J. Inst. Math. Jussieu (2015) 14(3), 577–588
doi:10.1017/S1474748014000097

ON THE BASE-POINT-FREE THEOREM OF 3-FOLDS IN POSITIVE CHARACTERISTIC

CHENYANG XU

Beijing International Center of Mathematical Research, 5 Yiheyuan Road, Beijing, 100871, China (cyxu@math.pku.edu.cn)

(Received 19 November 2013; revised 16 March 2014; accepted 16 March 2014; first published online 31 March 2014)

Abstract Let (X, Δ) be a projective klt (standing for Kawamata log terminal) three-dimensional pair defined over an algebraically closed field k with $\mathrm{char}(k) > 5$. Let L be a nef (numerically eventually free) and big line bundle on X such that $L - K_X - \Delta$ is big and nef. We show that L is indeed semi-ample.

1. Introduction

Throughout the paper, the ground field will be an algebraically closed field k of characteristic $p > 0$. The main purpose of this paper is to prove the following theorem.

Theorem 1.1. *Assume k to be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional klt pair which is projective over a quasi-projective variety U. Assume that L is a relatively big and nef $\mathbb{Q}$-divisor such that $K_X + \Delta + L$ is big and nef over U. Then $K_X + \Delta + L$ is semi-ample over U.*

For $k = \overline{\mathbb{F}}_p$, this is proved by Keel in [16]. In fact, Keel proved that in general $K_X + \Delta + L$ is *endowed with a map* (EWM), i.e., there exists a morphism $f : X \to Z$ to an algebraic space Z such that a curve vertical over U is contracted by f if and only if its intersection with $K_X + \Delta + L$ is 0. Furthermore, to check that $K_X + \Delta + L$ is semi-ample over U, it suffices to show that the restriction $(K_X + \Delta + L)|_{\mathrm{Ex}(f)}$ is semi-ample over U where $\mathrm{Ex}(f)$ is the exceptional locus of $f : X \to Z$.

Besides using Keel's results, our approach also relies heavily on the recent results on the minimal model program (MMP) in dimension 3 in positive characteristic (see [5, 15]). More precisely, when $p > 5$, by combining with Shokurov's idea (cf. [22]) of reduction to a special MMP and the recent development of lifting sections coming from Frobenius image initiated in [13] (see [21] and references therein), it is proved that an MMP sequence can be run in a generalized sense in [15] (cf. Section 3), where the coefficients are also assumed to be contained in the standard set

$$\left\{\frac{n-1}{n} | n \in \mathbb{N}\right\}.$$

Later, using Shokurov's reduction technique again, [5] removes the restriction on the coefficients by reducing the general case to the one in [15]. Using this existence of a sequence of a generalized MMP, after passing to an étale covering of the algebraic space Z given by [16], a standard trick of running an MMP can change the model X over Z such that the exceptional locus $\mathrm{Ex}(f)$ is contained in $\lfloor\Delta\rfloor$ for a plt (purely log terminal) pair $K_X+\Delta$. Then Theorem 1.1 follows from Keel's theorem in the semi-ample case and the abundance for surfaces.

Now we want to apply Theorem 1.1 to get the standard consequences on birational contractions when we run an MMP for a klt pair (X,Δ). Let R be a $(K_X+\Delta)$-negative extremal ray over U, and $L=K_X+\Delta+H$ be a nef divisor for some ample $\mathbb{Q}$-divisor H such that

$$L^{\perp}\cap\overline{NE}(X/U)=R.$$

Assume that L is big over U; then it follows from [16] that L is relatively EWM (see below) over U with a birational contraction $f:X\to Z$ to an algebraic space.

By Theorem 1.1, we obtain the contraction theorem, i.e., we can run the minimal model program (MMP) in the original sense.

Theorem 1.2. *Under the above notation. Assume that k is an algebraically closed field of characteristic $p>5$. Assume $h:(X,\Delta)\to U$ to be a klt pair projective over a quasi-projective variety U. Then*

(1) *Z is a quasi-projective variety, $\rho(X)-\rho(Z)=1$, and*
(2) *if f is small, then there exists a flip $f^+:X^+\to Z$.*

Therefore, we see that the concept of a 'generalized MMP' is no longer needed. But it is still conceptually important as an intermediate step. In other words, unlike in characteristic 0, where the base-point-free theorem was established much earlier than the existence of flips, to prove the base-point-free theorem in characteristic $p>0$, experience suggests that we might need to first establish everything for the MMP of 'special type' in the sense of Shokurov.

The note is organized in the following way. We discuss some preliminary results in § 2; in particular, we state Keel's theorems in [16] for the relative case. Then in § 3, we survey the results of [5, 15] on running a generalized MMP that we will need. Finally, we finish the proof of Theorem 1.1 in § 4.

