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TWO ARGUMENTS AGAINST THE GENERIC MULTIVERSE

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Abstract. This paper critically examines two arguments against the generic multiverse, both of which are due to W. Hugh Woodin. Versions of the first argument have appeared a number of times in print, while the second argument is relatively novel. We shall investigate these arguments through the lens of two different attitudes one may take toward the methodology and metaphysics of set theory; and we shall observe that the impact of these arguments depends significantly on which of these attitudes is upheld. Our examination of the second argument involves the development of a new (inner) model for Steel's multiverse theory, which is delivered in the Appendix.

Since the advent of forcing with Cohen's [4], there has been a recurring concern that the language of set theory may not have a unique interpretation. While philosophers and mathematicians had been familiar with unintended independence since Gödel's [9], the limitations drawn out by forcing appeared more confronting and challenging. Initial forays exploring this concern led to a vague idea of bifurcation in the meaning of the membership relation [13, 16, 21]. However, this initial wave arguably deteriorated into metaphysical pessimism and a species of formalism [5]. Nevertheless, as set theory matured through the last quarter of the twentieth century and into the twenty-first, this idea has persisted in a number of forms. While much of this remained below the surface in the form of philosophical conversations rather than published papers, more recently a number of set theorists have laid their pluralistic cards on the table.

This paper is concerned with a species of pluralism tailored very specifically to the problem of forcing: the Generic Multiverse. Versions of this idea have been discussed in a number of places [23, 24, 28–30]. I will discuss these later, however, the underlying motivation should be familiar to philosophers: in the face of the purported semantic indeterminacy exposed by forcing, we supervaluate over the diversity in search of common ground. The coherence of the resulting picture is controversial and it presents a radical departure from the usual metaphysical accounts of set theory presented in philosophy and mathematics classes.

I am going to present critical analyses of two arguments against the generic multiverse, both of which are due to W. Hugh Woodin. The first argument centres around a fascinating corollary of the Ω-conjecture and the second emerges out of considerations related to Ultimate-L. While the underlying mathematics of both arguments involves sophisticated mathematical techniques belonging to long-standing traditions in set theory, my goal here is to focus on the high-level structure of these

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arguments, rather than getting too mired in the details. This strategy has certain limitations as an appreciation of the strength of some of the claims seems to demand a more meticulous understanding of the mathematical details. However, my contention is that a more superficial survey of the arguments—with appropriately selected black boxes—will be of sufficient philosophical interest for our purposes.

The first of these arguments has been discussed a number of times by Woodin, while the second receives its first outing here. As such, I will spend more time delivering the second argument. However, my ultimate purpose in this paper is to highlight an interesting feature emerging out of the practice of contemporary set theory: its wide diversity of underlying philosophical agendas and the effects these have on the direction of its research. This will be illustrated by considering who the intended audiences of our target arguments are and how effective they are in those forums. The results are, I think, somewhat surprising and provide a fascinating case study highlighting the challenges of philosophical persuasion.

The paper is divided into three main sections. In the first, we provide some introductory discussion to contemporary set theory with a view to understanding why the generic multiverse could be appealing. In the second, we deliver a schematic version of Woodin's argument via the Ω-conjecture. Here we find that the generic multiverse is—in a certain sense—unfaithful to our intuitions about the cumulative hierarchy of sets. And in the third section, we deliver an argument leaning on considerations arising from the Ultimate-L project. Here we find that the theory of the generic multiverse is—in a certain sense—practically the same theory as ZFC or its appropriate augmentations.

§1. Motivating the generic multiverse. In order to demonstrate the effect of the philosophical diversity I claim exists in set theory, I want to start by carving our a couple of philosophical attitudes one could take. They are not intended to be in any way exhaustive, but they are particularly pertinent to the discussion below.

- *Univerism*—a hard commitment to ZFC being a theory with a unique *intended* interpretation.
- *Pragmatism*—a commitment to ZFC or something *like*[1] it being the best way to understand and organize strong mathematical theories.

I'm going to assume that universism is relatively familiar. It's the standard picture in that this is what one is taught when one initially learns set theory and it is probably safe to say that most philosophers and mathematicians sit on this side of the fence. Pragmatism, on the other hand, might seem a little vague and unfamiliar. The ensuing discussion of contemporary set theory motivating the Generic Multiverse is intended to alleviate this. That said, it would probably be fair to say that it lies somewhere between the poles of *thin realism* and *arealism* as introduced in Maddy's [18]. The key point is that the pragmatist selects their foundation on the basis of mathematical expediency rather than claiming some kind primal grasp of the underlying subject matter.

I should also note that these attitudes aren't mutually exclusive. One can be a universist and a pragmatist; for example, one might argue that ZFC is the most

[1] By "like" here, I mean something like theories that are mutually interpretable with ZFC, but I don't want to be too specific for now. See the discussion in §3.1.

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expedient foundation of mathematics and believe the language of set theory has a unique interpretation. One might place Maddy pretty close to this box [18]. Or, one might be neither a pragmatist nor a universist. For example, Hamkins claims to be a platonist with regard to a multiverse conception of sets [10].[2] In this paper, we won't be concerned so much with those positions. Rather, we shall be interested in the universist who is open to appeals that go beyond pragmatic considerations. The effectiveness of the argument from §2 seems to require this attitude. We shall also be interested in the pragmatist who is not a universist. They might not be a multiversist either: they could be on the fence. For the remainder of this section, we'll explore some reasons why such a swinging pragmatist could be swayed toward the multiverse. Then in §3, we'll examine an argument which aims to deflate those reasons.

Our discussion will now proceed in three stages:

(1) Deep incompleteness.
(2) Large Cardinals: successes and shortcomings.
(3) Turning the tables toward the Generic Multiverse.

In the interests of brevity and accessibility, I will tell an extremely abridged and simplified version of the history that brings the road to the generic multiverse into clear focus. This approach is loosely based on Steel's [23], but for a more detailed account of the relevant points the reader is recommended to consult Kanamori's [14].

1.1. Deep incompleteness. Sometime near the dawn of set theory, Georg Cantor showed:

THEOREM 1. *There is no surjection from the natural numbers onto the real numbers.*

There are different sizes of infinity and the reals are larger than the naturals. This surprising discovery and others opened up the realm of the transfinite and what became known as Cantor's paradise. But as soon as one understands the meaning of the theorem above a natural and obvious question emerges:

> *If the reals are larger than the naturals, is there any size which fits strictly between them?*

A negative answer to this question is, of course, the Continuum Hypothesis (CH). While Cantor showed that CH holds—in some sense—for the closed sets of reals, little initial progress was made on this question.

Sometime later, Hilbert—a convert to Cantor's paradise—wondered ambitiously whether all mathematical questions could be settled. Indeed, in response to Bois-Reymond's pessimistic maxim *ignoramus et ignorabimus* (we do not know and we will not know), Hilbert famously retorted, *wir mossen wissen—wir werden wissen* (we must know—we will know).

[2] I'd like to stress that neither of these examples is perfect. Maddy would likely object to my talk of interpretations as a means of distinguishing the universist from the multiversist. Hamkins would likely object that his form of platonism has a distinctively naturalistic and perhaps pragmatic flavor. Nonetheless, I hope I have illustrated to the reader that the while the scope of this paper is quite circumspect, there are other positions to occupy. For a more thorough picture of the conceptual landscape here see [15].

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Alas, a short while later Gödel showed that:

THEOREM 2. *If T is a reasonable theory of mathematics, then there is a sentence γ such that*:

$$T \nvdash \gamma \text{ and } T \nvdash \neg\gamma.$$

At first blush, this result seems devastating to even the possibility of completing mathematics. However, after a while a common feature of *almost all* the undecidable questions became clear: they were cooked up by logicians [1]! Given this, mathematicians might suspect that undecidability is just an aberration that will eventually be ironed out and as such, these questions can be mostly ignored for they hardly ever arise in ordinary mathematical inquiry.

1.2. Large cardinals: successes and shortcomings. There is, however, a problem with this workaround. While this—ignore the problem—approach might seem to operate relatively well for number theory, there were plenty of seemingly natural questions of analysis that remained unanswered (and were indeed, undecidable). Moreover, these questions did not appear to bear the usual stamps of the dark arts of logic: self-reference, coding trickery or some clever transformation of those tricks into a seemingly combinatorial question.

At this point, the Large Cardinals enter our story. In a nutshell, a Large Cardinal is a cardinal such that claiming its existence strengthens our set theoretic foundation. For our purposes, we shall concentrate on three tasks addressed by them:

(1) solving more problems;
(2) dissolving disagreement; and
(3) the Continuum Problem.

The Large Cardinals have been extremely successful in the first two tasks, yet failed in the final one.

1.2.1. Solving more problems. In the field of analysis, the Large Cardinals solved more problems. To take a classic and early example:

THEOREM 3 (Martin). *If there is a measurable cardinal, then every $\mathbf{\Sigma}^1_2$ set $A \subseteq \mathbb{R}$ is such that*:

(1) *A has the size of either the reals or the naturals*; *and*
(2) *A is Lebesgue measurable.*

With (1), we see that with respect to the $\mathbf{\Sigma}^1_2$ sets there is no size strictly between that of the naturals and the reals.[3] So these sets satisfy the Continuum Hypothesis. With the Lebesgque measurability of (2), we obtain an important mathematical property for the $\mathbf{\Sigma}^1_2$ sets, which makes them amenable to integration and probability.

[3] A $\mathbf{\Sigma}^1_2$ set is a set of reals Z that can be defined by a formula of second order arithmetic which is of the form $\exists X \forall Y \varphi(Z, X, Y, A)$ where $A \subseteq \omega$ is a parameter. See [12] for detailed definitions.

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Moreover, with stronger Large Cardinals, these results can be extended. For example:

THEOREM 4 (Martin, Steel). *If there are infinitely many Woodin cardinals, then every set $A \subseteq \mathbb{R}$ definable in the theory of analysis is such that*:

(1) *A has the size of either the reals or the naturals*; *and*
(2) *A is Lebesgue measurable.*

and

THEOREM 5 (Woodin). *If there are unboundedly many Woodin caridnals, then every set $A \subseteq \mathbb{R}$ definable in $L(\mathbb{R})$ is such that*:[4]

(1) *A has the size of either the reals or the naturals*; *and*
(2) *A is Lebesgue measurable.*

The mechanism behind the proofs of these theorems lies in establishing the determinacy of infinite games played on the relevant subsets of the real numbers. If all the games on the sets definable in analysis are determined, there are no known problems—beyond Gödelian questions—that remain undecided. In light of this, it has been argued that Large Cardinals have been able to provide the complete theory of the real numbers, albeit in this restricted sense.

Beyond the ability to solve more problems, the Large Cardinals are well-ordered according to their strength. For example, a measurable cardinal is smaller than a strong cardinal, which is smaller than a Woodin cardinal, which is smaller than a supercompact cardinal, and so on.[5] This makes them an ideal measuring stick for the strength of other natural extensions of set theory; and indeed, we find that all natural extensions of *ZFC* appear to be calibrated by the hierarchy of Large Cardinals according to their consistency strength. For example,

THEOREM 6. (1) (*Solovay, Shelah*) *Con*(*ZFC* + *there is an inaccessible cardinal*) *iff Con*(*ZFC* + *All sets definable in analysis are Lebesgue measurable*);
(2) (*Martin, Steel, Woodin*) *Con*(*ZFC* + *there is a Woodin cardinal*) *iff Con*(*ZFC* + *games on* Π^1_2 *sets are determined*).[6]

The upshot of this might be described informally as follows:

> *The road beyond ZFC is the one with Large Cardinals marking out the way.*

1.2.2. Dissolving disagreement. With regard to the project of strengthening *ZFC* to solve more problems, the Large Cardinals are not the only show in town. However, they do have another trick up their sleeves when it comes to comparing their wares

[4] $L(\mathbb{R})$ is—very briefly—the class of sets which are naturally constructible from the reals, $\mathbb{R}$, and the ordinals. The definition relies on a relativisation of Gödels constructible hierarchy. See [7] for further detail.

