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Estimates for solutions of a low-viscosity kick-forced generalized Burgers equation

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We consider a non-homogeneous generalized Burgers equation

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = \eta^\omega, \quad t \in \mathbb{R},\ x \in S^1.$$

Here, ν is small and positive, f is strongly convex and satisfies a growth assumption, while η^ω is a space-smooth random 'kicked' forcing term. For any solution u of this equation, we consider the quasi-stationary regime, corresponding to $t \geqslant 2$. After taking the ensemble average, we obtain upper estimates and time-averaged lower estimates for a class of Sobolev norms of u. These estimates are of the form $C\nu^{-\beta}$ with the same values of β for bounds from above and from below. They depend on η and f, but do not depend on the time t or the initial condition.

1. Notation

Consider a zero mean value smooth function w on S^1. For $p \in [1, +\infty]$, we denote its L_p-norm by $|w|_p$. The L_2-norm will be denoted by $|w|$, and $\langle \cdot, \cdot \rangle$ stands for the L_2 scalar product. Hereafter, L_p, $p \in [1, +\infty]$, stands for the space of zero mean value functions in $L_p(S^1)$.

For a non-negative integer n and $p \in [1, +\infty]$, $W^{n,p}$ stands for the Sobolev space of zero mean value functions w on S^1 with the norm

$$|w|_{n,p} = |w^{(n)}|_p,$$

where

$$w^{(n)} = \frac{\mathrm{d}^n w}{\mathrm{d}x^n}.$$

In particular, $W^{0,p} = L_p$ for $p \in [1, +\infty]$. For $p = 2$, we denote $W^{n,2}$ by H^n, and the corresponding norm is abbreviated as $\|w\|_n$.

We recall a version of the classical Gagliardo–Nirenberg inequality (see [11, p. 125]).

Lemma 1.1. *For a smooth zero mean value function w on S^1,*

$$|w|_{\beta,r} \leqslant C|w|^{\theta}_{m,p}|w|^{1-\theta}_{q},$$

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where $m > \beta$, and r is defined by

$$\frac{1}{r} = \beta + \theta\left(\frac{1}{p} - m\right) + (1-\theta)\frac{1}{q},$$

under the assumption that $\theta = \beta/m$ if $p = 1$ or $p = +\infty$, and $\beta/m \leqslant \theta < 1$ otherwise. Here $C = C(m,p,q,\beta,\theta) > 0$ is a constant.

For a smooth function $v(t,x)$ defined on $[0,+\infty)\times S^1$, v_t, v_x, and v_{xx} respectively mean $\partial v/\partial t$, $\partial v/\partial x$ and $\partial^2 v/\partial x^2$.

2. Introduction

The generalized one-dimensional space-periodic Burgers equation (note that the classical Burgers equation corresponds to $f(u) = u^2$)

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \quad \nu > 0, \tag{2.1}$$

appears in different domains of science, ranging from cosmology to traffic modelling (see [1]). It is sometimes called a viscous scalar conservation law. Historically, it has received most attention as a model for the Navier–Stokes equation (NSE). Indeed, it has a nonlinear term analogous to the nonlinearity $(u\cdot\nabla)u$ in the incompressible NSE. The dissipation term in (2.1) is also similar that of the NSE. We note that the classical Burgers equation is explicitly solvable. This is done using the Cole–Hopf transformation (see [3]).

In [2], Biryuk considered equation (2.1) with f strongly convex, i.e. satisfying

$$f''(x) \geqslant \sigma > 0, \quad x \in \mathbb{R}. \tag{2.2}$$

He studied the behaviour of the Sobolev norms of solutions u for small values of ν and obtained the following estimates:

$$\|u\|_m^2 \leqslant C\nu^{-(2m-1)/2}, \quad \frac{1}{T}\int_0^T \|u\|_m^2 \geqslant c\nu^{-(2m-1)/2}, \quad m \geqslant 1,\ \nu \leqslant \nu_0.$$

Note that the exponents of ν in lower and upper estimates are the same. The quantities ν_0, C, c and T depend on the deterministic initial condition u_0 and on m. To get results independent from the initial data, it is natural to introduce random forcing and to estimate ensemble-averaged norms of solutions.

We consider (2.1) with a random kick force on the right-hand side. In § 3 we recall classical existence and uniqueness results and introduce the probabilistic setting needed to define the kick force. Then, we estimate from above the moments of the $W^{1,1}$-norm of u. These estimates, valid after a certain damping time, are proved using ideas similar to those in [7]. Remarkably, this damping time and the estimate do not depend on the initial condition. This is the main result of this paper.