Notation and conventions: We follow the notation as in [20]. For any divisor Γ on a normal variety X and birational map $X\dashrightarrow X'$ to a normal variety X', we will denote by $\Gamma_{X'}$ its birational transform on X'. For $f:X\to Z$ a birational morphism, we denote by $\mathrm{Ex}(f)$ or $\mathrm{Ex}(X/Z)$ the exceptional set.

2. Preliminaries

In this section, we discuss some background. We note that the resolution of singularities is known in dimension 3 in arbitrary characteristic (see [1, 6, 7, 9]).

2.1. Basic facts

Lemma 2.1 (The Negativity Lemma)**.** *Let $f:X\to X'$ be a proper birational morphism from a quasi-projective normal variety to a normal algebraic space. Let E be an effective f-nef $\mathbb{Q}$-divisor which is exceptional over X'; then $E=0$.*

https://doi.org/10.1017/S1474748014000097 Published online by Cambridge University Press

Proof. It suffices to observe that the usual argument (see [20, 3.39]) of cutting X by hyperplanes still holds in this situation. □

Lemma 2.2. *Let (X, D) be a simple normal crossing pair, and $|H|$ a very ample linear system. Then there exists a prime divisor $A \in |H|$ such that $(X, D + A)$ is simple normal crossing.*

Proof. Write $D = \sum_{i=1}^{N} D_i$, where $D_1, \ldots, D_N$ are prime divisors. For any $I \subset \{1, 2, \ldots, N\}$, let

$$D_I^j \subset \bigcap_{i \in I} D_i$$

be an irreducible component of positive dimension. We denote by $|H|_{D_I^j}$ the restricted linear system, i.e., the one corresponding to the image of the restriction map

$$h_I^j : H^0(X, H) \to H^0(D_I^j, H|_{D_I^j}).$$

By Bertini's theorem (see [12, II 8.18]), we know that there exists an open subset $V_I^j \subset |H|_{D_I^j}$ such that any element $A_I^j \in V_I^j$ corresponds to a smooth hypersurface of D_I^j. For any D_I^j, by definition

$$f_I^j : |H| \setminus Z_I^j \to |H|_{D_I^j}$$

is surjective, where Z_I^j is the proper linear subsystem given by the kernel of h_I^j. Thus, we can take

$$H \in \bigcap_{I,j} (f_I^j)^{-1}(V_I^j).$$ □

Proposition 2.3 (The Bertini Theorem). *Let (X, Δ) be a quasi-projective klt pair and A an ample $\mathbb{Q}$-divisor. Assume that $f : Y \to (X, \Delta)$ is a log resolution such that the exceptional locus $\mathrm{Ex}(f)$ supports a relatively anti-ample divisor E. Then there exists a $\mathbb{Q}$-divisor Δ' such that $\Delta' \sim_{\mathbb{Q}} \Delta + A$ and (X, Δ') is klt. Furthermore, in dimension 3, we can show the same result only assuming A to be big and nef.*

Proof. By the negativity lemma (see [20, 3.39]), we know that $E \geqslant 0$. Since it is relatively anti-ample, we have $\mathrm{Supp}(E) = \mathrm{Ex}(f)$.

Write $f^*(K_X + \Delta) = K_Y + \Delta_Y$ (Δ_Y may not be effective). Then by the assumption we know that $(Y, \mathrm{Supp}(\Delta_Y + \mathrm{Ex}(f)))$ is simple normal crossing. Since (X, Δ) is klt, if we write $\Delta_Y = \sum_i a_i E_i$, then $a_i < 1$. There exists a sufficiently small $\epsilon > 0$ such that $f^*A - \epsilon E$ is ample on Y and the coefficients of $\Delta_Y + \epsilon E$ are strictly less than 1.

Then we can choose $H \sim_{\mathbb{Q}} f^*A - \epsilon E$ such that the coefficients of $\Delta_Y + \epsilon E + H$ are strictly less than 1 and $(Y, \mathrm{Supp}(\Delta_Y + E + H))$ is simple normal crossing (cf. Lemma 2.2). Thus for any divisorial valuation v of $K(X)$, its discrepancy $a(v, Y, \Delta_Y + \epsilon E + H) > -1$. Therefore, we can choose $\Delta' = f_*(\Delta_Y + \epsilon E + H)$.

In dimension 3, the log resolution of any pair exists and as it is obtained by a sequence of blow-ups with smooth centers, we know that the exceptional locus of such a log resolution

https://doi.org/10.1017/S1474748014000097 Published online by Cambridge University Press

always supports a relatively anti-ample divisor E. We can write $A \sim_{\mathbb{Q}} A' + G$ for an ample $\mathbb{Q}$-divisor A' and a sufficiently small effective $\mathbb{Q}$-divisor G such that $(X, \Delta + G)$ is still klt. Then we only need to apply the case of A being ample. □

Proposition 2.4. *Let X be a connected variety. If $\pi : X_1 \to X$ is a finite flat morphism, and π^*L is trivial for a line bundle L on X, then L is a torsion in* $\mathrm{Pic}(X)$.

