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A STOCHASTIC CLEARING MODEL WITH A BROWNIAN AND A COMPOUND POISSON COMPONENT

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We consider a stochastic input–output system with additional total clearings at certain random times determined by its own evolution (and specified by a controller). Between two clearings, the stock level process is a superposition of a Brownian motion with drift and a compound Poisson process with positive jumps, reflected at zero. We introduce meaningful cost functionals for this system and determine them explicitly under several (classical and new) clearing policies.

1. INTRODUCTION

We consider a storage system with a clearing mechanism (i.e., a model characterized by stochastic inputs and outputs and an additional total "clearing" at certain random

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times determined by its own evolution). The associated content process is assumed to be regenerative, starting anew at level zero at every clearing time. The examples of real-world applications we have in mind are queues with triggered bulk services and/or with "catastrophic" clearings of their workload and various other demand-responsive service systems, as well as inventory systems with backlog removal policies. In order to assess the functioning of such an inventory model, three types of cost are essential: the setup cost for the clearings, the holding cost for the stock, and the cost due to unsatisfied demand. The main objective of this article is to develop techniques to compute meaningful measures of these costs for several clearing mechanisms. For background on clearing models, see Stidham [22–24], Serfozo and Stidham [21], Whitt [25], and Kella [15]. Further uses of these models include the control of epidemics, in which the quantity of interest is the number of "susceptibles" and clearing corresponds to mass vaccination, and the quality control in industrial processes, in which one is interested in some measure of deviation from a norm for the process and clearing corresponds to resetting the process to the norm. Another point of view on clearing models has been recently developed in the queuing literature in which, in addition to regular customers, so-called negative arrivals are considered. A negative arrival has the effect of deleting some customers (or some amount of the workload) from the queue. These queues were first studied by Gelenbe et al. [11] and Harrison and Pitel [13,14]. Recently, Boucherie and Boxma [6] and Boucherie, Boxma, and Sigman [7] generalized this concept by allowing the removal of random amounts of work, or indeed of all work, as is the case at a time of clearing. The asymptotic distribution of the workload in an $M/G/1$ clearing system was studied by Boxma, Perry, and Stadje [8]. Further ramifications can be found in Kim and Seila [17], Chao [10], and Artalejo and Gomez-Corral [1].

In our model, we assume that the stock level process $W = (W(t))_{t \geq 0}$ starts anew from zero after each clearing and has two components:

1. There are continuous "small" inflows and outflows which, together, form a Brownian motion with drift.
2. In addition, batches of random size arrive at random times, forming a Poisson process of instantaneous big inflows.

We model this situation by taking W between clearings as an independent superposition of a Brownian motion (BM) with drift and a compound Poisson process. To deal with unsatisfied demand, we use the local time process, which is the minimal process that has to be added in order to keep the underlying process from becoming negative, so that the stock level process is reflected at zero.

The inflow and the demand never stop and all arriving batches are admitted to the system so that, from the operational research point of view, the clearing policy is the only decision variable. In earlier models, the clearing times are taken as the instances when the stock level process crosses some critical level q (which is then taken as the decision variable). We also consider this traditional policy $T_q = \inf\{t \geq 0 | W(t) \geq q\}$ ($q > 0$ fixed) for our model; Figure 1 shows a typical sample path of W under T_q.

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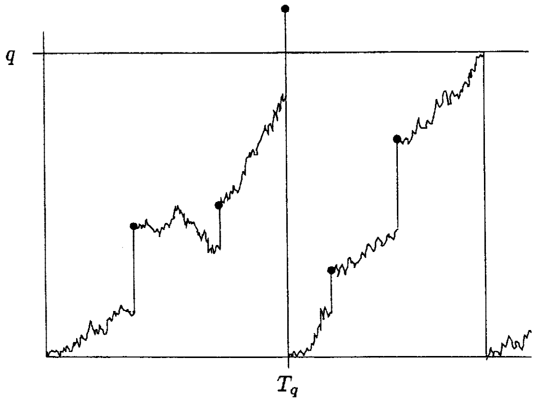

FIGURE 1. A typical sample path of W under the clearing policy T_q.

Additionally, we study some other useful clearing policies. If the controller who clears the system is not continuously available, policy T_q is not possible. In our second core model, we assume that the clearing times form a Poisson process which is independent of W so that the clearing policy for the first cycle is an $\exp(\xi)$-distributed random time, say $T(\xi)$, which can, for example, be interpreted as the waiting time for the controller's first arrival or the time of the first "catastrophic" clearing due to exogeneous causes. We also deal with combinations of the above clearing policies:

(a) $T_q + T(\xi)$, a policy with lead time $T(\xi)$, meaning that once level q is reached or exceeded (at time T_q), the controller arrives $T(\xi)$ time units later;
(b) $\min(T_q, T(\xi))$, with the interpretation that clearing takes place at time $T(\xi)$ unless the level q is reached before, in which case the controller gets an emergency call, arrives immediately, and clears the system.

Our objective is to obtain tractable formulas for the appropriate cost functionals under all of these clearing policies. We consider cost for setup, holding and unsatisfied demand and a large class of phase-type distributions for the jump sizes. To derive the functionals for T_q, $T(\xi)$, and their combinations, it is required to solve a system of linear equations whose coefficients depend on the roots of a certain polynomial. The number of equations and unknowns and the degree of this polynomial are both equal to $N + 2$, where N is the total number of phases involved. Thus, already for exponential jumps, the polynomial is cubic; in this case, the linear equations can, however, be solved in closed form; we will discuss these derivations (and the special distributions involved) in some detail. If more than two phases are possible, no fully explicit solutions are available. It is clear that closed-form expressions will be intricate and some numerical and algebraic work needs to be done in any concrete example. Using a second approach (applicable to T_q), the functionals of

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interest turn out to be expressible in terms of a complete system of linearly independent eigenvectors of a certain generating matrix.

Diffusion models have often been used successfully as approximations of classical discrete models of random-walk type in various applications under conditions of heavy traffic. As examples, we mention Browne and Zipkin [9], Bar-Ilan and Sulem [5], Asmussen and Perry [4], Kella [15], and Perry and Stadje [19,20]. Our specific problem is similar to buffer flow applications in Harrison [12, Chap. 5] and in Newell [18, Chap. 2]. The basic process in this article is, however, a combination of a BM diffusion and a compound Poisson jump process. As will be seen, our entire approach, the techniques, the problems, and the results, are different.

In this article, no optimization issues are treated: We only determine the cost functionals as explicitly as possible. Starting from this analysis, one can minimize the cost of the system with respect to the parameters of the feasible clearing policies.

The article is organized as follows. In Section 2, we define the clearing process and the cost functionals in a formal manner. The crucial tool of the analysis is a martingale introduced by Kella and Whitt [16] and extended to a multivariate setting by Asmussen and Kella [3]. The basic technique is expounded in Section 3. In Sections 4 and 5, we present two approaches to clearing under T_q, based on the one-dimensional and the multidimensional martingales, respectively. In Sections 5–8, we derive all cost functionals under the different clearing policies.

2. THE STOCK LEVEL PROCESS AND THE COST FUNCTIONALS

Let τ_n, $n \geq 1$, be the length of the nth cycle (i.e., the time between the $(n-1)$st and the nth clearing). Let $W_n = (W_n(s))_{s\geq 0}$ be the stock level process for the nth cycle. We assume that the pairs (τ_n, W_n), $n \geq 1$, are independent and identically distributed (i.i.d.). Let $S_0 = 0$, $S_n = \tau_1 + \cdots + \tau_n$ for $n \geq 1$, and $N(t) = \sup\{n \in \mathbb{R}_+ | S_n \leq t\}$. Then, the stock level process $W = (W(t))_{t\geq 0}$ is defined by $W(0) = 0$ and

$$W(t) = W_n(t - S_n) \quad \text{if } n \in \mathbb{Z}_+, S_n < t \leq S_{n+1}.$$

Clearly, W is a regenerative process with respect to the clearing times $S_0, S_1, S_2, \ldots$. Within each cycle, we assume that the stock level process is a Lévy process reflected at zero and composed of a BM with drift and a compound Poisson process with positive phase-type distributed jumps, so that the only negative jump in a cycle is the clearing at its end. The paths of W are right-continuous within each cycle $(S_n, S_{n+1}]$. Note that $N(t)$ is the number of cycles completed in $[0, t]$.