In §§ 4 and 5 this result allows us to obtain lower and upper estimates that are, up to taking the ensemble average, of the same type as in [2], for time $t \geqslant 2$. These estimates will only depend on the function f and the forcing. Let us emphasize that, for $t \geqslant 2$, we are in a quasi-stationary regime: all estimates hold independently of the initial condition. In § 6 we give some additional estimates for the Sobolev norms.

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We use the methods introduced by Kuksin [8, 9] and developed by Biryuk [2].

Equation (2.1) with $\nu \ll 1$ is a popular one-dimensional model for the theory of hydrodynamic turbulence. In § 7 we present an interpretation of our results in terms of this theory.

3. Preliminaries

In this section we review the properties of solutions of (2.1) used in our proof.

Physically, t corresponds to the time variable, whereas x corresponds to the one-dimensional space variable, and the constant $\nu > 0$ corresponds to a viscosity coefficient. The real-valued function $u(t, x)$ is defined on $[0, +\infty) \times \mathbb{R}$ and is L-periodic in x. The function f is C^∞-smooth and strongly convex, i.e. it satisfies the condition (2.2) for some constant σ. Moreover, we assume that f (as well as its derivatives) has at most polynomial growth, i.e. for all $m \geqslant 0$, there exists $n \geqslant 0$ and $C_m > 0$ such that

$$|f^{(m)}(x)| \leqslant C_m(1 + |x|)^n, \quad x \in \mathbb{R}, \tag{3.1}$$

where $n = n(m)$. We now fix $L = 1$, which involves studying the problem on $[0, +\infty) \times S^1$. We note that L-periodic solutions of (2.1) with any L reduce, by means of scaling in x, to 1-periodic solutions with scaled f and ν.

Since we are mostly interested in the asymptotics of solutions of (2.1) as $\nu \to 0^+$, we assume that

$$\nu \in (0, 1].$$

Moreover, it is sufficient to study the special case

$$\int_{S^1} u_0(y)\, \mathrm{d}y = 0. \tag{3.2}$$

Indeed, if the mean value of u_0 on S^1 equals b, we may consider

$$v(t, x) = u(t, x + bt) - b.$$

Then v satisfies (3.2) and is a solution of (2.1) with $f(y)$ replaced with $g(y) = f(y + b) - by$.

Given a C^∞-smooth initial condition $u_0 = u(0, \cdot)$, equation (2.1) has a unique classical solution u, C^∞-smooth in both variables (see [6, chapter 5]). Condition (3.2) implies that the mean value of a solution for (2.1) vanishes identically in t.

Now provide each space $W^{n,p}(S^1)$ with the Borel σ-algebra. Consider a random variable ζ on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with values in $L^2(S^1)$, such that $\zeta^\omega \in C^\infty(S^1)$ for almost every ω. We suppose that ζ satisfies the following three properties.

(i) Non-triviality:

$$\mathbb{P}(\zeta \equiv 0) < 1.$$

(ii) Finiteness of exponential moments for Sobolev norms: for every $m \geqslant 0$ there exist constants $\alpha = \alpha(m) > 0, \beta = \beta(m)$ such that

$$\mathbb{E} \exp(\alpha \|\zeta\|_m^2) \leqslant \beta.$$

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In particular,

$$I_m = \mathbb{E}\|\zeta\|_m^2 < +\infty, \quad \forall m \geqslant 0.$$

(iii) Vanishing of the expected value:

$$\mathbb{E}\zeta \equiv 0.$$

It is not difficult to explicitly construct ζ satisfying (i)–(iii). For example, we could consider the real Fourier coefficients of ζ, defined for $k > 0$ by

$$a_k(\zeta) = \sqrt{2}\int_{S^1} \cos(2\pi kx)u(x), \qquad b_k(\zeta) = \sqrt{2}\int_{S^1} \sin(2\pi kx)u(x) \tag{3.3}$$

as independent random variables with zero mean value and exponential moments tending to 1 sufficiently fast as $k \to +\infty$.

Now let ζ_i, $i \in \mathbb{N}$, be independent identically distributed random variables having the same distribution as ζ. The sequence $(\zeta_i)_{i\geqslant 1}$ is a random variable, defined on a probability space that is a countable direct product of copies of Ω. From now on, this space will itself be called Ω. The meaning of $\mathbb{F}$ and $\mathbb{P}$ changes accordingly.

For $\omega \in \Omega$ and a time period $\theta > 0$, the kick force η^ω is a C^∞-smooth function in the variable x, with values in the space of distributions in the variable t, defined by

$$\eta^\omega(x) = \sum_{i=1}^{+\infty} \delta_{t=i\theta}\zeta_i^\omega(x),$$

where $\delta_{t=i\theta}$ denotes the Dirac measure at a time moment $i\theta$.

