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# BIHOLOMORPHIC MAPPINGS ON BANACH SPACES

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*Abstract* We present an infinite-dimensional version of Cartan's theorem concerning the existence of a holomorphic inverse of a given holomorphic self-map of a bounded convex open subset of a dual Banach space. No separability is assumed, contrary to previous analogous results. The main assumption is that the derivative operator is power bounded, and which we, in turn, show to be diagonalizable in some cases, like the separable Hilbert space.

*Keywords:* automorphism; biholomorphic mapping; power-bounded operator

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## 1. Introduction and preliminaries

H. Cartan's theorem stating that a holomorphic self-map of a bounded domain in $\mathbb{C}^n$ with a fixed point at which the derivative is the identity has to be the identity was widened by Cima *et al.* [**3**] to separable Hilbert spaces and then to separable dual Banach spaces by the authors of [**2**]. It is key in all these instances to still follow Cartan's iteration idea that the sequence of iterates of the derivative is bounded. Here we aim at getting rid of the separability. Thus we are led to study power-bounded operators, which is the topic of § 2, while in § 3 we provide some sufficient conditions for a mapping to be biholomorphic and show that the well-known group of automorphisms of the unit ball of a Hilbert space is a connected topological group with respect to the uniform convergence.

All Banach spaces considered throughout this paper are complex. Let $E$ be a Banach space and $V$ an open subset of $E$. We will denote by $\mathcal{H}(V, E)$ the space of all holomorphic, that is, Fréchet differentiable, mappings from $V$ into $E$. If $f \in \mathcal{H}(V, E)$, we denote the derivative of $f$ at the point $p \in V$ by $\mathrm{d}f_p$. We refer to [**15**] for non-explained notation regarding holomorphic mappings and to [**4**] for functional analysis background.

Let $T \in \mathcal{L}(E)$ be a bounded linear operator. As usual, $\sigma(T)$ denotes its spectrum and $\sigma_p(T)$ its point-spectrum, that is, the set of eigenvalues. The immediate precedents of this research dealt with triangularizable derivative operators. We call $T$ triangularizable if there is a total, that is, with dense span, linearly independent sequence $\{e_1, e_2, \dots e_n, \dots\}$ of $E$ such that for every $n \in \mathbb{N}$

$$T(x) \in span\{e_1, e_2, \dots, e_n\} \qquad \text{for all } x \in span\{e_1, e_2, \dots, e_n\}.$$

In this case, $T(e_k) = \sum_{j=1}^{k} \beta_j^k e_j$, for all $k = 1, 2, \dots, n$. So, the matrix of $T$ when restricted to the subspace generated by $\{e_1, e_2, \dots, e_k\}$ is an upper-triangular matrix, whose main diagonal is given by $\beta_1^1, \beta_2^2, \dots \beta_k^k$. We shall refer to the sequence $(\beta_k^k)$ as the diagonal entries.

Notice that the existence of a triangularizable operator on $E$ requires that $E$ be separable. As usual, $I_E$ denotes the identity operator on the Banach space $E$.

## 2. Power-bounded operators

Recall that an operator is said to be *power bounded* if $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$. If $E$ is a Banach space, this is equivalent to $\sup_{n \in \mathbb{N}} \|T^n(x)\| < \infty$ for each $x \in E$ according to the Uniform Boundedness Principle.

**Theorem 1.** *Let $E$ be a Banach space. Let $T \in \mathcal{L}(E)$ be a power-bounded closed range operator. Assume there is a subsequence $(T^{n(k)})_k \subset (T^n)_n$ which converges pointwise to an operator $S \in \mathcal{L}(E)$. If either $S$ is onto and $T$ is one-to-one, or $S$ is invertible, then $T$ is an invertible operator and $\sigma(T) \subset S(0,1)$. Further, if $m(k) = n(k+1) - n(k)$, then $(T^{m(k)})_k$ converges pointwise to the identity mapping.*

*In particular, if $E$ is a Hilbert space, then $T$ is similar to a unitary operator.*

**Proof.** First, notice that in the case where $S$ is invertible, $T$ is also injective, since if $T(x) = 0$, $T^{n(k)}(x) = 0$, so $S(x) = 0$. Thus, $T$ is an into isomorphism under any of the assumptions.

Let $y \in E$ be arbitrary and consider $x \in E$ such that $Sx = y$. Then $y = \lim T^{n(k)}x = \lim T(T^{n(k)-1}x)$. Now, for $y_k = T^{n(k)-1}x$, the sequence $(y_k)$ is a Cauchy sequence since $T$ is an isomorphism. Let $y_0 \in E$ be the limit of $(y_k)$. Then $T(y_0) = \lim T(y_k) = \lim T((T^{n(k)-1})(x)) = \lim T^{n(k)}(x) = y$. Therefore, $T$ is an onto mapping and thus is invertible by the Open Mapping Principle.