Using regenerative theory we can express all cost functionals in terms of $(W(t))_{t\leq\tau_1} = (W_1(t))_{t\leq\tau_1}$, the stock level process in the first cycle (for all clearing policies under consideration). For $t \leq \tau_1 = T$, the process W can be written as

$$W(t) = X(t) + J(t) + L(t), \qquad 0 \leq t \leq T,$$

where $X = (X(t))_{t\geq 0}$ and $J = (J(t))_{t\geq 0}$ are independent, X is a Brownian motion with drift μ and variance σ^2, J is a compound Poisson process with rate λ and positive phase-type jump sizes, $X(0) = J(0) = 0$, and $L(t) = -\min_{0\leq s\leq t}(X(s) + J(s))$ is

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the local time process. Without loss of generality, we assume that $\sigma^2 = 2$. Note that $X + J$ is a Lévy process with drift $\mu + (\lambda/\nu)$, ν^{-1} being the mean jump size, and that $W \geq 0$ is its reflection at zero.

Let us now introduce meaningful *discounted functionals* to measure the main cost factors of the clearing system.

(a) *Setup cost for clearings.* Recall that $N(t)$ is the number of clearings up to time t. Thus,

$$C_1(\beta) = KE\left(\int_0^\infty e^{-\beta t}\, dN(t)\right)$$

is an appropriate clearing cost functional. Here, K is the cost of one clearing and $\beta > 0$ is a discount factor. Clearly, $C_1(\beta)$ can be expressed in terms of one cycle:

$$C_1(\beta) = KE\left(\sum_{n=1}^{\infty} e^{-\beta S_n}\right) = K\,\frac{\theta(\beta)}{1-\theta(\beta)}, \tag{2.1}$$

where $\theta(\beta) = E(e^{-\beta T})$ is the Laplace–Stieltjes transform (LST) of the clearing time T.

(b) *Holding cost.* The total expected discounted holding cost can be expressed as

$$C_2(\beta) = hE\left(\int_0^\infty e^{-\beta t} W(t)\, dt\right),$$

where $h\, dt$ is the holding cost for a unit of stock during a time interval of length dt. In terms of the first cycle, we have

$$\begin{aligned} C_2(\beta) &= hE\left(\int_0^{\tau_1} e^{-\beta t} W(t)\, dt + e^{-\beta\tau_1}\int_0^{\tau_2} e^{-\beta t} W(\tau_1 + t)\, dt + \cdots\right) \\ &= h\,\frac{E\left(\int_0^T e^{-\beta t} W(t)\, dt\right)}{1-\theta(\beta)}. \end{aligned} \tag{2.2}$$

(c) *Unsatisfied demands.* The local time process L is nondecreasing and increases only when $W = 0$. Therefore, an appropriate functional for the cost of unsatisfied demands is

$$C_3(\beta) = \rho E\left(\int_0^\infty e^{-\beta t}\, dL(t)\right),$$

where we consider $\rho\, dL(t)$ as the penalty for the unsatisfied demand that occurs in the time interval $(t, t + dt)$. The constant $\rho > 0$ is the penalty per unit of unsatisfied demand. In terms of the first cycle, we can write $C_3(\beta)$ as

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$$C_3(\beta) = \frac{\rho E\left(\int_0^T e^{-\beta t}\, dL(t)\right)}{1-\theta(\beta)}. \tag{2.3}$$

The simplest way to combine the functionals in (a)–(c) in one cost measure is to add them and consider

$$C(\beta) = C_1(\beta) + C_2(\beta) + C_3(\beta).$$

If one is interested in the long-run average cost, a possible indicator could be

$$\bar{C} = \lim_{\beta\downarrow 0} \beta C(\beta) = hE(W_e) + \rho E(L_e) + \frac{K}{E(T)}. \tag{2.4}$$

Here, $E(T)$ is, of course, the expected cycle length and $E(W_e)$ and $E(L_e)$ denote the expected values of the steady-state stock level and amount of unsatisfied demands, respectively.

Let

$$\Gamma(\beta) = E\left(\int_0^T e^{-\beta t} W(t)\, dt\right), \qquad \eta(\beta) = \left(\int_0^T e^{-\beta t}\, dL(t)\right).$$

In the following, we will compute the three functions $\theta(\beta)$, $\Gamma(\beta)$, and $\eta(\beta)$ for the different clearing policies. In (2.1)–(2.3), the discounted cost functionals are expressed in terms of $\theta(\beta)$, $\Gamma(\beta)$, and $\eta(\beta)$. Regarding $\bar{C}$ in (2.4), we note that

$$E(T) = -\theta'(0), \tag{2.5}$$

$$E(W_e) = \lim_{\beta\downarrow 0} \frac{E\left(\int_0^T e^{-\beta t} W(t)\, dt\right)}{E(T)} = \frac{E\left(\int_0^T W(t)\, dt\right)}{E(T)} = -\frac{\Gamma(0)}{\theta'(0)}, \tag{2.6}$$

$$E(L_e) = \lim_{\beta\downarrow 0} \frac{E\left(\int_0^T e^{-\beta t}\, dL(t)\right)}{E(T)} = \frac{E(L(T))}{E(T)} = -\frac{\eta(0)}{\theta'(0)}. \tag{2.7}$$

3. THE BASIC IDENTITY

A central tool of our analysis is a martingale which was introduced by Kella and Whitt [16]. If $(X(t)+J(t))_{t\geq 0}$ is a Lévy process with no negative jumps and *exponent* $\varphi(\alpha) = \log E(e^{-\alpha(X(1)+J(1))})$, $(Y(t))_{t\geq 0}$ is an adapted process with bounded expected variation on finite intervals, and $Z(t) = X(t) + J(t) + Y(t)$, then the process

$$M(t) = \varphi(\alpha)\int_0^t e^{-\alpha Z(s)}\, ds + e^{-\alpha Z(0)} - e^{-\alpha Z(t)} - \alpha \int_0^t e^{-\alpha Z(s)}\, dY(s) \tag{3.1}$$

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is a martingale. In our case, we take $X(t)$ and $J(t)$ as in Section 2 and set $Y(t) = L(t) + (\beta/\alpha)t$ for arbitrary $\beta \geq 0$. Clearly, Y is adapted and has paths of bounded expected variation, since $L(t)$ is nondecreasing and bounded above by the local time of a Brownian motion, which is known to have a finite expected value for every t. By (3.1), we obtain the martingale

$$M(t) = \varphi(\alpha)\int_0^t e^{-\alpha W(s)-\beta s}\,ds + e^{-\alpha W(0)} - e^{-\alpha W(t)-\beta t} - \alpha\int_0^t e^{-\alpha W(s)-\beta s}\,d\left(L(s) + \left(\frac{\beta}{\alpha}\right)s\right). \tag{3.2}$$

(Recall that $\sigma^2 = 2$.) It is straightforward to see that the conditions of the martingale stopping theorem are satisfied for all clearing times T under consideration. Thus, we have $E(M(T)) = E(M(0))$ or, equivalently,

$$\varphi(\alpha)E\left(\int_0^T e^{-\alpha W(s)-\beta s}\,ds\right) = -E(e^{-\alpha W(0)}) + E(e^{-\alpha W(T)-\beta T}) + \alpha E\left(\int_0^T e^{-\alpha W(s)-\beta s}\,d\left(L(s) + \left(\frac{\beta}{\alpha}\right)s\right)\right). \tag{3.3}$$

Since L increases only when $W = 0$, it is clear that

$$E\left(\int_0^T e^{-\alpha W(s)-\beta s}\,dL(s)\right) = E\left(\int_0^T e^{-\beta s}\,dL(s)\right) = \eta(\beta). \tag{3.4}$$

Now, using $W(0) = 0$ and $d(L(s) + (\beta/\alpha)s) = dL(s) + (\beta/\alpha)\,ds$, and inserting (3.4) in (3.3), we find that

$$(\varphi(\alpha) - \beta)E\left(\int_0^T e^{-\alpha W(s)-\beta s}\,ds\right) = -1 + E(e^{-\alpha W(T)-\beta T}) + \alpha\eta(\beta). \tag{3.5}$$

In the sequel, the cost functionals of all proposed clearing times are derived using the basic identity (3.5).

