ are both reducible. Consequently, there is no $\lambda \in \Lambda^n$ making $T = (\lambda, \pi)$ self-induced. Observe that we can choose π with k flips, $2 \le k \le n$.

Also take $\tau \in S_n^{\sigma,*}$ satisfying $\tau(n-1) = n$, $\tau(n) = -1$. Then it is a simple task to show that $b(\tau) = \tau$, $b(\alpha) \neq \tau$ for any $\alpha \in S_n^{\sigma,*}\backslash\{\tau\}$ and $a(\alpha) \neq \tau$ for any $\alpha \in S_n^{\sigma,*}$. Then the existence of $\lambda \in \Lambda^n$ making $T = (\lambda, \pi)$ self-induced would imply that $R^j(T)$ is always of type b, but this is a contradiction with [**1**, Lemma 6]. Observe that τ can be chosen with k flips, $1 \le k \le n-1$.

4. *Proof of main theorem for $n = 2m \ge 8$*

We divide the proof of this case into three subsections. In §4.1 we build a periodic Rauzy graph, and we show that its associated matrix is positive in §4.2. Finally, in §4.3, we include the proof of main theorem for $n = 2m$.

4.1. *The periodic Rauzy graph.*

We divide the proof of this section into several lemmas. The idea is to construct a periodic Rauzy graph of period $p = 4m + (2m-3)(m-1)$ and to apply Theorem 25. Let $\mathbf{z}^i = (z^i_r)_{r=1}^{L_i} \in \{a, b\}^{L_i}$, with $L_i \in \mathbb{N}$ and $1 \le i \le h$, for some integer $h \ge 2$. We define the *concatenation vector* $\mathbf{z} = \mathbf{z}^1 * \mathbf{z}^2 * \cdots \mathbf{z}^h \in \{a, b\}^{L_1+L_2+\cdots+L_h}$ by $z_j = z^i_{j-\sum_{t<i} L_t}$ if $\sum_{t<i} L_t < j \le \sum_{t<i+1} L_t$ for some i.

Let us take

$$\pi_0 = (-3, -4, -5, \ldots, -[2m-1], 2m, 1, -2), \tag{13}$$

$$\mathbf{v}^1 = (a, a, a, \ldots, a) \in \{a, b\}^{2m-3}, \tag{14}$$

$$\mathbf{v}^2 = (b, a, b, b, a, b) \in \{a, b\}^6, \tag{15}$$

$$\mathbf{v}^3 = (a, b, a, b, a, b \ldots, a, b) \in \{a, b\}^{2m-6}, \tag{16}$$

$$\mathbf{v}^4 = (a, \underbrace{b, b}_{2}, \underbrace{a, a, a}_{3}, \underbrace{b, b, b, b}_{4}, \ldots, \underbrace{b, \ldots, b}_{2m-4}, \underbrace{a, \ldots, a}_{2m-3}) \in \{a, b\}^{(2m-3)(m-1)}, \tag{17}$$

$$\mathbf{v}^5 = (b, a, b) \in \{a, b\}^3, \tag{18}$$

$$\mathbf{v} = \mathbf{v}^1 * \mathbf{v}^2 * \mathbf{v}^3 * \mathbf{v}^4 * \mathbf{v}^5 \in \{a, b\}^p, \quad p = 4m + (2m-3)(m-1). \tag{19}$$

† Observe that if π is a vertex of $\mathcal{G}^{\pi_1,\mathbf{v}}$, then $\pi \in S_n^{\sigma,*}$ due to the periodicity of the graph, taking into account that a reducible permutation is sent to another reducible permutation by the maps a and b.

We can also define the vector $\mathbf{w}$ as the *periodic concatenation* of $\mathbf{v}$, that is, the vector

$$\mathbf{w} = (w_j)_j \in \{a, b\}^{\mathbb{N}}, \quad \text{such that } \mathbf{w}_{i+kp} = \mathbf{v}_i \text{ for any } 1 \le i \le p \text{ and } k \in \mathbb{N}.$$

LEMMA 27. *Let $\mathcal{G}^{\pi_0,\mathbf{v}^1}$ be the graph of vertices $\{\pi_i\}_{i=0}^{2m-3}$, $m \ge 4$. Then*

$$\pi_1 = (-4, -5, -6, \ldots, -[2m-1], -2m, -2, 1, -3),$$
$$\pi_2 = (-5, -6, -7, \ldots, -2m, 3, -2, 1, -4),$$

and, for any $3 \le j \le 2m-3$, we have†

$$\pi_j = (-[3+j], -[4+j], \ldots, \underbrace{-2m}_{2m-2-j}, j+1, j, \ldots, 3, -2, 1, -[2+j]).$$

Proof. To compute the different values of the vertices of the graph, we must recall the definition of $a(\cdot)$; see (2). Then it is easy to see that

$$\begin{aligned}\pi_1 &= a(-3, -4, -5, \ldots, -[2m-1], 2m, 1, -2)\\ &= (-4, -5, -6, \ldots, -[2m-1], -2m, -2, 1, -3),\\ \pi_2 &= a(-4, -5, -6, \ldots, -[2m-1], -2m, -2, 1, -3)\\ &= (-5, -6, -7, \ldots, -2m, 3, -2, 1, -4).\end{aligned}$$

The rest of the result follows by applying recurrence on j. For $j = 3$, we have

$$\begin{aligned}\pi_3 &= a(-5, -6, -7, \ldots, -[2m-1], -2m, 3, -2, 1, -4)\\ &= (-6, -7, \ldots, -2m, 4, 3, -2, 1, -5).\end{aligned}$$

Now we assume that, for some $j \ge 3$,

$$\pi_j = (-[3+j], -[4+j], \ldots, \underbrace{-2m}_{2m-2-j}, j+1, j, j-1, \ldots, 3, -2, 1, -[2+j]),$$

and we obtain

$$\begin{aligned}\pi_{j+1} = a(\pi_j) = (&-[4+j], -[5+j], \ldots, \underbrace{-2m}_{2m-3-j},\\ &j+2, j+1, j, j-1, \ldots, 3, -2, 1, -[3+j]),\end{aligned}$$

which ends the proof. □

From the relations $b(\pi_{2m-3}) = \pi_{2m-2}$, $a(\pi_{2m-2}) = \pi_{2m-1}$, $b(\pi_{2m-1}) = \pi_{2m}$, $b(\pi_{2m}) = \pi_{2m+1}$, $a(\pi_{2m+1}) = \pi_{2m+2}$ and $b(\pi_{2m+2}) = \pi_{2m+3}$, the next result easily follows (use the definitions (2) and (3)).

LEMMA 28. *Let $\mathcal{G}^{\pi_{2m-3},\mathbf{v}^2}$ be the graph of vertices $\{\pi_i\}_{i=2m-3}^{2m+3}$, with $m \ge 4$. Then:*

$$\begin{aligned}\pi_{2m-2} &= (2m-1, -2m, 2m-2, 2m-3, \ldots, 4, 3, -2, 1),\\ \pi_{2m-1} &= (2m, -2, 2m-1, 2m-2, \ldots, 5, 4, -3, 1),\\ \pi_{2m} &= (2m, 1, -2, 2m-1, 2m-2, \ldots, 4, -3),\\ \pi_{2m+1} &= (2m, -3, 1, -2, 2m-1, 2m-2, \ldots, 5, 4),\\ \pi_{2m+2} &= (5, -3, 1, -2, 2m, 2m-1, 2m-2, \ldots, 7, 6, 4),\\ \pi_{2m+3} &= (5, -3, 1, -2, 2m, 4, 2m-1, 2m-2, \ldots, 7, 6).\end{aligned}$$

† The underbrace in the following formula indicates the position where the value $-2m$ is placed.

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LEMMA 29. *Let $\mathcal{G}^{\pi_{2m+3},\mathbf{v}^3}$ be the graph of vertices $\{\pi_i\}_{i=2m+3}^{4m-3}$, $m \geq 5$. Then, for any $1 \leq j \leq m-4$, we have*†

$$\pi_{(2m+3)+(2j-1)} = (5, -3, 1, -2, 7, 4, \ldots, 2k+5, 2k+2, \ldots, 2j+5, 2j+2, \underbrace{2m}_{7+2(j-1)}, 2m-1, 2m-2, \ldots, 2j+7, 2j+6, 2j+4)$$

and

$$\pi_{(2m+3)+2j} = (5, -3, 1, -2, 7, 4, \ldots, 2k+5, 2k+2, \ldots, 2j+5, 2j+2, \underbrace{2m}_{7+2(j-1)}, 2j+4, 2m-1, 2m-2, \ldots, 2j+7, 2j+6),$$

where, in both cases, $1 \leq k \leq j$. Moreover, for $m \geq 4$,

$$\pi_{4m-4} = \pi_{4m-3} = (5, -3, 1, -2, 7, 4, \ldots, 2s+5, 2s+2, \ldots, 2m-1, 2m-4, 2m, 2m-2),$$

with $1 \leq s \leq m-3$.

Proof. If the first equation is true then the second follows since $\pi_{(2m+3)+2j} = b(\pi_{(2m+3)+(2j-1)})$. Let us prove the first equality by induction on j. For $j = 1$ the formula is valid because

$$\pi_{(2m+3)+1} = a(\pi_{2m+3}) = (5, -3, 1, -2, 7, 4, 2m, 2m-1, 2m-2, \ldots, 9, 8, 6).$$

Assume now that the equation is true for some j (recall that the second one is also valid). Then

$$\begin{aligned}\pi_{(2m+3)+(2j+1)} &= a(\pi_{(2m+3)+2j}) \\ &= (5, -3, 1, -2, 7, 4, 9, 6, 11, 8, 13, 10, \ldots, 2j+5, 2j+2, \\ &\quad \underbrace{2j+7}_{7+2(j-1)}, 2j+4, \underbrace{2m}_{7+2j}, 2m-1, 2m-2, \ldots, 2j+9, 2j+8, 2j+6),\end{aligned}$$

which proves that the equalities of the statement are true for $\pi_{(2m+3)+(2j-1)}$ and $\pi_{(2m+3)+(2j)}$, $1 \leq j \leq m-4$. Finally, since $\pi_{4m-4} = a(\pi_{4m-5})$ and $\pi_{4m-3} = b(\pi_{4m-4})$, it is easy to see that both permutations are equal to

$$(5, -3, 1, -2, 7, 4, \ldots, 2s+5, 2s+2, \ldots, 2m-1, 2m-4, 2m, 2m-2),$$

with $1 \leq s \leq m-3$. □

We already know that

$$\pi_{4m-3} = (5, -3, 1, -2, 7, 4, \ldots, 2k+5, 2k+2, \ldots, 2m-3, 2m-6, 2m-1, 2m-4, 2m, 2m-2),$$

with $1 \leq k \leq m-3$.

† Observe that the definitions of $\pi_{(2m+3)+(2j-1)}$ and $\pi_{(2m+3)+(2j)}$ do not apply in the case $m = 4$. In this situation, only the expressions for π_{4m-4} and π_{4m-3} make sense.

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In order to give a new representation of the vector $\mathbf{v}^4$ (see (19) and (17)) and to simplify the writing in next lemma, let us introduce some notation. For $j \in \mathbb{N}$, we define the blocks

$$\mathbf{a}^j := (a, \ldots, a) \in \{a, b\}^j, \tag{20}$$

$$\mathbf{b}^j := (b, \ldots, b) \in \{a, b\}^j, \tag{21}$$

and also

$$\mathbf{c}^j := \mathbf{a}^{2j-1} * \mathbf{b}^{2j} \in \{a, b\}^{4j-1}. \tag{22}$$

In this way, it is an easy task to check that

$$\mathbf{v}^4 = \mathbf{c}^1 * \mathbf{c}^2 * \mathbf{c}^3 * \cdots * \mathbf{c}^{m-3} * \mathbf{c}^{m-2} * \mathbf{a}^{2m-3},$$

where the symbol $*$ denotes the concatenation of vectors.

We will write

$$\mathbf{c}^j := (c_1^j, c_2^j, \ldots, c_{2j-1}^j, c_{2j}^j, \ldots, c_{4j-1}^j), \tag{23}$$

so c_i^j is the ith coordinate of the vector $\mathbf{c}^j$.

It is an easy task to check that $\mathbf{v}^4 = \mathbf{c}^1 * \mathbf{c}^2 * \mathbf{c}^3 * \cdots * \mathbf{c}^{m-3} * \mathbf{c}^{m-2} * \mathbf{a}^{2m-3}$. We will construct the graph, $\mathcal{G}^{\pi_{4m-3},\mathbf{v}^4}$, of vertices $\{\pi_i\}_{i=4m-3}^{4m-3+(2m-3)(m-1)}$. We will also denote the vertices by

$$\{\delta_{j,i} : 1 \le j \le m-2 \text{ and } 1 \le i \le 4j-1\} \cup \{\varepsilon_i : 1 \le i \le 2m-3\}, \tag{24}$$

where

- $\delta_{1,1} = c_1^1(\pi_{4m-3}) = a(\pi_{4m-3})$ and $\delta_{1,i} = c_i^1(\delta_{1,i-1}) = b(\delta_{1,i-1})$ if $i = 2, 3$ (recall that $\mathbf{c}^1 = (a, b, b)$);
- $\delta_{j,1} = c_1^j(\delta_{j-1,4j-5})$, $2 \le j \le m-2$ (that is, $\delta_{j,1} = a(\delta_{j-1,4j-5})$);
- $\delta_{j,i} = c_i^j(\delta_{j,i-1})$ for $2 \le i \le 4j-1$, $2 \le j \le m-2$, so $\delta_{j,i} = a(\delta_{j,i-1})$ if $2 \le i \le 2j-1$ and $\delta_{j,i} = b(\delta_{j,i-1})$ if $2j \le i \le 4j-1$;
- $\varepsilon_1 = a(\delta_{m-2,4m-9})$ and $\varepsilon_i = a(\varepsilon_{i-1})$ if $2 \le i \le 2m-3$.

Roughly speaking, $\delta_{j,i}$ is the permutation obtained from π_{4m-3} by applying the vector of operators $\mathbf{c}^1, \ldots, \mathbf{c}^{j-1}$ and the first i components of $\mathbf{c}^j$; ε_i is the permutation obtained from π_{4m-3} by applying $\mathbf{c}^1, \ldots, \mathbf{c}^{m-2}$ and i times the operator a.

LEMMA 30. *The components of the Rauzy graph $\mathcal{G}^{\pi_{4m-3},\mathbf{v}^4}$ are given in the following items.*

C$_1$. *For $m \ge 4$, the permutations appearing when applying the operators of $\mathbf{c}^1$ on π_{4m-3} are*†

$$\begin{aligned}\pi_{4m-2} = \delta_{1,1} &= a(\pi_{4m-3}) \\ &= (5, -3, 1, -2, \underbrace{7, 4, \ldots, 2k+5, 2k+2, \ldots, 2m-3, 2m-6}_{2m-8 \text{ components}}, \\ &\qquad 2m, 2m-4, 2m-1, 2m-2), \\ \pi_{4m-1} = \delta_{1,2} &= b(\pi_{4m-2})\end{aligned}$$

† The block of $2m-8$ components obviously disappears when $m = 4$.