[5] See Kanamori's [14] for a detailed discussion of underlying mathematics here. For a quick overview, see the diagram on page 473.

[6] For discussion of the first result see Chapter 11 of [14]; and for the second, see Chapter 32.

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with that of other approaches. To introduce this, we start with the following theorem template:

THEOREM TEMPLATE A: *For all natural theories, T and U extending ZFC, we have:*

$$T \subseteq_{\Sigma^1_2} U \text{ or } U \subseteq_{\Sigma^1_2} T$$

where $T \subseteq_{\Sigma^1_2} U$ *means that* $\{\varphi \in \Sigma^1_2 \mid T \vdash \varphi\} \subseteq \{\varphi \in \Sigma^1_2 \mid U \vdash \varphi\}$; *i.e., the* Σ^1_2 *consequences of T are a subset of those of U.*

REMARK. I'm calling this a *theorem template* rather than a *theorem* as it is not articulated with sufficient mathematical precision to be called a theorem. In particular, we have not said what a *natural theory* is. A full discussion of this is outside the scope of this paper, however, we might think of a natural theory extending *ZFC* as an extension which appears combinatorial (rather than syntactic) in content and provides fruitful consequences for outstanding set theoretic problems.[7]

The intuitive upshot of this is that natural theories extending *ZFC* cannot disagree about Σ^1_2 sentences. This means that they are forced to agree upon a significant chuck of the theory of analysis.

Using Large Cardinals, this phenomenon can be generalised significantly as the following illustrates:

THEOREM TEMPLATE B:

For any two natural theories extending	$ZFC+1$ *measurable*	*we have*	$T \subseteq_{\Sigma^1_3} U$	*or*	$U \subseteq_{\Sigma^1_3} T$
	$ZFC+$ *infinite Woodins*		$T \subseteq_{\Sigma^1_\omega} U$		$U \subseteq_{\Sigma^1_\omega} T$
	$ZFC+$ *infinite Woodins & a measurable above them*		$T \subseteq_{Th(L(\mathbb{R}))} U$		$U \subseteq_{Th(L(\mathbb{R}))} T$

So as we increase the strength of our mathematical theory using Large Cardinals more and more concrete mathematics is removed from the realm of controversy.

Putting this together with the first task we might say:

> *Not only is the road beyond ZFC marked out by the large cardinals, there is less and less room for theories to disagree as we move along that road.*

These facts provide evidence for an interesting argument for the use of the Large Cardinals over other alternatives. The Large Cardinals provide natural benchmarks for the measurement of alternative strengthenings of *ZFC*. Moreover, with regard to

[7] Natural theories are introduced in [23]. See [20], for a detailed discussion.

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an increasing domain of problems, the diversity of strengthenings beyond are forced into agreement. Now suppose we ask ourselves the question: what strengthenings of *ZFC* should I use? The answer seems obvious. Use the Large Cardinals. No *practical* advantage will come from taking up the alternatives. In other words, by the lights of this argument, large cardinals are intended to be maximally mathematically expedient.

1.2.3. The continuum problem. From the discussion so far, the prospects for the Large Cardinal programme look bright indeed. However, at this point we must return the problem of the continuum. Through Gödel and Cohen, we learned that *ZFC* would offer no help.

THEOREM 7. *If ZFC is consistent, then*

$$ZFC \nvdash \neg CH \text{ and } ZFC \nvdash CH.$$

The discussion above might cause us to think that the Large Cardinals could assist us. But for quite elementary reasons, the following can be seen to hold.

THEOREM TEMPLATE C: *Let Φ be a Large Cardinal assumption.*[8] *Then if $ZFC + \Phi$ is consistent, we have*

$$ZFC + \Phi \nvdash \neg CH \text{ and } ZFC + \Phi \nvdash CH.$$

Thus, despite the successes of *ZFC* in the theory of analysis and restricted questions beyond, the Large Cardinals are of no help at all for the solution of the Continuum Hypothesis. Moreover, the Continuum Hypothesis also lies outside the region of agreement that can be wrought via the Large Cardinals.

1.3. Turning the tables into the Generic Multiverse. Taking stock, it looks like the initial promise of the Large Cardinal programme has turned out disappointing. While the Large Cardinals succeeded impressively in solving more problems and removing disagreement between natural theories, they have no effect at all on our core problem, the Continuum Hypothesis.

This is the point at which the Generic Multiverse enters the story. I think the easiest way to motivate this position is through a question:

> *What if the Continuum Hypothesis is left undecided by the Large Cardinals, not because the Large Cardinals have failed, but because the Continuum Hypothesis was a faulty question*?

If we take seriously the success of the Large Cardinals in addressing the tasks of solving problems and obtaining agreement between natural theorems, then perhaps the reason that the Continuum Hypothesis can be disagreed upon by natural theories, is that natural theories have nothing to say about the Continuum Hypothesis. And if natural theories have nothing to say on the matter, then perhaps there is nothing sensible to say.

[8] This is also a theorem template as the term, "Large Cardinal," cannot be given a mathematically precise definition.

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Taking this line of thought seriously, we might then think that ZFC is somehow defective in that it allows questions beyond the ken of natural theories to be treated as worthwhile. As such we might search for a theory which respects the limits of what natural theories can agree upon. The Generic Multiverse presents one way of achieving this.

There are two main versions of this idea in play: Steel's and Woodin's [23, 30]. Here is the axiomatisation provided by Steel. First we move into a language with two sorts of variables: sets and worlds. We then say:

(1) Every axiom of ZFC is true at every world W.
(2) Any two worlds W and U have the same ordinals.
(3) (Generic Extension) If $\mathbb{P} \in W$ is a poset, then there is a world $U = W[G]$ where G is $\mathbb{P}$-generic over W.
(4) (Generic Refinement) If $W = U[G]$ where G is $\mathbb{P}$-generic over U for some poset $\mathbb{P} \in U$, then U is a world.
(5) (Generic Amalgamation) If W and U are worlds, then there exist $\mathbb{P} \in W$, G $\mathbb{P}$-generic over W, $\mathbb{Q} \in U$ and H $\mathbb{Q}$-generic over U such that $W[G] = U[H]$ is a world.

Call this theory GMV. Informally speaking, the idea is to form a multiverse of worlds that are closed under forcing extensions and the inverse of that operation.

Woodin's approach is a little different. Rather than providing an axiomatisation, he externally defines a generic multiverse based on a countable transitive model M of ZFC. He then keeps adding worlds until every generic extension and refinement of every world N in the multiverse is also a world in the multiverse. Woodin denotes this $\mathbb{V}_M$ and the overall effect is a little different. For a world W in Woodin's generic multiverse, *every* generic extension of W via $\mathbb{P} \in W$ is added; but for Steel, we only require *a* witness for each $\mathbb{P} \in W$. This means that Woodin's system cannot satisfy Generic Amalgamation.[9]

1.3.1. Why does this do the job? The obvious question that emerges is why GMV provides a good representation of what natural theories agree upon. This is a large question that requires a paper in itself to properly explain and then defend. Moreover, in the context of the subsequent arguments, this may now be of limited value. However, we can say something which should be sufficient to complete our motivating picture.

Theorem Template C showed us that Large Cardinals cannot solve the Continuum Hypothesis. The proof of that theorem involves two facts:

(1) Large Cardinals are not affected by (small) forcings;[10]
(2) (Small) forcings are all that is required to make CH either true or false.

So the impotence of Large Cardinals on the Continuum Hypothesis is brought on by their imperviousness to (small) forcing; indeed this fact is also crucial for their ability to resolve disagreement between natural theories. Given this, we start to see a picture in which Large Cardinals—while they can solve many problems and remove much disagreement—are powerless in the face of the forcings they leave open. As such, a multiverse which contains all of the worlds accessible by forcing and its inverse

[9] See [20] for a detailed discussion on this matter.
[10] By a small forcing we mean a poset $\mathbb{P}$ with $|\mathbb{P}| < \kappa$ where κ is the Large Cardinal in question.

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may provide the key to neutralising the effects of forcing and removing the apparent limitations of the Large Cardinals. For example, following Woodin we might consider the multiverse truth (or the worthwhile questions) to be those which have a single answer across the multiverse. In a nutshell, we may be tempted by a picture in which the Large Cardinals and forcing are two sides of much the same coin.

1.4. Revisiting philosophical motivations. Before we move on to the arguments against this position, I want to put a little more meat on the bones of the position I described as *pragmatism* and try to link it up with the Generic Multiverse. I want to claim that a pragmatist could easily come to accept something like GMV as their best theory for the foundations of mathematics.

Recall that our pragmatist is less wed to a preexisting view of the ontology of set theory and is more committed to developing the best theory for the organization and understanding of strong mathematical theories. As such, the ontological revision required by GMV is not necessarily a major hurdle. Now when we look at the Large Cardinal program, we see an extremely effective tool for developing stronger mathematical theories and understanding them. However, we also see a significant limitation. GMV provides a way of turning the tables on this scenario: we revise the ontology and obtain a theory which suggests a completed picture in which Large Cardinals provide the right way to understand strong mathematical theories and the apparent limitations are removed as involving improper questions.

Of course, this isn't a compelling argument that every pragmatist should adopt GMV, but the appeal to at least some pragmatists should now be clearer.

§2. Argument 1: Transfinite fidelity. Our first argument against the generic multiverse comes from W. Hugh Woodin. Versions of it have been presented in a number of places [28–30]. A full presentation of the argument is mathematically deep involving a tour of universally Baire sets, generic invariance and Ω-logic. However, we shall content ourselves with a relatively superficial version of the argument which highlights its over-arching structure and which will suffice for our purposes.

2.1. The strategy. The underlying strategy of the argument can be described as follows:

(1) Let M be a countable transitive model of ZFC.
(2) Let $\mathbb{V}_M$ be Woodin's generic multiverse as generated from M.
(3) Find a sentence $\varphi \in \mathcal{L}_\in$ such that:
    (a) $M \models \neg\varphi$;
    (b) $\mathbb{V}_M \models \varphi^\dagger$ where $\varphi^\dagger$ is the *adaptation* of φ into the multiverse framework.
    (c) If $\mathbb{V}_M$ is *reasonable*, then we have

$$M \models \neg\varphi \;\Rightarrow\; \mathbb{V}_M \models \neg\varphi^\dagger.$$

Then we see that given 3(a) to (c) $\mathbb{V}_M$ cannot be reasonable. This is the target conclusion of Woodin's argument. The key point is that we have a sentence whose falsity in the ground universe should be preserved into the multiverse, but this does not happen.

The two *italicized* terms in the template above are a little loose. We'll first do a little tidying of the notion of *adaptation* and we'll come back to what a *reasonable* representation is toward the end of the section.

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2.1.1. Tidying "adaptation.". Let M and $\mathbb{V}_M$ be as above:

(1) Find some formula $\Phi(\cdot) \in \mathcal{L}_\in$ such that:

(a) $\neg\Phi(\{M\})$;
(b) $\Phi(\mathbb{V}_M)$; and
(c) If $\mathbb{V}_M$ is *reasonable*, then

$$\neg\Phi(\{M\}) \Rightarrow \neg\Phi(\mathbb{V}_M).$$

The idea here is that with 1(a), we use $\{M\}$ as a singleton multiverse, so that we can use the same formula to talk about M and $\mathbb{V}_M$ without having to make any adaptation.

2.1.2. A problematic example. Let's first consider an example of this strategy that does not work. After that we'll be ready for Woodin's argument. Let $\mathbb{V}$ be a variable representing an arbitrary multiverse and let

$$\Phi(\mathbb{V}) := \exists W_1 \in \mathbb{V} \exists W_2 \in \mathbb{V}(W_1 \models CH \wedge W_2 \models \neg CH).$$

Then we clearly have:

(i) $\neg\Phi(\{M\})$ since there is only one world $M \in \mathbb{V}$; and
(ii) $\Phi(\mathbb{V}_M)$ since CH can always be forced on and off.