The kick-forced version of (2.1) corresponds to the case where, on the right-hand side, 0 is replaced with the kick force. This means that, for integers $i \geqslant 1$, at the moments $i\theta$, the solution $u(x)$ instantly increases by the kick $\zeta_i^\omega(x)$, and that, between these moments, u solves (2.1). The equation is written as follows:

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = \eta^\omega. \tag{3.4}$$

Derivatives are taken in the sense of distributions.

When studying solutions of (3.4), we will always assume that the initial condition $u_0 = u(0,\cdot)$ is C^∞-smooth. Moreover, we normalize those solutions to be right-continuous in time at the kick moments $i\theta$. Such a solution is uniquely defined for a given value of u_0, for almost every ω.

For a given initial condition u_0, the function $u(t,x)$ will always denote such a solution of (2.1). The value of u before the ith kick will be denoted by $u(i\theta^-,\cdot)$, or shortly u_i^-. We will also use the notation $u_i = u(i\theta,\cdot)$ and denote the function $u(t,\cdot)$ by $u(t)$. Finally, for a solution of (3.4), we consider time derivatives at the kick moments in the sense of right-sided time derivatives. Those derivatives are right-continuous in time.

Since space averages of the kicks vanish and $u_0(x)$ satisfies (3.2), the space average of $u(t)$, $t \geqslant 0$, vanishes identically. For the sake of simplicity, we normalize the kick period: from now on $\theta = 1$.

We observe that, since the kicks are independent and between the kicks (3.4) is deterministic, the solutions of (3.4) make a random Markov process. For details, see [10], where a kick force is introduced in a similar setting.

Agreements. All constants denoted C with sub- or super-indexes are strictly positive. Unless otherwise stated, they depend only on f, on the distribution of the kicks, and on the parameters $a_1, \dots, a_k$ if they are denoted $C(a_1, \dots, a_k)$. u always denotes a solution of (3.4) with any initial condition u_0. Averaging in ensemble corresponds to averaging in $\mathbb{P}$. All our estimates hold independently of the value of u_0.

We observe that, for every integer i, we have the following energy dissipation identity on the maximal kick-free intervals:

$$A_i = |u_i|^2 - |u_{i+1}^-|^2, \tag{3.5}$$

where

$$A_i = 2\nu \int_i^{(i+1)} \|u(t)\|_1^2 \, \mathrm{d}t. \tag{3.6}$$

Indeed, for any $t \in (i, i+1)$, u satisfies

$$\begin{aligned} 2\nu \|u(t)\|_1^2 &= -2\nu \int_{S^1} u u_{xx} \, \mathrm{d}x \\ &= -2 \int_{S^1} u f'(u) u_x \, \mathrm{d}x - 2 \int_{S^1} u u_t \, \mathrm{d}x. \end{aligned}$$

The first term on the right-hand side vanishes since its integrand is a full derivative. The second term equals $-\mathrm{d}/\mathrm{d}t |u|^2$. Integrating in time, we get (3.5). We note that energy dissipation between kicks A_i is always non-negative: energy can be added only at the kick points. We also note that an analogue of (3.5) holds on every kick-free time interval.

The following two lemmas are proved using the maximum principle in the same way as in [7].

Lemma 3.1. *We have the estimate*

$$u_x(t,x) \leqslant 2\sigma^{-1}, \quad t \in [k + \tfrac{1}{2}, k+1), \ k \in \mathbb{N}, \ x \in S^1,$$

where σ is the constant in the assumption (2.2).

Proof. Consider the equation (3.4) on the kick-free time interval $[0, 1-\epsilon]$ for arbitrarily small ϵ and differentiate it once in space. We obtain

$$\frac{\partial u_x}{\partial t} + f''(u) u_x^2 + f'(u) \frac{\partial u_x}{\partial x} - \nu \frac{\partial^2 u_x}{\partial x^2} = 0. \tag{3.7}$$

Consider $v(t,x) = t u_x(t,x)$. For $t > 0$, v verifies

$$\frac{\partial v}{\partial t} + t^{-1}(-v + f''(u) v^2) + f'(u) \frac{\partial v}{\partial x} - \nu \frac{\partial^2 v}{\partial x^2} = 0. \tag{3.8}$$