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$$= (5,\ -3,\ 1,\ -2,\ \underbrace{7,\ 4,\ \ldots,\ 2k+5,\ 2k+2,\ \ldots,\ 2m-3,\ 2m-6}_{2m-8 \text{ components}},$$
$$2m,\ 2m-2,\ 2m-4,\ 2m-1),$$
$$\pi_{4m} = \delta_{1,3} = b(\pi_{4m-1})$$
$$= (5,\ -3,\ 1,\ -2,\ \underbrace{7,\ 4,\ \ldots,\ 2k+5,\ 2k+2,\ \ldots,\ 2m-3,\ 2m-6}_{2m-8 \text{ components}},$$
$$2m,\ 2m-1,\ 2m-2,\ 2m-4),$$

with $1 \le k \le m-4$.

$\mathbf{C}_j$. *For* $m \ge 5$, *the permutations obtained by applying the operators of the vector* $\mathbf{c}^j$, *with* $2 \le j \le m-3$, *are the following*†:

$$\delta_{j,s} = (5,\ -3,\ 1,\ -2,\ \underbrace{7,\ 4,\ \ldots,\ 2k+5,\ 2k+2,\ \ldots,\ 2m-2j-1,\ 2m-2j-4}_{2m-2j-6 \text{ components}},$$
$$2m-2j+s+1,\ 2m-2j-2,$$
$$\underbrace{2m-2j+s,\ \ldots,\ 2m-2j+s-l,\ \ldots,\ 2m-2j+1}_{s \text{ components}},$$
$$\underbrace{2m}_{2m-2j+s+1},\ \underbrace{2m-1,\ 2m-2,\ \ldots,\ 2m-2j+s+3,\ 2m-2j+s+2}_{2j-s-2 \text{ components}},$$
$$2m-2j)$$

(where $0 \le l \le s-1$*)*,

$$\delta_{j,2j-2} = (5,\ -3,\ 1,\ -2,\ \underbrace{7,\ 4,\ \ldots,\ 2k+5,\ 2k+2,\ \ldots,\ 2m-2j-1,\ 2m-2j-4}_{2m-2j-6 \text{ components}},$$
$$2m-1,\ 2m-2j-2,\ 2m-2,\ 2m-3,\ \ldots,\ 2m-2j+2,\ 2m-2j+1,$$
$$\underbrace{2m}_{2m-1},\ 2m-2j),$$

$$\delta_{j,2j-1} = (5,\ -3,\ 1,\ -2,\ \underbrace{7,\ 4,\ \ldots,\ 2k+5,\ 2k+2,\ \ldots,\ 2m-2j-1,\ 2m-2j-4}_{2m-2j-6 \text{ components}},$$
$$\underbrace{2m}_{2m-2j-1},\ 2m-2j-2,\ 2m-1,\ 2m-2,\ \ldots,\ 2m-2j),$$

$$\delta_{j,2j+i} = (5,\ -3,\ 1,\ -2,\ \underbrace{7,\ 4,\ \ldots,\ 2k+5,\ 2k+2,\ \ldots,\ 2m-2j-1,\ 2m-2j-4}_{2m-2j-6 \text{ components}},$$
$$\underbrace{2m}_{2m-2j-1},\ 2m-2j+i,\ \ldots,\ 2m-2j+i-q,\ \ldots,\ \underbrace{2m-2j}_{2m-2j+i},$$
$$2m-2j-2,\ 2m-1,\ 2m-2,\ \ldots,\ 2m-2j+i+1)$$

(where $0 \le q \le i$*)*,

$$\delta_{j,4j-1} = (5,\ -3,\ 1,\ -2,\ \underbrace{7,\ 4,\ \ldots,\ 2k+5,\ 2k+2,\ \ldots,\ 2m-2j-1,\ 2m-2j-4}_{2m-2j-6 \text{ components}},$$

† In the case $\mathbf{C}_{m-3}$ it is necessary to take into account that, in the expressions for $\delta_{j,s}$, $\delta_{j,2j-2}$, $\delta_{j,2j-1}$, $\delta_{j,2j+i}$ and $\delta_{j,4j-1}$, the group $\underbrace{7,\ 4,\ 9,\ 6,\ 11,\ 8,\ \ldots,\ 2m-2j-1,\ 2m-2j-4}_{2m-2j-6 \text{ components}}$ disappears.

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$$\underbrace{2m}_{2m-2j-1}, 2m-1, 2m-2, \ldots, \underbrace{2m-2j}_{2m-1}, 2m-2j-2),$$

with $1 \le s < 2j-2$, $0 \le i \le 2j-2$ *and* $1 \le k \le m-j-3$.

$\mathbf{C}_{m-2}$. *For* $m \ge 4$, *the permutations obtained by applying the vector* $\mathbf{c}^{m-2}$ *are given by*†

$$\delta_{m-2,s} = (s+5, -3, 1, -2, s+4, \ldots, s+4-h, \ldots, 5, \underbrace{2m}_{s+5}, \underbrace{2m-1, \ldots, s+7, s+6}_{2m-s-6 \text{ components}}, 4),$$

$$\delta_{m-2,2m-5} = (2m, -3, 1, -2, 2m-1, 2m-2, \ldots, 7, 6, 5, 4),$$

$$\delta_{m-2,2m-5+i} = (2m, \underbrace{3+i, \ldots, 3+i-p, \ldots, 4}_{i \text{ components}}, \underbrace{-3}_{2+i}, 1, -2, \underbrace{2m-1}_{5+i}, \underbrace{2m-2, \ldots, i+4}_{2m-i-5 \text{components}}),$$

$$\delta_{m-2,4m-9} = (2m, 2m-1, 2m-2, \ldots, 4, -3, 1, -2),$$

where $1 \le s \le 2m-6$, $0 \le h \le s-1$, *and* $1 \le i \le 2m-5$, $0 \le p \le i-1$.

$\mathbf{A}_{2m-3}$. *Finally, for* $m \ge 4$ *we apply* $\mathbf{a}^{2m-3}$ *to obtain*

$$\varepsilon_i = (\underbrace{-2, \ldots, -(2+j), \ldots, -[i+1]}_{i \text{ components}}, \underbrace{2m}_{i+1}, \underbrace{2m-1, 2m-2, \ldots, i+4}_{2m-i-4 \text{components}}, -[i+3], 1, -[i+2]) \quad \text{with } 1 \le i \le 2m-4, 0 \le j \le i-1,$$

$$\varepsilon_{2m-3} = (-2, -3, -4, \ldots, -[2m-3], -[2m-2], \underbrace{-2m}_{2m-2}, 1, -[2m-1]).$$

Proof. It is straightforward to see that the result is true for $\mathbf{C}_1$ and it is a simple matter to prove that $\mathbf{C}_2$ holds, so we omit omit the proofs‡. Next, we prove by induction that $\mathbf{C}_j$ holds true for $2 < j < m-3$ (then we implicitly assume that $m > 6$). Let us suppose that $\mathbf{C}_j$ holds for some $2 \le j < m-4$, $m > 6$, and prove the validity of $\mathbf{C}_{j+1}$. Since

$$\delta_{j,4j-1} = (5, -3, 1, -2, \underbrace{7, 4, 9, 6, \ldots, 2m-2j-3, 2m-2j-6, 2m-2j-1, 2m-2j-4}_{2m-2j-6 \text{ components}}, \underbrace{2m}_{2m-2j-1}, 2m-1, 2m-2, \ldots, \underbrace{2m-2j}_{2m-1}, 2m-2j-2),$$

according to the definition of a, with $\sigma(\pi(|\pi|^{-1}(n))) = +1$, we find

$$\delta_{j+1,1} = a(\delta_{j,4j-1}) = (5, -3, 1, -2, \underbrace{7, 4, 9, 6, \ldots, 2m-2j-3, 2m-2j-6, \underline{2m-2j}, 2m-2j-4}_{2m-2j-6 \text{ components}}, \underbrace{2m-2j-1}_{2m-2j-1}, 2m, 2m-1, \ldots, \underbrace{2m-2j+1}_{2m-1}, 2m-2j-2)$$

† Observe that the blocks of $2m-i-5$ and $2m-s-6$ components can have length 0.

‡ Observe that if $m = 4$ then the item $\mathbf{C}_j$ does not appear and we are implicitly assuming $m > 4$ when proving the validity of $\mathbf{C}_j$.

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$$= (5,\ -3,\ 1,\ -2,\ \underbrace{7,\ 4,\ 9,\ 6,\ \ldots,\ 2m-2j-3,\ 2m-2j-6}_{2m-2(j+1)-6 \text{ components}},$$
$$\underline{2m-2(j+1)+1+1},\ 2m-2(j+1)-2,\ 2m-2(j+1)+1,\ \underbrace{2m}_{2m-2(j+1)+1+1},$$
$$2m-1,\ 2m-2,\ \ldots,\ 2m-2(j+1)+1+3,\ 2m-2(j+1)+1+2,$$
$$2m-2(j+1)),$$

therefore $\delta_{j+1,1}$ follows the pattern of the induction. This allows us to apply induction on s, so we will prove that if the formula $\delta_{j+1,s}$ holds for $1 \le s < 2(j+1)-3$, then the formula remains valid for $\delta_{j+1,s+1}$. Indeed, by definition of a,

$$\delta_{j+1,s+1} = a(\delta_{j+1,s}) = a(5,\ -3,\ 1,\ -2,\ \underbrace{7,\ 4,\ \ldots,\ 2m-2j-3,\ 2m-2j-6}_{2m-2(j+1)-6 \text{ components}},$$
$$2m-2(j+1)+s+1,\ \underline{2m-2(j+1)-2},$$
$$\underbrace{2m-2(j+1)+s,\ \ldots,\ 2m-2(j+1)+s-l,\ \ldots,\ 2m-2(j+1)+1}_{s \text{ components}}$$
$$\underbrace{2m}_{2m-2(j+1)+s+1},$$
$$\underbrace{2m-1,\ 2m-2,\ \ldots,\ 2m-2(j+1)+s+3,\ 2m-2(j+1)+s+2}_{2(j+1)-s-2 \text{ components}},$$
$$2m-2(j+1))$$
$$= (5,\ -3,\ 1,\ -2,\ \underbrace{7,\ 4,\ \ldots,\ 2m-2j-3,\ 2m-2j-6}_{2m-2(j+1)-6 \text{ components}},$$
$$2m-2(j+1)+(s+1)+1,\ \underline{2m-2(j+1)-2},$$
$$2m-2(j+1)+(s+1),\ \ldots,\ 2m-2(j+1)+(s+1)-l,\ \ldots,$$
$$2m-2(j+1)+1+1\ \underbrace{2m-2(j+1)+1}_{2m-2(j+1)+s+1},$$
$$\underbrace{2m,\ 2m-1,\ \ldots,\ 2m-2(j+1)+(s+1)+3,\ 2m-2(j+1)+(s+1)+2}_{2(j+1)-s-2 \text{ components}},$$
$$2m-2(j+1)),$$

that is,

$$\delta_{j+1,s+1} = (5,\ -3,\ 1,\ -2,\ \underbrace{7,\ 4,\ \ldots,\ 2m-2j-3,\ 2m-2j-6}_{2m-2(j+1)-6 \text{ components}},$$
$$2m-2(j+1)+(s+1)+1,\ \underline{2m-2(j+1)-2},$$
$$\left.\begin{array}{l} 2m-2(j+1)+(s+1),\ \ldots,\ 2m-2(j+1)+(s+1)-l,\ \ldots \\ \ldots,\ 2m-2(j+1)+1+1,\ 2m-2(j+1)+1,\ \underbrace{2m-2(j+1)+1}_{2m-2(j+1)+s+1}, \end{array}\right\} s+1 \text{ components}$$
$$\underbrace{2m}_{2m-2(j+1)+(s+1)+1},\ \underbrace{2m-1,\ \ldots,\ 2m-2(j+1)+(s+1)+3,\ 2m-2(j+1)+(s+1)+2}_{2(j+1)-(s+1)-2 \text{ components}},$$
$$2m-2(j+1)),$$

so $\delta_{j+1,s+1}$ satisfies the corresponding formula for the first part of $\mathbf{C}_{j+1}$. Thus we finish

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the induction on s. In particular, for $s = 2(j+1)-3$ we have proved that

$$\begin{aligned}\delta_{j+1,2(j+1)-3} = (5, -3, 1, -2, &\underbrace{7, 4, \ldots, 2m-2j-3, 2m-2j-6}_{2m-2(j+1)-6 \text{ components}}, \\ &2m-2, 2m-2(j+1)-2, \\ &\underbrace{2m-3, 2m-4, \ldots, 2m-2j, 2m-2j-1}_{2(j+1)-3 \text{ components}}, \\ &\underbrace{2m}_{2m-2}, \underbrace{2m-1}_{1 \text{ component}}, 2m-2(j+1)).\end{aligned}$$

Applying the operator a twice, we find

$$\begin{aligned}\delta_{j+1,2(j+1)-2} = (5, -3, 1, -2, &\underbrace{7, 4, \ldots, 2m-2j-3, 2m-2j-6}_{2m-2(j+1)-6 \text{ components}}, \\ &2m-1, 2m-2(j+1)-2, 2m-2, 2m-3, \ldots, 2m-2j+1, \\ &2m-2j, 2m-2(j+1)+1, \underbrace{2m}_{2m-1}, 2m-2(j+1))\end{aligned}$$

and

$$\begin{aligned}\delta_{j+1,2(j+1)-1} = (5, -3, 1, -2, &\underbrace{7, 4, \ldots, 2m-2j-3, 2m-2j-6}_{2m-2(j+1)-6 \text{ components}}, \\ &\underbrace{2m}_{2m-2(j+1)-1}, 2m-2(j+1)-2, 2m-1, 2m-2, \ldots, \\ &2m-2(j+1)+4, 2m-2(j+1)+3, \\ &2m-2(j+1)+2, 2m-2(j+1)+1, 2m-2(j+1)),\end{aligned}$$

which ends part of the inductive process of the proof for $\mathbf{C}_{j+1}$, with $2 \le j < m-4$. It remains to obtain the corresponding formulas for $\delta_{j+1,2(j+1)+i}$ $(0 \le i \le 2(j+1)-2)$ and for $\delta_{j+1,4(j+1)-1}$. To this purpose, we now apply consecutively the operator b and do induction on i. To start,

$$\begin{aligned}\delta_{j+1,2(j+1)} = b(\delta_{j+1,2(j+1)-1}) = b(5, -3, 1, -2, &\underbrace{7, 4, \ldots, 2m-2j-3, 2m-2j-6}_{2m-2(j+1)-6 \text{ components}}, \\ &\underbrace{2m}_{2m-2(j+1)-1}, 2m-2(j+1)-2, 2m-1, 2m-2, \ldots, \\ &2m-2(j+1)+4, 2m-2(j+1)+3, \\ &2m-2(j+1)+2, 2m-2(j+1)+1, \underline{2m-2(j+1)}) \\ = (5, -3, 1, -2, &\underbrace{7, 4, \ldots, 2m-2j-3, 2m-2j-6}_{2m-2(j+1)-6 \text{ components}}, \\ &\underbrace{2m}_{2m-2(j+1)-1}, \underline{2m-2(j+1)}, 2m-2(j+1)-2, \\ &2m-1, 2m-2, \ldots, 2m-2(j+1)+4, \\ &2m-2(j+1)+3, 2m-2(j+1)+2, 2m-2(j+1)+1).\end{aligned}$$