We now prove that $\sup_m \|T^{-m}\| < \infty$. Suppose that for some sequence, $\lim_k \|T^{-l(k)}\| = \infty$. Let $x \in E$, and pick $y \in E$ such that $Sy = x$. So for every $l(k)$ there is $n(k) > l(k)$ such that $\|Sy - T^{n(k)}y\| < \|T^{-l(k)}\|^{-1}$. Therefore,

$$\begin{aligned} \|T^{-l(k)}x\| &= \|T^{-l(k)}Sy\| \\ &= \|T^{-l(k)}(Sy - T^{n(k)}y) + T^{n(k)-l(k)}y\| \\ &\le 1 + \|T^{n(k)-l(k)}y\| \le 1 + M\|y\|, \end{aligned}$$

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where $M \geq \sup_{n\in\mathbb{N}} ||T^n||$. Then $\sup_k \|T^{-l(k)}x\| < \infty$, so by the Uniform Boundedness Principle $(T^{-l(k)})$ would be uniformly bounded, a contradiction. Therefore, we have

$$C := \sup_{n\in\mathbb{Z}} \|T^n\| < \infty.$$

Then $||x|| \leq C||T^m(x)||$ for all $m \in \mathbb{Z}$ and $x \in E$.

Let $m(k) = n(k+1) - n(k)$. We check that $\lim_k T^{m(k)}x = x$. Since $S(x) = \lim_k T^{n(k)}(x)$, the sequence $\{T^{n(k)}(x)\}$ is a Cauchy sequence. Thus, for given $\varepsilon > 0$, we have $||T^{n(k+1)}(x) - T^{n(k)}(x)|| \leq \varepsilon$ for $k$ big enough. Hence, for $k$ big enough,

$$\varepsilon \geq ||T^{n(k+1)}(x) - T^{n(k)}(x)|| = ||T^{n(k)}(T^{m(k)}(x) - x)|| \geq \frac{1}{C}||(T^{m(k)}(x) - x)||.$$

Thus $\lim_k T^{m(k)}(x) - x = 0$.

Now, since $||T^n|| \leq C$ for all $n \in \mathbb{Z}$, we have $r(T) \leq 1$ and $r(T^{-1}) \leq 1$. So $\sigma(T) \subset \Delta(0,1)$ and $\sigma(T^{-1}) \subset \Delta(0,1)$ and, moreover, by the Functional Calculus for operators, $\sigma(T^{-1}) = 1/\sigma(T)$. Thus $\sigma(T) \subset S(0,1)$.

If $E$ is a Hilbert space it is proved in [**16**] that for every power-bounded operator $T$ there is an invertible self-adjoint operator $Q$ such that $QTQ^{-1} = U$ is unitary. □

**Example 2.** The backward shift operator $B((x_n)) := (x_2, x_3, \ldots)$ acting on $\ell_2$ is power bounded since $\|B\| \leq 1$ and the sequence of iterates $B^m((x_n)) = (x_{m+1}, x_{m+2}, \ldots)$ converges pointwise to a non-surjective operator, the null one. So in Theorem 1 some assumptions about $S$ are necessary.

**Example 3.** Let $(\lambda_n)_n \subset S(0,1)$. Consider the Hilbert space $E = \ell_2(\mathbb{R})$. Put $\mathbb{R} \equiv I \times \mathbb{N}$ for an uncountable set $I$. For every element in the canonical basis $\{e_{i,n} : I \times \mathbb{N}\}$ define $T(e_{i,n}) = \lambda_n e_{i,n}$. Clearly, $T \in \mathcal{L}(E)$, $\|T\| = 1$ and hence $\|T^m\| \leq 1$. According to [**2**, Lemma 2.1.3], there is a sequence of positive integers $(m_k)$ such that $\lim_k \lambda_n^{m_k} = 1$ for all $n \in \mathbb{N}$. It turns out that $T^{m_k}$ converges pointwise to $Id_{|E}$ since for all pairs $(i,n)$, $\lim_k T^{m_k}(e_{i,n}) = \lim_k (\lambda_n)^{m_k} e_{i,n} = e_{i,n}$. Notice that $T$ is not triangularizable, since $E$ is not separable.

**Example 4.** Let $H^\infty$ be the uniform algebra of bounded analytic functions on the open unit disc $\Delta(0,1) \subset \mathbb{C}$. Every analytic self-map $\varphi$ of $\Delta(0,1)$ leads to the composition operator $C_\varphi : x \in H^\infty \to x \circ \varphi \in H^\infty$. Suppose $\varphi$ is not constant. Since $C_\varphi$ is clearly one-to-one and has norm not greater than 1, according to Theorem 1, if $C_\varphi$ has closed range and there is a subsequence $(C_\varphi)^{n(k)} = (C_{\varphi^{n(k)}})$ that converges pointwise to an onto operator in $\mathcal{L}(H^\infty)$, then $C_\varphi$ is invertible. The same holds for the generalization $H^\infty(B_E)$, the uniform algebra of bounded analytic functions on the open unit ball of a Banach space $E$. A similar situation occurs for the disc algebra $A$ or its generalization $A(B_E)$.

**Remark 5.** In the above Theorem 1, the assumption of $T$ having closed range may be replaced by the weak compactness of $T$.

Indeed, let $y \in E$ be arbitrary and consider $x \in E$ such that $Sx = y$. Then $y = \lim T^{n(k)}x = \lim T(T^{n(k)-1}x)$. Put $y_k = T^{n(k)-1}x = T(T^{n(k)-2}x)$. Since the sequence

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$\{y_k\}$ is contained in the relatively weakly compact set $T(M||x||B_E)$, where $M$ is such that $||T^n|| \leq M$, there is a subsequence $(y_{k_l}) = (T^{n(k_l)-1}x)$ which weakly converges to $y_0 \in E$. Since $T$ is weakly continuous, we have that $T(y_0) = \omega - \lim T(T^{n(k_l)-1}) = \omega - \lim T^{n(k_l)}(x) = y$. Therefore, $T$ is onto.