Now observe that, if $v > 0$ somewhere on the domain $S_\epsilon = [0, 1-\epsilon] \times S^1$, then v attains its maximum M on S_ϵ at a point (t_1, x_1) such that $t_1 > 0$. At (t_1, x_1) we have $\partial v/\partial t \geqslant 0$, $\partial v/\partial x = 0$, and $\partial^2 v/\partial x^2 \leqslant 0$. Therefore, (3.8) yields that

$$t_1^{-1}[-v(t_1, x_1) + f''(u(t_1, x_1)) v^2(t_1, x_1)] \leqslant 0.$$

Since, by (2.2), $f'' \geqslant \sigma > 0$, then

$$-M + \sigma M^2 \leqslant 0,$$

and therefore

$$M \leqslant \sigma^{-1}.$$

Thus, we have proved that $v \leqslant \sigma^{-1}$ everywhere on S_ϵ for every $\epsilon > 0$. In particular, by definition of v and S_ϵ, we obtain that

$$u_x(t,x) \leqslant 2\sigma^{-1}, \quad x \in S^1, \ t \in [\tfrac{1}{2}, 1).$$

Repeating the same argument on all the intervals $[k, k+1), k \in \mathbb{N}$, we obtain the lemma's assertion. □

LEMMA 3.2. *There are constants C', C such that*

$$\mathbb{E} \exp\Big(C' \sup_{t\in[k,k+1)} \max u_x(t,\cdot)\Big) \leqslant C, \quad k \geqslant 1.$$

Proof. Fix $k \geqslant 1$. Since the $W^{1,\infty}$-norm is dominated by the H^2-norm, then, for $C' > 0$, we get

$$\exp(C' u_x(k,x)) \leqslant \exp(C' u_x(k^-,x) + C' \|\zeta_k\|_2), \quad x \in S^1.$$

The same inequality holds when we maximize in x. Now denote by X_k the random variable

$$\max u_x(k,\cdot).$$

By lemma 3.1 and property (ii) of the kicks, for $C' = \alpha(2)$ we obtain

$$\mathbb{E} \exp(C' X_k) \leqslant \exp(2C'\sigma^{-1}) \mathbb{E} \exp(C' \|\zeta_k\|_2) \leqslant C, \tag{3.9}$$

for some constant C. Now consider equation (3.7). An application of the maximum principle to the function u_x, which cannot be negative everywhere, yields

$$\max u_x(t,\cdot) \leqslant \max u_x(k,\cdot), \quad t \in [k, k+1).$$

Therefore, in (3.9), we can replace X_k by $\sup_{t\in[k,k+1)} \max u_x(t,\cdot)$. This proves the lemma's assertion. □

COROLLARY 3.3. *For the same C' and C as in lemma 3.2, we have*

$$\mathbb{E} \exp\Big(\tfrac{1}{2} C' \sup_{t\in[k,k+1)} |u(t)|_{1,1}\Big) \leqslant C, \quad k \geqslant 1.$$

Proof. Since the mean value of $u_x(t)$ is 0, then

$$\int_{S^1} |u_x(t)| = 2 \int_{S^1} \max(u_x(t), 0).$$

□

COROLLARY 3.4. *For the same C' and C as in lemma 3.2, we have*

$$\mathbb{E} \exp\Big(C' \sup_{t\in[k,k+1)} |u(t)|_p\Big) \leqslant C, \quad k \geqslant 1, \ p \in [1, +\infty].$$

Note that C' and C do not depend on p.

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4. Lower estimates of H^m-norms

For a solution u of (3.4), the first quantity that we estimate from below is the expected value of

$$\frac{1}{N}\int_1^{N+1}\|u(t)\|_1^2 = \frac{1}{N}(2\nu)^{-1}\sum_{i=1}^{N} A_i, \tag{4.1}$$

where N is a fixed natural number chosen later, and A_i is the same as in (3.6).

LEMMA 4.1. *There exists a natural number $N \geqslant 1$, independent from u_0, such that*

$$\frac{1}{N}\int_1^{N+1}\mathbb{E}\|u(s)\|_1^2 \geqslant C\nu^{-1}.$$

Proof. For $N \geqslant 1$, we have

$$\begin{aligned}
\mathbb{E}|u_{N+1}^-|^2 &\geqslant \mathbb{E}(|u_{N+1}^-|^2 - |u_1^-|^2)\\
&= \mathbb{E}\sum_{i=1}^{N}(|u_{i+1}^-|^2 - |u_i|^2) + \mathbb{E}\sum_{i=1}^{N}(|u_i|^2 - |u_i^-|^2)\\
&= -\mathbb{E}\sum_{i=1}^{N}A_i + \mathbb{E}\sum_{i=1}^{N}(|u_i^- + \zeta_i|^2 - |u_i^-|^2)\\
&= -\mathbb{E}\sum_{i=1}^{N}A_i + 2\mathbb{E}\sum_{i=1}^{N}\langle u_i^-, \zeta_i\rangle + \mathbb{E}\sum_{i=1}^{N}|\zeta_i|^2.
\end{aligned}$$