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Hence, the formula $\delta_{j+1,2(j+1)+i}$ is correct for $i=0$. The induction for any other value of i follows a direct (and tedious) procedure, so we omit it. Consequently,

$$\begin{aligned}\delta_{j+1,2(j+1)+2(j+1)-2} = (5,\, -3,\, 1,\, -2,\, &\underbrace{7,\, 4,\, \ldots,\, 2m-2j-3,\, 2m-2j-6}_{2m-2(j+1)-6 \text{ components}},\\ &\underbrace{2m}_{2m-2(j+1)-1},\, 2m-2,\, 2m-3,\, \ldots,\, \underbrace{2m-2(j+1)}_{2m-2},\\ &2m-2(j+1)-2,\, 2m-1).\end{aligned}$$

Finally,

$$\begin{aligned}\delta_{j+1,2(j+1)+2(j+1)-1} &= b(\delta_{j+1,2(j+1)+2(j+1)-2})\\ &= (5,\, -3,\, 1,\, -2,\, \underbrace{7,\, 4,\, \ldots,\, 2m-2(j+1)-1,\, 2m-2(j+1)-4}_{2m-2(j+1)-6 \text{ components}},\\ &\quad \underbrace{2m}_{2m-2(j+1)-1},\, 2m-1,\, 2m-2,\, 2m-3,\, \ldots,\, \underbrace{2m-2(j+1)}_{2m-1},\\ &\quad 2m-2(j+1)-2).\end{aligned}$$

This ends the inductive proof of the effect of the vectors $\mathbf{c}^j$ for $j=2,\ldots,m-4$.

Concerning the effect of $\mathbf{c}^{m-3}$, we initiate the computations with the permutation

$$\delta_{m-4,4(m-4)-1} = (5,\, -3,\, 1,\, -2,\, 7,\, 4,\, 2m,\, 2m-1,\, 2m-2,\, \ldots,\, 10,\, 9,\, 8,\, 6)$$

to firstly obtain

$$\delta_{m-3,1} = a(\delta_{m-4,4(m-4)-1}) = (5,\, -3,\, 1,\, -2,\, 8,\, 4,\, 7,\, 2m,\, 2m-1,\, \ldots,\, 10,\, 9,\, 6),$$

which coincides with the formula for the statement for $j=m-3$. The rest of the proof for the effect of the block $\mathbf{c}^{m-3}$ is similar to that of $\mathbf{c}^j$, $1\le j\le m-4$; merely take into account that the block $7, 4, \ldots, 2m-2j-1, 2m-2j-4$ disappears. This ends definitively the proof of the part $\mathbf{C}_j$, $j=2,\ldots,m-3$.

To prove $\mathbf{C}_{m-2}$ we first assume $m\ge 5$. Our initial permutation is $\delta_{m-3,4(m-3)-1}$, namely

$$\delta_{m-3,4(m-3)-1} = (5,\, -3,\, 1,\, -2,\, 2m,\, 2m-1,\, 2m-2,\, \ldots,\, 8,\, 7,\, 6,\, 4).$$

Then the first components of this part are

$$\begin{aligned}\delta_{m-2,1} &= a(\delta_{m-3,4(m-3)-1}) = (6,\, -3,\, 1,\, -2,\, 5,\, 2m,\, \underbrace{2m-1,\, \ldots,\, 8,\, 7}_{2m-7 \text{ components}},\, 4),\\ \delta_{m-2,2} &= a(\delta_{m-2,1}) = (7,\, -3,\, 1,\, -2,\, 6,\, 5,\, 2m,\, \underbrace{2m-1,\, \ldots,\, 9,\, 8}_{2m-8 \text{ components}},\, 4),\\ \delta_{m-2,3} &= a(\delta_{m-2,2}) = (8,\, -3,\, 1,\, -2,\, 7,\, 6,\, 5,\, 2m,\, \underbrace{2m-1,\, \ldots,\, 10,\, 9}_{2m-9 \text{ components}},\, 4),\end{aligned}$$

and it is a short exercise to check by induction the formulas of the elements appearing in $\mathbf{C}_{m-2}$. A special case is $m=4$ for which the effect of $\mathbf{c}^j$ does not apply. Then we provide

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the exact effect of $\mathbf{c}^{m-2}$ on $\pi_{16} = \delta_{1,3} = (5, -3, 1, -2, 8, 7, 6, 4)$ in order to prove this particular case, avoiding the proof by recurrence:

$$\begin{array}{ccccc}
(5, -3, 1, -2, 8, 7, 6, 4) & \xrightarrow{a} & (6, -3, 1, -2, 5, 8, 7, 4) & \xrightarrow{a} & (7, -3, 1, -2, 6, 5, 8, 4) \\
 & & & & \downarrow a \\
(8, 5, 4, -3, 1, -2, 7, 6) & \xleftarrow{b} & (8, 4, -3, 1, -2, 7, 6, 5) & \xleftarrow{b} & (8, -3, 1, -2, 7, 6, 5, 4) \\
\downarrow b & & & & \\
(8, 6, 5, 4, -3, 1, -2, 7) & \xrightarrow{b} & (8, 7, 6, 5, 4, -3, 1, -2) & &
\end{array}$$

Once we have proved the parts $\mathbf{C}_1$–$\mathbf{C}_j$–$\mathbf{C}_{m-2}$, $\mathbf{A}_{2m-3}$ follows immediately; it suffices to apply the definition of the operator a. Indeed,

$$\begin{aligned}
\varepsilon_1 &= a(\delta_{m-2,4m-9}) = a((2m, 2m-1, 2m-2, \ldots, 4, -3, 1, -2)) \\
&= (-2, 2m, 2m-1, \ldots, 5, -4, 1, -3), \\
\varepsilon_2 &= a(\varepsilon_1) = a((-2, 2m, 2m-1, \ldots, 5, -4, 1, -3)) \\
&= (-2, -3, 2m, 2m-1, \ldots, 6, -5, 1, -4),
\end{aligned}$$

which proves the validity of the proposed expressions for ε_1 and ε_2. Moreover, for any $2 \le i \le 2m-5$, from

$$\begin{aligned}
\varepsilon_{i+1} &= a(\varepsilon_i) \\
&= a((\underbrace{-2, \ldots, -(2+j), \ldots, -[i+1]}_{i \text{ components}}, \underbrace{2m}_{i+1}, \underbrace{2m-1, 2m-2, \ldots, i+4}_{2m-i-4 \text{ components}}, \\
&\quad -[i+3], 1, -[i+2])) \\
&= (\underbrace{-2, \ldots, -(2+j), \ldots, -[i+1], -[i+2]}_{i+1 \text{ components}}, \underbrace{2m}_{i+2}, \\
&\quad \underbrace{2m-1, 2m-2, \ldots, i+5}_{2m-i-5 \text{ components}}, -[i+4], 1, -[i+3])
\end{aligned}$$

we deduce that that the expressions for ε_i remain true for $2 \le i \le 2m-4$. Finally, ε_{2m-3} also applies:

$$\begin{aligned}
\varepsilon_{2m-3} &= a(\varepsilon_{2m-4}) \\
&= a((-2, -3, -4, \ldots, -[2m-3], \underbrace{2m}_{2m-3}, -[2m-1], 1, -[2m-2])) \\
&= (-2, -3, -4, \ldots, -[2m-3], -[2m-2], \underbrace{-2m}_{2m-2}, 1, -[2m-1]). \qquad \square
\end{aligned}$$

LEMMA 31. *Let $r = 4m-3+(2m-3)(m-1)$ and let $\mathcal{G}^{\pi_r, \mathbf{v}^5}$ be the graph of vertices $\{\pi_i\}_{i=r}^{r+3}$, where $\pi_r = \pi_{\varepsilon_{2m-3}}$. Then:*

$$\begin{aligned}
\pi_{r+1} &= (-2, -3, -4, -5, \ldots, -[2m-2], 2m-1, -2m, 1), \\
\pi_{r+2} &= (-3, -4, -5, \ldots, -[2m-1], 2m, -2, 1), \\
\pi_{r+3} &= (-3, -4, -5, \ldots, -[2m-1], 2m, 1, -2).
\end{aligned}$$

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Proof. This follows by checking that $b(\pi_r) = \pi_{r+1}$, $a(\pi_{r+1}) = \pi_{r+2}$, $b(\pi_{r+2}) = \pi_{r+3}$. □

As a consequence of the previous lemmas we obtain the following result.

PROPOSITION 32.

(1) *The Rauzy subgraph $\mathcal{G}^{\pi_0,\mathbf{v}}$ is periodic, where π_0 and $\mathbf{v}$ are given by (13) and (19), respectively.*

(2) *In the vertices of $\mathcal{G}^{\pi_0,\mathbf{v}}$ there are permutations with f flips, $2 \le f \le 2m-1$, satisfying the conditions $\pi(j+1) - \pi(j) \neq 1$ and $\pi(j+1) - \pi(j) \not\equiv 1 \bmod (n)$, for any $j \in \{1, 2, \ldots, n\}$, in Remarks 2 and 3, respectively. In particular, these permutations can be found in the set $\{\pi_j\}_{j=1}^{2m-2}$.*

Proof. The first statement follows easily from Lemmas 27–31 since $\pi_{4m+(2m-3)(m-1)} = \pi_0$.

Concerning the second statement, observe that from Lemma 27 the permutations π_j, $1 \le j \le 2m-3$, are in $\mathcal{G}^{\pi_0,\mathbf{v}}$. Moreover, π_j has $2m - j$ flips, $1 \le j \le 2m-3$, and then in the graph $\mathcal{G}^{\pi_0,\mathbf{v}}$ there are permutations with f flips, $3 \le f = 2m - j \le 2m - 1$. Note finally that π_{2m-2} has two flips and that π_j satisfies the conditions in Remarks 2 and 3 for any $1 \le j \le 2m-2$. □

In the Rauzy subgraph $\mathcal{G}^{\pi_0,\mathbf{v}}$ there are no permutations with $2m$ flips or 1 flip. The next results give such permutations and show that they belong to the same Rauzy class as $\mathcal{G}^{\pi_0,\mathbf{v}}$. The proofs are obtained simply by applying the operators a and b.

LEMMA 33. *Let $\tau_1, \tau_2 \in S_{2m}^{\sigma}$ be the following signed permutations*

$$\tau_1 = (-2, -3, -4, \ldots, -[2m-2], -2m, -1, -[2m-1]),$$
$$\tau_2 = (-2, -3, -4, \ldots, -[2m-2], 2m-1, -2m, -1).$$

Then we have the Rauzy subgraph

$$\tau_1 \xrightarrow{b} \tau_2 \xrightarrow{a} \pi_0 = (-3, -4, -5, \ldots, -[2m-1], 2m, 1, -2).$$

LEMMA 34. *Let $\alpha_1, \alpha_2 \in S_{2m}^{\sigma}$ be the signed permutations*

$$\alpha_1 = (2m-1, 1, 2m, 2m-2, 2m-3, \ldots, 6, 5, 4, 3, -2),$$
$$\alpha_2 = (2m, 1, -2, 2m-1, 2m-2, \ldots, 7, 6, 5, 4, -3).$$

Then, we have the Rauzy subgraph

$$\alpha_1 \xrightarrow{a} \alpha_2 \xrightarrow{b} \pi_{2m+1} = (2m, -3, 1, -2, 2m-1, 2m-2, \ldots, 7, 6, 5, 4).$$

4.2. *Study of the matrix associated to the graph $\mathcal{G}^{\pi_0,\mathbf{v}}$ of §4.1.* It is a difficult task to build the whole matrix $M^{\mathcal{G}}_{\pi_0,\mathbf{v}}$ explicitly. We only need to prove that a certain power of this matrix is positive. Thus we are only interested in showing that some entries of $M^{\mathcal{G}}_{\pi_0,\mathbf{v}}$ are non-zero.

Since $M^{\mathcal{G}}_{\pi_0,\mathbf{v}} = M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1} M^{\mathcal{G}}_{\pi_{2m-3},\mathbf{v}^2} M^{\mathcal{G}}_{\pi_{2m+3},\mathbf{v}^3} M^{\mathcal{G}}_{\pi_{4m-3},\mathbf{v}^4} M^{\mathcal{G}}_{\pi_r,\mathbf{v}^5}$, with $r = 4m - 3 + (2m-3)(m-1)$, we divide this section into several subsections to build, step by step, some relations between $M^{\mathcal{G}}_{\pi_0,\mathbf{v}}$ and a known matrix. It is useful to revise Notation 15 in order to clarify the meaning of the superindices, + and −, appearing in some columns of the following matrices.

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4.2.1. *The matrix $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1}$.* By definition

$$M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1} = W \cdot M_a(\pi_0) \cdot M_a(\pi_1) \cdot M_a(\pi_2) \cdot \cdots \cdot M_a(\pi_{2m-5}) \cdot M_a(\pi_{2m-4}),$$

where W is the $n \times n$ identity matrix $W = (\mathrm{e}_1; \mathrm{e}_2; \ldots; \mathbf{e}^+_{\mathbf{2m-2}}; \mathrm{e}_{2m-1}; \mathrm{e}_{2m})$.

LEMMA 35. $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1} = (\mathbf{e}^-_{\mathbf{1}}; \mathrm{e}_{2,2m}; \mathrm{e}_{3,2m}; \mathrm{e}_{4,2m}; \ldots; \mathrm{e}_{2m-2,2m}; \mathrm{e}_{2m-1}; \mathrm{e}_{2m})$.