**Proposition 6.** *Let $E$ be a separable dual Banach space. If $T$ is a triangularizable operator with diagonal entries in $S(0,1)$ such that $\sup_n \|T^n\| < \infty$, then $T$ is invertible and $\sigma_p(T) \supset diag(T)$.*

**Proof.** According to [2, Lemma 2.3] there is a sequence $(T^{n(k)})_k$ that converges pointwise to the identity $Id_E$. By using [2, Lemma 2.7], we can find a subsequence such that $(T^{n(k_i)-1})_i$ is compact-open-weak* convergent to some linear operator $S \in \mathcal{L}(E)$ that is actually onto, because for $y \in E$, $y = \lim T^{n(k_i)}y = \lim T^{n(k_i)-1}(T(y)) = w^* - \lim T^{n(k_i)-1}(T(y)) = S(T(y))$. In addition, $S \circ T = Id_E$. We verify also that $T \circ S = Id_E$ by checking that $(T \circ S)(x_n) = x_n$. This will follow from observing that such an identity holds on $E_k := span\{x_1, \ldots, x_k\}$ for all $k \in \mathbb{N}$. Indeed, notice that since for $T$ the subspace $E_m$ is invariant, all iterates $T^m$ also have $E_k$ as an invariant subspace. That is, for every $x \in E_k$, $S(x) \in \overline{E_k}^{w^*} = E_k$. This means that when restricting the equality $T \circ S = Id_E$ to the finite-dimensional space $E_k$, we have $T_{|E_k} \circ S_{|E_k} = Id_{|E_k}$ and hence also $S_{|E_k} \circ T_{|E_k} = Id_{|E_k}$.

In addition, the matrix representing the linear operator $T_{|E_k}$ is a $k \times k$ triangular matrix, thus the diagonal terms are characteristic values and hence eigenvalues for $T$. $\square$

Without the power-boundedness, none of the conclusions of Proposition 6 hold. Let $T : \ell_p \to \ell_p$ be given by

$$T = \begin{pmatrix} -1 & 2 & 0 & 0 & \cdots \\ 0 & -1 & 2 & 0 & \cdots \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

If $(\alpha_i) \in \ell_p$, then for $(\beta_i) := T(\alpha_i)$, we have that $\beta_i = -\alpha_i + 2\alpha_{i+1}$, $(\beta_i) \in \ell_p$ and $\|(\beta_i)\| \leq 3\|(\alpha_i)\|$. So $T \in \mathcal{L}(\ell_p)$ and $T(1, 1/2, 1/4, \ldots) = (0, 0, \ldots)$. Thus $T$ is not one-to-one. Further, the sequence $((-1)^i/2^i)$ is an eigenvector with eigenvalue $-2$.

**Proposition 7.** *Let $Y$ be a dense subspace of the Banach space $E$ and $Q \in \mathcal{L}(E)$ such that the orbit of every $x \in Y$ is eventually constant. If $\sup_{n\in\mathbb{N}} \|(I+Q)^n\| < \infty$, then $Q = 0$.*

**Proof.** Let $x \in Y$ and $m_x \in \mathbb{N}$ be such that $Q^n(x) = v$ is constant for $n \geq m_x$. So

$$(I+Q)^n(x) = x + \sum_{j=1}^{m_x} \binom{n}{j} Q^j x + v \cdot \sum_{j=m_x+1}^{n} \binom{n}{j}.$$

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Suppose that for some $j \in \{1, 2, ..., m_x\}$ we have that $Q^j x \neq 0$. Since the space generated by $\{v, Q^1 x, Q^2 x, \ldots, Q^{m_x}\}$ is finite dimensional, then

$$\limsup_n \left\| \sum_{j=1}^{m_x} \binom{n}{j} Q^j x + v \sum_{j=m_x+1}^{n} \binom{n}{j} \right\| = \infty.$$

On the other hand,

$$\left\| \sum_{j=1}^{m_x} \binom{n}{j} Q^j x + v \sum_{j=m_x+1}^{n} \binom{n}{j} \right\| = \|(I+Q)^n x - x\| \leq C\|x\| + 1 < \infty.$$

This is a contradiction. So $Q^j(x) = 0$ for $j = 1, 2, \ldots m_x$, and $Q_{|_Y} = 0$. As a consequence, $Q = 0$ in $E$. □

We recall that an operator $T \in \mathcal{L}(E)$ is said to be nilpotent if $T^n = 0$ for some $n \in \mathbb{N}$, and pseudonilpotent if for each $x \in E$ there exists $n_x \in \mathbb{N}$ such that $T^{n_x} x = 0$. The operator $T$ is said to be quasinilpotent or topologically nilpotent if its spectrum $\sigma(T) = \{0\}$ or, equivalently, if its spectral radius $r(T) = \lim \|T\|^{1/n} = 0$.

Obviously, Proposition 7 holds if $Q$ is a nilpotent operator. We do not know whether Proposition 7 is true if $Q$ is a quasinilpotent operator.