This gives us (a) and (b) of our template strategy, but now consider (c). If it were correct, we'd be have it that if $\mathbb{V}_M$ is reasonable, then

$$\neg\Phi(\{M\}) \Rightarrow \neg\Phi(\mathbb{V}_M).$$

Given (i) and (ii) immediately above, this would then tell us that $\mathbb{V}_M$ is not reasonable. But this cannot be right. It would be question begging to reject the generic multiverse on the basis that it did exactly what it was intended to do; i.e., make questions like the Continuum Hypothesis true in some worlds and false in others. This tell us that whatever *reasonableness* means in this context it cannot work like this: we are going to need some independent reasons to justify (c).

2.2. Woodin's argument. We need a little terminology and a few facts to get things moving. However, I'll try to provide some intuitive explanation of what is going on in the *black boxes*.

Let $X, Y \subseteq \omega$. We shall say that X is *Turing reducible* to Y, $X \leq_T Y$ if there is a Turing machine $\psi : \omega \rightarrow \omega$ that draws on information from Y as an oracle and which decides X; i.e., computes its characteristic function. The intuitive idea is that given the set Y there is a simple way of figuring out if something belongs to X; so in this sense Y is at least as complex as X.

Given that sets of sentences in the language of set theory can be represented by natural numbers, we can ask whether theories are Turing reducible to each other. Here is an example that will be useful for Woodin's argument.

FACT 8. *If α is definable by a Σ_2 formula, then*[11]

$$\{\varphi \in \mathcal{L}_\in \mid V_\alpha \models \varphi\} \leq_T \{\varphi \in \Pi_2 \mid \varphi\}.$$

[11] The right hand side is a little sloppy. Strictly, we should write something like $\{\varphi \in \Pi_2 | \ V \models_{\Pi_2} \varphi\}$. This is well-defined since we may define the Π_2 truth predicate for V within V.

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Informally, this tells us that the truths about V_α can be calculated from the Π_2 truths of the universe. In essence, this works because being true in V_α is just a Π_2 fact. Thus the set of Π_2 facts on the right hand side is sufficient to calculate the truths of V_α. We might then wonder whether the other direction holds. But we have the following:

FACT 9. *If α is definable by a Σ_2 formula, then*

$$\{\varphi \in \Pi_2 \mid \varphi\} \not\leq_T \{\varphi \in \mathcal{L}_\in \mid V_\alpha \models \varphi\}.$$

The intuitive reason why this fails is that the Π_2 truths of the universe are not only strong enough to calculate the truths of V_α but they can also calculate its truth predicate. This would allow us to define the truth predicate of V_α in V_α which is impossible by Tarski's theorem.

Now using Fact 9 we can define our formula $\Phi(\cdot)$. We let $\Phi(\mathbb{V}) :=$

$$\{\varphi \in \Pi_2 \mid \forall W \in \mathbb{V}\ W \models \varphi\} \leq_T \{\varphi \in \mathcal{L}_\in \mid \forall W \in \mathbb{V}\ W \models (\varphi)^{V_{\delta_0+1}}\}$$

where δ_0 is a term denoting the first Woodin cardinal in any world.

The basic idea here is that we generalise the statement from Fact 9 into a multiverse setting. We then see that since the statement that a Woodin cardinal exists is Σ_2, Fact 9 straightforwardly gives us (a) from our template strategy. With regard to (b), we have the following:

THEOREM 10. *Suppose there is a proper class of Woodin cardinals and that the Ω-conjecture holds. Then we have*[12]

$$\{\varphi \in \Pi_2 \mid \forall W \in \mathbb{V}_M\ W \models \varphi\} \leq_T \{\varphi \in \mathcal{L}_\in \mid \forall W \in \mathbb{V}_M\ W \models (\varphi)^{V_{\delta_0+1}}\}.$$

The proof of this is quite technical. However, we can say a little something about why it holds. In essence given the assumptions of the theorem, V_{δ_0+1} has enough room to calculate which Π_2 sentences will be true in every model in the generic multiverse. In a simpler context, we see that V_ω has enough room to calculate which sentences of first order logic will be true in every model. An analogous, although more sophisticated phenomenon is conjectured to be occurring here.

It should now be clear that we have (a) and (b) which leaves us with (c): this time we can make a more compelling case. Let's consider what Theorem 10 tell us a little more informally. We might put it as follows:

> *The Π_2 truths across the multiverse can be calculated in the V_{δ_0+1} of any world in that multiverse.*

According to our motivations for the Generic Multiverse, this means that the meaningful Π_2 questions about the *entire universe* can be established in a *mere initial segment* of any universe. So why would this be a bad thing? This is the point at which philosophy makes its entrance. Here is what Woodin says:

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12 The Ω-conjecture is—very briefly—the claim that the Π_2 sentences true across the generic multiverse (the Ω-validities) can be witnessed by proofs in the form of universal Baire sets. For an excellent introduction to these notions, see [2].

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> This amounts to a rejection of the transfinite beyond $H_{\delta_0^+}$ and constitutes in effect the unacceptable brand of formalism alluded to earlier. ([30], p18)[13]

The underlying problem is that $\mathbb{V}_M$ seems to promote a kind of unfaithfulness to our fundamental understanding of the transfinite. But why does it do this? Woodin continues:

> It is a fairly common (informal) claim that the quest for truth about the universe of sets is analogous to the quest for truth about the physical universe. However I am claiming there is an important distinction. While physicists would rejoice in the discovery that the conception of the physical universe reduces to the conception of some simple fragment or model, in my view the set theorist must reject the analogous possibility for truth about the universe of sets. By the very nature of its conception, the set of all truths of the transfinite universe (the universe of sets) cannot be reduced to the set of truths of some explicit fragment of the universe of sets. ([30], p17)

So Woodin claims that unlike physics our understanding of the universe of sets leads us to the belief that the truths of the universe cannot be obtained from a mere fragment of that universe. He summarizes:

> The essence of the argument against the generic-multiverse position is that assuming the Ω Conjecture is true (and there is a proper class of Woodin cardinals) then the position is simply a brand of formalism that denies the transfinite by reducing truth about the universe of sets to truth about a simple fragment such as the integers or, in this case, the collection of all subsets of the least Woodin cardinal. ([30], p17)

The breed of formalism alluded to here is certainly an exotic one, however, the point is well taken. Under the assumptions of Theorem 10, a complex class of truths about the universe are reduced to truth about some restricted fragment of the universe.

REMARK. At this point it's worth mentioning a common confusion regarding this argument. It is well-known that if λ is a supercompact cardinal, then V_λ is a Σ_2 elementary submodel of V. Thus, the Π_2-truths are Turing reducible to those of such a V_λ (see Proposition 22.3 in [14]). Given this, we may be tempted to try to run an analogous argument using this reduction instead of that of Theorem 10. However, this will not work. The statement that there is a supercompact cardinal is Σ_3 and thus does not fit the requirements of Fact 9. Thus, we do not get (a) from our template strategy and this analogy cannot get off the ground.

2.3. Evaluating this argument. Let's take a little stock:

- Woodin claims that an intrinsic fact about set theory is that the truths (even the Π_2 truths) cannot be obtained in simple fragments of the universe.

[13] Here Woodin uses $H_{\delta_0^+}$ the set of sets whose transitive closure has cardinality less that δ_0^+, but for our purposes it is much the same as V_{δ_0+1}.

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- But the generic multiverse allows the (worthwhile) Π_2 truths to be calculated in V_{δ_0+1} (assuming Woodin cardinals and the Ω-conjecture).
- So the generic multiverse is unreasonable in the sense that it is unfaithful to our understanding of the transfinite.

But how seriously should we take this argument? Our intuitions certainly have limited traction in this territory, perhaps this is a bullet we can bite.

I think this is a juncture where our distinction between the universist and the pragmatist can be brought to bear. If one has universist sympathies, as I believe Woodin does, then clearly bullets are off the menu. Rather the universist should see this argument as providing further evidence for their own view. They might claim that their belief in a unique interpretation for the language of set theory has opened up the insight that the truths of the universe should not be accessible in restricted fragments of that universe. The fact that the Generic Multiverse view appears to challenge that insight is then another reason to reject that view.

The pragmatist, on the other hand, can view these matters from quite a different perspective. Given that they see set theory as the study of strong mathematical theories, they might—as we have seen—be attracted by the Generic Multiverse approach. As such, they are able to vindicate the successes of the Large Cardinal program and use them to delimit the worthwhile mathematical questions by removing disagreement. They might then argue that GMV provides the right framework for this approach to set theory. Given this, the pragmatist might be tempted to turn the tables again. Rather than seeing Woodin's argument as short drive off a tall cliff, we might take this as a fascinating result: the logic of the Π_2 truths across the generic multiverse can be calculated in V_{δ_0+1}! We have a deep and interesting feature rather than a pathological bug.

§3. Argument 2: Pragmatic indifference. Our second argument against the generic multiverse emerges from considerations arising from Woodin's Ultlimate-L programme [31]. This is an extremely technical programme, however, we shall be able to extract a couple of salient features which are sufficient to mount an argument against the Generic Multiverse. This argument has not appeared in print before, however, I think it's fair to say that it's been part of the folk-knowledge among set theorists for some time.

Here is a rough version of the argument. We start with the following fact:

FACT 11. *If $V =$ Ultimate-L, then the Generic Multiverse has a unique core; i.e., there is some world in the Generic Multiverse which has no generic refinements and such that every world is a generic extension of it. Moreover, Ultimate – L itself is that core.*

REMARK. It is worth noting that multiverses based on more ordinary L-like models like L, $L[0^\sharp]$ and $L[\mu]$ also have cores, so this feature is not unexpected.

Now given that Ultimate-L is the core of its multiverse, one might argue that this is where all the action is really taking place. So why bother with the generic multiverse? Why not just stick with Ultimate-L?

Of course, it's the phrase "where the action is really taking place" that is doing all the work here, but how should we unpack it? Here's one way of doing this. Given

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that Ultimate-L is the core of its multiverse, we might be able to show that—assuming $V = \text{Ultimate} - L$—anything we could do in the Generic Multiverse could already be done in Ultimate-L. In this way, we would be arguing that the Generic Multiverse is superfluous. But how do we substantiate this claim? We might argue that the theories $GMV + \exists W(V = \text{Ultimate} - L)^W$ and $ZFC + V = \text{Ultimate} - L$ are equivalent in some salient sense. If we could show that the two theories are sufficiently close then we could well be in a position to argue that the Generic Multiverse is redundant.

In the remainder of this section, I want to elucidate some useful senses in which theories can be equivalent; and then show that the theories above are equivalent in a particularly natural one of these senses. I will proceed by:

(1) providing a hierarchy of different ways in which theories could be considered equivalent;
(2) discussing some problems for fitting our theories into this picture; and
(3) isolating the principles required for the target equivalence and describing the result.

A proof of the main result will be deferred until the Appendix. We shall close the section with a critical examination of the philosophical force of the resultant argument.

3.1. A hierarchy of theoretic equivalences. Let's start with the the most commonly used form of relationship between theories in set theoretic research—relative consistency; and let's dig a little more deeply into the underlying ideas. We shall make heavy use of Visser's work on interpretability in setting out a framework for this discussion [26]. Given two theories T and S we say that T is *consistent relative to* S, abbreviated $T \leq_{Con} S$, if we can show—often using a weaker theory—that $Con(S) \to Con(T)$. So in a loose sense we are saying that T is no more risky than S in terms of their consistency. We then say that T and S are *equiconsistent*, abbreviated $T \equiv_{Con} S$, if $T \leq_{Con} S$ and $S \leq_{Con} T$. Intuitively speaking, we are saying that T and S are as risky as each other. Here is a classic example:

THEOREM 12 (Gödel, Cohen). $ZFC \equiv_{Con} ZFC + V = L \equiv_{Con} ZFC + \neg V = L$.