Proof. By Lemma 27, $|\pi_j|^{-1}(-2m) = 2m - 2 - j$, $1 \le j \le 2m - 4$; moreover, $|\pi_0|^{-1}(2m) = 2m - 2$. Applying Lemma 13(2) concludes the proof. □

4.2.2. *The matrix $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2}$.* By the definition of $\mathbf{v}^2$ we have

$$M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2} = U \cdot M_b(\pi_{2m-3}) \cdot M_a(\pi_{2m-2}) \cdot M_b(\pi_{2m-1}) \cdot M_b(\pi_{2m}) \cdot M_a(\pi_{2m+1}) \cdot M_b(\pi_{2m+2}),$$

where U is the $n \times n$ matrix

$$U := (\mathbf{u}^-_{\mathbf{1}}; \mathrm{u}_2; \ldots; \mathrm{u}_{2m}) = (\mathbf{e}^-_{\mathbf{1}}; \mathrm{e}_{2,2m}; \mathrm{e}_{3,2m}; \mathrm{e}_{4,2m}; \ldots; \mathrm{e}_{2m-2,2m}; \mathrm{e}_{2m-1}; \mathrm{e}_{2m}). \quad (25)$$

LEMMA 36. $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2} = (\mathrm{u}_{1,2m,2m-3}; \mathrm{u}_{1,2m,2m-2}; \mathrm{u}_{1,2m,2m-1}; \mathrm{u}_{1,2m-1}; \mathbf{u}^+_{\mathbf{2}}; \mathrm{u}_{2,2m-3}; \mathrm{u}_3; \mathrm{u}_4; \ldots; \mathrm{u}_{2m-4})$.

Proof. It is necessary to use Lemmas 27 and 28 to locate $|\pi_j|^{-1}(2m)$, $2m - 3 \le j \le 2m + 2$ (also notice that $\sigma(\pi_{2m-3}(|\pi_{2m-3}|^{-1}(2m))) = -1$ whereas $\sigma(\pi_j(|\pi_j|^{-1}(2m))) = +1$ if $j = 2m - 1, 2m, 2m + 2$). By using Lemma 13,

$$U_1 = U \cdot M_b(\pi_{2m-3}) = (\mathrm{u}_{1,2m}; \mathbf{u}^-_{\mathbf{1}}; \mathrm{u}_2; \mathrm{u}_3; \ldots; \mathrm{u}_{2m-1}),$$
$$U_2 = U_1 \cdot M_a(\pi_{2m-2}) = (\mathbf{u}^+_{\mathbf{1,2m}}; \mathrm{u}_{1,2m-1}; \mathrm{u}_2; \mathrm{u}_3; \ldots; \mathrm{u}_{2m-1}),$$
$$U_3 = U_2 \cdot M_b(\pi_{2m-1}) = (\mathbf{u}^+_{\mathbf{1,2m}}; \mathrm{u}_{1,2m,2m-1}; \mathrm{u}_{1,2m-1}; \mathrm{u}_2; \mathrm{u}_3; \ldots; \mathrm{u}_{2m-2}),$$
$$U_4 = U_3 \cdot M_b(\pi_{2m}) = (\mathbf{u}^+_{\mathbf{1,2m}}; \mathrm{u}_{1,2m,2m-2}; \mathrm{u}_{1,2m,2m-1}; \mathrm{u}_{1,2m-1}; \mathrm{u}_2; \mathrm{u}_3; \ldots; \mathrm{u}_{2m-3}),$$
$$U_5 = U_4 \cdot M_a(\pi_{2m+1}) = (\mathrm{u}_{1,2m,2m-3}; \mathrm{u}_{1,2m,2m-2}; \mathrm{u}_{1,2m,2m-1}; \mathrm{u}_{1,2m-1}; \mathbf{u}^+_{\mathbf{2}}; \mathrm{u}_3; \ldots; \mathrm{u}_{2m-3}),$$

and finally

$$T = U_6 = U_5 \cdot M_b(\pi_{2m+2}) = (\mathrm{u}_{1,2m,2m-3}; \mathrm{u}_{1,2m,2m-2}; \mathrm{u}_{1,2m,2m-1}; \mathrm{u}_{1,2m-1}; \mathbf{u}^+_{\mathbf{2}}; \mathrm{u}_{2,2m-3}; \mathrm{u}_3; \ldots; \mathrm{u}_{2m-4}).$$ □

4.2.3. *The matrix $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3}$.* In this case

$$M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3} = T \cdot M_a(\pi_{2m+3}) \cdot M_b(\pi_{2m+4}) \cdot M_a(\pi_{2m+5}) \cdot M_b(\pi_{2m+6}) \cdot \cdots \cdot M_a(\pi_{4m-5}) \cdot M_b(\pi_{4m-4}),$$

where T is the $n \times n$ matrix

$$T = (\mathrm{t}_1; \mathrm{t}_2; \mathrm{t}_3; \mathrm{t}_4; \mathbf{t}^+_{\mathbf{5}}; \mathrm{t}_6; \ldots; \mathrm{t}_{2m}) := (\mathrm{u}_{1,2m,2m-3}; \mathrm{u}_{1,2m,2m-2}; \mathrm{u}_{1,2m,2m-1}; \mathrm{u}_{1,2m-1}; \mathbf{u}^+_{\mathbf{2}}; \mathrm{u}_{2,2m-3}; \mathrm{u}_3; \ldots; \mathrm{u}_{2m-4}). \quad (26)$$

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Remark 37. Notice that in terms of the values of the identity matrix, T can be written as

$$\begin{aligned}T &= (\mathrm{u}_{1,2m,2m-3};\ \mathrm{u}_{1,2m,2m-2};\ \mathrm{u}_{1,2m,2m-1};\ \mathrm{u}_{1,2m-1};\ \mathbf{u}_{\mathbf{2}}^{+};\ \mathrm{u}_{2,2m-3};\ \mathrm{u}_{3};\ \ldots;\ \mathrm{u}_{2m-4})\\ &= (\mathrm{e}_{1,2m}+\mathrm{e}_{2m-3,2m};\ \mathrm{e}_{1,2m}+\mathrm{e}_{2m-2,2m};\ \mathrm{e}_{1,2m,2m-1};\ \mathrm{e}_{1,2m-1};\ \mathrm{e}_{2,2m};\\ &\quad \mathrm{e}_{2,2m}+\mathrm{e}_{2m-3,2m};\ \mathrm{e}_{3,2m};\ \mathrm{e}_{4,2m};\ \ldots;\ \mathrm{e}_{2m-4,2m})\\ &\geq (\mathrm{e}_{1,2m,2m-3};\ \mathrm{e}_{1,2m,2m-2};\ \mathrm{e}_{1,2m,2m-1};\ \mathrm{e}_{1,2m-1};\ \mathrm{e}_{2,2m};\ \mathrm{e}_{2,2m,2m-3};\ \mathrm{e}_{3,2m};\\ &\quad \mathrm{e}_{4,2m};\ \ldots;\ \mathrm{e}_{2m-4,2m}). \qquad \square \end{aligned} \tag{27}$$

LEMMA 38.

$$\begin{aligned}M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3} = (&\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\\ &\mathrm{t}_{7,2m-1};\ \mathrm{t}_{7,2m};\ \ldots;\ \mathrm{t}_{5+l,2m+1-l};\ \mathrm{t}_{5+l,2m+2-l};\ \ldots;\ \mathrm{t}_{m+2,m+4};\ \mathrm{t}_{m+2,m+5};\\ &\mathbf{t}^{+}_{\mathbf{m+3}};\ \mathrm{t}_{m+3,m+4}),\end{aligned}$$

with $l = 2, 3, \ldots, m-3$.

Proof. We proceed by induction. Recall that $\sigma(\pi_j(|\pi_j|^{-1}(2m))) = +1$ for all $j \in \{2m+3, \ldots, 4m-4\}$; see Lemmas 28 and 29. We will prove the following formulas for any $2 \leq j \leq m-3$:

$$\begin{aligned}A_j &:= T \cdot \left(\prod_{h=0}^{j-2} M_a(\pi_{2m+3+2h}) M_b(\pi_{2m+3+2h+1})\right) \cdot M_a(\pi_{2m+3+2(j-1)}) \qquad (28)\\ &= (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathrm{t}_{7,2m-1};\ \mathrm{t}_{7,2m};\ \ldots;\ \mathrm{t}_{5+s,2m+1-s};\ \mathrm{t}_{5+s,2m+2-s};\ \ldots;\\ &\quad \mathrm{t}_{5+j,2m+1-j};\ \mathrm{t}_{5+j,2m+2-j};\ \underbrace{\mathbf{t}^{+}_{\mathbf{6+j}}}_{5+2j};\ \mathrm{t}_{7+j};\ \ldots;\ \mathrm{t}_{2m+1-j}),\end{aligned}$$

$$\begin{aligned}B_j &:= T \cdot \left(\prod_{h=0}^{j-1} M_a(\pi_{2m+3+2h}) M_b(\pi_{2m+3+2h+1})\right) = A_j \cdot M_b(\pi_{2m+3+2j-1}) \qquad (29)\\ &= (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathrm{t}_{7,2m-1};\ \mathrm{t}_{7,2m};\ \ldots;\ \mathrm{t}_{5+s,2m+1-s};\ \mathrm{t}_{5+s,2m+2-s};\ \ldots;\\ &\quad \mathrm{t}_{5+j,2m+1-j};\ \mathrm{t}_{5+j,2m+2-j};\ \underbrace{\mathbf{t}^{+}_{\mathbf{6+j}}}_{5+2j};\ \mathrm{t}_{6+j,2m+1-j};\ \mathrm{t}_{7+j};\ \mathrm{t}_{8+j};\ \ldots;\ \mathrm{t}_{2m-j}), \qquad (30)\end{aligned}$$

where the range of the value s is $2, \ldots, j$. Additionally, by using Lemmas 28, 29 and 13 (observe that $|\pi_{2m+3}|^{-1}(2m) = 5$ and $|\pi_{2m+4}|^{-1}(2m) = 7$), we define and compute

$$A_1 := T \cdot M_a(\pi_{2m+3}) = (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathbf{t}^{+}_{\mathbf{7}};\ \underbrace{\mathrm{t}_8;\ \ldots;\ \mathrm{t}_{2m}}_{2m-7 \text{ columns}}),$$

$$B_1 := A_1 \cdot M_b(\pi_{2m+4}) = (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathbf{t}^{+}_{\mathbf{7}};\ \mathrm{t}_{7,2m};\ \underbrace{\mathrm{t}_8;\ \mathrm{t}_9;\ \ldots;\ \mathrm{t}_{2m-1}}_{2m-8 \text{ columns}}).$$

We now prove equations (28) and (29) for $j = 2$ and $m > 4$. Observe that $A_2 = B_1 \cdot M_a(\pi_{2m+5})$, with $|\pi_{2m+5}|^{-1}(2m) = 7$ and $|\pi_{2m+6}|^{-1}(2m) = 9$; see Lemma 29. By Lemma 13,

$$A_2 = (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathrm{t}_{7,2m-1};\ \mathrm{t}_{7,2m};\ \underbrace{\mathbf{t}^{+}_{\mathbf{8}};}_{9}\ \mathrm{t}_9;\ \ldots;\ \mathrm{t}_{2m-1}),$$

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$$B_2 = A_2 \cdot M_b(\pi_{2m+6})$$
$$= (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathrm{t}_{7,2m-1};\ \mathrm{t}_{7,2m};\ \underbrace{\mathbf{t}_8^+;}_{9}\ \mathrm{t}_{8,2m-1};\ \mathrm{t}_9;\ \ldots;\ \mathrm{t}_{2m-2}).$$

Therefore, equations (28) and (29) hold for $j = 2$.

Assume now that equations (28) and (29) hold for some $2 \le l < m - 3$, that is,

$$A_l = (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathrm{t}_{7,2m-1};\ \mathrm{t}_{7,2m};\ \mathrm{t}_{8,2m-2};\ \mathrm{t}_{8,2m-1};\ \ldots;$$
$$\mathrm{t}_{5+l,2m+1-l};\ \mathrm{t}_{5+l,2m+2-l};\ \underbrace{\mathbf{t}_{6+l}^+;}_{5+2l}\ \mathrm{t}_{7+l};\ \ldots;\ \mathrm{t}_{2m+1-l}),$$
$$B_l = (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathrm{t}_{7,2m-1};\ \mathrm{t}_{7,2m};\ \mathrm{t}_{8,2m-2};\ \mathrm{t}_{8,2m-1};\ \ldots;$$
$$\mathrm{t}_{5+l,2m+1-l};\ \mathrm{t}_{5+l,2m+2-l};\ \underbrace{\mathbf{t}_{6+l}^+;}_{5+2l}\ \mathrm{t}_{6+l,2m+1-l};\ \mathrm{t}_{7+l};\ \mathrm{t}_{8+l};\ \ldots;\ \mathrm{t}_{2m-l}).$$

We will prove that equations (28) and (29) hold for $j = l + 1$. By Lemma 29 we know that $|\pi_{2m+3+2l}|^{-1}(2m) = 7 + 2(l - 1) = 5 + 2l$ and $|\pi_{2m+3+2l+1}|^{-1}(2m) = 7 + 2l = 5 + 2(l + 1)$. Then, according to Lemma 13, we find

$$A_{l+1} = B_l \cdot M_a(\pi_{2m+3+2l})$$
$$= (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathrm{t}_{7,2m-1};\ \mathrm{t}_{7,2m};\ \mathrm{t}_{8,2m-2};\ \mathrm{t}_{8,2m-1};\ \ldots;$$
$$\mathrm{t}_{5+l,2m+1-l};\ \mathrm{t}_{5+l,2m+2-l};\ \mathrm{t}_{6+l,2m-l};\ \mathrm{t}_{6+l,2m+1-l};\ \underbrace{\mathbf{t}_{7+l}^+;}_{7+2l}\ \mathrm{t}_{8+l};\ \ldots;\ \mathrm{t}_{2m-l}),$$
$$B_{l+1} = B_l \cdot M_a(\pi_{2m+3+2l}) \cdot M_b(\pi_{2m+3+2l+1})$$
$$= (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathrm{t}_{7,2m-1};\ \mathrm{t}_{7,2m};\ \mathrm{t}_{8,2m-2};\ \mathrm{t}_{8,2m-1};\ \ldots;$$
$$\mathrm{t}_{5+l,2m+1-l};\ \mathrm{t}_{5+l,2m+2-l};\ \mathrm{t}_{6+l,2m-l};\ \mathrm{t}_{6+l,2m+1-l};$$
$$\underbrace{\mathbf{t}_{7+l}^+;}_{7+2l}\ \mathrm{t}_{7+l,2m-l};\ \mathrm{t}_{8+l};\ \mathrm{t}_{9+l};\ \mathrm{t}_{10+l};\ \ldots;\ \mathrm{t}_{2m-1-l}).$$

Finally, observe that $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3} = B_{m-3}$. This concludes the proof. □

4.2.4. *The matrix* $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4}$. In this section we are not interested in computing the components of the matrix $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4}$ exactly. We will fix our interest on some of its components.