It is known that if $\liminf \sqrt{n}\|Q^n\|^{1/n} = 0$, then $Q$ is a pseudonilpotent and quasinilpotent operator (see [**1**, Proposition 4.4]). Proposition 7 holds if $Q$ is a projection, but it is not true in general for compact operators. To see this, we take a compact operator $Q \neq 0$ with $\sigma(Q) \subset [-2, 0]$ and observe that $(I+Q)^n$ is a self-adjoint operator for all $n$ and therefore $\|(I+Q)^n\| = \sup\{|1+\lambda|^n : \lambda \in \sigma(Q)\} \leq 1$. So $\sup_{n\in\mathbb{N}} \|(I+Q)^n\| < \infty$.

In [**3**, Theorem 1.1], it is shown for a bounded convex domain $\Omega$ in a separable Hilbert space $H$ that if $f : \Omega \to \Omega$ is a holomorphic mapping with fixed point $p \in \Omega$, such that $\mathrm{d}f_p$ is triangularizable and $\sigma(\mathrm{d}f_p) \subset S(0, 1)$, then $f$ is biholomorphic. In fact, under these conditions, $\mathrm{d}f_p$ is similar to a diagonal matrix. This is a consequence of the following result.

**Corollary 8.** *Let $H$ be a separable Hilbert space and $T \in \mathcal{L}(H)$ a triangular operator with respect to the basis $\{e_i : i = 1, 2..\}$ of $H$. If $T$ is similar to a unitary $U \in \mathcal{L}(H)$, then $T$ is diagonalizable.*

**Proof.** By hypothesis there is an operator $M \in \mathcal{L}(H)$ such that $MTM^{-1} = U$. This implies that $\sigma(T) = \sigma(U) \subset S(0, 1)$ and $U$ is also triangularizable with respect to the basis $\{\varepsilon_i := Me_i : i \in \mathbb{N}\}$. Let $\{\mu_i : i \in \mathbb{N}\}$ be the orthonormal basis of $H$ obtained from $\{\varepsilon_i : i \in \mathbb{N}\}$ by the Gram–Schmidt procedure. Then $span[\varepsilon_1, \varepsilon_2, ...\varepsilon_k] = span[\mu_1, \mu_2 \ldots, \mu_k]$ for all $k = 1, 2 \ldots$. This implies that the matrix representation of $U$ with respect to the orthonormal basis $\{\mu_i : i \in \mathbb{N}\}$ is upper triangular, so we can write

$$MTM^{-1} = (D + Q),$$

where $D$ is a diagonal matrix with entries in $S(0, 1)$ and $Q$ is a pseudonilpotent matrix in $\bigcup_{n\geq 1} span[\mu_1, \mu_2, \ldots, \mu_n]$. Obviously, $D \in \mathcal{L}(H)$ is invertible and $\|Dx\| = \|D^{-1}x\| = \|x\|$

for all $x \in H$. So $\|D^{-1}\| = 1$. Moreover,

$$\|(I + D^{-1}Q)^n\| = \|(D^{-1}U)^n\| \leq 1,$$

hence, by Proposition 7, $D^{-1}Q = 0$, so $Q = 0$. Therefore, $T = M^{-1}DM$ is diagonalizable. □

In general, for Banach spaces with unconditional basis we get a similar result; however, it is more restrictive.

**Corollary 9.** *Let $E$ be a Banach space with unconditional Schauder basis $\{e_i : i \in \mathbb{N}\}$ and basis constant 1. Suppose that $U \in \mathcal{L}(E)$ is a triangularizable isometry with respect to $\{e_i : i \in \mathbb{N}\}$. Then $U$ is diagonal.*

**Proof.** Put $E_k := span\{e_1, \ldots, e_k\}$. Since $U_{|E_k} \subset E_k$ and $U$ is one-to-one, we have $U_{|E_k} = E_k$. Thus $U$ has dense range, hence it is an isometric isomorphism. Observe also that $U^{-1}$ is an isometric isomorphism and that $U^{-1}(E_k) = E_k$ for all $k = 1, 2, \ldots$. So the matrix representation of $U^{-1}$ with respect to the basis $\{e_i : i \in \mathbb{N}\}$ is upper triangular. We write $U = D + Q$, where $D$ is an infinite matrix with diagonal entries $\theta_i \in S(0, 1)$, because they are eigenvalues of the invertible isometry $U_{|E_i}$. By [**14**, Proposition. 1.c.8] we have that $D \in \mathcal{L}(E)$ and $\|D\| \leq \|U\| \leq 1$. In fact, we show that $D$ is invertible and $\|D\| = \|D^{-1}\| = 1$. Consider $U^{-1} = \bar{D} + \bar{Q}$ where $\bar{D}$ is diagonal with entries in $S(0, 1)$ and $\bar{Q}$ is upper triangular with null diagonal entries. Then

$$I = UU^{-1} = (D + Q)(\bar{D} + \bar{Q}) = D\bar{D} + D\bar{Q} + Q\bar{D} + Q\bar{Q}.$$

As $D\bar{Q} + Q\bar{D} + Q\bar{Q}$ is an upper triangular matrix with null diagonal, $D\bar{Q} + Q\bar{D} + Q\bar{Q} = 0$ and $D\bar{D} = I$. So $D$ is invertible and $D^{-1} = \bar{D}$. Observe that $1 \leq \|DD^{-1}\| \leq \|D\|\|D^{-1}\| \leq \|D^{-1}\|$. Now using again [**14**, Proposition 1.c.8], we have that $\|D^{-1}\| \leq \|U^{-1}\| = 1$. So $\|D^{-1}\| = 1$.