Equiconsistency can be seen as a kind of equivalence between theories, but it isn't very strong. As the Theorem above shows—even though these theories are equiconsistent, they don't even agree on whether or not there is a constructible set. Equivalence of this kind would be too weak for the purposes of our main argument above.

3.1.1. Relative interpretability. To get a better understanding of the weakness of relative consistency it will help to consider more closely how we establish part of result above: $ZFC + V = L \leq_{Con} ZFC$.[14] To do this we define a translation function $\tau : \mathcal{L}_\in \to \mathcal{L}_\in$ from the language of set theory to itself such that for all $\varphi \in \mathcal{L}_\in$ we have

$$ZFC + V = L \vdash \varphi \;\Rightarrow\; ZFC \vdash \tau(\varphi).$$

The translation in question is, of course, just the relativisation of all the quantifiers in φ to L. To complete the proof, we then suppose $ZFC + V = L$ is inconsistent; i.e.,

14 Note that the other direction is trivial.

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$ZFC + V = L \vdash \bot$. Then our translation tells us that $ZFC \vdash \tau(\bot)$ where $\tau(\bot)$ is just $\bot$; hence ZFC is inconsistent and the result is established.

More generally, this kind of translation is known as a *relative interpretation*. Given theories T and S in languages $\mathcal{L}_T$ and $\mathcal{L}_S$, we say that T can be *interpreted* by S, abbreviated $T \leq_{Int} S$, if there are formulas of $\mathcal{L}_S$ defining a domain and counterparts for the non-logical vocabulary of $\mathcal{L}_T$ such that the resultant translation, $\tau : \mathcal{L}_T \to \mathcal{L}_S$, yields for all $\varphi \in \mathcal{L}_S$

$$S \vdash \varphi \ \Rightarrow \ T \vdash \tau(\varphi).$$

We then say that two theories T and S are *mutually interpretable*, abbreviated $T \equiv_{Int} S$, if $T \leq_{Int} S$ and $T \leq_{Int} S$. We say that T *faithfully interprets* S if there is a some relative interpretation $\tau : \mathcal{L}_T \to \mathcal{L}_S$ such that

$$S \vdash \varphi \ \Leftrightarrow \ T \vdash \tau(\varphi).$$

Given that relative interpretability is used to establish parts of the results above, we see that mutual interpretability is also quite a weak form of equivalence. We'll be able to enrich this picture soon, however, there is an alternative—more semantic—perspective which will make this much clearer.

Observe that when we have $T \leq_{Int} S$ as witnessed by some $\tau : \mathcal{L}_T \to \mathcal{L}_S$ we are—in essence—using the translation, τ, to define a model of S within any model of T. We'll call this an *inner model* although we shall take care to distinguish this from the notion of inner model employed in set theory; a definable inner model whose membership relation is just a restriction of the actual membership relation and which contains all of the ordinals.[15]

We describe this more formally below.

Let $Mod(T)$ and $Mod(S)$ denote the classes of models of T and S respectively. Then

THEOREM 13. *Let $T \leq_{Int} S$ as witnessed by $\tau : \mathcal{L}_T \to \mathcal{L}_S$. Then there is a function $\tau^* : Mod(S) \to Mod(T)$.*[16]

So intuitively speaking, we see that a relative interpretation from T to S gives rise to a function from models of S to models of T. We shall call this the *mod-functor*. This gives us a more picturesque idea of what occurs in relative interpretation and we shall use this to enrich our taxonomy of theoretical equivalences.

3.1.2. A hierarchy of equivalences. Suppose T and S are mutually interpretable as witnessed by mod-functors $\tau^* : Mod(S) \to Mod(T)$ and $\sigma^* : Mod(T) \to Mod(S)$. It is interesting to consider what happens when we compose them. We then get $\tau^* \circ \sigma^* : Mod(T) \to Mod(T)$ and $\sigma^* \circ \tau^* : Mod(S) \to Mod(S)$. Just considering the first of these, we now have a new mod-functor which takes us from models of T back to models of T. This raises an interesting question: is there any relationship between the

15 In ordinary set theoretic parlance an inner model is a transitive class defined by a formula which (inwardly) restricts the domain and retains the ordinary membership relation therein. I will write "*Inner Model*" for the set theoretic interpretation of this term; and "*inner model*" for the interpretation of the term used in the relative interpretability literature.

16 See §2.3 of [26]. Note that the converse does not hold in general.

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initial model and the model that results? Different answers to this question will yield our hierarchy of equivalences.

The most obvious relationship is identity. It would be ideal if we were able to take a model of T turn it into a model of S and then turn it back the same model of T that we started with. If this happens in both directions—i.e., $\tau^* \circ \sigma^* = id$ and $\sigma^* \circ \tau^* = id$—then we say that the theories are *synonymous*or *definitionally equivalent*, which we shall abbreviate $T \equiv_{Syn} S$. Intuitively speaking, when we have this relationship between theories they are so close as to be identical. We might think T and S as merely two different ways of describing the same stuff which are such that anything said by one can be said—via translation—by the other.

For a simple example, we might consider two axiomatisations of group theory in two different languages: $\mathcal{L}_0 = \{e, \circ, \cdot^{-1}\}$ and $\mathcal{L}_1 = \{S\}$ where S is a ternary relation. The obvious translations between these theories yields mod-functors whose respective compositions are just the identity. This is because there is nothing substantively different between these theories, they are equivalent ways of describing the same class of structures.

There are other relationships which also provide significant degrees of closeness. For example, we might suppose that the compositions of the mod-functors give us isomorphism or elementary equivalence. In the former case, where $\tau^* \circ \sigma^*(\mathcal{M}) \cong \mathcal{M}$ and $\sigma^* \circ \tau^*(\mathcal{M}) \cong \mathcal{M}$, we say T and S are *bi-interpretable*, abbreviated $T \equiv_{Bi-Int} S$. In the latter case, where $\tau^* \circ \sigma^*(\mathcal{M}) \equiv \mathcal{M}$ and $\sigma^* \circ \tau^*(\mathcal{M}) \equiv \mathcal{M}$,[17] we say that T and S are *sententially equivalent*, abbreviated $T \equiv_{Sent} S$.

We summarize this information in the following table:

$$\begin{array}{c} T \equiv_{Syn} S \\ \Downarrow \\ T \equiv_{Bi-Int} S \\ \Downarrow \\ T \equiv_{Sent} S \\ \Downarrow \\ T \equiv_{Int} S \\ \Downarrow \\ T \equiv_{Con} S \end{array}$$

The downward arrows, "$\Downarrow$" represent implications which hold because the notions of equivalence between the models are stronger. The implications do not work in the opposite directions.

THEOREM 14. *There exist theories T and S such that*:

(1) $T \equiv_{Con} S$ *but* $T \not\equiv_{Int} S$;
(2) $T \equiv_{Int} S$ *but* $T \not\equiv_{Sent} S$;
(3) $T \equiv_{Int} S$ *but* $T \not\equiv_{Bi-Int} S$; *and*
(4) $T \equiv_{Bi-Int} S$ *but* $T \not\equiv_{Syn} S$.[18]

[17] Here we use "$\equiv$" to mean elementary equivalence not a relation between theories.

[18] For (1), *GBN* and *ZFC* suffice using a reflection argument. For (2), consider *ZFC* and $ZFC + V = L$ recalling the comments above. For (3), we may use *ZF* and *ZFC* recalling results from Cohen's [6]. A proof can be found at [8]. And for (4), see Friedman and Visser's

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3.2. Fitting forcing into this picture. In the previous section we outlined a hierarchy of refinements of relative interpretability which give us some measure of how close two theories are. Crucial to this account is the ability of the stronger theory to be able to define an inner model of the weaker theory. In set theory, we can easily apply this story to arguments for interpretability based on Inner Models—in the strong set-theoretic sense of that term. However as we have seen, many arguments for relative consistency in set theory depend on a different technique: generic extension. Moreover, given our target is a comparison between the Generic Multiverse theory and the standard set theory, it is clear that we need to be able to account for this technique too.

Unfortunately, generic extensions cannot yield Inner Models in the strong sense. In fact, the generic object from which the generic extension is constructed cannot exist in the actual universe, V. Thus, if *per impossible* we could find such a model it would have to be an outer model. From the universist perspective this is, of course, absurd.

One way around this is to use small models M—known as countable transitive models—inside V which are small enough that although they cannot contain their own generic elements, the generic elements can still be found in V. This allows us to define the generic extension $M[G]$ in V and is sufficient for the purposes of relative consistency arguments. It will not, however, be sufficient to facilitate the comparisons we want in the framework we have outlined above. Fortunately, there is a way around this, which I will now outline.

3.2.1. Boolean valued ultrapowers. The remainder of this subsection will be somewhat technical, however, I will endeavour to sign-post the salient high-points that are pertinent to the main philosophical discussion. Recall the Boolean valued model approach to forcing. Rather than using a poset, we use a complete Boolean algebra, $\mathbb{B}$, and define a class of $\mathbb{B}$-names $V^{\mathbb{B}}$. For standard resources on this, see [3], [12] or [19].

Following [11], we may use an ordinary ultrafilter $U \subseteq \mathbb{B}$ rather than a generic ultrafilter in order to define a class model $V^{\mathbb{B}}/U$ which is a model of ZFC. Such ultrafilters exist in V by the ultrafilter theorem which is slight weakening of the Axiom of Choice. Moreover, we can define another model $\check{V}_U$ as the class of $\mathbb{B}$-names, τ, such that there is some $u \in U$ for which the set

$$D = \{b \in \mathbb{B} \mid \exists x \in V \; b \Vdash \tau = \check{x}\}$$

is dense below u. $\check{V}_U$ has some pleasing properties:

THEOREM 15 (Hamkins and Seabold). *There is an elementary embedding $j_U : V \to_{\Sigma_\omega} \check{V}_U$ such that $j_U(x) = [\check{x}]_U$. Moreover, if we let $G = [\dot{G}]_U$ then*

(1) *G is $j_U(\mathbb{B})$-generic over $\check{V}_U$; and*
(2) *$V^{\mathbb{B}}/U$ is the generic extension of $\check{V}_U$ by G; i.e., $V^{\mathbb{B}}/U = \check{V}_U[G]$.*[19]

Given that such an ultrafilter exists in V, this means that each of these models can be defined in V via the ultrafilter. Putting this together, this theorem then tells us that,

[27]. I'm not aware of an example of two theories where we have sentential equivalence but bi-interpretability fails.

[19] Here $\dot{G} = \{\langle \check{b}, b\rangle \mid b \in \mathbb{B}\}$.

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although we cannot define a *real* generic extension of V, we can define the generic extension of an elementary extension of V.

Pertinently, for our current goals this means that we can define an inner model (in the weaker sense) which behaves very much like a generic extension of V. In terms of the the theory of relative interpretability, this tells us that forcing extensions can be understood as *interpretations in a parameter* [26].

3.3. The result and its requirements. With this in hand, we are almost ready to show how to move between our theories. Recall our goal is provide definitions giving rise to inner models and the following mod-functors:

- $\tau^* : Mod(GMV+?) \to Mod(ZFC+?)$; and
- $\sigma^* : Mod(ZFC+?) \to Mod(GMV+?)$.[20]

For the first of these, we'll make use of the results described in the previous subsection. The latter is a little trickier. With GMV we have a plurality of ZFC models but nothing to distinguish them. In fact, we don't really have a way of comparing these two theories as they are. We shall need to extend them with the following principles:

(1) The universe has no generic refinements; and
(2) The universe has a definable well-ordering.