It is strongly recommended to revise the notation of §4.1 (see (20)–(24)) because we need it in the following definitions:

$$C_1 := M_a(\pi_{4m-3}) \cdot M_b(\delta_{1,1}) \cdot M_b(\delta_{1,2}),$$
$$C_j := M_a(\delta_{j-1,4j-5}) \cdot \prod_{l=1}^{4j-2} M_{c^j_{l+1}}(\delta_{j,l}), \quad 2 \le j \le m-2,$$
$$D_j := S \cdot C_1 \cdot C_2 \cdots C_j, \quad 1 \le j \le m-2,$$
$$D_{m-1} := D_{m-2} \cdot M_a(\delta_{m-2,4m-9}) \cdot \prod_{l=1}^{2m-4} M_a(\varepsilon_l).$$

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Moreover, S is the $n \times n$ matrix

$$S = (\mathrm{s}_1;\ \mathrm{s}_2;\ \mathrm{s}_3;\ \ldots;\ \mathrm{s}_{2m-2};\ \mathbf{s}^{+}_{\mathbf{2m-1}};\ \mathrm{s}_{2m}) \tag{31}$$
$$= (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathrm{t}_{7,2m-1};\ \mathrm{t}_{7,2m};\ \mathrm{t}_{8,2m-2};\ \mathrm{t}_{8,2m-1};\ \ldots;$$
$$\mathrm{t}_{m+2,m+4};\ \mathrm{t}_{m+2,m+5};\ \mathbf{t}^{+}_{\mathbf{m+3}};\ \mathrm{t}_{m+3,m+4})$$
$$= M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3}.$$

Remark 39. According to the definition of S and Remark 37, we can express and compare the entries of S in terms of the entries of the identity matrix W as follows:

$$S = (\mathrm{t}_1;\ \mathrm{t}_2;\ \mathrm{t}_3;\ \mathrm{t}_4;\ \mathrm{t}_{5,2m};\ \mathrm{t}_6;\ \mathrm{t}_{7,2m-1};\ \mathrm{t}_{7,2m};\ \mathrm{t}_{8,2m-2};\ \mathrm{t}_{8,2m-1};\ \ldots;$$
$$\mathrm{t}_{m+2,m+4};\ \mathrm{t}_{m+2,m+5};\ \mathbf{t}^{+}_{\mathbf{m+3}};\ \mathrm{t}_{m+3,m+4})$$
$$= (\mathrm{e}_{1,2m} + \mathrm{e}_{2m-3,2m};\ \mathrm{e}_{1,2m} + \mathrm{e}_{2m-2,2m};\ \mathrm{e}_{1,2m,2m-1};\ \mathrm{e}_{1,2m-1};$$
$$\mathrm{e}_{2,2m} + \mathrm{e}_{2m-4,2m};\ \mathrm{e}_{2,2m} + \mathrm{e}_{2m-3,2m};$$
$$\mathrm{e}_{3,2m} + \mathrm{e}_{2m-5,2m};\ \mathrm{e}_{3,2m} + \mathrm{e}_{2m-4,2m};\ \mathrm{e}_{4,2m} + \mathrm{e}_{2m-6,2m};\ \mathrm{e}_{4,2m} + \mathrm{e}_{2m-5,2m};\ \ldots$$
$$\ldots;\ \mathrm{e}_{m-2,2m} + \mathrm{e}_{m,2m};\ \mathrm{e}_{m-2,2m} + \mathrm{e}_{m+1,2m};\ \mathrm{e}_{m-1,2m};\ \mathrm{e}_{m-1,2m} + \mathrm{e}_{m,2m})$$
$$\geq (\mathrm{e}_{1,2m,2m-3};\ \mathrm{e}_{1,2m,2m-2};\ \mathrm{e}_{1,2m,2m-1};\ \mathrm{e}_{1,2m-1};\ \mathrm{e}_{2,2m,2m-4};\ \mathrm{e}_{2,2m,2m-3};$$
$$\mathrm{e}_{3,2m,2m-5};\ \mathrm{e}_{3,2m,2m-4};\ \mathrm{e}_{4,2m,2m-6};\ \mathrm{e}_{4,2m,2m-5};\ \ldots;$$
$$\mathrm{e}_{m-2,2m,m};\ \mathrm{e}_{m-2,2m,m+1};\ \mathrm{e}_{m-1,2m};\ \mathrm{e}_{m-1,2m,m}).$$
□

Observe that $D_{m-1} = M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4}$.

LEMMA 40. $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4} \geq (\mathrm{s}_1;\ \mathrm{s}_5;\ \mathrm{s}_7;\ \ldots;\ \mathrm{s}_{2m-3};\ \mathrm{s}_{2m-1,2m};\ \mathrm{s}_{2m};\ \mathrm{s}_{2m-2};\ \mathrm{s}_{2m-4};\ \ldots;$ $\mathrm{s}_8;\ \mathrm{s}_6;\ \mathbf{s}^{-}_{\mathbf{2}};\ \mathrm{s}_3;\ \mathrm{s}_4)$.

Proof. We begin by calculating an inequality concerning D_1. We use Lemma 13(2)–(3) and Corollary 14 (observe that $|\pi_{4m-3}|^{-1}(2m) = 2m-1$, $|\pi_{4m-2}|^{-1}(2m) = |\pi_{4m-1}|^{-1}(2m) = 2m-3$ and $\sigma(\pi_j(|\pi_j|^{-1}(2m))) = +1$ for $j = 4m-2, 4m-1$) to find

$$S \cdot M_a(\pi_{4m-3}) = (\mathrm{s}_1;\ \mathrm{s}_2;\ \mathrm{s}_3;\ \ldots;\ \mathbf{s}^{+}_{\mathbf{2m-3}};\ \mathrm{s}_{2m-2};\ \mathrm{s}_{2m-1,2m};\ \mathrm{s}_{2m}),$$
$$S \cdot M_a(\pi_{4m-3}) \cdot M_b(\delta_{1,1}) \geq (\mathrm{s}_1;\ \mathrm{s}_2;\ \mathrm{s}_3;\ \ldots;\ \mathbf{s}^{+}_{\mathbf{2m-3}};\ \mathrm{s}_{2m};\ \mathrm{s}_{2m-2};\ \mathrm{s}_{2m-1,2m}),$$
$$D_1 = S \cdot M_a(\pi_{4m-3}) \cdot M_b(\delta_{1,1}) \cdot M_b(\delta_{1,2})$$
$$\geq (\mathrm{s}_1;\ \mathrm{s}_2;\ \mathrm{s}_3;\ \ldots;\ \mathbf{s}^{+}_{\mathbf{2m-3}};\ \mathrm{s}_{2m-3} + \mathrm{s}_{2m-1,2m};\ \mathrm{s}_{2m};\ \mathrm{s}_{2m-2})$$
$$\geq (\mathrm{s}_1;\ \mathrm{s}_2;\ \mathrm{s}_3;\ \ldots;\ \mathbf{s}^{+}_{\mathbf{2m-3}};\ \mathrm{s}_{2m-1,2m};\ \mathrm{s}_{2m};\ \mathrm{s}_{2m-2}) =: E_1.$$

Now, for $2 \leq j \leq m-3$, and obviously $m > 4$, we will prove the following inequality by recurrence:

$$D_j \geq (\mathrm{s}_1;\ \mathrm{s}_2;\ \mathrm{s}_3;\ \mathrm{s}_4;\ \mathrm{s}_5;\ \ldots;\ \mathbf{s}^{+}_{\mathbf{2m-2j-1}};\ \mathrm{s}_{2m-2j+1};\ \mathrm{s}_{2m-2j+3};\ \mathrm{s}_{2m-2j+5};\ \ldots; \tag{32}$$
$$\mathrm{s}_{2m-3};\ \mathrm{s}_{2m-1,2m};\ \mathrm{s}_{2m};\ \mathrm{s}_{2m-2};\ \mathrm{s}_{2m-4};\ \ldots;\ \mathrm{s}_{2m-2j}) =: E_j.$$

Since $D_1 \geq E_1$, Corollary 14(4) yields $D_2 \geq E_1 \cdot C_2$. Now, by Corollary 14(1,4),

$$E_1 \cdot C_2 = E_1 \cdot M_a(\delta_{1,3}) \cdot M_a(\delta_{2,1}) \cdot M_a(\delta_{2,2}) \cdot M_b(\delta_{2,3}) \cdot M_b(\delta_{2,4}) \cdot M_b(\delta_{2,5}) \cdot M_b(\delta_{2,6})$$
$$\geq E_1 \cdot M_b(\delta_{2,3}) \cdot M_b(\delta_{2,4}) \cdot M_b(\delta_{2,5}) \cdot M_b(\delta_{2,6}).$$

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Observe that, by Lemma 30($\mathbf{C}_j$)†, $|\delta_{2,j}|^{-1}(2m) = 2m-5$ and $\sigma(\delta_{2,j}(|\delta_{2,j}|^{-1}(2m))) = +1$ for $j \in \{3, 4, 5, 6\}$. Apply Corollary 14(4,2) repeatedly to obtain

$$\begin{aligned} D_2 &\geq E_1 \cdot M_b(\delta_{2,3}) \cdot M_b(\delta_{2,4}) \cdot M_b(\delta_{2,5}) \cdot M_b(\delta_{2,6}) \\ &= (\mathrm{s}_1;\ \mathrm{s}_2;\ \ldots;\ \mathbf{s}^+_{\mathbf{2m-5}};\ \mathrm{s}_{2m-4};\ \mathrm{s}_{2m-3};\ \mathrm{s}_{2m-1,2m};\ \mathrm{s}_{2m};\ \mathrm{s}_{2m-2}) \\ &\quad \cdot M_b(\delta_{2,3}) M_b(\delta_{2,4}) M_b(\delta_{2,5}) M_b(\delta_{2,6}) \\ &\geq (\mathrm{s}_1;\ \mathrm{s}_2;\ \ldots;\ \mathbf{s}^+_{\mathbf{2m-5}};\ \mathrm{s}_{2m-3};\ \mathrm{s}_{2m-1,2m};\ \mathrm{s}_{2m};\ \mathrm{s}_{2m-2};\ \mathrm{s}_{2m-4}), \end{aligned}$$

thus inequality (32) holds for $j = 2$ if $m > 4$. Assuming (32) holds for some $2 \leq p < m-3$, we will prove it holds for $p+1$. By the hypothesis and Corollary 14(1,4), we have

$$\begin{aligned} D_{p+1} = D_p \cdot C_{p+1} \geq E_p \cdot C_{p+1} &= E_p \cdot M_a(\delta_{p,4p-1}) \cdot \prod_{l=1}^{4p+2} M_{l+1}^{p+1}(\delta_{p+1,l}) \\ &= E_p \cdot M_a(\delta_{p,4p-1}) \cdot \prod_{l=1}^{2p} M_a(\delta_{p+1,l}) \cdot \prod_{l=2p+1}^{4p+2} M_b(\delta_{p+1,l}) \\ &\geq E_p \cdot \prod_{l=2p+1}^{4p+2} M_b(\delta_{p+1,l}). \end{aligned}$$

Lemma 30($\mathbf{C}_j$) guarantees $|\delta_{p+1,l}|^{-1}(2m) = 2m-2p-3$ for any $2p+1 \leq l \leq 4p+2$ (even more, $\sigma(\delta_{p+1,l}(|\delta_{p+1,l}|^{-1}(2m))) = +1$), which, together with Corollary 14(2) and the induction hypothesis, implies

$$\begin{aligned} &E_p \cdot \prod_{l=2p+1}^{4p+2} M_b(\delta_{p+1,l}) \\ &= (\mathrm{s}_1;\ \mathrm{s}_2;\ \mathrm{s}_3;\ \mathrm{s}_4;\ \mathrm{s}_5;\ \ldots;\ \mathbf{s}^+_{\mathbf{2m-2p-3}};\ \mathrm{s}_{2m-2p-2};\ \mathrm{s}_{2m-2p-1};\ \mathrm{s}_{2m-2p+1}; \\ &\quad \mathrm{s}_{2m-2p+3};\ \mathrm{s}_{2m-2p+5};\ \ldots;\ \mathrm{s}_{2m-3};\ \mathrm{s}_{2m-1,2m};\ \mathrm{s}_{2m};\ \mathrm{s}_{2m-2};\ \mathrm{s}_{2m-4};\ \ldots;\ \mathrm{s}_{2m-2p}) \\ &\quad \cdot \prod_{l=2p+1}^{4p+2} M_b(\delta_{p+1,l}) \\ &\geq (\mathrm{s}_1;\ \mathrm{s}_2;\ \mathrm{s}_3;\ \mathrm{s}_4;\ \mathrm{s}_5;\ \ldots;\ \mathbf{s}^+_{\mathbf{2m-2p-3}};\ \mathrm{s}_{2m-2};\ \mathrm{s}_{2m-4};\ \ldots;\ \mathrm{s}_{2m-2p-2}; \\ &\quad \mathrm{s}_{2m-2p-1};\ \mathrm{s}_{2m-2p+1};\ \mathrm{s}_{2m-2p+3};\ \mathrm{s}_{2m-2p+5};\ \ldots;\ \mathrm{s}_{2m-3};\ \mathrm{s}_{2m-1,2m};\ \mathrm{s}_{2m}) \\ &\quad \cdot \prod_{l=2p+p+1}^{4p+2} M_b(\delta_{p+1,l}) \\ &\geq (\mathrm{s}_1;\ \mathrm{s}_2;\ \mathrm{s}_3;\ \mathrm{s}_4;\ \mathrm{s}_5;\ \ldots;\ \mathbf{s}^+_{\mathbf{2m-2p-3}};\ \mathrm{s}_{2m-2p+1};\ \mathrm{s}_{2m-2p+3};\ \ldots;\ \mathrm{s}_{2m-3}; \\ &\quad \mathrm{s}_{2m-1,2m};\ \mathrm{s}_{2m};\ \mathrm{s}_{2m-2};\ \mathrm{s}_{2m-4};\ \ldots;\ \mathrm{s}_{2m-2p-2};\ \mathrm{s}_{2m-2p-1}) \cdot M_b(\delta_{p+1,4p+2}) \\ &\geq (\mathrm{s}_1;\ \mathrm{s}_2;\ \mathrm{s}_3;\ \mathrm{s}_4;\ \mathrm{s}_5;\ \ldots;\ \mathbf{s}^+_{\mathbf{2m-2p-3}};\ \mathrm{s}_{2m-2p-1};\ \mathrm{s}_{2m-2p+1};\ \mathrm{s}_{2m-2p+3};\ \mathrm{s}_{2m-2p+5};\ \ldots \\ &\quad \ldots;\ \mathrm{s}_{2m-3};\ \mathrm{s}_{2m-1,2m};\ \mathrm{s}_{2m};\ \mathrm{s}_{2m-2};\ \mathrm{s}_{2m-4};\ \ldots;\ \mathrm{s}_{2m-2p};\ \mathrm{s}_{2m-2p-2}). \end{aligned}$$

Then

$$D_{p+1} \geq (\mathrm{s}_1;\ \mathrm{s}_2;\ \mathrm{s}_3;\ \mathrm{s}_4;\ \mathrm{s}_5;\ \ldots;\ \mathbf{s}^+_{\mathbf{2m-2p-3}};\ \mathrm{s}_{2m-2p-1};\ \mathrm{s}_{2m-2p+1};\ \mathrm{s}_{2m-2p+3};$$

† This only applies if $m > 4$.