Since $D^{-1}Q$ is pseudonilpotent in $Y := \bigcup_{n \geq 1} span[e_1, e_2, \ldots, e_n]$ and

$$\sup_{n \in \mathbb{N}} \|(I + D^{-1}Q)^n\| = \sup_{n \in \mathbb{N}} \|(D^{-1}U)^n\| \leq \|D^{-1}\|^n \|U\|^n \leq 1,$$

we may apply Proposition 7 to get that $D^{-1}Q = 0$, so $Q = 0$. □

## 3. Biholomorphic functions

Recall that an open set $A \subset E$ has the separation property if for every $u \in \overline{A} \setminus A$, there is an analytic function $h$ in a neighbourhood of $\overline{A}$ such that $h(u) = 1$ and $|h(x)| < 1$ for all $x \in A$. Any convex domain $\Omega$ has the separation property. Other examples of open sets with the separation property can be found in [**2**].

The coming Theorem 10 yields and extends [**2**, Theorem 2.9] because of Proposition 6 and because no separability is assumed. Although the proof is very similar, we include it for the reader's convenience.

**Theorem 10.** *Let $E$ be a Banach dual space and let $\Omega \subset E$ be a bounded domain with the separation property such that its weak* closure coincides with its norm closure.*

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*Let $f \in \mathcal{H}(\Omega, \Omega)$ and $p \in \Omega$ such that $f(p) = p$. Assume that $\mathrm{d}f_p$ is a one-to-one closed range operator such that there is a subnet $(\mathrm{d}f_p^{n(k)})$ which converges pointwise to an onto operator. Then $f$ is a biholomorphic mapping.*

**Proof.** By [2, Lemma 2.1(2)] there is a constant $C > 0$ such that $\left\|\mathrm{d}f_p^n\right\| < C$ for all $n$. Theorem 1 shows that $\mathrm{d}f_p$ is invertible, which implies that $\mathrm{d}f_p^{-1}$ does exist, and also that there is a subsequence $(\mathrm{d}f_p^{n(k)})$ pointwise convergent to the identity.

Let $A = \{f^{n(k)} : k \in I\}$. By [2, Lemma 2.7] there is a subnet $(f^{n(k_i)})_i$ compact-open-weak* convergent to some function, say $g \in \mathcal{H}(\Omega, E)$. Then $g(z) \in \overline{\Omega}^{w^*} = \overline{\Omega}$ for all $z \in \Omega$ and $g(p) = p$. Using [2, Proposition 2.6] we have that $g(\Omega) \subset \Omega$. Moreover, we have that $((\mathrm{d}f_p)^{n(k)})_k$ is compact-open-weak* convergent to $\mathrm{d}g_p$. Therefore $\mathrm{d}g_p = I_E$. Using Cartan's theorem, $g = I_\Omega$.

Again by [2, Lemma 2.7], the bounded subnet $(f^{n(k_i)-1})_i$ has a subnet that converges to a holomorphic function $h \in \mathcal{H}(\Omega, E)$ in the compact-open-weak* topology; there is no harm in assuming that it is the subnet $(f^{n(k_i)-1})_i$ itself. Clearly, $h(p) = p$ and $h(z) \in \overline{\Omega}^{w^*} = \overline{\Omega}$ for all $z \in \Omega$, so, again by [2, Proposition 2.6], we have that $h \in \mathcal{H}(\Omega, \Omega)$. Moreover, for $z \in \Omega$,

$$z = \lim f(f^{n(k_i)-1})(z) = \lim f^{n(k_i)-1}(f(z)) = h(f(z)),$$

which shows that $h \circ f = I_\Omega$.

Finally, we show that $f \circ h = I_\Omega$. Since $h \circ f = I_\Omega$, we have $\mathrm{d}h_p \circ \mathrm{d}f_p = I_H$ and, since $\mathrm{d}f_p^{-1}$ exists, it follows that $\mathrm{d}h_p = \mathrm{d}f_p^{-1}$. Therefore, $\mathrm{d}f_p \circ \mathrm{d}h_p = I_H$, and using Cartan's theorem we obtain $f \circ h = I_\Omega$. □

**Remark 11.** Observe that Theorem 10 does not provide a necessary condition for $f$ to be biholomorphic: Consider $E = l_2(\mathbb{Z})$ and $f$ the shift operator in $E$, $f((x_n)) = (x_{n+1})$. Clearly, $f$ is an automorphism of the unit ball, $f = \mathrm{d}f_0$, and $f^m(e_n) = e_{n+m}$, so the sequence $(f^k(e_n))_k$ does not have a Cauchy subsequence, and therefore $(f^k)_k$ cannot have a pointwise convergent subsequence. In addition, $f$ is known not to be a triangularizable operator.