Since $\mathbb{E}\zeta_i \equiv 0$ (property (iii) of the kicks), and since u_i^- and ζ_i are independent, then $\mathbb{E}\langle u_i^-, \zeta_i\rangle = 0$. Therefore, by (3.6), we have

$$\mathbb{E}|u_{N+1}^-|^2 \geqslant -2\nu\mathbb{E}\int_1^{N+1}\|u(s)\|_1^2 + 0 + NI_0.$$

On the other hand, by corollary 3.4 ($p = 2$), there is a constant C_1 such that

$$\mathbb{E}|u_{N+1}^-|^2 \leqslant C_1.$$

Consequently,

$$\frac{1}{N}\int_1^{N+1}\mathbb{E}\|u(s)\|_1^2 \geqslant \frac{NI_0 - C_1}{2N}\nu^{-1}.$$

Choosing the smallest possible integer N verifying

$$N \geqslant \max\left(1, \frac{C_1 + 1}{I_0}\right),$$

we obtain the lemma's assertion. □

We have reached our first goal of estimating the expected value of (4.1) from below. Thus, we have a time-averaged lower estimate of the H^1-norm, which enables us to obtain similar estimates of H^m-norms for $m \geqslant 2$.

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Lemma 4.2. *We have*

$$\frac{1}{N}\int_1^{N+1}\mathbb{E}\|u(s)\|_m^2 \geqslant C(m)\nu^{-(2m-1)}, \quad m \geqslant 1,$$

where N is the same as in lemma 4.1.

Proof. This statement is already proved in the previous lemma for $m = 1$, so we may assume that $m \geqslant 2$. By lemma 1.1 and Hölder's inequality, we have

$$(\mathbb{E}\|u(s)\|_1^2)^{2m-1} \leqslant C'(m)\mathbb{E}\|u(s)\|_m^2(\mathbb{E}|u(s)|_{1,1}^2)^{2m-2}. \tag{4.2}$$

Since, by corollary 3.3,

$$\mathbb{E}|u(s)|_{1,1}^2 \leqslant K, \quad t \in [1, N+1],$$

where $K > 0$ is a constant, then, integrating (4.2) in time, we obtain

$$\frac{1}{N}\int_1^{N+1}\mathbb{E}\|u(s)\|_m^2 \geqslant \frac{\int_1^{N+1}[\mathbb{E}(\|u(s)\|_1^2)]^{2m-1}}{NC'(m)K^{2m-2}}.$$

By Hölder's inequality,

$$\int_1^{N+1}[\mathbb{E}(\|u(s)\|_1^2)]^{2m-1} \geqslant \left(\int_1^{N+1}\mathbb{E}\|u(s)\|_1^2\right)^{(2m-1)} N^{2-2m},$$

and then

$$\begin{aligned}\frac{1}{N}\int_1^{N+1}\mathbb{E}\|u(s)\|_m^2 &\geqslant \frac{(\int_1^{N+1}\mathbb{E}\|u(s)\|_1^2)^{2m-1}N^{2-2m}}{NC'(m)K^{2m-2}}\\ &= \frac{((1/N)\int_1^{N+1}\mathbb{E}\|u(s)\|_1^2)^{2m-1}}{C'(m)K^{2m-2}}.\end{aligned}$$

Now the assertion follows from lemma 4.1. □

Since we impose no conditions on u_0, we can consider a different positive integer 'starting time'. We may also consider a different averaging time interval of length $T \geqslant N$. Finally, we obtain a general result for a non-integer starting time $t \geqslant 1$ by considering the maximal interval $[m_1, m_2] \subset [t, t+T]$ such that m_1 and m_2 are positive integers.

Theorem 4.3. *We have*

$$\frac{1}{T}\int_t^{t+T}\mathbb{E}\|u(s)\|_m^2 \geqslant \tfrac{1}{4}C(m)\nu^{-(2m-1)}, \quad t \geqslant 1,\ T \geqslant N+1,\ m \geqslant 1,$$

where N and $C(m)$ are the same as in lemma 4.2.

5. Upper estimates of H^m-norms

To estimate from above a Sobolev norm $\|u\|_m$, $m \geqslant 1$, of a solution u for (3.4), we differentiate between the kicks the quantity $\|u(t)\|_m^2$.

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Denote by $B(u)$ the nonlinearity $2f'(u)u_x$, and by L the operator $-\partial_{xx}$. Integrating by parts, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m^2 &= 2\langle u^{(m)}, u_t^{(m)}\rangle \\ &= -2\nu\|u\|_{m+1}^2 - \langle L^m u, B(u)\rangle. \end{aligned} \tag{5.1}$$

We will need a standard estimate for the nonlinearity $\langle L^m u, B(u)\rangle$.