Proof. Since by our assumption $\pi_* N$ is locally free on X for any line bundle N on X, then we can define (see [10, 6.5.1, hypothesis I])

$$\mathrm{Nm} : \mathrm{Pic}(X_1) \to \mathrm{Pic}(X),$$

with the property that $\mathrm{Nm} \circ \pi^*(L) = L^{\otimes d}$, where $d = \deg(\pi)$. As $\mathrm{Nm}(\mathcal{O}_{X'}) = \mathcal{O}_X$, we conclude that L is a torsion in $\mathrm{Pic}(X)$. □

2.2. The relative version of Keel's theorem

For the relative setting of a proper morphism between quasi-projective varieties, we can take a projectivization and then work in the category of projective varieties. For our purpose, we would like to directly treat it, which requires us to generalize Keel's theorems to the relative case for $X \to Z$ a proper morphism between quasi-projective varieties. The argument is essentially verbatim.

Definition 2.5. Let $f : X \to Z$ be a proper morphism between quasi-projective schemes. Let L be a relatively nef line bundle. For a subvariety $W \subset X$, we denote the Stein factorization as $W \to V \to Z$. We say that W is *relatively exceptional* (for L) if $L^{\dim W_\eta}|_{W_\eta} = 0$, where η is the generic point of V. We denote by $\mathbb{E}(L/Z)$ the Zariski closure of the union of all relatively exceptional varieties (with reduced structure).

Definition 2.6. With the above notation, we say that L is *endowed with a map over* Z (or *EWM* over Z), if there is a morphism $g : X \to Y$ to an algebraic space over Z with the property that a subvariety W is contracted by g if and only if it is relatively exceptional.

Then we have the following relative version of Keel's theorem.

Proposition 2.7. *Let $f : X \to Z$ be a proper morphism between quasi-projective schemes. Let L be a nef line bundle on X. Then L is EWM (resp. semi-ample) over Z if and only if $L|_{\mathbb{E}(L)}$ is EWM (resp. semi-ample) over Z.*

Proof. The proof just follows Keel's original one. We will briefly sketch it for the reader's convenience. First, as the natural morphism $X_{\mathrm{red}} \to X$ can factor through the iterated geometric Frobenius

$$F^q : X \to X^{(q)}_{\mathrm{red}} \to X^{(q)}$$

for some $q = p^r$ $(r \gg 0)$ (see [18, 6.6]) and $(F^q)^*(L) = L^{\otimes q}$, we can assume that X is reduced.

https://doi.org/10.1017/S1474748014000097 Published online by Cambridge University Press

We write $X = X_1 \cup X_2$, where $X_1 = \mathbb{E}(L/Z)$ and X_2 is the union of components of X on which the restriction of L is big. If $X_2 \neq X$, from the induction, we can assume that $L|_{X_2}$ is EMW (resp. semi-ample). Now the proof of [16, 2.6] also works in this relative setting.

Therefore, we can assume that L is big on X, i.e.,

$$nL \sim_Z A + D$$

for some $n \in \mathbb{N}$, an ample divisor A and an effective divisor D. As in the proof of [16, 1.6], we can use [3, 3.1, 6.2] to construct an algebraic space Z such that $f : X \to Z$ is the endowed map for L.

In the case of $L|_{\mathbb{E}(L)}$ being semi-ample, the same argument as in [16, 1.10] implies that L is also semi-ample. □

Remark 2.8. When Z is an algebraic space, the part for concluding that f is EWM if $f|_{\mathbb{E}(L)}$ is EMW follows from a standard descent argument. But we do not know whether the statement for semi-ampleness holds or not.

We need the following statement.

Corollary 2.9. *Let X and Y be quasi-projective normal varieties which are projective over a quasi-projective Z with a birational morphism $f : X \to Y$ over Z. Let Δ be a $\mathbb{Q}$-divisor on X such that (X, Δ) is a dlt (divisorial log terminal) pair. Let $S \subset \lfloor \Delta \rfloor$ be a normal prime divisor such that $-(K_X + \Delta)|_S$ is f-ample and $C \cdot S < 0$ for any contracted curve C. If L is a line bundle on X such that $L \cdot C = 0$ for any contracted curve C, then $L \sim_{\mathbb{Q}} f^* L_Y$ for some $\mathbb{Q}$-line bundle L_Y on Y.*

Proof. As we assume S to be normal, we can write $(K_X + \Delta)|_S = K_S + \mathrm{Diff}_S \Delta$. Thus

$$L|_S - S - \mathrm{Diff}_S \Delta$$

is ample over Y and $(S, \mathrm{Diff}_S \Delta)$ is dlt. Since $S \to f(S)$ has connected fibers, we know that the normalization morphism $f(S)^{\mathrm{n}} \to f(S)$ is a finite and universal homeomorphism. Therefore, it follows from the abundance theorem for log canonical surfaces (see e.g [23, 15.2]) that $L|_S \sim_{\mathbb{Q}, f(S)^{\mathrm{n}}} 0$, which implies that $L|_S \sim_{\mathbb{Q}, f(S)} 0$ by [18, 6.6]. Thus by Proposition 2.7, L is $\mathbb{Q}$-linearly equivalent to 0 over Y, i.e.