**Corollary 12.** *Let $E$ be a reflexive Banach space and let $\Omega \subset E$ be a convex bounded domain. Let $f \in \mathcal{H}(\Omega, \Omega)$ and $p \in \Omega$ such that $f(p) = p$. If $\mathrm{d}f_p$ is a one-to-one operator and there is a subsequence $((\mathrm{d}f_p)^{m_k})_k$ that converges pointwise to an onto operator in $\mathcal{L}(E)$, then $f$ is biholomorphic.*

**Proof.** Since $\mathrm{d}f_p$ is weakly compact, the result follows from Remark 5 and Theorem 10. □

**Theorem 13.** *Let $\Omega \subset \ell_p$ $(p > 1,\ p \neq 2)$ be a bounded convex subset and $f : \Omega \to \Omega$, holomorphic with a fixed point $a$. Suppose that $\mathrm{d}f_a$ is an isometric isomorphism and that there is an eigenvector $v = (v_1, v_2, \ldots, v_i, \ldots) \in \ell_p$ such that $v_i \neq 0$ for all $i \in \mathbb{N}$. Then $f$ is biholomorphic.*

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**Proof.** Denote $T = \mathrm{d}f_a \in \mathcal{L}(\ell_p)$. Since $p \neq 2$, by [**14**, Proposition 2.f.14] or [**18**, Lemma 3.1], there is a permutation $\pi$ of $\mathbb{N}$ and a sequence $(\theta_i) \in S(0,1)$ such that

$$T\left(\sum_{i\geq 1} a_i e_i\right) = \sum_{i\geq 1} \theta_i a_{\pi(i)} e_i. \tag{1}$$

Let $\mu \in \sigma_p(T)$ such that $T(v) = \mu v$.

Now $T(\sum_{i\geq 1} v_i e_i) = \sum_{i\geq 1} \theta_i v_{\pi(i)} e_i$ and for each $n \in \mathbb{N}$ we have

$$T^n\left(\sum_{i\geq 1} v_i e_i\right) = \sum_{i\geq 1} \theta_i \theta_{\pi(i)} \theta_{\pi^2(i)} \cdots \theta_{\pi^{n-1}(i)} v_{\pi^n(i)} e_i. \tag{2}$$

So

$$\mu^n \sum_{i\geq 1} v_i e_i = \sum_{i\geq 1} \theta_i \theta_{\pi(i)} \theta_{\pi^2(i)} \cdots \theta_{\pi^{n-1}(i)} v_{\pi^n(i)} e_i. \tag{3}$$

As $|\mu^n| = |\theta_i| = 1$ for all $i, n \in \mathbb{N}$, the identity (3) implies that for every $i$, $|v_i| = |v_{\pi^n(i)}|$ and, further, that for every $i$, the orbit $\{\pi^n(i) : n \in \mathbb{N}\}$ is finite; indeed, if it were not, the set $\{\pi^n(i) : |v_{\pi^n(i)}| = |v_i|\}$ would be infinite and this would imply that $\|v\| = \infty$. It is easy to show that $\pi$ restricted to the orbit $F_i = \{\pi^n(i) : n \in \mathbb{N}\}$ is a cycle and, that if $F_i \cap F_j \neq \emptyset$, then $F_i = F_j$. By the identity (2), it follows that $T(e_i) = \theta_{\pi^{-1}(i)} e_{\pi^{-1}(i)}$ for all $i \in \mathbb{N}$. Therefore, if $n_i$ is the permutation order of $\pi$ restricted to $F_i$, then the representing matrix of $T$ restricted to the subspace $M_i = \mathit{span}\{e_j : j \in F_i\}$ is a matrix in $M_{n_i \times n_i}(\mathbb{C})$, all of whose row and column entries are null, except in a position where the entry is in $S(0,1)$. In particular, $T$ restricted to the subspace $M_i$ is an isometric isomorphism. To use Corollary 12, we need to show that there is a sequence of powers of $T$ that converges pointwise to an invertible operator.

Observe that $T_{|M_i}$ is an isometry and $\dim(M_{n_i \times n_i}(\mathbb{C})) = n_i^2 < \infty$ for each $i \in \mathbb{N}$. Therefore, we have that the set $\{T^n_{|M_i} : n \in \mathbb{N}\} \subset M_{n_i \times n_i}(\mathbb{C})$ is relatively compact with respect to the norm of $\mathcal{L}(\ell_p)$ restricted to $M_{n_i \times n_i}(\mathbb{C})$. Thus, for $i = 1$ there is a subsequence $(m_1(n))$ such that $\lim T^{m_1(n)}_{|M_1} = S_1$ exists. Furthermore, if $x \in M_1$, we have that $\|T^n x\| = \|x\|$ and so $\|S_1(x)\| = \lim \|T^n x\| = \|x\|$. Therefore, the operator $S_1 : M_1 \to M_1$ is also an isometric isomorphism. Now, for $i = 2$ the set $\{T^{m_1(n)}_{|M_2} : n \in \mathbb{N}\}$ is also relatively compact in $M_{n_2 \times n_2}(\mathbb{C})$ and therefore there is a subsequence $(m_2(n)) \subset (m_1(n))$ such that $\lim T^{m_2(n)}_{|M_2} = S_2$ exists and $S_2$ is also an isometry in $M_2$. Proceeding inductively and by a diagonalization process, we obtain a sequence $(m_k(k))$ such that for each fixed $i$ we have $\lim_k T^{m_k(k)}_{|M_j} = S_j$ for all $1 \leq j \leq i$. Define $U \in \mathcal{L}(\ell_p)$ according to $U(e_j) = S_i e_j$ if $j \in F_i$. Since $F_i \cap F_j \neq \emptyset$ implies $F_i = F_j$, $U$ is well defined and $\|U(x)\| = \|x\|$. Obviously, $U$ is invertible. If $x \in \mathit{span}\{e_1, e_2, \ldots, e_i\}$, then $x \in \bigcup_{j=1}^{i} M_j$, and so $\lim_k T^{m_k(k)} x = Ux$. The equicontinuity of the family $\{T^{m_k(k)}\}_k$ and the density of $\mathit{span}\{\bigcup_{i=1} M_i\}$ in $\ell_p$ lead to $\lim_k T^{m_k(k)}(x) = Ux$ for all $x \in \ell_p$, as we wanted. □