Lemma 5.1. *For a zero mean value smooth function w such that $|w|_\infty \leqslant M$, we have*

$$|\langle L^m w, B(w)\rangle| \leqslant C\|w\|_m\|w\|_{m+1}, \quad m \geqslant 1,$$

with C satisfying

$$C \leqslant C_m(1+M)^n, \tag{5.2}$$

where C_m, as well as the natural number $n = n(m)$, depend only on m.

Proof. Let C' denote various positive constants satisfying an estimate of the type (5.2). Then we have

$$\begin{aligned}|\langle L^m w, B(w)\rangle| &= 2|\langle w^{(2m)}, (f(w))^{(1)}\rangle| \\ &= 2|\langle w^{(m+1)}, (f(w))^{(m)}\rangle| \\ &\leqslant C' \sum_{k=1}^{m} \sum_{\substack{1\leqslant a_1\leqslant\cdots\leqslant a_k\leqslant m \\ a_1+\cdots+a_k=m}} \int_{S^1} |w^{(m+1)} w^{(a_1)}\cdots w^{(a_k)} f^{(k)}(w)| \\ &\leqslant C'|f|_{C^m[-M,M]} \sum_{k=1}^{m} \sum_{\substack{1\leqslant a_1\leqslant\cdots\leqslant a_k\leqslant m \\ a_1+\cdots+a_k=m}} \int_{S^1} |w^{(a_1)}\cdots w^{(a_k)} w^{(m+1)}|.\end{aligned}$$

By (3.1), $|f|_{C^m[-M,M]}$ satisfies an estimate of the type (5.2). By Hölder's inequality, we obtain

$$|\langle L^m w, B(w)\rangle| \leqslant C'\|w\|_{m+1} \sum_{\substack{1\leqslant a_1\leqslant\cdots\leqslant a_k\leqslant m \\ a_1+\cdots+a_k=m}} (|w^{(a_1)}|_{2m/a_1}\cdots|w^{(a_k)}|_{2m/a_k}). \tag{5.3}$$

Finally, the Gagliardo–Nirenberg inequality yields

$$\begin{aligned}&|\langle L^m w, B(w)\rangle| \\ &\leqslant C'\|w\|_{m+1} \sum_{k=1}^{m} \sum_{\substack{1\leqslant a_1\leqslant\cdots\leqslant a_k\leqslant m \\ a_1+\cdots+a_k=m}} [(\|w\|_m^{a_1/m}|w|_\infty^{(m-a_1)/m})\cdots(\|w\|_m^{a_k/m}|w|_\infty^{(m-a_k)/m})] \\ &\leqslant C'|w|_\infty^{m-1}\|w\|_m\|w\|_{m+1} \\ &\leqslant C'\|w\|_m\|w\|_{m+1},\end{aligned}$$

which proves the lemma's assertion. □

Theorem 5.2. *For any natural numbers m and n, we have*

$$\mathbb{E}\Big(\sup_{t\in[k,k+1)} \|u(t)\|_m^n\Big) \leqslant C(m,n)\nu^{-(2m-1)n/2}, \quad k \geqslant 2.$$

Proof. Fix $k \geqslant 2$ and $m \geqslant 1$. In this proof, Θ denotes various positive random constants which depend on m, such that all their moments are finite, and C denotes various positive deterministic constants, depending only on m.

We begin by noting that corollary 3.3 and property (ii) of the kicks imply the inequalities

$$|u(t)|_{1,1}, \|\zeta_k\|_m \leqslant \Theta, \quad t \in [k-1, k+1). \tag{5.4}$$

We claim that when $\|u\|_m^2$ is too large, it decreases at least as fast as a solution of the differential equation

$$y' + (2m-1)y^{2m/(2m-1)} = 0,$$

i.e. as $t^{-(2m-1)}$. More precisely, we wish to prove that, for $t \in [k-1, k+1)$, we have

$$\begin{aligned} &\|u(t)\|_m^2 \geqslant \Theta_1 \nu^{-(2m-1)} \\ &\qquad \implies \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_m^2 \leqslant -(2m-1)\|u(t)\|_m^{4m/(2m-1)}, \end{aligned} \tag{5.5}$$

where Θ_1 is a random positive constant to be chosen later. Random constants Θ below do not depend on Θ_1.