(1) is arguably the crux of the argument here. It tells us that the universe is the core of any generic multiverse generated from it. Without this principle, we just have an ocean of different set theoretic universes and no natural way to return to a particular universe. We require (2) to make the appropriate generalisation of the results in the previous section.

We then need a way to expand our theories in order to obtain these principles. This is where Ultimate-L comes in.

THEOREM 16 ([31]). *Ultimate-L is the core of its generic multiverse and it has a definable well-ordering.*

We shall write $V = UL$ for the sentence stating that the universe is Ultimate-L. We now proceed to define the required mod-functors.

3.3.1. $\tau^* : Mod(GMV + \exists W(V = UL)^W) \to Mod(ZFC + V = UL)$. Given the existence of a core, the translation τ giving rise to the appropriate mod-functor is easy to describe. Given a sentence $\varphi \in \mathcal{L}_\in$ we let

$$\varphi \overset{\tau}{\mapsto} \forall W(W = UL \to \varphi^W).$$

The translation just tells us to ask what is going on at the core of the multiverse: where the real action is taking place. The resultant mod-functor, τ^*, takes a Generic Multiverse and returns the inner model consisting of just its core: it forgets the other worlds.

[20] I've added the " +?" as we are going to need a little extra to get the result beyond mere equiconsistency, which is easier to obtain.

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3.3.2. $\sigma^* : Mod(ZFC + V = UL) \to Mod(GMV + \exists W(V = UL)^W)$. Our goal here is to show how to define a model of GMV within an arbitrary model of ZFC. We saw above that given a model V of ZFC and some $\mathbb{B} \in V$ we can—with the help of an ultrafilter, U, define an elementary extension $\bar{V}$ of V and a generic extension $\bar{V}[G]$ of $\bar{V}$. We'd like to be able to take this further and define an elementary extension $\bar{V}$ of V from which every complete Boolean algebra has a generic extension.

There is, however, a hurdle. Suppose we have complete Boolean algebras $\mathbb{B}_0$ and $\mathbb{B}_1$ with ultrafilters U_0 and U_1 on each of them respectively. Then although we have $\check{V}_{U_0}[G_0]$ generically extending $\check{V}_{U_0}$ and $\check{V}_{U_1}[G_1]$ generically extending $\check{V}_{U_1}$, we do not—in general – have $\check{V}_{U_0}[G_0]$ extending $\check{V}_{U_1}$. There is a way around this too:

Theorem 17 (Hamkins and Seabold). *Let $\mathbb{B}_0$ be a complete subalgebra of $\mathbb{B}_1$ with U_1 an ultrafilter on $\mathbb{B}_1$. Then $U_0 = U_1 \cap \mathbb{B}_0$ is an ultrafilter on $\mathbb{B}$. Letting $V_0 = \check{V}_{U_0}$ and $V_1 = \check{V}_{U_1}$ let $k : V_0[G_0] \to V_1[G_1]$ be such that for $\tau \in V^{\mathbb{B}_0}$*

$$k([\tau]_{U_0}) = [\tau]_{U_1}.$$

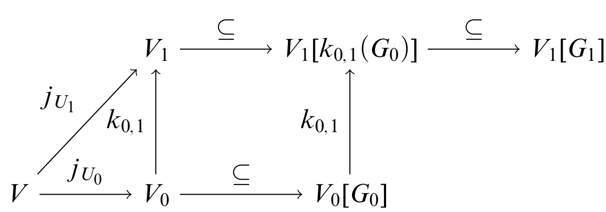
Then:

- *$V_1[k(G_0)]$ is the set of $[\tau]_{U_1}$ for $\tau \in V^{\mathbb{B}_1}$ such that there is some $u \in U_1$ for which the set*

$$D = \{b \in \mathbb{B}_1 \mid \exists \sigma \in V^{\mathbb{B}_0}\ b \Vdash \tau = \sigma\}$$

is dense below u.
- $k : V_0[G_0] \to_{\Sigma_\omega} V_1[k(G_0)]$.

This can be nicely summarized in the following diagram.



Informally speaking, this gives us a partial solution to the problem described above. Provided $\mathbb{B}_0$ is a complete subalgebra of $\mathbb{B}_1$ and U_0 is the restriction of U_1 to $\mathbb{B}_0$ we can use $V_1 = \check{V}_{U_1}$ as the ground model from which we can can produce both generic extensions $V_1[k(G_0)]$ and $V_1[G_1]$ which are respectively $j_{U_1}(\mathbb{B}_0)$-generic and $j_{U_1}(\mathbb{B}_1)$ generic over V_1.

Moreover, we can generalise this to longer finite chains as illustrated in the following diagram.

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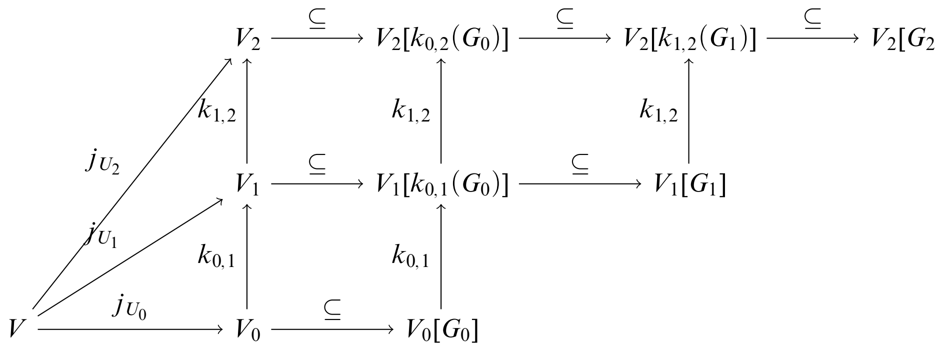

So for any finite amount of forcing we can always find a ground model—provided each of the complete Boolean algebras are all sub-algebras of one among them.

Of course, this is not enough either. A generic multiverse will need to be closed under an infinite amount of forcing not merely a finite amount. The theorem above does not help us here but the diagram above provides the crucial hint. The full proof will be provided in the Appendix, but I give a rough sketch here. In the diagram above we see that as we go higher we get better and better ground model in the sense that the higher we go the more generic extensions can be accommodated. So if we want to get a model that can accommodate enough forcing to satisfy the GMV axioms, we want a kind of limit of this tower construction. The standard construction of a direct limit of these models provides this. Looking at the diagram above, we are – loosely speaking— taking the limit of the tower of models $V_0 \xrightarrow{k_{0,1}} V_1 \xrightarrow{k_{1,2}} V_2 \xrightarrow{k_{2,3}}$ to form our universal ground model. The key to our construction is our use of a definable well-ordering of the universe to design a sequence of ultrafilters ensuring that the amalgamation axiom is satisfied.

Using the same technique we can define each of the worlds in our generic multiverse uniformly in an ordinal indexing the ultrafilter used to give the world in question. A little more formally, each world is a generic refinement of something of the form:

$$V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)] = \varinjlim\{\langle V_{(\beta)}[k_{\alpha,\beta}(G_\alpha)]\rangle_{\beta\in[\alpha,Ord)}, \langle k_{\gamma,\beta}\rangle_{\alpha\leq\gamma\leq\beta<Ord}\}.$$

3.3.3. The result. We may now state the result.

THEOREM 18. *There exist translations giving rise to mod-functors*:

- $\sigma^* : Mod(ZFC + V = UL) \to Mod(GMV + \exists W(V = UL)^W)$; *and*
- $\tau^* : Mod(GMV + \exists W(V = UL)^W) \to ZFC + V = UL$

such that

(1) $M \precsim \tau^* \circ \sigma^*(M)$ *for* $M \in Mod(ZFC + V = UL)$;
(2) $M \equiv \sigma^* \circ \tau^*(M)$ *for* $M \in Mod(GMV + \exists W(V = UL)^W)$.

Proof. See Appendix. □

3.3.4. Summary and philosophical commentary. This tells us that the theories $ZFC + V = UL$ and $GMV + \exists W(V = UL)^W$ are very close. While not synonymous,

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we see that they give us essentially the same first order theories modulo their respective translations. Their equivalence fits in between that of sentential equivalence and bi-interpretability. To emphasize the importance of this proximity we note the following result:

PROPOSITION 19. *Let T and S be sententially equivalent via translations $\tau : \mathcal{L}_S \to \mathcal{L}_T$ and $\sigma : \mathcal{L}_T \to \mathcal{L}_S$. And suppose $T \cup \Gamma$ is a consistent extension of T. Then $S \cup \sigma''\Gamma$ is also consistent.*

Proof. Suppose $T \cup \Gamma$ is consistent, but $S \cup \sigma''\Gamma$ is inconsistent. Then we may fix φ a conjunction of sentences from Γ such that $S \vdash \neg\sigma(\varphi)$. From this we see that $T \vdash \neg\tau \circ \sigma(\varphi)$. Since $T \cup \Gamma$ is consistent, we see that $T \cup \{\varphi\}$ is consistent, so we may fix a model $\mathcal{M}$ of T such that $\mathcal{M} \models \varphi$. Since T and S are sententially equivalent, we see that $\sigma^* \circ \tau^*(\mathcal{M}) \models \varphi$. From this we see that $\tau^*(\mathcal{M}) \models \sigma(\varphi)$ and so $\mathcal{M} \models \tau \circ \sigma(\varphi)$. But this means $T \nvdash \neg\tau \circ \sigma(\varphi)$, which is a contradiction. □

The upshot here is that if we are invested in the project of extending our foundations to solve more problems, then nothing about that project will give us a reason to choose between the two options. If we want to add something to one theory, we can always add appropriate counterpart to the other theory using the translations. Let us say that such theories are *co-extendible*. For a situation where this breaks down, consider the theories ZFC and $ZFC + V = L$ and the usual translations between them: $\varphi \mapsto \varphi^L$; and $\varphi \mapsfrom \varphi$. They are then equiconsistent and indeed mutually interpretable. However, while it is thought that we can consistently extend ZFC with the assumption of the existence of a measurable cardinal, we know that we cannot extend $ZFC + V = L$ with this assumption. Thus, these theories are not on a par with the project of exploring stronger foundational theories. We might say that $ZFC + V = L$ is *restrictive with respect to ZFC*.

So practically speaking this tells that GMV and ZFC[21] are about even with regard to the considerations of our pragmatist position. Nonetheless, there is a sense in which the evidence being provided here is weaker than that of our first argument. There it was claimed that the acceptance of GMV violated a fundamental principle about the transfinite realm of set theory. Whereas here we are merely arguing that there is little to be gained from taking up GMV instead of ZFC.

This might give us reason to think that our second argument should be less persuasive to the generic multiverse adherent. I think this is misleading. Recall that we motivated our acceptance of the generic multiverse on what we called pragmatic grounds. We did not take up the theory on the basis of its faithfulness to our philosophical conceptions about the membership relation or infinity. Rather we selected the theory on the basis of its ability to account for the existing data. So at this point it could seem that our theories are on the same wicket.

But recall our defence of the Large Cardinal hierarchy. We did not argue that there are no other strengthenings of ZFC which could address its incompleteness. We merely argued that the Large Cardinals provided the most convenient means of achieving this. Given that the Large Cardinals provide a natural way of ordering strong foundational theories and that they force agreement with regard to concrete questions, we might as

[21] Strictly, we must add something like the Ultimate-L assumption to both theories.

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well use them to strengthen ZFC. But this is where a problem emerges for the pragmatic defence in that the same argumentative strategy seems to apply here. If we come to accept Ultimate-L, then we should also accept that Ultimate-L is a very convenient place to work. But if that's the case, why don't we just stick work stick with ZFC and work inside Ultimate-L. We can always consider what's happens in generic extensions via our inner model, but really UItimate-L *is where the action is*! Despite the fact that this argument could look weaker that our first argument, I think it is ultimately more persuasive for the pragmatist.