In Sections 4, 7, and 8, we assume that the distribution function G of the jumps has a LST of the form

$$G^*(\alpha) = \sum_{i=1}^{n} p_i \prod_{j=1}^{k_i} \frac{\mu_{ij}}{\mu_{ij} + \alpha}, \tag{3.6}$$

where $n, k_1, \ldots, k_n \in \mathbb{N}$, $p_1, \ldots, p_n$ are positive, $\sum_{i=1}^{n} p_i = 1$, and $\mu_{ij} > 0$. This class covers a wide range of phase-type distributions. For example, Coxian and hyper-exponential distributions have LSTs of this type (see Asmussen [2, p. 74]). For simplicity, we assume that all μ_{ij} are distinct (otherwise we obtain the desired functionals of W by taking a simple limit). As a LST, $G^*(\alpha)$ is defined for $\alpha \in (0,\infty)$, but

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we will consider its (unique) analytic extension to $\mathbb{C}\backslash\{\mu_{ij} \mid j=1,\ldots,k_i \text{ and } i=1,\ldots,n\}$, which is simply given by the right-hand side of (3.6).

The exponent of $X+J$ is, in this case,

$$\varphi(\alpha)=\alpha^2-\mu\alpha-\lambda\left(1-\sum_{i=1}^{n}p_i\prod_{j=1}^{k_i}\frac{\mu_{ij}}{\mu_{ij}+\alpha}\right). \tag{3.7}$$

For values of α satisfying $\varphi(\alpha)-\beta=0$, the left-hand side of (3.5) becomes zero. It turns out that there are exactly $k_1+\cdots+k_n+2$ solutions, all of them real. To see this, note that, by (3.7), the equation $\varphi(\alpha)=\beta$ can be written in the form

$$\lambda+\beta-\alpha^2+\mu\alpha=\lambda\sum_{i=1}^{n}p_i\prod_{j=1}^{k_i}\frac{\mu_{ij}}{\mu_{ij}+\alpha}. \tag{3.8}$$

Set $N=k_1+\cdots+k_n$ and let $0<m_1<\cdots<m_N$ be the μ_{ij} in ascending order. Since (3.8) is equivalent to a polynomial equation of degree $N+2$ for α, there are exactly $N+2$ solutions, each counted with its multiplicity.

First, consider $\beta>0$. Equation (3.8) has a real solution in every interval $(-m_\ell,-m_{\ell-1})$, $\ell=2,\ldots,N$, because the function on the right-hand side of (3.8) runs from ∞ to $-\infty$, and it has one solution in $(-m_1,0)$ and one in $(0,\infty)$, because the right-hand function decreases on $(-m_1,\infty)$ from ∞ to 0 and is smaller than $\lambda+\beta$ at $\alpha=0$, whereas the left-hand function is equal to $\lambda+\beta$ at $\alpha=0$. Finally, there is a solution in $(-\infty,-m_N)$, since the left-hand function runs from $-\infty$ to some real value (to $\lambda+\beta-m_N^2-\mu m_N$) on this interval and the right-hand function from 0 to $-\infty$. Thus there are exactly $N+2$ roots $\alpha_1(\beta),\ldots,\alpha_{N+2}(\beta)$ of (3.8); they are real and can be ordered so as to satisfy $\alpha_1(\beta)>0>\alpha_2(\beta)>-m_1$, $\alpha_{\ell+1}(\beta)\in(-m_\ell,-m_{\ell-1})$, $\ell=2,\ldots,N$, and $\alpha_{N+2}(\beta)\in(-\infty,-m_N)$.

Now, let $\beta=0$. The above arguments again yield negative roots $\alpha_3(\beta)>\cdots>\alpha_{N+2}(\beta)$ in $(-\infty,-m_1)$. Clearly, $\alpha=\alpha_2(0)=0$ is also a solution. On $(0,\infty)$, the right-hand function in (3.8) decreases and has the range $(0,\lambda)$, whereas the parabola $\alpha\mapsto\lambda-\alpha^2+\mu\alpha$ on $(0,\infty)$ increases until it reaches its maximum at $\mu/2$ and then decreases to $-\infty$. Hence, there is one more root $\alpha=\alpha_1(0)$ of (3.8) in $(\mu/2,\infty)$.

In the following, we will insert these roots in (3.5) and derive the desired LSTs and functionals from the resulting equations. For this, it is required that all terms in (3.5) be well defined at $\alpha=\alpha_i(\beta)$, $i=1,\ldots,N+2$. First, note that $\varphi(\alpha)-\beta$ is an analytic function of α on $\mathbb{C}\backslash\{-m_1,\ldots,-m_N\}$ (for fixed $\beta>0$). For $T=T_q$, the integral $\int_0^T e^{-\alpha W(s)-\beta s}\,ds$ is bounded by $e^{|\alpha|q}T_q$ and thus the expected value $E(\int_0^T e^{-\alpha W(s)-\beta s}\,ds)$ is an analytic function of α for all $\alpha\in\mathbb{C}$. For $T=T(\xi)$, the clearing rule studied in Section 7, this expected value is shown to be equal to $\xi^{-1}E(e^{-\alpha W(T(\xi))-\beta T(\xi)})$; see (7.1). In both cases, $E(e^{-\alpha W(T)-\beta T})$ is analytic in α for Re $\alpha>0$, which, by (3.5), coincides with a meromorphic function on $\mathbb{C}$, which has no poles in $\alpha_1(\beta),\ldots,\alpha_{N+2}(\beta)$. By the identity theorem for meromorphic functions, the analytic continuation of $E(e^{-\alpha W(T)-\beta T})$ satisfies (3.5) at $\alpha = \alpha_i(\beta)$, $i = 1, \ldots, N+2$. The reasoning for the clearing rules analyzed in Section 8 is similar.

4. THE DISCOUNTED FUNCTIONALS UNDER T_q

In this section, we use the basic identity (3.5) to derive all of the discounted functionals for the system under T_q, assuming that the jumps have LST (3.6). By the structure of $G^*(\cdot)$, any jump of W can be thought of as being generated by first selecting an index $i \in \{1, \ldots, n\}$ according to the probability distribution $(p_1, \ldots, p_n)$ and then carrying out k_i successive phases which are independent and exponentially distributed with means $1/\mu_{i1}, \ldots, 1/\mu_{ik_i}$. Let C_{ij} be the event that the level q is first crossed by the phase with distribution $\exp(\mu_{ij})$. At time T_q, there is, of course, also the possibility to hit q exactly due to the Brownian component; let C be the event that this happens [i.e., that $W(T_q) = q$]. Let $\eta(\beta) = E\left(\int_0^{T_q} e^{-\beta s}\, dL(s)\right)$, $h(\beta) = E(e^{-\beta T_q} 1_C)$, and $h_{ij}(\beta) := E(e^{-\beta T_q} 1_{C_{ij}})$, $i = 1, \ldots, n$, $j = 1, \ldots, k_i$. Given C_{ij}, the overshoot $W(T_q) - q$ is $\exp(\mu_{ij})$ distributed and independent of T_q. Using the formula of total probability, we can rewrite the right-hand side of (3.5) as

$$-1 + \sum_{i=1}^{n} \sum_{j=1}^{k_i} P(C_{ij}) e^{-\alpha q} \frac{\mu_{ij}}{\mu_{ij} + \alpha} E(e^{-\beta T_q} | C_{ij}) + P(C) e^{-\alpha q} E(e^{-\beta T_q} | C) + \alpha \eta(\beta)$$

$$= -1 + \sum_{i=1}^{n} \sum_{j=1}^{k_i} e^{-\alpha q} \frac{\mu_{ij}}{\mu_{ij} + \alpha} h_{ij}(\beta) + e^{-\alpha q} h(\beta) + \alpha \eta(\beta). \quad \textbf{(4.1)}$$