Indeed, assume that

$$\|u(t)\|_m^2 \geqslant \Theta_1 \nu^{-(2m-1)}. \tag{5.6}$$

We begin by observing that, by lemma 1.1, we have

$$\|u\|_m \leqslant C\|u\|_{m+1}^{(2m-1)/(2m+1)} |u|_{1,1}^{2/(2m+1)},$$

hence,

$$\begin{aligned} \|u\|_{m+1} &\geqslant C|u|_{1,1}^{-2/(2m-1)} \|u\|_m^{(2m+1)/(2m-1)} \\ &\geqslant \Theta^{-1}\|u\|_m^{(2m+1)/(2m-1)}, \end{aligned} \tag{5.7}$$

where we used (5.4). Now, (5.1), (5.4) and lemma 5.1 imply that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m^2 &\leqslant -2\nu\|u\|_{m+1}^2 + \Theta\|u\|_m\|u\|_{m+1} \\ &= (-2\nu\|u\|_{m+1}^{2/(2m+1)} + \Theta\|u\|_m\|u\|_{m+1}^{-(2m-1)/(2m+1)})\|u\|_{m+1}^{4m/(2m+1)}. \end{aligned} \tag{5.8}$$

Combining (5.8) and (5.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m^2 \leqslant (-2\nu\|u\|_{m+1}^{2/(2m+1)} + \Theta)\|u\|_{m+1}^{4m/(2m+1)}. \tag{5.9}$$

Therefore, by (5.7) and (5.6), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m^2 &\leqslant (-\nu\Theta^{-1}\|u\|_m^{2/(2m-1)} + \Theta)\|u\|_{m+1}^{4m/(2m+1)} \\ &\leqslant (-\Theta^{-1}\Theta_1^{1/(2m-1)} + \Theta)\|u\|_{m+1}^{4m/(2m+1)}. \end{aligned}$$

Now we choose Θ_1 in such a way that the quantity in the parentheses is negative. Under this assumption, we obtain from (5.7) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m^2 \leqslant (-\Theta^{-1}\Theta_1^{1/(2m-1)} + \Theta)\Theta^{-1}\|u\|_m^{4m/(2m-1)}. \tag{5.10}$$

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This relation implies (5.5) if, for Θ_1, we choose a sufficiently large random constant with all moments finite.

Now we claim that

$$\|u_k^-\|_m^2 \leqslant \Theta_2 \nu^{-(2m-1)}, \tag{5.11}$$

where

$$\Theta_2 = \max(\Theta_1, 1)$$

has finite moments. Indeed, if $\|u(t)\|_m^2 \leqslant \Theta_1 \nu^{-(2m-1)}$ for some $t \in [k-1, k)$, then (5.5) ensures that $\|u(t)\|_m^2$ remains under this threshold up to $t = k^-$. Otherwise, we consider the function

$$y(t) = \|u(t)\|_m^{-2/(2m-1)}, \quad t \in [k-1, k).$$

By (5.5), since $\|u(t)\|_m^2 > \Theta_1 \nu^{-(2m-1)}$, $y(t)$ increases at least as fast as t. Indeed,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} y(t) &= -\frac{1}{2m-1} (\|u(t)\|_m^2)^{-2m/(2m-1)} \frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|_m^2 \\
&\geqslant \frac{1}{2m-1} \|u(t)\|_m^{-4m/(2m-1)} (2m-1) \|u(t)\|_m^{4m/(2m-1)} \\
&\geqslant 1.
\end{aligned}$$

Therefore, $\|y(k^-)\|_m^2 \geqslant 1$. Since $\nu \leqslant 1$, in this case we also have (5.11).

In exactly the same way, using (5.4), we obtain that, for $t \in [k, k+1)$,

$$\begin{aligned}
\|u(t)\|_m^2 &\leqslant \max(\Theta_2 \nu^{-(2m-1)}, \|u(k)\|_m^2) \\
&\leqslant \max[\Theta_2, (\Theta + \sqrt{\Theta_2})^2] \nu^{-(2m-1)} \\
&\leqslant (\Theta + \sqrt{\Theta_2})^2 \nu^{-(2m-1)}.
\end{aligned}$$

Therefore, $\|u(t)\|_m^2 \nu^{2m-1}$ is uniformly bounded by $(\Theta + \sqrt{\Theta_2})^2$ for $t \in [k, k+1)$. Since all moments of this random variable are finite, the lemma's assertion is proved. □

6. Estimates of other Sobolev norms

The results in the three previous sections enable us to find upper and lower estimates for a large class of Sobolev norms. Unfortunately, while lower estimates extend to the whole Sobolev scale for $m \geqslant 0$ and $p \in [1, +\infty]$, there is a gap, corresponding to the case $m \geqslant 2$ and $p = 1$, for upper estimates.