3.3.5. What if we don't have a definable well-ordering of the universe? This all said, Ultimate-L is a relative newcomer to the world of set theory. There are number of open questions whose answers could see it cut off at the knees. Perhaps this could give succor to the GMV enthusiast. This section provides some discussion of what happens without Ultimate-L. A recent result by Toshimichi Usuba demonstrates that there may still be cause for concern [25].

THEOREM 20 (Usuba). *If there is a hyper-huge cardinal, then the generic multiverse has a core.*

This gives us one of the ingredients required for our translations. It allows us to define a particular world within any generic multiverse. In the other direction, we don't have a definable well-ordering of the universe, which we can use to define a generic multiverse within any universe. This means that we cannot get the full result above, although we can get something very close to it. For this we need the following result of Steel from [23].

THEOREM 21. *There a recursive* $\rho : \mathcal{L}_{GMV} \to \mathcal{L}_{\in}$ *such that*:

$$GMV \vdash \varphi \Leftrightarrow ZFC \vdash \rho(\varphi).$$

This is not the function σ described above. Rather than exploiting Hamkins' work on Boolean valued ultrapowers, we simply use the syntactic forcing relation. The resultant translation is not a genuine relative interpretation. It is not compositional in the sense that we do not define domains and relations. As such, the translation does not fit easily into our hierarchy of relative interpretations for the simple reason that it does not yield one. There is, however, a syntactic counterpart to sentential equivalence as the following Fact illustrates:

FACT 22. *The following are equivalent*:

(1) *There are relative translations* $\tau : \mathcal{L}_S \to \mathcal{L}_T$ *and* $\sigma : \mathcal{L}_T \to \mathcal{L}_S$ *such that*:
  - $S \vdash \varphi \leftrightarrow \sigma \circ \tau(\varphi)$;
  - $T \vdash \varphi \leftrightarrow \tau \circ \sigma(\varphi)$.

(2) *S and T are sententially equivalent.*

We might then have hoped that to retain the syntactic version of sentential equivalence in this new scenario, but this is not available. First note the following facts from Steel's [23]:

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FACT 23 (Steel). (1) *For* $\varphi \in \mathcal{L}_\in$, $ZFC \vdash \varphi$ *iff* $GMV \vdash \forall U\ \varphi^U$.
(2) *For all* $\varphi \in \mathcal{L}_{GMV}$, $GMV \vdash \forall U(\varphi \leftrightarrow \rho(\varphi)^U)$.

THEOREM 24. *Let* Φ *say that a proper class of hyper-huge cardinals exists and let* Ψ *say that this universe is not the core. Then,*
(1) *if* $\varphi \in \mathcal{L}_{GMV}$ *and* $GMV \cup \{\forall W \Phi^W, \varphi\}$ *is consistent, then* $ZFC \cup \{\Phi, \rho(\varphi)\}$ *is consistent; but*
(2) *if the existence of unbounded hyper-huges is consistent, then* $ZFC \cup \{\Phi, \Psi\}$ *is consistent, while* $GMV \cup \{\forall W \Phi^W, \tau(\Psi)\}$ *is inconsistent.*

Proof. (1) Suppose not. Then $ZFC + \Phi \vdash \neg\rho(\varphi)$. Thus using (1) from Fact 23 and working in $GMV + \forall W \Phi^W$ we have

$$\begin{aligned} &\forall U\ (\neg\rho(\varphi))^U \\ \Leftrightarrow &\forall U\ \neg\rho(\varphi)^U \\ \Leftrightarrow &\neg\exists U\ \rho(\varphi)^U \\ \Leftrightarrow &\neg\varphi \end{aligned}$$

where the first $\Leftrightarrow$ follows by relative interpretation, the second follows from logic and the third $\Leftrightarrow$ follows from (2) of Fact 23. Thus, $GMV \cup \{\forall W \Phi^W, \varphi\}$ is inconsistent.

(2) Let $\mathcal{M} \models ZFC + \Phi$. Then force to add a Cohen real thus obtaining $\mathcal{M}[G] \models ZFC + \Phi$ and $\mathcal{M}[G] \models \Psi$ since $\mathcal{M}[G]$ cannot be the core. Then working in $GMV + \forall W \Phi^W$, note that for $\chi \in \mathcal{L}_\in$, $\tau(\chi)$ says that χ is true at the core. Thus $\tau(\Psi)$ says, "I'm not the core" is true at the core. Laver's theorem[22] shows that a model of ZFC is correct about whether or not it is a generic extension, so we have a contradiction. □

Part (2) tells us that even in the presence of hyper-huge cardinals coextendibility is lost using these translations. So while the theories are still close, it is possible to drive a wedge between them.

COROLLARY 25. $ZFC + \Phi$ *and* $GMV + \exists W \Phi^W$ *are not sententially equivalent via* τ *and* ρ.

Proof. This follows from Proposition 19 and Theorem 24. □

Given we've now jettisoned the project of providing a genuine relative interpretation, we might go even further and avoid hyper-huge cardinals. This means we lose the existence of a core, however, we can get around this issue by employing a simpler translation $\pi : \mathcal{L}_\in \to \mathcal{L}_{GMV}$. Given a sentence $\varphi \in \mathcal{L}_\in$, we just let $\pi(\varphi) = \forall W\ \varphi^W$; i.e., we translate it to saying that φ is true in every world. So rather than checking a particular world, we supervaluate over all of them. We then see that this gives us another faithful interpretation:

LEMMA 26. *For* $\varphi \in \mathcal{L}_\in$, *we see that*

$$ZFC \vdash \varphi \ \Leftrightarrow\ GMV \vdash \pi(\varphi).$$

[22] See Reitz's [22] for an excellent discussion of this theorem and its consequences.

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Proof. ($\rightarrow$) Suppose $ZFC \vdash \varphi$. Then let $\mathcal{W}$ be a model of GMV. Since every world in $\mathcal{W}$ is a model of ZFC, we see that $\mathcal{W} \models \forall W\ \varphi^W$. Thus, $GMV \vdash \pi(\varphi)$.

($\leftarrow$) Suppose $ZFC \nvdash \varphi$. Fix $\mathcal{M} \models ZFC \cup \{\neg\varphi\}$. Let $\mathcal{W}$ be a generic multiverse containing $\mathcal{M}$. Then $\mathcal{W} \models \neg\forall W\ \varphi^W$ and so $GMV \nvdash \pi(\varphi)$. □

There is a sense in which ρ is a better translation than τ was in the context of hyper-huge cardinals. With τ we lacked faithfulness: the $\Leftarrow$ direction failed since GMV proves that "I'm the core" is true at the core, but ZFC doesn't prove "I'm the core." Despite this, we still do not get co-extendibility between ZFC and GMV.

THEOREM 27. *If ZFC is consistent, then there is a sentence φ such that $ZFC + \varphi$ is consistent, but $GMV + \pi(\varphi)$ is inconsistent.*

Proof. Let φ be CH. Then by our assumption and Theorem 7, we see that $ZFC + \varphi$ is consistent. Now let $\mathcal{W}$ be an arbitrary model of GMV. Let $\mathcal{M}$ be an arbitrary world in $\mathcal{W}$. Suppose $\mathcal{M} \models CH$. Then we may fix a generic extension $\mathcal{N}$ of $\mathcal{M}$ in $\mathcal{W}$ such that $\mathcal{N} \models \neg CH$. Similarly, if $\mathcal{M} \models \neg CH$ there will be a generic extension $\mathcal{N}$ of $\mathcal{M}$ in $\mathcal{W}$ such that $\mathcal{N} \models CH$. Thus, we see that $\mathcal{W} \not\models \forall W\ CH^W$ and so $\mathcal{W} \not\models \pi(\varphi)$. Thus we see that $GMV + \pi(\varphi)$ is inconsistent. □

3.3.6. Summary and discussion. We have offered three ways of translating between theories extending ZFC and GMV:

(1) Mutual, faithful, relative interpretation via σ (the *ultrafilter construction*) and τ (the *core finder*) using Ultimate-L.
(2) Mutual, faithful, non-relative interpretation via ρ (the *Steel translation*) and τ (the *core finder*) using hyper-huge cardinals.
(3) Mutual, faithful, non-relative interpretation via ρ (the *Steel translation*) and π (the *supervaluator*) using no extension.

Given that the project of the pragmatist is to explore stronger theories of mathematics, we see that co-extendibility is important: we don't want to be bound unnecessarily. Moreover, given that our default theory is ZFC or some extension thereof, it is important that if we extend ZFC by a corresponding extension of GMV can also be offered. This is only available using the first approach. The latter two only allow us to take a GMV extension and find a corresponding ZFC extension.[23]

3.3.7. What is the cost of abandoning relative interpretation? The issues in the previous sections stemmed from facets of the translation from GMV to ZFC. However, we also offer alternative translations in the other direction. They were contrasted by the fact that the σ translation gives a genuine relative translation while Steel's ρ translation does not. Thus, we now consider the costs associated with abandoning relative interpretation. We consider two responses:

(1) Nothing substantive is lost;
(2) Something important is lost.

[23] Indeed the results above show that if we use the latter approaches, we might even think of GMV as restrictive with respect to ZFC as is discussed with regard to $ZF + V = L$ above (§3.3.4).

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In support of (1), we make the following argument. First we note that Steel's translation gives up on trying to describe a particular model of GMV using ordinary set theory. Rather we exploit the forcing relation to give partial information about a family of possible ways the GMV multiverse could be. Rather than pinning down a particular multiverse, we end up describing a family of them. The underlying reason for this is that the forcing relation is not negation complete. I.e., we do not have $\Vdash \varphi$ or $\Vdash \neg\varphi$ for all $\varphi \in \mathcal{L}_{\in}$: completeness is the property we get from a generic set or ultrafilter.

At this point, it could appear that something substantial has been lost and we have been relegated to a realm of partial information and indeterminacy. However, the following fact about our translations reveals an alternative perspective.

FACT 28. *(i) If* $ZFC \vdash \rho(\gamma)$ *for all* $\gamma \in \Gamma$ *and* $\Gamma \models \varphi$, *then* $ZFC \vdash \rho(\varphi)$.
(ii) $ZFC \vdash \rho(\gamma)$ *for all* $\gamma \in GMV$.

This tells us that although our translations don't give us specific models, logical consequence is still preserved through each of them. Moreover, both translations give us the the theories that we are trying to emulate. This means that if we are using ρ to describe what happens in the generic multiverse, then we can just reason within the context of what ZFC can prove about ρ and lose nothing; i.e., we just do we set theory inside the context: $ZFC \vdash \rho(\cdot)$. So even though we cannot procure the generic multiverse, we can behave—to to speak—as though there is one.[24] A similar story works for π in the other direction.

This then leaves the question: how big a problem is the inability to procure the generic multiverse? For the universist, I think this is a serious issue. The generic multiverse cannot be procured for the simple reason that it contains universes that are bigger than the universe itself. As such, these kinds of worlds can be—at best—instrumental fictions. But for the pragmatist, the issue is more subtle. The pragmatist is inclined to accept the ontology of set theory on the basis that set theory gives them their best theory of mathematics and infinity. As such, pre-theoretic commitments to notions about membership and the transfinite play a much smaller role. Given that we can reason within the context of translation functions in exactly the same way we ordinarily reason about set theory, the pragmatist might take the plunge and accept that there are interpretations which fit with what is described within those contexts.

This brings us back around to (2). Although the pragmatist could indulge in a commitment to the generic multiverse without the use of Ultimate-L, something may still be lost in the move away from relative translation. In particular, we note that the mod-functors τ^* used in the Theorem 18 give use more information than the translation functions ρ from Theorem 24. Beyond sentential equivalence, we also obtain elementary embeddings between models of one theory and the other. Following Steel, we might be tempted to say that these embeddings preserve meaning in the sense that we are able to move between models of one theory and the other and identify objects across that interface.[25] In more philosophical terms, we have established a *de re*

24 This kind of virtual account is closely related to that provided in §IV.5.2 in [17]
25 Given the nature of the ultrapower model described in the Appendex, I suspect it is likely that Steel would not endorse the claim that this translation is *meaning preserving*. However, in the absence of a thoroughgoing account of meaning preservation in these contexts, I think

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connection between models of these theories. This connection is absent in our second version.