Among the spaces of biholomorphic mappings, the group of automorphisms $Aut(B_{\ell_2^n})$ of the Euclidean $n$-dimensional space is a prominent one. For general Hilbert spaces $H$ we consider the group $Aut(B_H)$ of the automorphisms of $B_H$. There has been extensive

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research on $Aut(B_H)$ when endowed with the topology $\tau_b$ of the local uniform convergence or, equivalently, of the uniform convergence on balls strictly contained in $B_H$. We refer to the survey [**12**] or the book [**10**] for this matter and much more. Here we consider $Aut(B_H)$ endowed with the (stronger) topology of uniform convergence on $B_H$ given by the sup-norm $\|\cdot\|_\infty$ naturally inherited from $H^\infty(B_H, H)$ and directly related to the uniform topology in $\mathcal{L}(H)$.

The group $I(H)_0$ of automorphisms of $B_H$ that fix the origin, called (see for instance [**6**]) the isotropy group of 0 for $B_H$, coincides with the subspace $\mathcal{U}$ of unitary operators in $\mathcal{L}(H)$. Such isotropy groups have been used to prove that the Euclidean $n$-ball is not biholomorphically equivalent to the $n$-polydisc (see [**13**]).

It is known that the subspace $\mathcal{U} \subset \mathcal{L}(H)$ is an arcwise connected set. We sketch a proof along the lines of [**17**, 12.37 Theorem] for the reader's convenience. Given a unitary operator $U$, its spectrum $\sigma(U)$ lies on the unit circle, so there is a real bounded Borel function $f$ on $\sigma(U)$ that satisfies $\exp(if(\lambda)) = \lambda$ , $\lambda \in \sigma(U)$. Put $Q = f(U)$. Then $Q$ is self-adjoint and $U = \exp(iQ)$. From this it follows easily that $\mathcal{U}$ is connected, for if $U_r$ is defined, as $0 \leq r \leq 1$, by $U_r = \exp(irQ)$ the mapping $r \in [0, 1] \mapsto U_r \in \mathcal{U}$ is continuous and $U_r \in \mathcal{U}$ as a consequence of the Spectral Theorem [**17**, 12.22 (b) Theorem] (see also [**4**, Exercise 22, Chapter IX]).

Recall [**6**] that every automorphism of $B_H$ is of the form $U \circ \varphi_a$, where $U$ is a unitary operator in $\mathcal{L}(H)$ and

$$\varphi_a(x) = (s_a Q_a + P_a)(m_a(x))$$

for some $a \in B_H$. Here, $s_a = \sqrt{1 - \|a\|^2}$, $m_a : B_H \to B_H$ is the analytic mapping

$$m_a(x) = \frac{a - x}{1 - \langle x, a\rangle}$$

and $P_a = 1/\|a\|^2 a \otimes a$, where $u \otimes v(x) = \langle x, u\rangle v$, and $Q_a = Id - P_a$ are the orthogonal projection on the one-dimensional subspace generated by $a$ and on its orthogonal complement, respectively. Denote for simplicity $L_a = s_a Q_a + P_a$ for $a \neq 0$ and $L_0 = Id_H$. If $a \neq 0$, $\varphi_a$ is holomorphic on the ball $B(0, 1/\|a\|)$. Thus, in all cases, every $m_a$ and every automorphism is a bounded holomorphic mapping on a ball of radius greater than 1 where it is uniformly continuous as well.

**Lemma 14.** *The subgroup $I(H)_0 \subset (Aut(B_H), \|\cdot\|_\infty)$ is connected. Consequently, $\{\psi \in Aut(B_H) : \psi(a) = a\}$ is connected as well for each $a \in B_H$.*

**Proof.** The additional statement follows from the fact that for a given $a$, there is an automorphism $\varphi_a$ such that $\varphi_a(a) = 0$. Then the mapping $U \in I(H)_0 \mapsto \varphi_a^{-1} \circ U \circ \varphi_a$ is a homeomorphism between $I(H)_0$ and $\{\psi \in Aut(B_H) : \psi(a) = a\}$. □

**Theorem 15.** *The group $(Aut(B_H), \circ, \|\cdot\|_\infty)$ is a connected topological group. The inversion mapping $\Phi \in Aut(B_H) \mapsto \Phi^{-1} \in Aut(B_H)$ is an isometry.*

**Proof.** Let $(f_n, g_n) \subset Aut(B_H) \times Aut(B_H)$ be a sequence converging to $(f, g) \in Aut(B_H) \times Aut(B_H)$. Fix $\epsilon > 0$. The uniform continuity of $f$ on $B_H$ leads us to $\delta > 0$ such that $\|f(u) - f(v)\| \leq \epsilon$ if $u, v \in B_H$ and $\|u - v\| < \delta$.