$$L \sim_{\mathbb{Q}} f^*(L_Y)$$

for some $\mathbb{Q}$-line bundle on L_Y. □

Remark 2.10. We also remark that Keel's cone theorem [16, 5.5.2] holds for the relative setting X/U, where $X \to U$ is a projective morphism to a quasi-projective variety. His original proof can be directly applied without any change.

3. Running a generalized MMP for 3-folds

In this section, we assume k to be an algebraically closed field with $\mathrm{char}(k) > 5$. We provide a short sketch of the proof for the results from [5, 15] that we will need. Since we

https://doi.org/10.1017/S1474748014000097 Published online by Cambridge University Press

still need to prove the base-point-free theorem now, *a priori* we cannot run the MMP in the original sense. In [15], a notion called the *generalized MMP* was invented. In fact, for a three-dimensional klt pair (X, Δ), using Shokurov's idea in [22] of reducing the MMP to a 'special' MMP, it is proved (see [5, 15]) that we can always run a generalized MMP over Z in one of the following two cases:

(I) when X is projective over some quasi-projective variety U, and $X \to Z$ is the relative endowed map for some big and nef divisor L over U;

(II) when Z is quasi-projective.

Definition 3.1. For a dlt pair $(X, \Delta)/U$, let $S = \lfloor \Delta \rfloor$. We call an extremal birational contraction $X \to Y$ induced by a $(K_X + \Delta)$-negative ray R of $NE(X/U)$ of *special type* if $R \cdot S_i < 0$ for some component $S_i \subset S$.

If follows from [16] (and Proposition 2.7) that if (X, Δ) is a three-dimensional dlt pair projective over a quasi-projective variety U and $K_X + \Delta \sim_{\mathbb{Q},U} E \geqslant 0$, then there exists an ample divisor H such that

$$(K_X + \Delta + H)^{\perp} \cap \overline{NE}(X/U) = R.$$

If we assume that X is $\mathbb{Q}$-factorial, then each component S_i of $\lfloor \Delta \rfloor$ is normal by [15, 4.1]. Furthermore, if $R \cdot S_i < 0$, Proposition 2.7 and the arguments used in Corollary 2.9 imply that the extremal contraction $X \to Z$ exists and we get Z as a quasi-projective variety. If $X \to Z$ is small, then it follows from [15, Theorem 1.1] that the flip $X^+ \to Z$ exists. Thus, if we start from a $\mathbb{Q}$-factorial dlt three-dimensional pair $(X, \Delta)/U$, and we assume that we run a sequence of an MMP of special type such that each time the contraction is birational, e.g., $X \to U$ is birational, then we will have a sequence of models

$$(X, \Delta) = (X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow (X_2, \Delta_2) \cdots \dashrightarrow (X_n, \Delta_n),$$

where Corollary 2.9 can be applied to show that each model X_i is $\mathbb{Q}$-factorial. Special termination (see [11, 4.2.1] or [5, 4.7]), which holds in this case, implies that the MMP will end with a relatively minimal model X_n over Z.

Let first introduce the definition of a generalized MMP.

Definition 3.2. Let (X, Δ) be a klt pair which is projective over a quasi-projective variety U and R an extremal ray. Let $f : X \to Z$ be the birational extremal contraction corresponding to a $(K_X + \Delta)$-negative extremal ray

$$R = \mathbb{R}_{\geqslant 0}[C] \subset \overline{NE}(X/U)$$

with the target space possibly being an algebraic space, i.e., Z is a normal algebraic space and $f : X \to Z$ is the endowed map (by Proposition 2.7) for a big and nef line bundle $L = K_X + \Delta + H$ satisfying

$$L^{\perp} \cap \overline{NE}(X/U) = R.$$

We say that (X^+, Δ^+) is *a step of the generalized MMP* if $X \dashrightarrow X^+$ is birational, $\Delta = \phi^+_* \Delta$ and $K_{X^+} + \Delta^+$ is nef over Z. When $\phi^+ : X \dashrightarrow X^+$ is isomorphic in codimension 1, we call it a *generalized flip*.

https://doi.org/10.1017/S1474748014000097 Published online by Cambridge University Press

Theorem 3.3 ([5, 15]). *Let k be an algebraically closed field with* $\mathrm{char}(k) > 5$. *Assume X to be a $\mathbb{Q}$-factorial 3-fold. Let $(X, \Delta) \to U$ be a klt pair projective over a quasi-projective variety U. Assume that $f : X \to Z$ is given as in Definition 3.2; then a step of the generalized MMP always exists, i.e., we can always find X^+ as in Definition 3.2.*

Proof. If the coefficients of Δ are contained in $\{\frac{n-1}{n}|n \in \mathbb{N}\} \cup \{1\}$, then this follows from [15, 5.6]. We remark that since we have the relative cone theorem 2.10, the relative contraction theorem 2.7 and the special flip, then the argument in [15, 5.6] holds in this relative setting.