§4. Conclusions. In this paper, we have critically examined two arguments against the Generic Multiverse. To motivate these arguments, we first described the underlying motivations that could lead someone toward a multiverse interpretation of set theory. In so doing, we exploited two different attitudes toward the content of set theory. The first was the traditional *universe* view that is familiar to anyone with elementary set theory training. The second *pragmatic* view could appear a little unorthodox, but was ultimately a variation on broadly Quinean themes. Interestingly, the pragmatic view is compatible with and arguably even a good fit with the Generic Multiverse discussed in §1.3.

Our first argument against the Generic Multiverse relied on antecedent—or at least extra-theoretical—intuitions about the nature of the transfinite. As such, it provided a kind of bolstering evidence for adherents of the universe view. However, for Generic Multiverse proponents the argument fell somewhat flat since the antecedent intuitions required for the argument fell into the pot of revisable propositions left open by our ongoing struggles with incompleteness.

The second argument pushed for a more modest conclusion: that ZFC and GMV (appropriately augmented) are for all practical purposes the same theory. As such, there is little to be gained from moving from ZFC to GMV. Although the argument's conclusion is weaker, its underlying strategy is pragmatic in nature. Moreover, it makes far less use of extra-theoretical intuitions about the nature of sets and infinity. The result is—perhaps ironically—a more serious argument for the Generic Multiverse adherent to address.

Appendix.

THEOREM 29 (ZFC). *Suppose there is a definable, set-like, well-ordering of the universe. Then there is a definable inner model of* GMV.

Proof. The proof proceed in three stages:

($STAGE$-1) Defining a sequence of ultrafilters which form the spine of the model.
($STAGE$-2) Defining the inner model through a direct limit of the corresponding ultrapowers.
($STAGE$-3) Showing that GMV is satisfied in the model.

(**STAGE-1**) First we make a few definitions to lighten the notation burden. For $S \subseteq Ord$, let

- $\mathbb{P}_S = Col(\omega, S)$;[26] and
- $\mathbb{B}_S = ro(Col(\omega, S))$.

it is worthwhile proposing this analogy even if only to lead to a more refined analysis of this idea.

26 For our convenience when $S \subseteq Ord$, we let $Col(\omega, S)$ be

$$\{p : S \times \omega \to S \mid |p| < \omega \wedge \forall \langle \alpha, n \rangle \in dom(p)\, p(\alpha, n) < \alpha\}.$$

This ensures that $Col(\omega, \leq \beta)$ is a complete subposet of $Col(\omega, \leq \alpha)$ whenever $\beta \leq \alpha$.

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Let $S \subseteq T \subseteq Ord$. Then we see that $Col(\omega, S)$ is a complete subposet of $Col(\omega, T)$. Moreover, this (trivial) embedding can be lifted to an embedding of the regular open algebras of these posets as follows. Let $r_{S,T} : ro(Col(\omega, S)) \to ro(Col((\omega, T))$ be such that for all $b \in ro(Col(\omega, S)$ we have

$$r_{S,T}(b) = int(cl(b))$$

where the right hand side is calculated in $ro(Col((\omega, T))$. This is a complete embedding. Moreover, it should be clear that whenever we have $S \subseteq T \subseteq U \subseteq Ord$, then

$$r_{T,U} \circ r_{S,T} = r_{S,U}.$$

For collapses of initial segments of the ordinals we'll write $\mathbb{P}_{\leq\alpha}$ and $\mathbb{P}_{<\alpha}$ for $\mathbb{P}_{[0,\alpha]}$ and $\mathbb{P}_{[0,\alpha)}$ respectively. We define $\mathbb{B}_{\leq\alpha}$ and $\mathbb{B}_{<\alpha}$ similarly.

We now define a sequence $\langle U_\alpha \mid \alpha \in Ord\rangle$ of ultrafilters without recourse to parameters.

Claim. *We may uniformly define a sequence of ultrafilters $\langle U_\alpha \mid \alpha \in Ord\rangle$ such that for all $\alpha \in Ord$:*

(1) *U_α is an ultrafilter on $\mathbb{B}_{\leq\alpha}$; and*
(2) *for all $\beta < \alpha$, U_α extends U_β; i.e., for all $\beta < \alpha$, we have*

$$U_\beta = r^{-1}_{\leq\beta,\leq\alpha} U_\alpha.$$

Proof. Let $\prec$ be a definable set-like, well-ordering of the universe. Suppose that we've established the claim for all $\beta < \alpha$. Let

$$\mathbb{B}^\dagger_{<\alpha} = \varinjlim \langle \langle \mathbb{B}_{\leq\beta} \mid \beta < \alpha\rangle, \langle r_{\leq\gamma,\leq\beta} \mid \gamma < \beta < \alpha\rangle\rangle.$$

Note that when $\alpha = \beta + 1$, $\mathbb{B}^\dagger_{<\alpha} \cong \mathbb{B}_\beta$; and if α is a limit, then $ro(\mathbb{B}^\dagger_{<\alpha}) \cong ro(\mathbb{P}_{<\alpha})$.

Let

$$U^\dagger_{<\alpha} = \{r^\dagger_{\beta,\alpha}(u) \mid \beta < \alpha \wedge u \in U_\beta\}$$

where $r^\dagger_{\beta,\alpha}$ is the canonical embedding from $\mathbb{B}_{\leq\beta}$ into $\mathbb{B}^\dagger_{<\alpha}$ given by the direct limit. It can be seen the $U^\dagger_{<\alpha}$ is a ultrafilter on $\mathbb{B}^\dagger_{<\alpha}$; and that $r^\dagger_{\beta,\alpha}$ is a complete embedding for all $\beta < \alpha$.

In order to obtain part (2) of the theorem we shall define some embeddings and make use of Lemma 30 below. First we let $f : \mathbb{B}^\dagger_{<\alpha} \to \mathbb{B}_{<\alpha}$ be such that for all $r^\dagger_{\beta,\alpha}(b) \in \mathbb{B}^\dagger_{<\alpha}$

$$f(r^\dagger_{\beta,\alpha}(b)) = r_{\leq\beta,<\alpha}(b).$$

where $\beta < \alpha$ and $b \in \mathbb{B}_{\leq\beta}$. It can be seen that f is a complete embedding; moreover see that $f \circ r^\dagger_{\beta,\alpha} = r_{\leq\beta,<\alpha}$ for all $\beta < \alpha$.

Let $k : \mathbb{B}^\dagger_{<\alpha} \to \mathbb{B}_{\leq\alpha}$ be the complete embedding where $k = r_{<\alpha,\leq\alpha} \circ f$. Let $F = k``U^\dagger_{<\alpha}$ which is a filter on $\mathbb{B}_{\leq\alpha}$. Let U_α be the $\prec$-least ultrafilter on $\mathbb{B}_{\leq\alpha}$ such that $U_\alpha \supseteq F$.

Then using Lemma 30, we see that:

(1) $U^\dagger_{<\alpha} = k^{-1} U_\alpha$; and
(2) $U_\beta = (r^\dagger_{\beta,\alpha})^{-1} U^\dagger_{<\alpha}$ for all $\beta < \alpha$;

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Thus we see that for all $\beta < \alpha$,

$$\begin{aligned} U_\beta &= (k \circ r^{\dagger}_{\beta,\alpha})^{-1} U_\alpha \\ &= (r_{<\alpha \leq \alpha} \circ r_{\leq \beta, <\alpha})^{-1} U_\alpha \\ &= r^{-1}_{\leq \beta, \leq \alpha} U_\alpha \end{aligned}$$

□

(**STAGE-2**) Before we define our inner model we recall few definitions and results from [11]. For $\mathbb{B}$ a complete Boolean algebra, U an ultrafilter on $\mathbb{B}$ and $\sigma, \tau \in V^{\mathbb{B}}$ we let:

$[\sigma]^{\mathbb{B}}_U$ be the set of $\mathbb{B}$-names τ of least rank such that $[\sigma = \tau]^{\mathbb{B}} \in U$.

We then define the model $V^{\mathbb{B}}/U$ as the model:

- whose domain is the class of $[\sigma]^{\mathbb{B}}_U$ for $\sigma \in V^{\mathbb{B}}$; and
- whose membership relation is defined such that

$$[\sigma]^{\mathbb{B}}_U \in^{\mathbb{B}}_U [\tau]^{\mathbb{B}}_U \Leftrightarrow [\sigma \in \tau]^{\mathbb{B}} \in U.$$

We then let $\check{V}_{(\mathbb{B})}/U$ be the submodel of $V^{\mathbb{B}}/U$ consisting of those $[\tau]^{\mathbb{B}}_U$ for which

$$\bigvee_{x \in V} [\tau = \check{x}] \in U.$$

Then if we let $i : V \to \check{V}_{(\mathbb{B})}/U$ be such that

$$i(x) = [\check{x}]^{\mathbb{B}}_U,$$

it can be seen that i is a cofinal elementary embedding.[27]

For each $\alpha \in Ord$, let i_α be the embedding given above where we use $\mathbb{B}_{\leq \alpha}$ and U_α for $\mathbb{B}$ and U.

Let $V_{(0)} = V$ and $V_{(\alpha)} = \check{V}_{(\mathbb{B}_{\leq \alpha})}/U_\alpha$ for all α.

Let $G_\alpha = [\dot{G}_{\mathbb{B}_{\leq \alpha}}]_{U_\alpha}$ where $\dot{G}_{\mathbb{B}_{\leq \alpha}} = \{\langle \check{b}, b \rangle \mid b \in \mathbb{B}_{\leq \alpha}\}$. Then it can be seen that:[28]

$$(V_{(\alpha)}[G_\alpha] = V^{\mathbb{B}_{\leq \alpha}}/U_\alpha)^{V^{\mathbb{B}_{\leq \alpha}/U_\alpha}}.$$

Given this, we shall denote $V^{\mathbb{B}_{\leq \alpha}}/U_\alpha$ as $V_{(\alpha)}[G_\alpha]$.

Let $\beta \geq \alpha$. Then using Theorem 31, we see that there is an elementary embedding.

$$k_{\alpha,\beta} : V_\alpha[G_\alpha] \to V_\beta[G_\beta \cap i_{0,\beta}(r_{\leq \alpha, \leq \beta}``\mathbb{B}_{\leq \alpha})]$$

where for $\tau \in V^{\mathbb{B}_{\leq \alpha}}$ we have

$$k_{\alpha,\beta}([\tau]^{\mathbb{B}_{\leq \alpha}}_{U_\alpha}) = [\tau^{r_{\leq \alpha, \leq \beta}}]^{\mathbb{B}_{\leq \beta}}_{U_\beta}$$

and where $r_{\leq \alpha, \leq \beta}$ is the canonical complete embedding of $\mathbb{B}_{\leq \alpha}$ into $\mathbb{B}_{\leq \beta}$. Moreover, we see that $G_\beta \cap i_{0,\beta}(r_{\leq \alpha, \leq \beta}``\mathbb{B}_{\leq \alpha}) = k_{\alpha,\beta}(G_\alpha)$.

For $\beta \geq \alpha$, let $V_{(\beta)}[k_{\alpha,\beta}(G_\alpha)]$ be defined using $k_{\alpha,\beta}$; i.e., we let $V_{(\beta)}[k_{\alpha,\beta}(G_\alpha)]$ be the generic extension of $V_{(\beta)}$ by $k_{\alpha,\beta}(G_\alpha)$ according to $V_{(\beta)}[G_\beta]$.