Inserting the $N + 2$ real zeros $\alpha_1(\beta), \ldots, \alpha_{N+2}(\beta)$ of the equation $\varphi(\alpha) - \beta = 0$ in (3.5) and using (4.1) yields

$$\sum_{i=1}^{n} \sum_{j=1}^{k_i} e^{-\alpha_k(\beta) q} \frac{\mu_{ij}}{\mu_{ij} + \alpha_k(\beta)} h_{ij}(\beta) + e^{-\alpha_k(\beta) q} h(\beta) + \alpha_k(\beta) \eta(\beta) = 1,$$

$$k = 1, \ldots, N+2. \quad \textbf{(4.2)}$$

Equation (4.2) is a system of $N + 2$ linear equations for the $N + 2$ unknowns $\eta(\beta)$, $h(\beta)$, and $h_{ij}(\beta)$, $i = 1, \ldots, n$, $j = 1, \ldots, k_i$. By (4.2), we obtain $\eta(\beta)$, $h(\beta)$, and $h_{ij}(\beta)$ as rational functions of the roots $\alpha_k(\beta)$ and the exponential functions $\exp(\alpha_k(\beta))$. Now, we can express all functionals of interest in terms of $\eta(\beta)$, $h(\beta)$, and $h_{ij}(\beta)$. First, note that

$$\theta(\beta) = E(e^{-\beta T_q}) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} h_{ij}(\beta) + h(\beta) \quad \textbf{(4.3)}$$

and

$$E(T_q) = -\sum_{i=1}^{n} \sum_{j=1}^{k_i} h'_{ij}(0) - h'(0). \quad \textbf{(4.4)}$$

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The derivatives $h'_{ij}(\beta)$ can be computed in terms of the $\alpha_k(\beta)$, noting that $\alpha'_k(\beta) = 1/\varphi'(\alpha_k(\beta))$.

The LST $\Delta(\alpha,\beta) = E(e^{-\alpha W_e - \beta T_e})$ of the steady-state joint distribution of the stock level and the time elapsed since the last clearing is

$$\Delta(\alpha,\beta) = \frac{E\left(\int_0^{T_q} e^{-\alpha W(s)-\beta s}\, ds\right)}{E(T_q)}. \tag{4.5}$$

By (3.5), (4.1), (4.3), and (4.4), we find that

$$\Delta(\alpha,\beta) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{k_i} e^{-\alpha q}\,\frac{\mu_{ij}}{\mu_{ij}+\alpha}\,h_{ij}(\beta) + e^{-\alpha q}h(\beta) + \alpha\eta(\beta) - 1}{(\beta-\varphi(\alpha))\left(\sum_{i=1}^{n}\sum_{j=1}^{k_i} h'_{ij}(0) + h'(0)\right)}. \tag{4.6}$$

Setting $\beta = 0$ in (4.6), we obtain the LST of the stationary stock level distribution:

$$E(e^{-\alpha W_e}) = -\frac{\sum_{i=1}^{n}\sum_{j=1}^{k_i} e^{-\alpha q}\,\frac{\mu_{ij}}{\mu_{ij}+\alpha}\,h_{ij}(0) + e^{-\alpha q}h(0) + \alpha\eta(0) - 1}{\varphi(\alpha)\left(\sum_{i=1}^{n}\sum_{j=1}^{k_i} h'_{ij}(0) + h'(0)\right)}. \tag{4.7}$$

Finally,

$$\begin{aligned}\Gamma(\beta) &= -E(T_q)\,\frac{\partial}{\partial\alpha}\,\Delta(\alpha,\beta)\big|_{\alpha=0}\\ &= -\frac{\partial}{\partial\alpha}\left((\varphi(\alpha)-\beta)^{-1}\left[\sum_{i=1}^{n}\sum_{j=1}^{k_i} e^{-\alpha q}\,\frac{\mu_{ij}}{\mu_{ij}+\alpha}\,h_{ij}(\beta)\right.\right.\\ &\qquad\left.\left. + e^{-\alpha q}h(\beta) + \alpha\eta(\beta) - 1\right]\right)\Bigg|_{\alpha=0}.\end{aligned} \tag{4.8}$$

We have computed all three functionals $\theta(\beta)$, $\eta(\beta)$, and $\Gamma(\beta)$.

5. AN ALTERNATIVE APPROACH TO T_q

In this section, we present a second approach to the cost functionals related to T_q, which uses a detour to the multivariate version of the martingale (3.2) introduced in Asmussen and Kella [3]. The jumps $U_1, U_2, \ldots$ of J are now assumed to have a general phase-type distribution of the form $P(U_i > t) = \pi e^{At}\mathbf{1}$ for $t \geq 0$, where π is a strictly positive n-dimensional probability row vector, A is a $(n \times n)$ rate transition matrix of a terminating (from any state) Markov chain, and $\mathbf{1}$ is a column vector of n ones. Let W_e be a random variable whose distribution is the stationary law of W.

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The LST of W_e under T_q is, by the ergodic theorem for regenerative processes, given by

$$E(e^{-\alpha W_e}) = \frac{E\left(\int_0^{T_q} e^{-\alpha W(t)}\,dt\right)}{E(T_q)}. \tag{5.1}$$

At time T_q, the process may either hit q (due to the Brownian part) or jump above it (due to the phase-type jumps). In order to avoid dealing with directly computing the distribution of the overshoot above q, we will use a trick that was also used in Asmussen and Kella [3] and first change the jumps into slopes of rate 1: Thus, if there is a jump of size y at time t, the modified process increases at rate 1 for y time units. Figure 2 shows how the sample path depicted in Figure 1 looks after this modification.

The new process becomes a Markov additive process $X' = \{X'(t) | t \geq 0\}$, which has continuous sample paths. Its characteristics are easily derived from those of the original process; that is, if the modulating states are $0, 1, \ldots, n$, then when at state zero, the process behaves like the BM X, and when in any of the other states, it has slope 1. The underlying modulating process $I = (I(t) | t \geq 0)$ is a continuous-time Markov chain with the rate transition matrix

$$Q = \begin{pmatrix} -\lambda & \lambda\pi \\ -A\mathbf{1} & A \end{pmatrix}. \tag{5.2}$$

Because we assumed that A is terminating from any state and that π is strictly positive, Q is irreducible. As in Asmussen and Kella [3], it is seen that the generating matrix for this process is

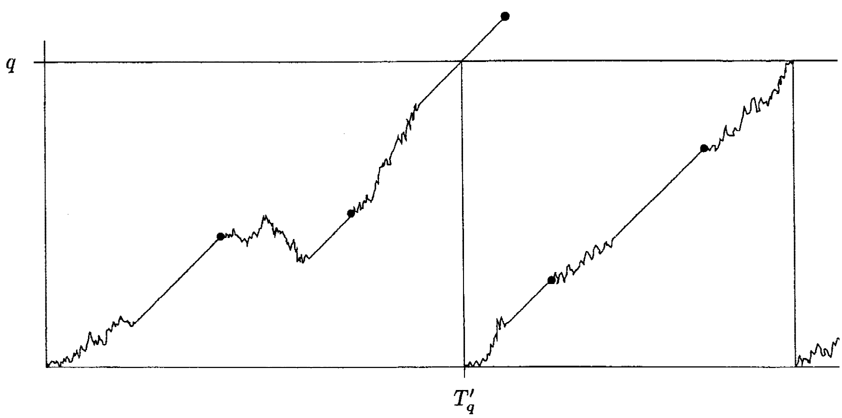

FIGURE 2. Sample path of W'.

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$$F(\alpha) = Q + \begin{pmatrix} \mu\alpha + \frac{1}{2}\alpha^2 & 0 \\ 0 & \alpha I_n \end{pmatrix}, \tag{5.3}$$

where I_n is the n-dimensional identity matrix.