In general, we repeat the argument of [5], which reduces the general case to the above case of standard coefficients as follows.

Write $\Delta = \sum_{i=1}^{m} a_i \Delta_i$, where $\Delta_1, \ldots, \Delta_m$ are distinct prime divisors. We will show that if we can run a generalized MMP for any klt pair (X', Δ') birational over Z with coefficients in

$$I_0 := \left\{\frac{n-1}{n}|n \in \mathbb{N}\right\} \bigcup \{a_1, \ldots, a_{m-1}\} \bigcup \{1\},$$

then we have a step of the generalized MMP for $K_X + \Delta$; hence the theorem follows from the induction.

As part of our induction, we also assume that a step of the generalized MMP for the coefficients in I_0 is given by a sequence of birational models:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \cdots \dashrightarrow X_n = X^+,$$

where $X_i \dashrightarrow X_{i+1}$ is one of the following two operations:

(1) $X_{i+1} \to (X_i, \Delta_i)$ is a log resolution for some divisor Δ_i;

(2) $X_i \dashrightarrow X_{i+1}$ is a sequence of the special MMP with standard coefficients, which can be run by [15].

Write $\Delta = \Delta_1 + \Delta_2$, where $\Delta_1 = \sum_{i=1}^{m-1} a_i \Delta_i$ and $\Delta_2 = a_m \Delta_m$.

Assume that $\Delta_m \cdot R \geqslant 0$. Then we know that $\Delta_2 \equiv_Z -t(K_X + \Delta_1)$ for some $t \in [0, 1)$. By induction, we can run a generalized MMP for (X, Δ_1) over Z, which provides a sequence of birational models:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_n = X^+.$$

Using induction, we also know that $X_i \dashrightarrow X_{i+1}$, where $X_i \dashrightarrow X_{i+1}$ is either a log resolution for some divisor Δ_i or a sequence of the special MMP with standard coefficients.

In particular, by repeatedly applying Corollary 2.9, we see that if there is a $\mathbb{Q}$-line bundle L on X such that $L \equiv_Z 0$, then there is a $\mathbb{Q}$-line bundle L^+ on X^+ such that if we pull back L and L^+ to a common resolution, we get two $\mathbb{Q}$-line bundles which are relatively $\mathbb{Q}$-linear equivalent to each other. Thus

$$K_{X^+} + (\Delta_1)_{X^+} + (\Delta_2)_{X^+} \equiv (1-t)(K_{X^+} + (\Delta_1)_{X^+}),$$

which is nef.

Now we assume that $\Delta_2 \cdot R < 0$. We can apply the argument of [15, 5.6] by choosing $T = \Delta_m$. Then we get a model W such that $K_W + (\Delta_1)_W + E_W + T_W$ is nef over Z, where E_W is the divisorial part of the exceptional locus $\mathrm{Ex}(W/Z)$. Now we run a generalized MMP of $K_W + (\Delta_1)_W + E_W$ with scaling of T_W over Z, which exists by our induction assumption. As argued in [15, 5.6], this is of special type. In particular, it terminates. By the definition of the MMP with scaling, we know that it provides a model

$$W \dashrightarrow W_r = X^+$$

such that

$$K_{X^+} + (\Delta_1)_{X^+} + E_{X^+} + a_m T_{X^+}$$

is nef, where E_{X^+} is the push-forward of E_W on X^+. Then the negativity lemma implies that $E_{X^+} = 0$.

In each case, we also see that the map $X \dashrightarrow X^+$ can be decomposed into maps of type (1) and type (2). □

Remark 3.4. As we already pointed out in the argument, by Corollary 2.9, we know that if we have a $\mathbb{Q}$-line bundle L on X such that $L \equiv_Z 0$, then there is a $\mathbb{Q}$-line bundle L^+ on X^+ such that the pull-backs of L and L^+ to a common resolution are $\mathbb{Q}$-linear equivalent to each other.

As a standard consequence of running a generalized MMP and special termination, we know the following:

Lemma 3.5. *Assume that k is an algebraically closed field with* $\mathrm{char}(k) > 5$. *If (X, Δ) is a log canonical three-dimensional pair, then a $\mathbb{Q}$-factorial dlt modification exists.*

If we assume that $K_X + \Delta$ is effective over U, the termination of the generalized MMP is proved in [5] following an idea from [4]. It is shown that the termination of a sequence of generalized flips in this case is implied by the three-dimensional ascending chain condition (ACC) of log canonical thresholds. Then the ACC of three-dimensional log canonical thresholds is a corollary of the two-dimensional global ACC (cf. [17, 18.21] or [14, Section 5]). The latter was proved by Alexeev in [2].