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$$s_{2m-2p+5};\ \ldots;\ s_{2m-3};\ s_{2m-1,2m};\ s_{2m};\ s_{2m-2};\ s_{2m-4};\ \ldots;\ s_{2m-2p};\ s_{2m-2p-2}),$$

and inequality (32) is true. Therefore, for $m \geq 4$†,

$$D_{m-3} \geq E_{m-3} = (s_1;\ s_2;\ s_3;\ s_4;\ \mathbf{s}_\mathbf{5}^+;\ \underbrace{s_7;\ \ldots;\ s_{2m-5};\ s_{2m-3};}_{m-4 \text{ columns}}\ s_{2m-1,2m};$$
$$\underbrace{s_{2m};\ s_{2m-2};\ s_{2m-4};\ \ldots;\ s_8;\ s_6}_{m-2 \text{ columns}}).$$

We now look for an inequality involving D_{m-2}. Apply Corollary 14(1,4) to obtain

$$\begin{aligned}
D_{m-2} = D_{m-3} \cdot C_{m-2} &= D_{m-3} \cdot M_a(\delta_{m-3,4m-13}) \cdot \prod_{l=1}^{4m-10} M_{c_{l+1}^{m-2}}(\delta_{m-2,l}) \\
&= D_{m-3} \cdot M_a(\delta_{m-3,4m-13}) \cdot \prod_{l=1}^{2m-6} M_a(\delta_{m-2,l}) \cdot \prod_{l=2m-5}^{4m-10} M_b(\delta_{m-2,l}) \\
&\geq E_{m-3} \cdot M_a(\delta_{m-3,4m-13}) \cdot \prod_{l=1}^{2m-6} M_a(\delta_{m-2,l}) \cdot \prod_{l=2m-5}^{4m-10} M_b(\delta_{m-2,l}) \\
&\geq E_{m-3} \cdot \prod_{l=2m-5}^{4m-10} M_b(\delta_{m-2,l}).
\end{aligned}$$

By Lemma 30($\mathbf{C}_{2m-2}$) we know that $|\delta_{m-2,l}|^{-1}(2m) = 1$ for any $2m-5 \leq l \leq 4m-10$, having positive signature $\sigma(\delta_{m-2,l}(|\delta_{m-2,l}|^{-1}(2m))) = +1$. Use Corollary 14(2,4) to get

$$\begin{aligned}
D_{m-2} \geq{}& (\mathbf{s}_\mathbf{1}^+;\ s_2;\ s_3;\ s_4;\ s_5;\ s_7;\ \ldots;\ s_{2m-3};\ s_{2m-1,2m}; \\
& s_{2m};\ s_{2m-2};\ s_{2m-4};\ \ldots;\ s_8;\ s_6) \cdot \prod_{l=2m-5}^{4m-10} M_b(\delta_{m-2,l}) \\
\geq{}& (\mathbf{s}_\mathbf{1}^+;\ s_{2m};\ s_{2m-2};\ \ldots;\ s_8;\ s_6;\ s_2;\ s_3;\ s_4; \\
& s_5;\ s_7;\ \ldots;\ s_{2m-3};\ s_{2m-1,2m}) \cdot \prod_{l=2m-5+(m-2)}^{4m-10} M_b(\delta_{m-2,l}) \\
\geq{}& (\mathbf{s}_\mathbf{1}^+;\ s_5;\ s_7;\ \ldots;\ s_{2m-3};\ s_{2m-1,2m};\ s_{2m};\ s_{2m-2};\ s_{2m-4};\ \ldots;\ s_8;\ s_6;\ s_2;\ s_3;\ s_4).
\end{aligned}$$

Finally, by Corollary 14(1), we have

$$\begin{aligned}
D_{m-1} ={}& D_{m-2} \cdot M_a(\delta_{m-2,4m-9}) \cdot \prod_{l=1}^{2m-4} M_a(\varepsilon_l) \geq D_{m-2} \\
\geq{}& (s_1;\ s_5;\ s_7;\ \ldots;\ s_{2m-3};\ s_{2m-1,2m};\ s_{2m};\ s_{2m-2};\ s_{2m-4};\ \ldots;\ s_8;\ s_6;\ \mathbf{s}_\mathbf{2}^-;\ s_3;\ s_4).
\end{aligned}$$

Observe that we have marked the position $2m-2$ as $\mathbf{s}_\mathbf{2}^-$ according to Lemma 30($\mathbf{A}_{2m-3}$). □

† The block of $m-4$ columns disappears when $m = 4$.

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4.2.5. *The matrix* $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4*\mathbf{v}^5}$. By the definition of $\mathbf{v}^5$ and Corollary 14,

$$M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4*\mathbf{v}^5} \geq R \cdot M_b(\pi_r) \cdot M_a(\pi_{r+1}) \cdot M_b(\pi_{r+2}), \quad \text{with } r = 4m-3+(2m-3)(m-1),$$

where R is the $n \times n$ matrix

$$\begin{aligned} R &= (r_1; r_2; \dots; r_{2m-3}; \mathbf{r}^-_{\mathbf{2m-2}}; r_{2m-1}; r_{2m}) \\ &:= (s_1; s_5; s_7; \dots; s_{2m-3}; s_{2m-1,2m}; s_{2m}; s_{2m-2}; s_{2m-4}; \dots; s_8; s_6; \mathbf{s}^-_{\mathbf{2}}; s_3; s_4). \end{aligned} \tag{33}$$

Remark 41. Similarly to Remarks 37 and 39, we can describe and compare the matrix R in terms of the identity matrix as follows†:

$$\begin{aligned} R &= (s_1; s_5; \underbrace{s_7; \dots; s_{2m-3}}_{m-4 \text{ columns}}; s_{2m-1,2m}; s_{2m}; s_{2m-2}; s_{2m-4}; \dots; s_8; s_6; s_2; s_3; s_4) \\ &= (\underbrace{e_{1,2m}+e_{2m-3,2m}}_{s_1}; \underbrace{e_{2,2m}+e_{2m-4,2m}}_{s_5}; \underbrace{e_{3,2m}+e_{2m-5,2m}}_{s_7}; \dots; \underbrace{e_{m-2,2m}+e_{m,2m}}_{s_{2m-3}}; \\ &\quad \underbrace{e_{m-1,2m}+e_{m-1,2m}+e_{m,2m}}_{s_{2m-1,2m}}; \\ &\quad \underbrace{e_{m-1,2m}+e_{m,2m}}_{s_{2m}}; \underbrace{e_{m-2,2m}+e_{m+1,2m}}_{s_{2m-2}}; \underbrace{e_{m-3,2m}+e_{m+2,2m}}_{s_{2m-4}}; \dots \\ &\quad \dots; \underbrace{e_{3,2m}+e_{2m-4,2m}}_{s_8}; \underbrace{e_{2,2m}+e_{2m-3,2m}}_{s_6}; \underbrace{e_{1,2m}+e_{2m-2,2m}}_{s_2}; \underbrace{e_{1,2m,2m-1}}_{s_3}; \underbrace{e_{1,2m-1}}_{s_4}) \end{aligned} \tag{34}$$

$$\begin{aligned} &\geq (e_{1,2m,2m-3}; e_{2,2m,2m-4}; \underbrace{e_{3,2m,2m-5}; \dots; \underbrace{e_{j,2m,2m-j-2}}_{j\text{th column}}; \dots; e_{m-2,2m,m}}_{m-4 \text{ columns}}; \\ &\quad e_{m-1,2m,m}; e_{m-1,2m,m}; e_{m-2,2m,m+1}; \\ &\quad e_{m-3,2m,m+2}; \dots; \underbrace{e_{m-l,2m,m+l-1}}_{(m+l-1)\text{th column}}; \dots; e_{3,2m,2m-4}; e_{2,2m,2m-3}; e_{1,2m,2m-2}; \\ &\quad e_{1,2m,2m-1}; e_{1,2m-1}), \end{aligned} \tag{35}$$

for $1 \leq j \leq m-2$ and $1 \leq l \leq m-1$. □

LEMMA 42.

$$M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4*\mathbf{v}^5} \geq (r_1; r_2; r_3; \dots; r_{2m-4}; r_{2m-3}; r_{2m-2,2m}; r_{2m-2,2m,2m-1}; r_{2m-2,2m-1}).$$

Proof. By Corollary 14 and Lemmas 13, 30 and 31, we have (take into account that $\sigma(\pi_r(|\pi_r|^{-1}(2m))) = -1$, $\sigma(\pi_{r+1}(|\pi_{r+1}|^{-1}(2m))) = -1$, $\sigma(\pi_{r+2}(|\pi_{r+2}|^{-1}(2m))) = +1$, and $|\pi_r|^{-1}(2m) = 2m-2$, $|\pi_{r+1}|^{-1}(2m) = 2m-1$, $|\pi_{r+2}|^{-1}(2m) = 2m-2$)

$$M^{\mathcal{G}}_{\pi_0,v^1*v^2*v^3*v^4*v^5} \geq R \cdot M_b(\pi_r) \cdot M_a(\pi_{r+1}) \cdot M_b(\pi_{r+2})$$

† Note that the blocks of $m-4$ columns disappears when $m = 4$.

$$= (\mathrm{r}_1;\, \mathrm{r}_2;\, \mathrm{r}_3;\, \ldots;\, \mathrm{r}_{2m-4};\, \mathrm{r}_{2m-3};\, \mathbf{r}^{-}_{\mathbf{2m-2}};\, \mathrm{r}_{2m-1};\, \mathrm{r}_{2m}) \cdot M_b(\pi_r) \cdot M_a(\pi_{r+1}) \cdot M_b(\pi_{r+2})$$
$$= (\mathrm{r}_1;\, \mathrm{r}_2;\, \mathrm{r}_3;\, \ldots;\, \mathrm{r}_{2m-4};\, \mathrm{r}_{2m-3};\, \mathrm{r}_{2m-2,2m};\, \mathbf{r}^{-}_{\mathbf{2m-2}};\, \mathrm{r}_{2m-1}) \cdot M_a(\pi_{r+1}) \cdot M_b(\pi_{r+2})$$
$$= (\mathrm{r}_1;\, \mathrm{r}_2;\, \mathrm{r}_3;\, \ldots;\, \mathrm{r}_{2m-4};\, \mathrm{r}_{2m-3};\, \mathbf{r}^{+}_{\mathbf{2m-2,2m}};\, \mathrm{r}_{2m-2,2m-1};\, \mathrm{r}_{2m-1}) \cdot M_b(\pi_{r+2})$$
$$= (\mathrm{r}_1;\, \mathrm{r}_2;\, \mathrm{r}_3;\, \ldots;\, \mathrm{r}_{2m-4};\, \mathrm{r}_{2m-3};\, \mathrm{r}_{2m-2,2m};\, \mathrm{r}_{2m-2,2m,2m-1};\, \mathrm{r}_{2m-2,2m-1}). \qquad \square$$

Finally, we need to write $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4*\mathbf{v}^5}$ in terms of the initial identity matrix $(\mathrm{e}_1;\, \mathrm{e}_2;\, \mathrm{e}_3;\, \ldots;\, \mathrm{e}_{2m})$.

PROPOSITION 43.

$$\begin{aligned} M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4*\mathbf{v}^5} &\geq (\mathrm{r}_1;\, \mathrm{r}_2;\, \mathrm{r}_3;\, \ldots;\, \mathrm{r}_{2m-4};\, \mathrm{r}_{2m-3};\, \mathrm{r}_{2m-2,2m};\, \mathrm{r}_{2m-2,2m,2m-1};\, \mathrm{r}_{2m-2,2m-1}) \\ &\geq (\mathrm{e}_{1,2m-3,2m};\, \mathrm{e}_{2,2m-4,2m};\, \mathrm{e}_{3,2m-5,2m};\, \mathrm{e}_{4,2m-6,2m}; \\ &\qquad \ldots;\, \mathrm{e}_{m-2,m,2m};\, \mathrm{e}_{m-1,m,2m};\, \mathrm{e}_{m,m-1,2m};\, \mathrm{e}_{m+1,m-2,2m}; \\ &\qquad \ldots;\, \mathrm{e}_{2m-5,4,2m};\, \mathrm{e}_{2m-4,3,2m};\, \mathrm{e}_{2m-3,2,2m}; \\ &\qquad \mathrm{e}_{2m-2,1,2m-1,2m};\, \mathrm{e}_{2m-1,1,2m-2,2m};\, \mathrm{e}_{2m,1,2m-2,2m-1}). \end{aligned}$$

Proof. By Lemma 42, we have

$$M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4*\mathbf{v}^5} \geq (\mathrm{r}_1;\, \mathrm{r}_2;\, \mathrm{r}_3;\, \ldots;\, \mathrm{r}_{2m-4};\, \mathrm{r}_{2m-3};\, \mathrm{r}_{2m-2,2m};\, \mathrm{r}_{2m-2,2m,2m-1};\, \mathrm{r}_{2m-2,2m-1}).$$

Moreover, by (35):

$$\begin{aligned} R \geq (&\mathrm{e}_{1,2m,2m-3};\, \mathrm{e}_{2,2m,2m-4};\, \mathrm{e}_{3,2m,2m-5};\, \mathrm{e}_{4,2m,2m-6,2m};\, \ldots \\ &\ldots;\, \mathrm{e}_{m-2,2m,m};\, \mathrm{e}_{m-1,2m,m};\, \mathrm{e}_{m-1,2m,m};\, \mathrm{e}_{m-2,2m,m+1};\, \mathrm{e}_{m-3,2m,m+2};\, \ldots \\ &\ldots;\, \mathrm{e}_{3,2m,2m-4};\, \mathrm{e}_{2,2m,2m-3};\, \mathrm{e}_{1,2m,2m-2};\, \mathrm{e}_{1,2m,2m-1};\, \mathrm{e}_{1,2m-1}). \end{aligned}$$

Finally, the result follows since the columns $\mathrm{r}_{2m-2,2m}$, $\mathrm{r}_{2m-2,2m,2m-1}$ and $\mathrm{r}_{2m-2,2m-1}$ are greater than $\mathrm{e}_{1,2m-2,2m-1,2m}$. $\square$

4.2.6. *Positive character of* $M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4*\mathbf{v}^5}$.