Let $L'(t) = -\inf_{0\le s\le t} X'(s)$, $W'(t) = X'(t) + L'(t)$, and $W' = (W'(t))_{t\ge 0}$. By $\mathbf{1}_i$ we denote an $(n+1)$-dimensional row vector with 1 in coordinate i and 0 elsewhere. It follows from Theorem 2.1 in Asmussen and Kella [3] that

$$\int_0^t e^{\alpha W'(s)} \mathbf{1}_{I(s)}\, ds\, F(\alpha) + e^{\alpha W'(0)} \mathbf{1}_{I(0)} - e^{\alpha W'(t)} \mathbf{1}_{I(t)} + \alpha \int_0^t \mathbf{1}_{I(s)}\, dL'(s) \tag{5.4}$$

is an $(n+1)$-dimensional martingale. Now, consider $T_q' = \inf\{t \ge 0 \mid W_t' = q\}$; due to sample path continuity, q will be hit exactly by W'. Note that $L'(t)$ can increase only when $I(t) = 0$ (i.e., during Brownian phases), that $I(0) = 0$, and that the process W' is bounded on $[0, T_q]$. By the optional stopping theorem and bounded convergence, we have that

$$E\left(\int_0^{T_q'} e^{\alpha W'(s)} \mathbf{1}_{I(s)}\, ds\right) F(\alpha) = e^{\alpha q} \boldsymbol{\pi}^q - \mathbf{1}_0 - \alpha \mathbf{1}_0 \ell_q, \tag{5.5}$$

where $\boldsymbol{\pi}^q = E(\mathbf{1}_{I(T_q)})$ and $\ell_q = E(L'(T_q))$. Thus, it is clear that if we are able to compute the $(n+1)$ vector $\boldsymbol{\pi}^q$ and the positive constant ℓ_q, then we also have $E(\int_0^{T_q'} e^{\alpha W'(s)} \mathbf{1}_{I(s)}\, ds)$. Obviously, the processes W and W' and the stopping times T_q and T_q' are related by

$$E\left(\int_0^{T_q} e^{\alpha W(s)}\, ds\right) = E\left(\int_0^{T_q'} e^{\alpha W'(s)} 1_{I(s)=0}\, ds\right) \tag{5.6}$$

and

$$E(T_q) = E\left(\int_0^{T_q'} 1_{I(s)=0}\, ds\right). \tag{5.7}$$

In particular, we obtain

$$E(e^{\alpha W_e}) = \frac{E\left(\int_0^{T_q'} e^{\alpha W'(s)} 1_{I(s)=0}\, ds\right)}{E\left(\int_0^{T_q'} 1_{I(s)=0}\, ds\right)}. \tag{5.8}$$

The determinant of $F(\alpha)$ is a polynomial of degree $n+2$ in α. Let $\alpha_0, \ldots, \alpha_{n+1}$ be such that $\det F(\alpha_j) = 0$ and let $H_{\cdot 0}, \ldots, H_{\cdot n+1}$ be $(n+1)$-dimensional nonzero column vectors such that $F(\alpha_j) H_{\cdot j} = 0$. It is clear that one of the roots, say α_0, is zero (since $\det Q = 0$) with corresponding eigenvector $H_{\cdot 0} = \mathbf{1}^T$ ($\mathbf{1}$ now being an

$(n+1)$-dimensional column vector of ones). Let us assume that $H_{\cdot 1}, \dots, H_{\cdot n+1}$ are linearly independent vectors. Set $H = (H_{\cdot 1}, \dots, H_{\cdot n+1})$ and $\Lambda = \mathrm{diag}(\alpha_1, \dots, \alpha_{n+1})$. Then, from (5.5), we have that

$$\pi^q H e^{\Lambda q} = \mathbf{1}_0 H + \mathbf{1}_0 H \Lambda \ell_q \tag{5.9}$$

so that

$$\pi^q = \mathbf{1}_0 H e^{-\Lambda q} H^{-1} + \mathbf{1}_0 H \Lambda e^{-\Lambda q} H^{-1} \ell_q, \tag{5.10}$$

where, for a given matrix C, e^C denotes the matrix exponential of C (which in our case is a diagonal matrix). Thus, we need one more equation, which is

$$\pi^q \mathbf{1}^T = 1. \tag{5.11}$$

Equation (5.11) follows from the definition of π^q and is also precisely what we obtain if we substitute $\alpha = \alpha_0 = 0$ in (5.5) and multiply by $H_{\cdot 0} = \mathbf{1}^T$. Hence, multiplying (5.10) by $\mathbf{1}^{\mathrm{T}}$, we obtain

$$\ell_q = \frac{1 - \mathbf{1}_0 H e^{-\Lambda q} H^{-1} \mathbf{1}^T}{\mathbf{1}_0 H \Lambda e^{-\Lambda q} H^{-1} \mathbf{1}^T} = \frac{\mathbf{1}_0 H (I - e^{-\Lambda q}) H^{-1} \mathbf{1}^T}{\mathbf{1}_0 H \Lambda e^{-\Lambda q} H^{-1} \mathbf{1}^T}. \tag{5.12}$$

Having computed ℓ_q we also have π^q via (5.10). Now, we can use (5.5) to obtain $E(\int_0^{T_q'} e^{\alpha W'(s)} \mathbf{1}_{I(s)}\, ds)$ for all α satisfying $\det F(\alpha) \neq 0$. Then, (5.6) yields $E(\int_0^{T_q} e^{\alpha W(s)}\, ds)$. To compute $E(T_q)$, we would like to set $\alpha = 0$ in (5.6). However, this is not viable, as $\det F(0) = 0$. Instead we have to use (5.5) to conclude that

$$(e^{\alpha q} \pi^q - \mathbf{1}_0 - \alpha \mathbf{1}_0 \ell_q) F(\alpha)^{-1} \to E\left(\int_0^{T_q'} \mathbf{1}_{I(s)}\, ds \right), \tag{5.13}$$

as $\alpha \to 0$ through values for which $\det F(\alpha) \neq 0$. Relation (5.13) amounts to applying L'Hôpital's rule in order to determine $E(\int_0^{T_q'} \mathbf{1}_{I(s)}\, ds)$ and, in particular, the first component of this row vector, which is $E(T_q)$. Equation (5.1) now yields the LST of the stationary distribution of W.

There is also a direct method for deriving $A(\alpha) \equiv E(\int_0^{T_q'} e^{\alpha W'(s)} \mathbf{1}_{I(s)}\, ds)$ for values of α such that $\det F(\alpha) = 0$. First, note that for $\alpha \neq \alpha_i$, we have

$$\begin{aligned} A(\alpha) \frac{F(\alpha) - F(\alpha_i)}{\alpha - \alpha_i} H_{\cdot i} &= A(\alpha) \begin{pmatrix} \mu + \dfrac{\alpha + \alpha_i}{2} & 0 \\ 0 & I_n \end{pmatrix} H_{\cdot i} \\ &= \left(\frac{e^{\alpha q} - e^{\alpha_i q}}{\alpha - \alpha_i} \pi^q - \mathbf{1}_0 \ell_q \right) H_{\cdot i}. \end{aligned} \tag{5.14}$$

In particular, by continuity, this holds for all α. Thus, setting $\alpha = \alpha_j$ for a given j, this gives a system of equations from which we can find $A(\alpha_j)$. In particular, this can be done for $\alpha_j = 0$, so that

$$A(0) = E\left(\int_0^{T_q'} \mathbf{1}_{I(s)}\, ds\right) \tag{5.15}$$

is computable. Clearly,

$$E(T_q) = A_1(0) = E\left(\int_0^{T_q'} 1_{I(s)=0}\, ds\right). \tag{5.16}$$

6. EXAMPLE: EXPONENTIAL JUMPS

We now consider in some detail the important special case of exponential jumps (i.e., $G^*(\alpha) = \nu/(\nu+\alpha)$) using the technique of Section 4. Then, there is only one ($\exp(\nu)$-distributed) type of phase for a possible overshoot. The three values $\alpha_1(\beta) > 0 \geq \alpha_2(\beta) > \alpha_3(\beta)$ are the roots of the cubic polynomial

$$p(\alpha) = \alpha^3 + \alpha^2(\nu - \mu) - \alpha(\mu\nu + \lambda + \beta) - \beta\nu. \tag{6.1}$$

Recall that $\alpha_2(\beta) < 0$ for all $\beta > 0$ and $\alpha_2(0) = 0$. There exist closed-form expressions or the roots of (6.1) via Cardano's formula; they are, however, rather complicated and not given here.