To summarize, we have the following result which we need later.

Theorem 3.6 ([5, 15]). *Assume k to be an algebraically closed field with* $\mathrm{char}(k) = 5$. *Let (X, Δ) be a klt three-dimensional quasi-projective pair with a proper morphism to Z such that one of the following cases holds:*

(I) *X is projective over some quasi-projective variety U, and $X \to Z$ is the relative endowed map for some big and nef divisor L over U;*

(II) *Z is quasi-projective and $K_X + \Delta \sim_{\mathbb{Q},Z} E \geqslant 0$.*

Then we can run a generalized MMP of (X, Δ) over Z to obtain a minimal model $(X^{\mathrm{m}}, \Delta^{\mathrm{m}})$ of (X, Δ) over Z.

Furthermore, if L is a $\mathbb{Q}$-line bundle on X such that $L \equiv_Z 0$, then there exists a $\mathbb{Q}$-line bundle $L_{X^{\mathrm{m}}}$ on X^{m} such that the pull-backs of L and $L_{X^{\mathrm{m}}}$ to a common resolution are $\mathbb{Q}$-linearly equivalent.

4. Proof

In this section, we always assume that k is an algebraically closed field with $\mathrm{char}(k) > 5$. Let (X, Δ) be a klt pair and L be a big and nef $\mathbb{Q}$-divisor such that $K_X + \Delta + L$ is also big and nef. By a theorem of Bertini type, Theorem 2.3, after replacing $\Delta + L$ by Δ', we can indeed assume that $K_X + \Delta$ is big and nef, and we aim to show that $K_X + \Delta$ is semi-ample. Let $X \to Z$ be the endowed map for $K_X + \Delta$ provided by [16].

Lemma 4.1. *Assume the above notation. There is a birational contraction $f : X \dashrightarrow Y$ over Z such that $(Y, \Delta_Y = f_*(\Delta))$ is klt and* $\dim \mathbb{E}(K_Y + \Delta_Y) = 1$.

Proof. Let S be the sum of divisorial components $\dim \mathbb{E}(K_X + \Delta)$. Let $\epsilon > 0$ be sufficiently small that $K_X + \Delta_X + \epsilon S$ is klt. By Theorem 3.6, we can run a generalized MMP of $(X, \Delta + \epsilon S)$ over Z to obtain a birational model $f : X \dashrightarrow Y$ such that $K_Y + \Delta_Y + \epsilon S_Y$ is nef over Z, where Δ_Y and S_Y are the push-forwards of Δ and S.

From the assumption that $S \subset \dim \mathbb{E}(K_X + \Delta)$, we know that S is exceptional over Z. By the negativity lemma, Lemma 2.1, we know that the components of S are precisely the divisors contracted by $X \dashrightarrow Y$. □

Proof of Theorem 1.1. Replacing X by Y, we can assume that $\dim \mathbb{E}(K_X + \Delta) = 1$. Replacing X by its $\mathbb{Q}$-factorialization, we can then assume that X is $\mathbb{Q}$-factorial. By Keel's theorem, if we define $L = K_X + \Delta$, then it suffices to show that $L|_{\mathbb{E}(K_X+\Delta)}$ is semi-ample, i.e., for any closed point $p \in Z$, assume that $C = f^{-1}(p)$ is a connected curve; then $L|_C$ is semi-ample.

We first assume that $p \in Z$ is contained in a quasi-projective open neighborhood of Z. After possibly replacing Z by a neighborhood, we can assume that Z is quasi-projective and C is the exceptional locus $\mathrm{Ex}(X/Z)$.

The following construction is standard in characteristic 0, and is a combination of running an MMP after using the X-method.

Proposition 4.2. *Assume Z to be quasi-projective. Locally over Z, we can find an effective $\mathbb{Q}$-divisor $H \sim_{\mathbb{Q}} L$ such that if we define*

$$c = \mathrm{lct}(X, \Delta; H)$$

to be the log canonical threshold along C, then:

(1) *$(X, \Delta + cH)$ is plt along C.*

(2) *There exists a $\mathbb{Q}$-factorial model W/Z and a prime divisor E on W such that $(W, E + \Delta_W + cH_W)$ is plt where Δ_W and H_W are the birational transforms of Δ and H on W, and if $p : U \to X$ and $q : U \to W$ is a common resolution, then*

$$p^*(K_X + \Delta + cH) = q^*(K_W + \Delta_W + cH_W + E).$$

(3) *$-E_W$ is nef over Z.*

https://doi.org/10.1017/S1474748014000097 Published online by Cambridge University Press

Proof. As L is big, we can write it as $A+B$ where $B \geqslant 0$ and A is ample. It is standard that after using A in order to tie-break (see e.g. [14, 3.2.3]), we can find a $\mathbb{Q}$-divisor $H \sim_{\mathbb{Q}} L$ such that for

$$c = \mathrm{lct}(X, \Delta; H)$$

the log canonical threshold along C, the pair $(X, \Delta + cH)$ is log canonical with precisely one divisorial valuation v such that the discrepancy $a(v; X, \Delta + H) = -1$.