27 See Theorem 15 of [11].

28 See Lemma 13(3) of [11].

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Let

$$V_{(\infty)} = \varinjlim \langle \langle V_{(\alpha)} \rangle_{\alpha \in Ord}, \langle k_{\alpha,\beta} \rangle_{\alpha \leq \beta \in Ord} \rangle$$

and we let i_∞ be the elementary embedding associated with this direct limit.

For all $\alpha \in Ord$, let

$$V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)] = \varinjlim \langle \langle V_{(\beta)}[k_{\alpha,\beta}(G_\alpha)] \rangle_{\beta \in [\alpha, Ord)}, \langle k_{\gamma,\beta} \rangle_{\alpha \leq \gamma \leq \beta \in Ord} \rangle;$$

and for all $\alpha \in Ord$, we let

$$k_{\alpha,\infty} : V_{(\alpha)}[G_\alpha] \to V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]$$

be the elementary embedding associated with this direct limit.

CLAIM. *For all $\beta < \alpha \in Ord$, we have*: □

(1) $(V_{(\alpha)}[k_{\beta,\alpha}(G_\beta)]$ is a generic extension of $V_{(\alpha)}$ over $i_\alpha(\mathbb{B}_{\leq \beta}))^{V_{(\alpha)}[G_\alpha]}$;
(2) $(V_{(\infty)}[k_{\beta,\infty}(G_\beta)]$ is a generic extension of $V_{(\infty)}$ over $i_\infty(\mathbb{B}_{\leq \beta}))^{V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]}$.

Proof. (1) This follows from Theorem 31 since $\mathbb{B}_{\leq \beta}$ is completely embedded in $\mathbb{B}_{\leq \alpha}$ and $r^{-1}_{\leq \beta, \leq \alpha} U_\alpha = U_\beta$. (2) By elementarity. □

We now define our inner model. A world in our multiverse is a generic refinement of some $V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]$. So using Theorem 32 (below) a world is something of the form $(W_r)^{V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]}$ for some $\alpha \in Ord$ and $r \in V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]$.

It should be clear that each world is uniformly definable from some $\langle \alpha, r \rangle \in Ord \times V$. (**STAGE-3**)

CLAIM. *MV holds in this model.* □

Proof. (Refinement) Trivial.

(Extension) Suppose $\mathbb{Q}$ is a poset in some world $(W_r)^{V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]}$ for some $\alpha \in Ord$ and $r \in V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]$.

Work in $V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]$. Let γ be the least ordinal such that

$$i_\infty(\gamma) > |\mathbb{Q}|.$$

Now working in $V_{(\infty)}[k_{\gamma,\infty}(G_\gamma)]$, we see that this universe thinks it is a generic extension of $V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]$ by

$$t^{-1}(k_{\gamma,\infty}(G_\gamma))$$

over the poset

$$(\mathbb{B}_{(i_\infty(\alpha), i_\infty(\gamma)])})^{V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]}$$

where $t = (r_{(i_\infty(\alpha), i_\infty(\gamma)]), \leq i_\infty(\gamma)})^{V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]}$ is the canonical embedding from

$$(\mathbb{B}_{(i_\infty(\alpha), i_\infty(\gamma)]})^{V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]} \text{ into } (\mathbb{B}_{\leq i_\infty(\gamma)})^{V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]}.$$

Thus $V_{(\infty)}[k_{\gamma,\infty}(G_\gamma)]$ thinks it is

$$V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)][t^{-1}(k_{\gamma,\infty}(G_\gamma))].$$

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Then we observe that since $V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]$ thinks

$$i_\infty(\gamma) > |\mathbb{Q}|$$

there is a complete embedding $s \in V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]$ from $\mathbb{Q}$ into $(\mathbb{B}_{(i_\infty(\alpha),(i_\infty(\gamma))})^{V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]}$. Thus $t \circ s : \mathbb{Q} \to i_\infty(\mathbb{B}_\gamma)$ is a complete embedding in $V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]$.

Working in $V_{(\infty)}[k_{\alpha,\gamma}(G_\gamma)]$, if we let $H = (t \circ s)^{-1}(k_{\gamma,\infty}(G_\gamma))$, we see that $V_{(\infty)}[k_{\gamma,\infty}(G_\gamma)]$ thinks that H is $\mathbb{Q}$-generic over $V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]$. Clearly H is also $\mathbb{Q}$-generic over $(W_r)^{V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]}$. Thus since $(W_r)^{V_{(\infty)}[k_{\alpha,\infty}(G_\alpha)]}[H]$ is a generic refinement of $V_{(\infty)}[k_{\gamma,\infty}(G_\gamma)]$, we see that it is a world.

(Amalgamation) Consider the worlds:

- $(W_{r_0})^{V_{(\infty)}[k_{\alpha_0,\infty}(G_{\alpha_0})]}$ for some $\alpha_0 \in Ord$ and $r_0 \in V_{(\infty)}[k_{\alpha_0,\infty}(G_{\alpha_0})]$; and
- $(W_{r_1})^{V_{(\infty)}[k_{\alpha_1,\infty}(G_{\alpha_1})]}$ for some $\alpha_1 \in Ord$ and $r_1 \in V_{(\infty)}[k_{\alpha_1,\infty}(G_{\alpha_1})]$.

Let $\delta = {}^{(}\{\alpha_0, \alpha_1\})$. Then we see that $V_{(\infty)}[i_{\delta,\infty}(G_\delta)]$ is a generic extension of both:

$$V_{(\infty)}[k_{\alpha_0,\infty}(G_{\alpha_0})] \text{ and } V_{(\infty)}[k_{\alpha_1,\infty}(G_{\alpha_1})],$$

which in turn means that it is a generic extension of both $(W_{r_0})^{V_{(\infty)}[k_{\alpha_0,\infty}(G_{\alpha_0})]}$ and $(W_{r_1})^{V_{(\infty)}[k_{\alpha_1,\infty}(G_{\alpha_1})]}$ as required. □

LEMMA 30. *Suppose $k : \mathbb{P} \to \mathbb{Q}$ is a complete embedding and $U \subseteq \mathbb{P}$ is an ultrafilter. Let $F \subseteq \mathbb{Q}$ be a filter such that $F \supseteq k``U$. Then*

$$k^{-1}F = U.$$

Proof. Clearly $U \subseteq k^{-1}F$ since $F \supseteq k``U$. So suppose there is some $p \in k^{-1}F$ such that $p \in (k^{-1}F)\setminus U$. Then since U is an ultrafilter of $\mathbb{P}$, we may fix $u \in U$ such that $u \perp p$. But since k is complete this means that $k(u) \perp k(p)$ which is impossible, since $k(u), k(p) \in F$ and F is a filter on $\mathbb{Q}$. □

THEOREM 31. *Suppose $r : \mathbb{B}_0 \to \mathbb{B}_1$ is a complete embedding between complete Boolean algebras. Let $\tau \mapsto \tau^r$ be the standard translation of $\mathbb{B}_0$-names into $\mathbb{B}_1$-names via r. Then $U_0 = r^{-1}U_1$ is an ultrafilter on $\mathbb{B}_0$. For $j \in 2$, let*

- $V_j = \check{V}_{(\mathbb{B}_j)}/U_j$;
- *$i_j : V \to V_j$ be the elementary embedding where $i_j(x) = [\check{x}]^{\mathbb{B}_i}_{U_i}$; and*
- $G_j = [\dot{G}_{\mathbb{B}_j}]^{\mathbb{B}_j}_{U_j}$ *where* $\dot{G}_{\mathbb{B}_j} = \{\langle \check{b}, b\rangle \mid b \in \mathbb{B}_j\}$.

Then $k : V_0[G_0] \to V_1[i_1(r``\mathbb{B}_0) \cap G_1]$ is an elementary embedding which make the following diagram commute:

$$\begin{array}{ccccccc} & & V_1 & \xrightarrow{\subseteq} & V_1[k(G_0)] & \xrightarrow{\subseteq} & V_1[G_1] \\ & {}^{i_0}\nearrow & \uparrow k & & \uparrow k & & \\ V & \xrightarrow{i_1} & V_0 & \xrightarrow{\subseteq} & V_0[G_0]. & & \end{array}$$

Moreover, in $V_1[G_1]$ *we have* $k(G_0) = i_1(r``\mathbb{B}_0) \cap G_1$.

Remark. This is an elementary generalization of Theorem 46 from [11].

Theorem 32 ([22]). *There is a class term* $W_{(\cdot)} \in \mathcal{L}_\in$ *such that the following are equivalent*:

(1) *N is an inner model whose generic extension is the universe*; *and*
(2) $N = W_r$ *for some* r.

Finally, we prove the main translation theorem, which we restate here for convenience.

Theorem 18. *There exist translations giving rise to mod-functors*:

- $\sigma^* : Mod(ZFC + V = UL) \to Mod(GMV + \exists W(V = UL)^W)$; *and*
- $\tau^* : Mod(GMV + \exists W(V = UL)^W) \to ZFC + V = UL$

such that

(1) $M \precsim \tau^* \circ \sigma^*(M)$ *for* $M \in Mod(ZFC + V = UL)$;
(2) $M \equiv \sigma^* \circ \tau^*(M)$ *for* $M \in Mod(GMV + \exists W(V = UL)^W)$.

Proof. (1) Let $M \models ZFC + V = UL$. Then $\sigma^*(M) \models GMV + \exists W(V = UL)^W$. Moreover, there exists $(i_\infty)^M$ an embedding from M into $(V_{(\infty)})^M$. Then since $(V_{(\infty)})^M$ is the core of $\sigma^*(M)$, we see that $\tau^* \circ \sigma^*(M) = (V_{(\infty)})^M$. Thus $(i_\infty)^M$ gives us the elementary embedding we were looking for.

(2) Suppose $\mathcal{W} \models GMV + \exists W(V = UL)^W$. Then we see that $\tau^*(\mathcal{W}) \models ZFC + V = UL$ and $\sigma^* \circ \tau^*(\mathcal{W}) \models GMV + \exists W(V = UL)^W$. Let $\mathcal{N} \in \sigma^* \circ \tau^*(\mathcal{W})$ be the world at which $V = UL$ is true. Then since $\mathcal{N}$ is the core of $\sigma^* \circ \tau^*(\mathcal{W})$, we see that $\mathcal{N} = (V_\infty)^{\tau^*(\mathcal{W})}$ and so there exists an elementary embedding $j : \tau^*(\mathcal{W}) \to \mathcal{N}$. Thus we see that

$$\tau^*(\mathcal{W}) \in \mathcal{W}, \mathcal{N} \in \sigma^* \circ \tau^*(\mathcal{W}), \text{ and } \tau^*(\mathcal{W}) \equiv \mathcal{N};$$

then we see by Lemma 33 that $\mathcal{W} \equiv \sigma^* \circ \tau^*(\mathcal{W})$ as required. □

Lemma 33. *Suppose* $\mathcal{M}_0 \in \mathcal{W}_0$ *and* $\mathcal{M}_1 \in \mathcal{W}_1$ *where both* $\mathcal{W}_0$ *and* $\mathcal{W}_1$ *are models of* GMV. *Then if* $\mathcal{M}_0 \equiv \mathcal{M}_1$, *we have* $\mathcal{W}_0 \equiv \mathcal{W}_1$.

Proof. Suppose we have $\mathcal{M}_0, \mathcal{W}_0, \mathcal{M}_1$ and $\mathcal{W}_1$ are as described, and assume without loss of generality that both $\mathcal{W}_0$ and $\mathcal{W}_1$ are countable. Then for all $\varphi \in \mathcal{L}_{GMV}$ we see,

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from Fact 23(2), that:

$$\begin{aligned} \mathcal{W}_0 \models \varphi &\Leftrightarrow \mathcal{M}_0 \models \rho(\varphi) \\ &\Leftrightarrow \mathcal{M}_1 \models \rho(\varphi) \\ &\Leftrightarrow \mathcal{W}_1 \models \varphi. \end{aligned}$$

□

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