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Next, we find $n_0$ such that $\|g_n(x) - g(x)\| \leq \delta$ for $n \geq n_0$ and $x \in B_H$. Now, we pick $n_1 \geq n_0$ such that $\|f_n(y) - f(y)\| \leq \epsilon$ for $n \geq n_1$ and $y \in B_H$. Therefore, if $x \in B_H$ and $n \geq n_1$, we have

$$\|f_n(g_n(x)) - f(g(x))\| \leq \|f_n(g_n(x)) - f(g_n(x))\| + \|f(g_n(x)) - f(g(x))\| \leq \epsilon + \epsilon.$$

Thus the composition operation is continuous.

If $\Phi = U \circ \varphi_a$, then $\Phi^{-1} = \varphi_a \circ U^{-1} = \varphi_a \circ U^*$ because $U$ is unitary and $\varphi_a \circ \varphi_a = Id_H$. Also, $(U^* \circ \varphi_a) \circ \Phi^{-1} = Id_H$. Further, notice that

$$\begin{aligned}\|\Phi - Id_H\| &= \|U \circ \varphi_a - Id_H\| = \|U^* \circ U \circ \varphi_a - U^*\| = \|\varphi_a - U^*\| \\ &= \sup_{x \in B_H} \|\varphi_a(\varphi_a(x)) - U^*(\varphi_a(x))\| \\ &= \sup_{x \in B_H} \|x - U^*(\varphi_a(x))\| = \|Id_H - U^* \circ \varphi_a\| \\ &= \sup_{x \in B_H} \|\Phi^{-1}(x) - U^* \circ \varphi_a(\Phi^{-1}(x))\| = \|\Phi^{-1} - Id_H\|, \qquad (4)\end{aligned}$$

and that

$$\begin{aligned}\|\Psi^{-1} - \Phi^{-1}\| &= \sup_{x \in B_H} \|\Psi^{-1}(\Psi(x)) - \Phi^{-1}(\Psi(x))\| \\ &= \sup_{x \in B_H} \|x - (\Phi^{-1} \circ \Psi)(x)\| = \|Id_H - \Phi^{-1} \circ \Psi\|. \qquad (5)\end{aligned}$$

Therefore, using (4) for the second equality and (5) for the third,

$$\|\Psi - \Phi\| = \|Id_H - \Phi \circ \Psi^{-1}\| = \|Id_H - \Psi^{-1} \circ \Phi\| = \|\Phi^{-1} - \Psi^{-1}\|.$$

This shows that the inversion mapping is an isometry.

The first step in the proof of the connectedness is to show that the mapping

$$a \in B_H \mapsto m_a \in H^\infty(B_H, H)$$

is continuous. Indeed, for $x \in B(H)$,

$$\begin{aligned}\left\|\frac{x-a}{1-\langle x,a\rangle} - \frac{x-a'}{1-\langle x,a'\rangle}\right\| &= \left\|\frac{a - a' - a\langle x,a'\rangle + \langle x,a\rangle a' - \langle x,a\rangle x + x\langle x,a'\rangle}{(\langle x,a'\rangle - 1)(\langle x,a\rangle - 1)}\right\| \\ &= \left\|\frac{(a-a') + x(\langle x,a'\rangle - \langle x,a\rangle) + \langle x,a\rangle(a'-a) + a(\langle x,a\rangle - \langle x,a'\rangle)}{(\langle x,a'\rangle - 1)(\langle x,a\rangle - 1)}\right\|.\end{aligned}$$

We also have that $|(\langle x,a'\rangle - \langle x,a\rangle)| = |\langle x, a'-a\rangle| \leq \|a'-a\|$, and $|\langle x,a\rangle - 1| \geq 1 - \|a\|$, $|\langle x,a'\rangle - 1| \geq 1 - \|a'\|$. Thus

$$\begin{aligned}\left\|\frac{x-a}{1-\langle x,a\rangle} - \frac{x-a'}{1-\langle x,a'\rangle}\right\| &\leq \frac{\|a-a'\| + \|a-a'\| + \|a\|\|a'-a\| + \|a\|\|a-a'\|}{(1-\|a\|)(1-\|a'\|)} \\ &\leq \frac{2(1+\|a\|)}{(1-\|a\|)(1-\|a'\|)}\|a-a'\|.\end{aligned}$$

Thus, we have shown the claimed continuity.

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Next, we see that the mapping $a \in B_H \mapsto L_a \in \mathcal{L}(H)$ is also continuous because for $a \neq 0$, $\lim_{b\to a} P_b = P_a$, hence also $\lim_{b\to a}(s_b Q_b + P_b) = (s_a Q_a + P_a)$, while it is immediate that $\lim_{b\to 0}(s_b Q_b + P_b) = Id_H$.

Now, we appeal to the fact that $m_a$ is a bounded mapping and to

$$\begin{aligned} L_b \circ m_b - L_a \circ m_a &= L_b \circ m_b - L_b \circ m_a + L_b \circ m_a - L_a \circ m_a \\ &= L_b \circ (m_b - m_a) + (L_b - L_a) \circ m_a \end{aligned}$$

to see that the mapping $a \in B_H \mapsto L_a \circ m_a \in Aut(B_H)$ is also continuous.

Finally, since $Aut(B_H)$ is a topological group, the mapping

$$(U, a) \in I(H)_0 \times B_H \mapsto U \circ L_a \circ m_a \in Aut(B_H)$$

is continuous. Moreover, this is an onto mapping, hence $Aut(B_H)$ is connected as the continuous image of the connected set $I(H)_0 \times B_H$. □

This theorem immediately yields Kaup's earlier result [**11**] that $(Aut(B_H), \tau_b)$ is a connected group.

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