After possibly shrinking Z, applying Lemma 3.5, we can find $h : V \to X$ which is a $\mathbb{Q}$-factorial dlt modification of $(X, \Delta + cH)$, i.e., V is $\mathbb{Q}$-factorial and $(V, \Delta_V + cH_V + E)$ is dlt where E is the divisorial part of $\mathrm{Ex}(h)$ which corresponds to the valuation v. Hence

$$h^*(K_X + \Delta + cH) = K_V + \Delta_V + cH_V + E.$$

Then by Theorem 3.6, we can run a generalized MMP for

$$K_V + \Delta_V + cH_V \equiv_Z -E$$

over Z as in § 3, and we end up with a relatively minimal model $f_W : W \to Z$. Since each step of the generalized MMP is E-positive, E is not contracted in $V \dashrightarrow W$. Then we define E_W to be the push-forward of E on W. □

It follows from Remark 3.4 that $K_W + \Delta_W + cH_W + E_W$ is numerically trivial over Z. By [15, 4.1] and the fact that W is $\mathbb{Q}$-factorial, we know that E_W is normal as (W, E_W) is plt. Thus, restricting on E_W, we have that

$$K_W + \Delta_W + cH_W + E_W|_{E_W} = K_{E_W} + \mathrm{Diff}_{E_W}(\Delta_W + cH_W)$$

is numerically trivial and $(E_W, \mathrm{Diff}_{E_W}(\Delta_W + cH_W))$ is klt. Therefore it follows from the abundance for surfaces (cf. e.g. [23]) that

$$K_{E_W} + \mathrm{Diff}_{E_W}(\Delta_W + cH_W) \sim_{\mathbb{Q}} 0.$$

Since $-E_W$ is nef over Z we know that

$$E_W = f_W^{-1}(f_W(E_W)) \supset \mathrm{Ex}(f_W),$$

and hence $K_W + \Delta_W + cH_W + E_W$ is $\mathbb{Q}$-linearly trivial over Z by Proposition 2.7, i.e.,

$$K_W + \Delta_W + cH_W + E_W \sim_{\mathbb{Q}} f_W^*(A)$$

for some $\mathbb{Q}$-line bundle A on Z. By Theorem 3.6, we know that

$$K_Y + \Delta_Y + cH_Y + E \sim_{\mathbb{Q}} K_X + \Delta + cH \sim_{\mathbb{Q}} f^*(A).$$

Thus $(K_X + \Delta)|_C \sim_{\mathbb{Q}} 0$.

Now let us consider the general case. For $p = f(C) \subset Z$, let $\pi : \tilde{Z} \to Z$ be the étale map from a quasi-projective étale neighborhood of p. Let

$$\pi^X : X \times_Z \tilde{Z} \to X$$

be the étale morphism. We define $C_1 = (\pi^X)^{-1}(C)$ which is finite étale over C. Thus it follows from Proposition 2.4 that $\phi^* nL$ is trivial for n sufficiently divisible. □

https://doi.org/10.1017/S1474748014000097 Published online by Cambridge University Press

Proof of 1.2. (1) The assertion that Z is an algebraic variety follows from Theorem 1.1. In particular, we conclude that $L = f^*L_Z$ for some ample divisor L_Z. We also have the exact sequence

$$0 \to \mathrm{Pic}(Z)_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}} \to \mathbb{Q} \to 0,$$

where the last morphism is given by the intersection with a curve class $[C]$ in R. In fact, if L' is a line bundle on X such that $L' \cdot [C] = 0$, then it follows from the cone theorem, for sufficiently small $\epsilon > 0$, that $L + \epsilon L'$ is still big and nef over U and

$$(L + \epsilon L')^{\perp} \cap \overline{NE}(X/U) = R.$$

Thus we conclude that $L + \epsilon L'$ is semi-ample, and its multiple will be a pull-back of an ample $\mathbb{Q}$-divisor on Z.