PROPOSITION 44. *There exists a power of* $A := M^{\mathcal{G}}_{\pi_0,\mathbf{v}^1*\mathbf{v}^2*\mathbf{v}^3*\mathbf{v}^4*\mathbf{v}^5}$ *which is positive.*

Proof. Observe first that, by Proposition 43, the diagonal of A is positive and then, by Lemma 13(5),

$$A^{k+1} \geq A^k, \quad \text{for any } k \geq 1. \tag{36}$$

Use Proposition 43 to realize that the $2m$th row of A is positive and $a_{1,2m} > 0$, $a_{2m-2,2m} > 0$, $a_{2m-1,2m} > 0$. Then we deduce that rows 1, $2m-2$, $2m-1$ and $2m$ of A^2 are positive.

We claim that if $1 \leq k \leq m-2$ and we assume that the kth row of some power A^j, $j \in \mathbb{N}$, is positive then A^{j+1} has positive row $2m-2-k$ since $a_{2m-2-k,k} > 0$ by Proposition 43. Indeed, take into account that if $A^{j+1} = (\alpha_{s,t})$, $A^j = (\beta_{s,t})$ and $A = (a_{s,t})$ with $1 \leq s,\, t \leq 2m$ then

$$\alpha_{2m-2-k,r} = \sum_{l=1}^{2m} a_{2m-2-k,l}\beta_{l,r} \geq a_{2m-2-k,k}\beta_{k,r} > 0, \quad r = 1, 2, \ldots, 2m.$$

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If, in exchange, $m-1 \le k \le 2m-2$ and the kth row of some power A^j is positive then row $2m-1-k$ of A^{j+1} is positive because Proposition 43 provides $a_{2m-1-k,k} > 0$ (the reasoning is similar to the previous one $1 \le k \le m-2$).

Now let $S^+(n) := \{k : A^n \text{ has } k\text{th row positive}\}$. By the first observation in this proof, $S^+(n) \subseteq S^+(n+1)$ and $\{2m-2, 2m-1, 2m, 1\} \subseteq S^+(2)$. The claim guarantees consecutively $\{2m-2, 2m-1, 2m, 1, 2m-3\} \subseteq S^+(3)$, $\{2m-2, 2m-1, 2m, 1, 2m-3, 2\} \subseteq S^+(4)$, $\{2m-2, 2m-1, 2m, 1, 2m-3, 2, 2m-4\} \subseteq S^+(5)$, $\{2m-2, 2m-1, 2m, 1, 2m-3, 2, 2m-4, 3\} \subseteq S^+(6)$, and recursively we will obtain

$$\begin{aligned}&\{2m-2, 2m-1, 2m, 1, 2m-3, 2, 2m-4, 3, 2m-5, \ldots,\\ &\quad m-3, m+1, m-2, m\} \subseteq S^+(2m-3),\\ &\{2m-2, 2m-1, 2m, 1, 2m-3, 2, 2m-4, 3, 2m-5, \ldots,\\ &\quad m-3, m+1, m-2, m, m-1\} \subseteq S^+(2m-2).\end{aligned}$$

Therefore A^{2m-2} is positive. □

4.3. *Proof of main theorem for* $n = 2m \ge 8$. Let $\mathcal{G}^{\pi_0,\mathbf{v}}$ be the Rauzy subgraph associated to π_0, defined by (13), and $\mathbf{v}$, introduced in (19). Propositions 32 and 44 guarantee that $\mathcal{G}^{\pi_0,\mathbf{v}}$ satisfies the hypothesis of Theorem 25, and then we obtain a minimal, uniquely ergodic, self-induced $(2m, 2m-2)$-IET, $T_0 = (\lambda^0, \pi_0)$, whose associated graph is $\mathcal{G}^{\pi_0,\mathbf{v}}$. Moreover, $R^j(T_0) = (\lambda^j, \pi_j)$ is minimal, uniquely ergodic and self-induced by Theorem 25(3), and Proposition 32(2) guarantees that in the set $\{R^j(T_0)\}_{j=1}^{2m-2}$ we can find $(2m, k)$-IETs, $2 \le k \le 2m-1$, which are proper since they satisfy the condition in Remark 2.

We now show the existence of minimal, proper, uniquely ergodic $(2m, 1)$-IETs. We apply Corollary 19 to the subgraph given in Lemma 34 and we obtain a $(2m, 1)$-IET, $U = (\gamma, \alpha_1)$, such that $R^2(U) = T_0$. Now Proposition 21 and Theorem 23(3) imply the unique ergodicity of U. Theorem 23(2) implies the minimality of U. Moreover, it is easy to check that α_1 satisfies the condition in Remark 2 and then U is also proper.

The existence of minimal, proper, uniquely ergodic $(2m, 2m)$-IETs can be proved by repeating the argument in the previous paragraph for the subgraph in Lemma 33.

5. *Proof of main theorem for* $n = 2m+1 \ge 9$

The procedure is similar to the even case. We will omit the proofs of the results since they follow a similar scheme used for $n = 2m \ge 8$. We only present the required Rauzy graph and some of its properties.

5.1. *The periodic Rauzy graph.* We construct a periodic Rauzy graph of period $p = 4m + 2 + (2m-1)(m-1)$.

Let us take

$$\widetilde{\pi}_0 = (-3, -4, -5, \ldots, -2m, 2m+1, 1, -2), \tag{37}$$

$$\mathbf{u}^1 = (a, a, a, \ldots, a) \in \{a, b\}^{2m-2}, \tag{38}$$

$$\mathbf{u}^2 = (b, a, b, b, a, b) \in \{a, b\}^6, \tag{39}$$

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$$\mathbf{u}^3 = (a, b, a, b, a, b \ldots, a, b, a, b) \in \{a, b\}^{2m-4}, \tag{40}$$

$$\mathbf{u}^4 = (\underbrace{a, a}_{2}, \underbrace{b, b, b}_{3}, \underbrace{a, a, a, a}_{4}, \ldots, \underbrace{b, \ldots, b}_{2m-3}, \underbrace{a, \ldots, a}_{2m-2}) \in \{a, b\}^{m(2m-3)} \tag{41}$$

$$\mathbf{u}^5 = (b, a, b) \in \{a, b\}^3, \tag{42}$$

$$\mathbf{u} = \mathbf{u}^1 * \mathbf{u}^2 * \mathbf{u}^3 * \mathbf{u}^4 * \mathbf{u}^5 \in \{a, b\}^p, \quad p = 4m + 3 + m(2m - 3) = 2m^2 + m + 3. \tag{43}$$

Moreover, we consider the vector $\mathbf{w} \in \{a, b\}^{\mathbb{N}}$ of infinite length given by $\mathbf{w}_{i+kp} = \mathbf{u}_i$ for any $1 \leq i \leq p$ and $k \in \mathbb{N}$.

Reasoning as in the even case it can be proved, after a long procedure that does not involve significant novelties, the following result.

PROPOSITION 45.
(1) *The Rauzy subgraph $\mathcal{G}^{\widetilde{\pi}_0,\mathbf{u}}$ is periodic.*
(2) *There exist in $\mathcal{G}^{\widetilde{\pi}_0,\mathbf{u}}$ permutations with f flips, $2 \leq f \leq 2m$. Moreover, for any $2 \leq f \leq 2m$ there exists a vertex in $\mathcal{G}^{\widetilde{\pi}_0,\mathbf{u}}$ which satisfies the conditions in Remarks 2 and 3.*
(3) *$A = M^{\mathcal{G}}_{\widetilde{\pi}_0,\mathbf{u}^1*\mathbf{u}^2*\mathbf{u}^3*\mathbf{u}^4*\mathbf{u}^5}$ has a power which is positive.*

In the Rauzy subgraph $\mathcal{G}^{\widetilde{\pi}_0,\mathbf{u}}$ there are no permutations with $2m + 1$ flips nor with 1 flip. The next two results give such permutations and show that they belong to the same Rauzy class as $\mathcal{G}^{\widetilde{\pi}_0,\mathbf{u}}$. Both results follows immediately by applying the definitions of the operators a and b.

LEMMA 46. *Let $\widetilde{\tau}_1, \widetilde{\tau}_2 \in S^{\sigma}_{2m+1}$ be the signed permutations*

$$\widetilde{\tau}_1 = (-2, -3, -4, \ldots, -[2m - 2], -[2m - 1], -[2m + 1], -1, -2m),$$
$$\widetilde{\tau}_2 = (-2, -3, -4, \ldots, -[2m - 2], -[2m - 1], 2m, -[2m + 1], -1).$$

Then we have the Rauzy subgraph

$$\widetilde{\tau}_1 \xrightarrow{b} \widetilde{\tau}_2 \xrightarrow{a} \widetilde{\pi}_0 = (-3, -4, -5, \ldots, -[2m - 1], -[2m], 2m + 1, 1, -2).$$

LEMMA 47. *Let $\widetilde{\alpha}_1, \widetilde{\alpha}_2 \in S^{\sigma}_{2m+1}$ be the signed permutations*

$$\widetilde{\alpha}_1 = (2m, 1, 2m + 1, 2m - 1, 2m - 2, \ldots, 6, 5, 4, 3, -2),$$
$$\widetilde{\alpha}_2 = (2m + 1, 1, -2, 2m, 2m - 1, \ldots, 7, 6, 5, 4, -3).$$

Then we have the Rauzy subgraph

$$\widetilde{\alpha}_1 \xrightarrow{a} \widetilde{\alpha}_2 \xrightarrow{b} \widetilde{\pi}_{2m+2} = (2m + 1, -3, 1, -2, 2m, 2m - 1, \ldots, 7, 6, 5, 4).$$

5.2. *Proof of main theorem for $n = 2m + 1 \geq 9$.* The proof follows the reasoning in §4.3 for the even case, taking into account the Rauzy subgraph $\mathcal{G}^{\widetilde{\pi}_0,\mathbf{u}}$ introduced in §5.1. Then we repeat the arguments using, in this case, Proposition 45 and Lemmas 46 and 47.

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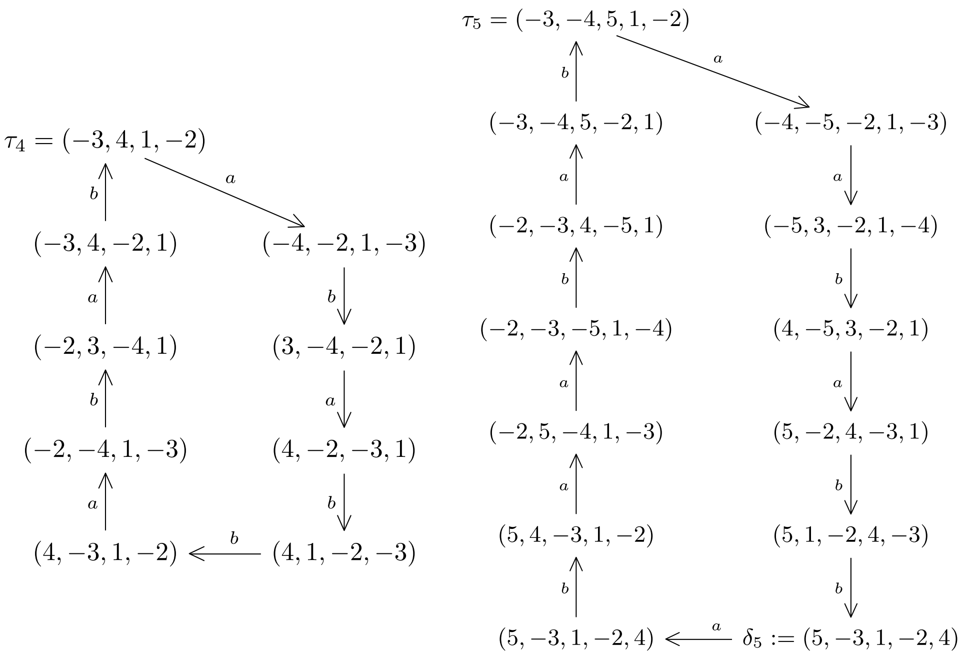

FIGURE 4. Complete Rauzy graphs presented in $\mathcal{G}_n$, $n = 4, 5$.

6. *The cases* $n = 4, 5, 6, 7$

In order to complete the proof of the main theorem we must present periodic Rauzy graphs in $\mathcal{G}_n$, $n = 4, 5, 6, 7$, because these have not been constructed in the general case. We will call them $\mathcal{H}_4$, $\mathcal{H}_5$, $\mathcal{H}_6$ and $\mathcal{H}_7$, respectively.

A periodic Rauzy graph in $\mathcal{G}_4$ is generated from the permutation $\tau_4 = (-3, 4, 1, -2)$ by applying the vector of operators $\tilde{\mathbf{v}}^4 = (\tilde{v}_1, \ldots, \tilde{v}_9) = (a, b, a, b, b, a, b, a, b)$. This graph, introduced in Figure 4, is taken from [**12**, Theorem 6.1]. Its associated matrix is $M_4 = \prod_{j=1}^{9} M_{\tilde{v}_j}(\sigma_j)$, with $\sigma_1 = \tau_4$, $\sigma_j = \tilde{v}_{j-1}(\sigma_{j-1})$ for $2 \le j \le 9$; see Table 1.

In $\mathcal{G}_5$ we take $\tau_5 = (-3, -4, 5, 1, -2)$ and the vector of operators $\tilde{\mathbf{v}}^5 = (a, a, b, a, b, b, a, b, a, a, b, a, b)$. See the complete graph in Figure 4. The associated matrix is M_5, given in Table 1.

The permutation $\tau_6 = (-3, -4, -5, 6, 1, -2)$ jointly with $\tilde{\mathbf{v}}^6 = (a, a, a, b, a, b, b, a, b, a, b, b, a, a, a, b, a, b)$ generates a periodic Rauzy graph in $\mathcal{G}_6$. See Figure 5 and its associated matrix M_6 in Table 1.

A periodic Rauzy graph in $\mathcal{G}_7$ is the one associated to the permutation $\tau_7 = (-3, -4, -5, -6, 7, 1, -2)$ and $\tilde{\mathbf{v}}^7 = (a, a, a, a, b, a, b, b, a, b, a, b, a, a, b, b, b, a, a, a, a, b, a, b)$. See the complete graph in Figure 6. The associated matrix is M_7; see Table 1.

Observe that M_4^2, M_5^2, M_6^3 and M_7^4 are positive. Then Theorem 25 provides minimal, uniquely ergodic, self-induced and proper (n, k)-IETs for any (n, k) with $4 \le n \le 7$ and $2 \le k \le n - 1$.