The transforms $h(\beta) = E(e^{-\beta T_q} 1_{\{W(T_q)=q\}})$, $h_1(\beta) = E(e^{-\beta T_q} 1_{\{W(T_q)>q\}})$, and $\eta(\beta)$ are found using some straightforward algebra to solve (4.2):

$$h(\beta) = \frac{(\alpha_3 y_1 - \alpha_1 y_3)(\alpha_2 - \alpha_1) - (\alpha_2 y_1 - \alpha_1 y_2)(\alpha_3 - \alpha_1)}{(\alpha_3 y_1 - \alpha_1 y_3)(\alpha_2 x_1 - \alpha_1 x_2) - (\alpha_2 y_1 - \alpha_1 y_2)(\alpha_3 x_1 - \alpha_1 x_3)}, \tag{6.2}$$

$$h_1(\beta) = \frac{(\alpha_3 x_1 - \alpha_1 x_3)(\alpha_2 - \alpha_1) - (\alpha_2 x_1 - \alpha_1 x_2)(\alpha_3 - \alpha_1)}{(\alpha_3 x_1 - \alpha_1 x_3)(\alpha_2 y_1 - \alpha_1 y_2) - (\alpha_2 x_1 - \alpha_1 x_2)(\alpha_3 y_1 - \alpha_1 y_3)}, \tag{6.3}$$

$$\eta(\beta) = \frac{(x_2 - x_1)(x_3 y_1 - x_1 y_3) - (x_3 - x_1)(x_2 y_1 - x_1 y_2)}{(\alpha_1 x_2 - \alpha_2 x_1)(x_3 y_1 - x_1 y_3) - (\alpha_1 x_3 - \alpha_2 x_1)(x_2 y_1 - x_1 y_2)}, \tag{6.4}$$

where for $i = 1, 2, 3$, we have set $\alpha_i = \alpha_i(\beta)$,

$$x_i = x_i(\beta) = e^{-\alpha_i(\beta) q},$$

$$y_i = y_i(\beta) = \frac{e^{-\alpha_i(\beta)}\nu}{\nu + \alpha_i(\beta)}.$$

From (4.8), we obtain (after some algebra)

$$\Gamma(\beta) = \frac{\beta[\eta(\beta) - q\theta(\beta) - \nu^{-1} h_1(\beta)] + (\mu + \nu^{-1}\lambda)(1 - \theta(\beta))}{\beta^2}. \tag{6.5}$$

By (4.2), $\theta(\beta) = h(\beta) + h_1(\beta)$. From (2.1)–(2.3) we obtain closed-form expressions for the discounted cost functionals $C_1(\beta)$, $C_2(\beta)$, and $C_3(\beta)$.

We will now compute several quantities of interest. We start with the hitting probability $P(W(T_q) = q)$. As this probability is equal to $h(0)$, we need $\alpha_i(0), i = 1, 2, 3$. Clearly,

$$\alpha_1(0) = \frac{1}{2}(\nu - \mu + [(\nu - \mu)^2 + 4(\mu\nu + \lambda)]^{1/2}) > 0,$$
$$\alpha_2(0) = 0,$$
$$\alpha_3(0) = \frac{1}{2}(\nu - \mu - [(\nu - \mu)^2 + 4(\mu\nu + \lambda)]^{1/2}) < 0.$$

Thus,

$$P(W(T_q) = q) = h(0) = \frac{\nu}{\alpha_1(0)\alpha_3(0)} \frac{\alpha_3(0)(1 - e^{-\alpha_1(0)q}) - \alpha_1(0)(1 - e^{-\alpha_3(0)q})}{e^{-\alpha_1(0)q} - e^{-\alpha_3(0)q}}.$$

The long-run average cost functionals $E(T_q)$, $E(L_e)$, and $E(W_e)$ are obtained by lengthy calculations:

$$E(T_q) = -\theta'(0) = \frac{1 - e^{-\alpha_1(0)q} \dfrac{\nu + h(0)\alpha_1(0)}{\nu + \alpha_1(0)} - \alpha_1(0)\left(q + \dfrac{1 + h(0)}{\nu}\right)}{\alpha_1(1)(\mu + \nu^{-1}\lambda)},$$

$$E(L_e) = \frac{\eta(0)}{E(T_q)} = \frac{h(0)q + 1 - h_1(0)(q + \nu^{-1})}{E(T_q)} - \mu - \frac{\lambda}{\nu},$$

$$E(W_e) = \frac{\Gamma(0)}{E(T_q)} = \frac{qh(0) + (q + \nu^{-1})h_1(0)}{E(T_q)}.$$

It is interesting that the stationary law of the stock level process can be determined explicitly. Its LST is obtained by setting $\beta = 0$ in (3.5) and then dividing both sides by $E(T_q)\varphi(\alpha)$. We find that

$$E(e^{-\alpha W_e}) = \frac{-1 + E(e^{-\alpha W(T_q)}) + \alpha E(L(T_q))}{\varphi(\alpha)E(T_q)}$$
$$= \frac{-1 + h(0)e^{-\alpha q} + h_1(0)e^{-\alpha q} \dfrac{\nu}{\nu + \alpha} + \alpha h(0)q + \alpha h_1(0)\left(q + \dfrac{1}{\nu}\right) - \alpha\left(\mu + \dfrac{\lambda}{\nu}\right)E(T_q)}{E(T_q)\dfrac{\alpha(\alpha - \alpha_1(0))(\alpha - \alpha_3(0))}{\nu + \alpha}}. \tag{6.6}$$

We can even derive the corresponding distribution function of W_e. Note that $E(L(T_q)) = E(W(T_q)) - E(X(T_q))$ and, by Wald's equation, $E(X(T_q)) =$

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$(\mu+\nu^{-1}\lambda)E(T_q)$. Recall that $\alpha_1(0)$ and $\alpha_3(0)$ are the positive and the negative root, respectively, of $\alpha^2-\alpha(\mu-\nu)-(\mu\nu+\lambda)=0$ so that

$$\alpha_1(0)|\alpha_3(0)| = \mu\nu+\lambda. \tag{6.7}$$

Therefore,

$$\begin{aligned}\frac{\alpha}{\varphi(\alpha)} &= \frac{\nu+\alpha}{(\alpha-\alpha_1(0))(\alpha-\alpha_3(0))}\\ &= -\frac{\nu+\alpha}{\nu}\,\frac{\alpha_1(0)}{\alpha_1(0)-\alpha}\,\frac{|\alpha_3(0)|}{\alpha+|\alpha_3(0)|}\,\frac{1}{\mu+(\lambda/\nu)}.\end{aligned} \tag{6.8}$$

From the first equation in (6.6) it follows that

$$\begin{aligned}E(e^{-\alpha W_e}) &= -\frac{1-E(e^{-\alpha W(T_q)})}{\varphi(\alpha)E(T_q)}+\frac{\alpha E(L(T_q))}{\varphi(\alpha)E(T_q)}\\ &= \frac{\alpha}{\varphi(\alpha)}\left[-\frac{1-E(e^{-\alpha W(T_q)})}{\alpha E(W(T_q))}\,\frac{E(W(T_q))}{E(T_q)}+\frac{E(W(T_q))-E(X(T_q))}{E(T_q)}\right].\end{aligned} \tag{6.9}$$

We need the distribution function A of $W(T_q)$. Clearly,

$$\begin{aligned}P(W(T_q)-q\le x) &= P(W(T_q)=q)\\ &\quad + P(W(T_q)>q)P(W(T_q)-q\le x|W(T_q)>q), \qquad x\ge 0.\end{aligned}$$

Given that $W(T_q)>q$, the overshoot $W(T_q)-q$ is $\exp(\nu)$ distributed. It follows that

$$A(x)=\begin{cases}0 & \text{if } x<q\\ h(0)+h_1(0)(1-e^{-\nu(x-q)}) & \text{if } x\ge q.\end{cases}$$

The function $H^*(\alpha)=[1-E(e^{-\alpha W(T_q)})]/\alpha E(W(T_q))$ is the LST of the distribution function H having the density $x\mapsto [E(W(T_q))]^{-1}(1-A(x))$. Let

$$a=\left[E(T_q)\left(\mu+\frac{\lambda}{\nu}\right)\right]^{-1}E(W(T_q)).$$

From (6.9), we obtain the remarkable relation

$$\begin{aligned}&\frac{\nu}{\nu+\alpha}E(e^{-\alpha W_e})\\ &\quad = aH^*(\alpha)\,\frac{\alpha_1(0)}{\alpha_1(0)-\alpha}\,\frac{|\alpha_3(0)|}{|\alpha_3(0)+\alpha}-(a-1)\,\frac{\alpha_1(0)}{\alpha_1(0)-\alpha}\,\frac{|\alpha_3(0)}{|\alpha_3(0)|+\alpha}.\end{aligned} \tag{6.10}$$