For (2), we know that there exists a generalized flip $f' : X' \to Z$ such that $K_{X'} + \Delta'$ is nef over Z where Δ' is the push-forward of Δ to X'. It follows from the cone theorem that we can find a sufficiently ample divisor H_Z on Z such that $K_{X'} + \Delta' + f'^* H_Z$ is big and nef and for which all $(K_{X'} + \Delta' + f'^* H_Z)$-trivial curves are vertical over Z. So $K_{X'} + \Delta' + f'^* H_Z$ is base-point-free by Theorem 1.1. Therefore, we can take

$$X^+ := \mathrm{Proj} \bigoplus_{m=0} f'_* \mathcal{O}_{X'}(m(K_{X'} + \Delta' + f'^* H_Z)),$$

which admits a morphism $f^+ : X^+ \to Z$ yielding the flip. □

Remark 4.3. The base-point-free theorem in characteristic 0 is proved for nef L, which can be written as $L \sim_{\mathbb{Q}} K_X + \Delta + A$ for a klt pair (X, Δ) and a big and nef $\mathbb{Q}$-divisor A. However, in characteristic $p > 0$, due to the existence of inseparable morphisms, it is still not known how one would deal with the case of L not big. There are partial results ($\kappa(L) = 1$ or 2) in [8] with more restrictive assumptions.

Acknowledgements. We would like to thank Caucher Birkar for communications. We started to write our preprint after Caucher Birkar sent us the first draft of his work [5], where the generalized MMP was extended to arbitrary coefficients. Later, when we finished this preprint, he sent us his second draft in which the main theorem, Theorem 1.1, is proved independently by a different method. We also want to thank Bhargav Bhatt and Hiromu Tanaka for helpful suggestions and the anonymous referee for many useful remarks on the exposition. CX was partially supported by the Chinese grant 'Recruitment Program of Global Experts' and Qiushi Outstanding Youth Scholarship.

References

1. S. S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces,* Second edn., Springer Monographs in Mathematics, (Springer-Verlag, Berlin, 1998).
2. Valery Alexeev, Boundedness and K^2 for log surfaces, *Int. J. Math.* **5**(6) (1994), 779–810.

3. M. Artin, Algebraization of formal moduli. II. Existence of modifications, *Ann. of Math. (2)* **91** (1970), 88–135.
4. C. Birkar, Ascending chain condition for log canonical thresholds and termination of log flips, *Duke Math. J.* **136**(1) (2007), 173–180.
5. C. Birkar, Existence of flips and minimal models for 3-folds in char p, 2013 (arXiv:1311.3098).
6. V. Cossart and O. Piltant, Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin–Schreier and purely inseparable coverings, *J. Algebra* **320**(3) (2008), 1051–1082.
7. V. Cossart and O. Piltant, Resolution of singularities of threefolds in positive characteristic. II, *J. Algebra* **321**(7) (2009), 1836–1976.
8. P. Cascini, H. Tanaka and Chenyang Xu, On base point freeness in positive characteristic, *Ann. Sci. École Norm. Sup.* (2013), (to appear) (arXiv:1305.3502).
9. S. D. Cutkosky, Resolution of singularities for 3-folds in positive characteristic, *Amer. J. Math.* **131**(1) (2009), 59–127.
10. A. Grothendieck, Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes, *Inst. Hautes Études Sci. Publ. Math.*(8) (1961), 222.
11. O. Fujino, Special termination and reduction to pl flips, in *Flips for 3-folds and 4-folds*, Oxford Lecture Ser. Math. Appl., Volume 35, pp. 63–75 (Oxford Univ. Press, Oxford, 2007).
12. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, pp. xvi+496 (Springer-Verlag, New York, 1977).
13. M. Hochster and C. Huneke, Tight closure, invariant theory, and the Briançon–Skoda theorem, *J. Amer. Math. Soc.* **3**(1) (1990), 31–116.
14. C. D. Hacon, J. McKernan and C. Xu, ACC for log canonical thresholds, *Ann. of Math.* (2012), (to appear) (arXiv:1208.4150).
15. C. D. Hacon and C. Xu,On the three dimensional minimal model program in positive characteristic, 2013 arXiv:1302.0298.
16. S. Keel, Basepoint freeness for nef and big line bundles in positive characteristic, *Ann. of Math. (2)* **149**(1) (1999), 253–286.
17. J. Kollár and 14 coauthors, Flips and abundance for algebraic threefolds, Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).
18. J. Kollár, Quotient spaces modulo algebraic groups, *Ann. of Math. (2)* **145**(1) (1997), 33–79.
19. J. Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, Volume 200 (Cambridge University Press, Cambridge, 2013). With a collaboration of Sándor Kovács.
20. J. Kollár and S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, Volume 134, pp. viii+254 (Cambridge University Press, Cambridge, 1998). With the collaboration of C. H. Clemens and A. Corti; Translated from the 1998 Japanese original.
21. K. Schwede, A canonical linear system associated to adjoint divisors in characteristic $p > 0$, *J. Reine Angew. Math.* (2011), (to appear) (arXiv:1107.3833v2).
22. V. V. Shokurov, Three-dimensional log perestroikas, *Izv. Ross. Akad. Nauk Ser. Mat.* **56**(1) (1992), 105–203.
23. H. Tanaka, Minimal models and abundance for positive characteristic log surfaces, *Nagoya Math. J.* (2012), (to appear) (arXiv:1201.5699).