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TABLE 1. Matrices of the periodic Rauzy graphs presented in $\mathcal{G}_n$, $n = 4, 5, 6, 7$.

$$M_4 = \begin{pmatrix} 2 & 2 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 3 \end{pmatrix} \qquad M_5 = \begin{pmatrix} 2 & 2 & 2 & 3 & 2 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 2 & 3 & 2 & 3 & 3 \end{pmatrix}$$

$$M_6 = \begin{pmatrix} 2 & 2 & 2 & 2 & 3 & 2 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 2 & 5 & 4 & 2 & 3 & 3 \end{pmatrix} \qquad M_7 = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 3 & 2 \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 2 & 6 & 7 & 4 & 2 & 3 & 3 \end{pmatrix}$$

We next construct proper (n, n)-IETs for any $4 \le n \le 7$. To this end we consider the Rauzy subgraphs

$$(-2, -4, -1, -3) \xrightarrow{b} (-2, 3, -4, -1) \xrightarrow{a} \tau_4 = (-3, 4, 1, -2)$$

$$(-2, -3, -5, -1, -4) \xrightarrow{b} (-2, -3, 4, -5, -1) \xrightarrow{a} \tau_5 = (-3, -4, 5, 1, -2)$$

$$(-2, -3, -4, -6, -1, -5) \xrightarrow{b} (-2, -3, -4, 5, -6, -1) \xrightarrow{a} \tau_6 = (-3, -4, -5, 6, 1, -2)$$

$$(-2, -3, -4, -5, -7, -1, -6) \xrightarrow{b} (-2, -3, -4, -5, 6, -7, -1) \xrightarrow{a} \tau_7 = (-3, -4, -5, -6, 7, 1, -2)$$

Then it suffices to apply Corollary 19 to the subgraphs and the corresponding IETs built in the previous paragraph. The minimality of the obtained IETs is guaranteed by Theorem 23(2), and the unique ergodicity by simultaneously applying Proposition 21 and Theorem 23(3).

We finally build minimal proper $(n, 1)$-IETs for $4 \le n \le 7$. Consider now the following Rauzy subgraphs (the elements δ_5, δ_6 and δ_7 belong to the periodic graphs $\mathcal{H}_5$, $\mathcal{H}_6$ and $\mathcal{H}_7$, respectively; see Figures 4–6):

$$(-4, 1, 3, 2) \xrightarrow{(b,b,a,b,a,b,b,a)} (4, -3, 1, -2) \xrightarrow{(a,b,a,b)} \tau_4 \text{ (this subgraph is taken from [12])}$$

$$(4, 1, 5, 3, -2) \xrightarrow{a} (5, 1, -2, 4, -3) \xrightarrow{b} \delta_5 = (5, -3, 1, -2, 4) \xrightarrow{a} \ldots \text{part of } \mathcal{H}_5 \ldots \xrightarrow{b} \tau_5$$

$$(5, 1, 6, 4, 3, -2) \xrightarrow{a} (6, 1, -2, 5, 4, -3) \xrightarrow{b} \delta_6 = (6, -3, 1, -2, 5, 4) \xrightarrow{a} \ldots \text{part of } \mathcal{H}_6 \ldots \xrightarrow{b} \tau_6$$

$$(6, 1, 7, 5, 4, 3, -2) \xrightarrow{a} (7, 1, -2, 6, 5, 4, -3) \xrightarrow{b} \delta_7 = (7, -3, 1, -2, 6, 5, 4) \xrightarrow{a} \ldots \text{part of } \mathcal{H}_7 \ldots \xrightarrow{b} \tau_7.$$

Then, repeating the previous reasoning for the case of (n, n)-IETs, we obtain the desired $(n, 1)$-IETs for $4 \le n \le 7$.

7. *Proof of Proposition A*

The 'only if part' is proved by Theorem 4: a minimal proper (n, k)-CET is automatically transitive. For the 'if part' we distinguish separately the cases of (n, k)-CETs with $n = 2m \ge 8$, $n = 2m + 1 \ge 9$, $n = 3$, $n = 4$, $n = 5$, $n = 6$ and $n = 7$.

Assume first $n = 2m \ge 8$. We consider now the proper, minimal and uniquely ergodic $(2m, k)$-IETs constructed in §4.3. In this case, observe that we obtained $\lambda \in \Lambda^n$ such

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$$
\begin{array}{ccccc}
\tau_6 = (-3,-4,-5,6,1,-2) & \xrightarrow{a} & (-4,-5,-6,-2,1,-3) & \xrightarrow{a} & (-5,-6,3,-2,1,-4) \\
 & & & & \downarrow a \\
(6,-2,5,4,-3,1) & \xleftarrow{a} & (5,-6,4,3,-2,1) & \xleftarrow{b} & (-6,4,3,-2,1,-5) \\
\downarrow b & & & & \\
(6,1,-2,5,4,-3) & \xrightarrow{b} & (6,-3,1,-2,5,4) =: \delta_6 & \xrightarrow{a} & (5,-3,1,-2,6,4) \\
 & & & & \downarrow b \\
(6,4,-3,1,-2,5) & \xleftarrow{b} & (6,-3,1,-2,5,4) & \xleftarrow{a} & (5,-3,1,-2,6,4) \\
\downarrow b & & & & \\
(6,5,4,-3,1,-2) & \xrightarrow{a} & (-2,6,5,-4,1,-3) & \xrightarrow{a} & (-2,-3,6,-5,1,-4) \\
 & & & & \downarrow a \\
(-3,-4,-5,6,-2,1) & \xleftarrow{a} & (-2,-3,-4,5,-6,1) & \xleftarrow{b} & (-2,-3,-4,-6,1,-5) \\
\downarrow b & & & & \\
(-3,-4,-5,6,1,-2) = \tau_6 & & & &
\end{array}
$$

FIGURE 5. Periodic Rauzy graph in $\mathcal{G}_6$.

that $T = (\lambda, \pi_0)$ is a minimal and uniquely ergodic $(2m, 2m-2)$-IET and in the set $\mathcal{E} = \{R^i(T)\} = \{(\lambda^i, \pi_i)\}_{i=1}^{2m-2}$ we found proper, minimal and uniquely ergodic $(2m, k)$-IETs, $2 \le k \le 2m-1$. Now it suffices to notice that each π_i, $1 \le i \le 2m-2$, satisfies the condition in Remark 3 by Proposition 32(2). Then, for any $1 \le i \le 2m-2$, after identifying the endpoints of the domain of $R^i(T)$, we obtain a proper, minimal and uniquely ergodic $(2m, k)$-CET $\widehat{R^i(T)}$ with the same flips as $R^i(T)$. This guarantees the proof of Proposition A for $(2m, k)$-CETs with $m \ge 4$ and $2 \le k \le 2m-1$. For $k = 1$ it is enough to note that the associated permutations of the $(n, 1)$- and (n, n)-IETs, constructed in §4.3, also satisfy the condition in Remark 3 (see Lemmas 34 and 33).

The case $n = 2m+1 \ge 9$ just requires the adaptation of the previous reasoning by exchanging Proposition 32 for Proposition 45, Lemma 34 for Lemma 46 and Lemma 33 for Lemma 47.

For $n = 3$ we note the existence of minimal, uniquely ergodic IETs T_1 and T_2 of type (4, 2) and (4, 4), with respective permutations $(4, 1, -2, -3)$ and $(-2, -4, -1, -3)$; see §6. Then $\widehat{T_1}$ and $\widehat{T_2}$ (for an IET T we use $\widehat{T}$ to denote the CET obtained from T after identifying the endpoints of the domain of T) are proper, minimal and uniquely ergodic (3, 2)- and (3, 3)-CETs, respectively.

Similarly, it is easy to check that the IETs with associated permutations $(-4, 1, 3, 2)$, $(3, -4, -2, 1)$, $(-4, -2, 1, -3)$ (again from §6) provide proper, minimal and uniquely ergodic (4, 1)-, (4, 2)- and (4, 3)-CETs, respectively (take into account Remark 3).

$$
\begin{array}{ccccc}
\tau_7 = (-3,-4,-5,-6,7,1,-2) & \xrightarrow{a} & (-4,-5,-6,-7,-2,1,-3) & \xrightarrow{a} & (-5,-6,-7,3,-2,1,-4) \\
 & & & & \downarrow{\scriptstyle a} \\
(6,-7,5,4,3,-2,1) & \xleftarrow{b} & (-7,5,4,3,-2,1,-6) & \xleftarrow{a} & (-6,-7,4,3,-2,1,-5) \\
\downarrow{\scriptstyle a} & & & & \\
(7,-2,6,5,4,-3,1) & \xrightarrow{b} & (7,1,-2,6,5,4,-3) & \xrightarrow{b} & \delta_7 := (7,-3,1,-2,6,5,4) \\
 & & & & \downarrow{\scriptstyle a} \\
(5,-3,1,-2,7,4,6) & \xleftarrow{a} & (5,-3,1,-2,7,4,6) & \xleftarrow{b} & (5,-3,1,-2,7,6,4) \\
\downarrow{\scriptstyle b} & & & & \\
(5,-3,1,-2,7,6,4) & \xrightarrow{a} & (6,-3,1,-2,5,7,4) & \xrightarrow{a} & (7,-3,1,-2,6,5,4) \\
 & & & & \downarrow{\scriptstyle b} \\
(7,6,5,4,-3,1,-2) & \xleftarrow{b} & (7,5,4,-3,1,-2,6) & \xleftarrow{b} & (7,4,-3,1,-2,6,5) \\
\downarrow{\scriptstyle a} & & & & \\
(-2,7,6,5,-4,1,-3) & \xrightarrow{a} & (-2,-3,7,6,-5,1,-4) & \xrightarrow{a} & (-2,-3,-4,7,-6,1,-5) \\
 & & & & \downarrow{\scriptstyle a} \\
(-3,-4,-5,-6,7,-2,1) & \xleftarrow{a} & (-2,-3,-4,-5,6,-7,1) & \xleftarrow{b} & (-2,-3,-4,-5,-7,1,-6) \\
\downarrow{\scriptstyle b} & & & & \\
(-3,-4,-5,-6,7,1,-2) = \tau_7 & & & &
\end{array}
$$

FIGURE 6. Periodic Rauzy graph in $\mathcal{G}_7$.

The IETs with associated permutations (4, 1, 5, 3, −2), (4, −5, 3, −2, 1), (−5, 3, −2, 1, −4), (−4, −5, −2, 1, −3) and (−2, −3, −5, −1, −4) provide in turn proper, minimal and uniquely ergodic (5, 1)-, (5, 2)-, (5, 3)-, (5, 4)- and (5, 5)-CETs (see §6 and apply Remark 3).

Following the same reasoning as in the previous cases, if we now consider the IETs with associated permutations (5, 1, 6, 4, 3, −2), (5, −6, 4, 3, −2, 1), (−6, 4, 3, −2, 1, −5), (−5, −6, 3, −2, 1, −4), (−4, −5, −6, −2, 1, −3) and (−2, −3, −4, −6, −1, −5), we obtain respectively proper, minimal and uniquely ergodic (6, 1)-, (6, 2)-, (6, 3)-, (6, 4)-, (6, 5)- and (6, 6)-CETs.

Analogously, if we take the IETs with associated permutations (6, 1, 7, 5, 4, 3, −2), (6, −7, 5, 4, 3, −2, 1), (−7, 5, 4, 3, −2, 1, −6), (−6, −7, 4, 3, −2, 1, −5), (−5, −6, −7, 3, −2, 1, −4), (−4, −5, −6, −7, −2, 1, −3) and (−2, −3, −4, −5, −7, −1, −6), then we obtain proper, minimal and uniquely ergodic (7, 1)-, (7, 2)-, (7, 3)-, (7, 4)-, (7, 5)-, (7, 6)- and (7, 7)-CETs.

It only remains to check the existence of a proper, minimal, uniquely ergodic (4, 4)-CET. The permutations considered in §6 do not generate (4, 4)-CETs. To overcome this difficulty we will show the existence of a minimal, uniquely ergodic, proper (5, 5)-IET with associated permutation $\pi := (-4, -1, -5, -2, -3)$. From here we will deduce the

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existence of a proper, minimal and uniquely ergodic (4, 4)-CET. Take into account the Rauzy graph

$$\begin{aligned}\pi = (-4, -1, -5, -2, -3) &\xrightarrow{a} (-5, -1, 3, -2, -4) \xrightarrow{b} (4, -5, -1, 3, -2)\\ &\xrightarrow{b} (4, 2, -5, -1, 3) =: \tau\end{aligned} \tag{44}$$

Also, it can be checked that $\tau = (4, 2, -5, -1, 3)$ jointly with the vector of operators $\mathbf{x} = (x_1, x_2, \ldots, x_{13}) := (b, a, b, a, a, b, b, a, b, a, b, b, a)$ generates a periodic Rauzy graph in $\mathcal{G}_5$ whose associated matrix has its second power positive. Indeed,

$$\begin{array}{ccccc}
\tau = (4, 2, -5, -1, 3) & \xrightarrow{b} & (4, 2, -3, -5, -1) & \xrightarrow{a} & (5, 3, -4, 1, -2)\\
 & & & & \Big\downarrow b\\
(3, -4, 5, -2, 1) & \xleftarrow{a} & (2, -3, 4, -5, 1) & \xleftarrow{a} & (5, -2, 3, -4, 1)\\
b \Big\downarrow & & & & \\
(3, -4, 5, 1, -2) & \xrightarrow{b} & (3, -4, 5, -2, 1) & \xrightarrow{a} & (4, -5, 2, -3, 1)\\
 & & & & \Big\downarrow b\\
(5, -4, -1, 3, 2) & \xleftarrow{b} & (5, -1, 3, 2, -4) & \xleftarrow{a} & (4, -1, -5, 2, -3)\\
b \Big\downarrow & & & & \\
(5, 2, -4, -1, 3) & \xrightarrow{a} & (4, 2, -5, -1, 3) = \tau & &
\end{array}$$

and the associated matrix $R := \prod_{j=1}^{13} M_{x_j}(\sigma_j)$ is given by (here $\sigma_1 = \tau$ and $\sigma_j = x_{j-1}(\sigma_{j-1})$, $2 \leq j \leq 13$)

$$R = \begin{pmatrix} 2 & 1 & 1 & 1 & 1\\ 1 & 2 & 1 & 1 & 1\\ 3 & 2 & 3 & 1 & 2\\ 3 & 2 & 2 & 2 & 2\\ 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

therefore R^2 is positive. Then we apply, as in §6, Theorem 25 to obtain a minimal, uniquely ergodic (n, k)-IET with associated permutation τ. Next, we apply Corollary 19 to the subgraph (44) and we obtain an IET, U, with associated permutation π. U is minimal by Theorem 23(2) and uniquely ergodic by Proposition 21 and Theorem 23(3). Finally, $\widehat{U}$ is a proper, minimal and uniquely ergodic (4, 4)-CET.

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