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Let F_e be the distribution function of W_e. Then, (6.10) states that the probability measure $\exp(\nu) * F_e$ is a linear combination of $H * (\exp(-\alpha_1(0)) * \exp(|\alpha_3(0)|)$ and $\exp(-\alpha_1(0)) * \exp(|\alpha_3(0)|)$, where $*$ denotes convolution and $\exp(-\alpha)$, $\alpha > 0$, is the distribution function $(1 - e^{\alpha x})1_{(-\infty,0)}(x)$, the "exponential distribution" on the negative real numbers. Note that $a > 1$, so that the right-hand side of (6.10) is *not* a convex combination of probability measures. In terms of random variables, we can rephrase (6.10) as follows. Let $W_e, Y, E_1, \ldots, E_5, Z$ be independent random variables, $Y \sim H$, $E_1, E_4 \sim \exp(-\alpha)$, $E_2, E_5 \sim \exp(|\alpha_3(0)|)$, $E_3 \sim \exp(\nu)$, $P(Z=1) = a^{-1}$, and $P(Z=0) = 1 - a^{-1}$. Then,

$$Y + E_1 + E_2 \stackrel{D}{=} (1-Z)(W_e + E_3) + Z(E_4 + E_5). \tag{6.11}$$

7. INDEPENDENT EXPONENTIAL CLEARING

We now consider the clearing process with exponential clearing times $\tau_1, \tau_2, \ldots \sim \exp(\xi)$, which are independent of the processes $(W_n(s))_{s \geq 0}, n \geq 1$. In general, if $Y = (Y(t))_{t \geq 0}$ is a measurable and nonnegative process, T is an $\exp(\xi)$-distributed random variable independent of Y, and $Y(T)$ is measurable, then

$$\begin{aligned} E(Y(T)) &= \int_0^\infty E(Y(t))\xi e^{-\xi t}\, dt = \xi \int_0^\infty E(Y(t))E(1_{\{T>t\}})\, dt \\ &= \xi E\left(\int_0^\infty Y(t) 1_{\{T>t\}}\, dt\right) = \xi E\left(\int_0^T Y(t)\, dt\right). \end{aligned}$$

Thus,

$$E(e^{-\alpha W(T(\xi)) - \beta T(\xi)}) = \xi E\left(\int_0^{T(\xi)} e^{-\alpha W(s) - \beta s}\, ds\right). \tag{7.1}$$

Inserting (7.1) in (3.5), we arrive at

$$\xi^{-1}(\varphi(\alpha) - (\beta + \xi))E(e^{-\alpha W(T(\xi)) - \beta T(\xi)}) = -1 + \alpha\eta(\beta). \tag{7.2}$$

Let $\hat{\beta} = \beta + \xi$. Setting $\alpha = \alpha_1(\hat{\beta}) > 0$ in (7.2), we obtain

$$\eta(\beta) = \frac{1}{\alpha_1(\hat{\beta})}. \tag{7.3}$$

Therefore,

$$E\left(\int_0^{T(\xi)} e^{-\alpha W(s) - \beta s}\, ds\right) = \frac{\alpha - \alpha_1(\hat{\beta})}{\alpha_1(\hat{\beta})(\varphi(\alpha) - \hat{\beta})}. \tag{7.4}$$

From (7.4), we can compute $\Gamma(\beta)$:

$$\Gamma(\beta) = -\frac{\partial}{\partial\alpha}\left(E\left(\int_0^{T(\xi)} e^{-\alpha W(s)-\beta s}\,ds\right)\right)\bigg|_{\alpha=0}$$

$$= \frac{\hat{\beta} + \mu + \lambda\sum_{i=1}^{n}\sum_{k=1}^{k_i}\mu_{ik}^{-1}}{\hat{\beta}^2\alpha_1(\hat{\beta})}. \tag{7.5}$$

The LST of $T(\xi)$ is, of course,

$$\theta(\beta) = \frac{\xi}{\xi+\beta}. \tag{7.6}$$

Relations (7.3), (7.5), and (7.6) provide closed-form expressions for the cost functionals under the clearing policy $T(\xi)$.

Example: Let $G(x) = 1 - e^{-\nu x}$, $x \geq 0$. Then, by (7.1) and (7.4),

$$E(e^{-\alpha W(T(\xi))-\beta T(\xi)}) = \frac{(\alpha-\alpha_1(\hat{\beta}))(\nu+\alpha)\xi}{\alpha_1(\hat{\beta})(\alpha-\alpha_1(\hat{\beta}))(\alpha-\alpha_2(\hat{\beta}))(\alpha-\alpha_3(\hat{\beta}))}$$

$$= \frac{|\alpha_2(\hat{\beta})|}{\alpha+|\alpha_2(\hat{\beta})|}\,\frac{|\alpha_3(\hat{\beta})|}{\alpha+|\alpha_3(\hat{\beta})|}\,\frac{\nu+\alpha}{\nu}\,\frac{\xi}{\hat{\beta}}; \tag{7.7}$$

the second equation in (7.7) follows from $\alpha_1(\hat{\beta})\alpha_2(\hat{\beta})\alpha_3(\hat{\beta}) = \hat{\beta}\nu$, $\alpha_2(\hat{\beta}) < 0$ and $\alpha_3(\hat{\beta}) < 0$. Furthermore, we find that

$$\Gamma(\beta) = \frac{\xi^2}{\xi+\beta}\left(\frac{1}{|\alpha_1(\beta+\xi)|} + \frac{1}{|\alpha_2(\beta+\xi)|} - \frac{1}{\nu}\right). \tag{7.8}$$

For the long-run average cost, we have $E(T(\xi)) = 1/\xi$ by definition and

$$E(L_e) = \frac{\eta(0)}{E(T(\xi))} = \left(\frac{\xi}{\alpha_1(\hat{\beta})}\right)\bigg|_{\beta=0} = \frac{\xi}{\alpha_1(\xi)}. \tag{7.9}$$

For the stationary stock level, we find from (7.4) and (7.7) (setting $\beta = 0$) that

$$E(e^{-\alpha W_e}) = \frac{|\alpha_2(\xi)|}{\alpha+|\alpha_2(\xi)|}\,\frac{|\alpha_3(\xi)|}{\alpha+|\alpha_3(\xi)|}\,\frac{\nu+\alpha}{\nu}. \tag{7.10}$$

Note that (7.10) can be written in the form

$$W_e + E_1 \stackrel{D}{=} E_2 + E_3, \tag{7.11}$$

where E_1, E_2, and E_3 are independent exponential random variables that are also independent of W_e and satisfy $E_1 \sim \exp(\nu)$, $E_2 \sim \exp(|\alpha_2(\xi)|)$, and $E_3 \sim \exp(|\alpha_3(\xi)|)$.

Remark: As a by-product of our approach, in the case of exponential jumps we can extend the known result that in the case $\lambda = 0$; that is, for a reflected BM, $W(T(\xi))$ and $L(T(\xi))$ are independent (see, e.g., Asmussen and Perry [4]). To see this, take $Y(t) = L(t)(1 + (\beta/\alpha))$ in (3.1). Then, $Z(t) = W(t) + (\beta/\alpha)L(t)$, and

$$M(t) = \varphi(\alpha)\int_0^t e^{-\alpha W(s)-\beta L(s)}\,ds + 1 - e^{-\alpha W(t)-\beta L(t)} - (\alpha+\beta)\int_0^t e^{-\beta L(s)}\,dL(s)$$

is a martingale. The basic identity becomes

$$\varphi(\alpha)E\left(\int_0^{T(\xi)} e^{-\alpha W(s)-\beta L(s)}\,ds\right) = -1 + E(e^{-\alpha W(T(\xi))-\beta L(T(\xi))}) + (\alpha+\beta)\hat{\eta}(\beta), \tag{7.12}$$

where $\hat{\eta}(\beta) = E(\int_0^{T(\xi)} e^{-\beta L(s)}\,dL(s))$. Let (W_e, L_e) be a pair of random variables whose joint law is the stationary distribution of the two-dimensional regenerative process $(W(t), L(t))$. We can use PASTA and the limit theorem for regenerative processes to obtain

$$E(e^{-\alpha W(T(\xi))-\beta L(T(\xi))}) = E(e^{-\alpha W_e-\beta L_e}) = E(T(\xi))^{-1}E\left(\int_0^{T(\xi)} e^{-\alpha W(s)-\beta L(s)}\,ds\right). \tag{7.13}$$

Thus,

$$(\varphi(\alpha)-\xi)E(e^{-\alpha W(T(\xi))-\beta L(T(\xi))}) = -1 + (\alpha+\beta)\hat{\eta}(\beta). \tag{7.14}$$

Inserting $\alpha_1(\xi)$ for α in (7.14) yields

$$\hat{\eta}(\beta) = (\beta + \alpha_1(\xi))^{-1}. \tag{7.15}$$

Now, factorize $\varphi(\alpha) - \xi$ in (7.14). From (7.14) and (7.15), we obtain, after some algebra (using $\alpha_1(\xi)\alpha_2(\xi)\alpha_3(\xi) = \nu\xi$),

$$E(e^{-\alpha W_e-\beta L_e}) = \frac{|\alpha_2(\xi)|}{\alpha+|\alpha_2(\xi)|}\,\frac{|\alpha_3(\xi)|}{\alpha+|\alpha_3(\xi)|}\,\frac{\nu+\alpha}{\nu}\,\frac{\alpha_1(\xi)}{\beta+\alpha_1(\xi)}. \tag{7.16}$$

Equation (7.16) is tantamount to saying that the steady-state random variables W_e and L_e are independent (because their joint LST factorizes), $L_e \sim \exp(\alpha_1(\xi))$, and W_e satisfies

$$W_e + E_1 \stackrel{D}{=} E_2 + E_3$$

where W_e, E_1, E_2, and E_3 are independent, $E_1 \sim \exp(\nu)$, $E_2 \sim \exp(|\alpha_2(\xi)|)$, and $E_3 \sim \exp(|\alpha_3(\xi)|)$. In the special case $\lambda = 0$, the underlying process is a reflected BM; we have $\alpha_2(\xi) = -\nu$, so that $W_e \sim \exp(|\alpha_3(\xi)|)$. This latter result was given in Asmussen and Perry [4].

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8. EXTENSIONS AND RAMIFICATIONS

It would be interesting to extend the analysis in Section 7 to nonexponential clearing times. This can be done in a straightforward manner for hyperexponential and Erlang clearing times, which are independent of the underlying processes W_n. For example, let $T = T_E \sim \mathrm{Erl}(\xi, n)$ for some $n \in \mathbb{N}$. Then, $T_E = T_1(\xi) + \cdots + T_n(\xi)$, where the $T_i(\xi)$ are independent and $\exp(\xi)$ distributed, and we can treat the n phases recursively. The stationary behavior during the first phase $[0, T_1(\xi))$ was studied in Section 7. Let $V(t) = W(T_1(\xi) + t)$, $t \geq 0$. By (7.4) and (7.1), we obtain for the initial value $V(0)$ of the second phase, the LST

$$E(e^{-\alpha V(0)}) = \frac{\xi(\alpha - \alpha_1(\xi))}{\alpha_1(\xi)(\varphi(\alpha) - \xi)}. \tag{8.1}$$

The basic identity (3.5) for the second phase of T_E (i.e., for the process V) conditional on $V(0) = v$ becomes

$$(\varphi(\alpha) - \beta)E\left(\int_0^{T_2(\xi)} e^{-\alpha V(s) - \beta s}\, ds | V(0) = 0\right)$$
$$= -e^{-\alpha v} + E(e^{-\alpha V(T_2(\xi)) - \beta T_2(\xi)} | V(0) = v) + \alpha\psi(\beta|v), \tag{8.2}$$

where $\psi(\beta|v) = E(\int_0^{T_2(\xi)} e^{-\beta s}\, dL_V(s)|V(0) = v)$, $L_V(\cdot)$ being the local time process of $V = (V(t))_{t\geq 0}$. Clearly, $\psi(\beta|v) = e^{-\alpha_1(\beta+\xi)v}/\alpha_1(\beta + \xi)$, and we can determine explicitly all expected values in (8.2). To derive the functionals for the second phase, we have to integrate the conditional expected values with respect to $P(V(0) \in dv)$. Then, we proceed to the third phase, and so on. Finally, one has to take a mixture over the n phases. The details are left to the reader.

As already explained in Section 1, several combinations of T_q and $T(\xi)$ are also clearing policies of practical interest. We will finally show how their analysis can be reduced to that of T_q and $T(\xi)$.

Let us first consider $T = T_q + T(\xi)$. Under this policy, once level q has been reached, it takes an $\exp(\xi)$-distributed time until the clearing operation. We can split the time until clearing in the two phases $[0, T_q]$ and $(T_q, T_q + T(\xi)]$. Clearly,

$$E\left(\int_0^T e^{-\alpha W(s) - \beta s}\, ds\right) = E\left(\int_0^{T_q} e^{-\alpha W(s) - \beta s}\, ds\right)$$
$$+ E(e^{-\beta T_q})E\left(\int_0^{T(\xi)} e^{-\alpha W_q(s) - \beta s}\, ds\right), \tag{8.3}$$

where $W_q(t)$ is the stock level process starting at $W_q(0) = q$. In (8.3), the expected values involving T_q have been determined in Section 4, and the functional of $W_q(\cdot)$ can be computed as earlier for the second phase in the Erlang case.

Now, let us turn to the more difficult case $T = \min[T_q, T(\xi)]$. Under this T, clearing takes place after an exponential time *or* at the next crossing of the threshold q, whichever occurs first. We start from the decomposition

$$E(e^{-\alpha W(T)-\beta T}) = E(e^{-\alpha W(T)-\beta T}1_{\{T_q<T(\xi)\}}) + E(e^{-\alpha W(T)-\beta T}1_{\{T_q\geq T(\xi)\}}). \quad \textbf{(8.4)}$$

The first term on the right-hand side of (8.4) is given by

$$\begin{aligned} E(e^{-\alpha W(T)-\beta T}1_{\{T_q<T(\xi)\}}) &= ((e^{-\alpha W(T_q)-\beta T_q}1_{\{T_q<T(\xi)\}}|T_q)) \\ &= E(\mathrm{E}(e^{-\alpha W(T_q)-\beta T_q}e^{-\xi T_q}|T_q)) \\ &= E(e^{-\alpha W(T_q)-(\beta+\xi)T_q}), \end{aligned} \quad \textbf{(8.5)}$$

and the right-hand side of (8.5) has been computed in Section 4. The second term on the right-hand side of (8.4) is equal to

$$E(e^{-\alpha W(T(\xi))-\beta T(\xi)}) - E(e^{-\alpha W(\xi))-\beta T(\xi)}1_{\{T_q<T(\xi)\}}). \quad \textbf{(8.6)}$$

The LST $E(e^{-\alpha W(T(\xi))-\beta T(\xi)})$ has already been derived in Section 7. Regarding the second term in (8.6), note that, by the memoryless property of the distribution of $T(\xi)$, the conditional distribution of the pair $(W(T(\xi)), T(\xi))$, given that $W(T_q) = w$ and $T_q = t < T(\xi)$, is equal to the unconditional distribution of $(W(T(\xi)), t + T(\xi))$, where $W(\cdot)$ starts at w. Hence,

$$\begin{aligned} &E(e^{-\alpha W(T(\xi))-\beta T(\xi)})1_{\{T_q<T(\xi)\}}) \\ &\quad = P(T_q < T(\xi))E(e^{-\alpha W(T(\xi))-\beta T(\xi)}|T_q < T(\xi)) \\ &\quad = \int_0^\infty e^{-\xi t}\, dP_{T_q}(t) \\ &\qquad \times \int_{\mathbb{R}_+^2} E(e^{-\alpha W(T(\xi))-\beta T(\xi)}|T_q < T(\xi), W(T_q) = w, T_q = t)\, dP_{(W(T_q),T_q)}(w,t) \\ &\quad = E(e^{-\xi T_q}) \int_{\mathbb{R}_+^2} E(e^{-\alpha W(T(\xi))-\beta(t+T(\xi))}|W(0) = w)\, dP_{(W(T_q),T_q)}(w,t). \end{aligned} \quad \textbf{(8.7)}$$

We have shown above how to compute the joint distribution of $W(T_q)$ and T_q as well as the integrand in (8.7). Once we know $E(e^{-\alpha W(T)-\beta T})$, we obtain $\eta(\beta)$ from (3.5) by setting $\alpha = \alpha_1(\beta)$. Then, $E(\int_0^T e^{-\alpha W(s)-\beta s}\, ds)$ is given by (3.5), and all cost functionals under T are available.

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