Lemma 6.1. *For $m \in \{0, 1\}$ and $p \in [1, +\infty]$, or, for $m \geqslant 2$ and $p \in (1, +\infty]$, we have*

$$\Big(\mathbb{E} \sup_{t \in [k, k+1)} |u(t)|_{m,p}^n \Big)^{1/n} \leqslant C(m, p, n) \nu^{-\gamma}, \quad n \geqslant 1,\ k \geqslant 2.$$

Here and later on,

$$\gamma = \gamma(m, p) = \max\left(0, m - \frac{1}{p}\right).$$

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Proof. We begin by considering the case $m = 1$ and $p \in [2, +\infty]$. Since, by lemma 1.1, we have

$$|u(t)|_{m,p} \leqslant C(m,p)\|u(t)\|_m^{1-\theta}\|u(t)\|_{m+1}^{\theta},$$

where

$$\theta = \frac{1}{2} - \frac{1}{p},$$

then theorem 5.2 and Hölder's inequality yield the desired result.

The case where $m = 1$ and $p \in [1, 2)$ is proved in exactly the same way, by combining corollary 3.3 and theorem 5.2 for $m = 1$. The same method is used to prove the case $m \geqslant 2$ and $p \in (1, 2)$, combining the case $p \in [2, +\infty]$ for a sufficiently large value of m and corollary 3.3. Unfortunately, it cannot be applied for $m \geqslant 2$ and $p = 1$, because lemma 1.1 only allows us to estimate a $W^{n,1}$-norm from above by other $W^{n,1}$-norms.

Finally, the case $m = 0$ follows from corollary 3.4. □

The first norm that we estimate from below is the L_2-norm.

LEMMA 6.2. *We have*

$$\left(\int_k^{k+1} \mathbb{E}|u(s)|^2\right)^{1/2} \geqslant C, \quad k \geqslant 2.$$

Proof. Using properties (i) and (iii) of the kicks (u_k^- and ζ_k being independent), we obtain

$$\begin{aligned}\mathbb{E}|u_k^+|^2 &= \mathbb{E}|u_k^-|^2 + 2\mathbb{E}\langle u_k^-, \zeta_k\rangle + \mathbb{E}|\zeta_k|^2 \\ &= \mathbb{E}|u_k^-|^2 + \mathbb{E}|\zeta_k|^2 \\ &\geqslant I_0.\end{aligned}$$

On the other hand, by theorem 5.2, we have

$$\mathbb{E}\|u(t)\|_1^2 \leqslant C'\nu^{-1}, \quad t \in (k, k+1).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2 = -2\nu\|u(t)\|_1^2, \quad t \in (k, k+1),$$

then, integrating in time and setting

$$d = \min\left(1, \frac{I_0}{4C'}\right),$$

we obtain that, for $s \in [k, k+d]$,

$$\mathbb{E}|u(s)|^2 \geqslant \mathbb{E}|u_k^+|^2 - 2(s-k)C' \geqslant I_0 - 2C'd \geqslant \tfrac{1}{2}I_0.$$

Therefore,

$$\int_k^{k+1} \mathbb{E}|u(s)|^2 \geqslant \min\left(\tfrac{1}{2}I_0, \frac{I_0^2}{8C'}\right) > 0,$$

which proves the lemma's assertion. □

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Now we can study the case where $m = 0$ and $p \in [1, +\infty]$.

Corollary 6.3. *We have*

$$\left(\int_k^{k+1} \mathbb{E}|u(s)|_p^2\right)^{1/2} \geqslant C, \quad k \geqslant 2,\ p \in [1, +\infty],$$

where C does not depend on p.

Proof. It suffices to prove the inequality for $p = 1$. Using Hölder's inequality and integrating in time and in ensemble, and then using the Cauchy–Schwarz inequality, we obtain

$$\int_k^{k+1} \mathbb{E}|u|_1^2 \geqslant \int_k^{k+1} \mathbb{E}|u|^4 |u|_\infty^{-2} \geqslant \left(\int_k^{k+1} \mathbb{E}|u|^2\right)^2 \left(\int_k^{k+1} \mathbb{E}|u|_\infty^2\right)^{-1}.$$

Lemma 6.2 and corollary 3.4 ($p = +\infty$) complete the proof. □

Since the $W^{1,1}$-norm dominates the L_∞-norm, we obtain the following.

Corollary 6.4. *We have*

$$\left(\int_k^{k+1} \mathbb{E}|u(s)|_{1,1}^2(t)\right)^{1/2} \geqslant C, \quad k \geqslant 2.$$

The cases $m \geqslant 2$ and $m = 1$, $p \geqslant 2$ follow from lemma 4.1 and lemma 1.1 by interpolation in the same way as lemma 4.2, for $p > 1$. The case $p = +\infty$ follows from the case $p = 1$, since $|u|_{m,1} \geqslant |u|_{m-1,\infty}$, and $\gamma(m, 1) = \gamma(m - 1, +\infty)$.

Lemma 6.5. *If either $m \geqslant 2$ and $p \in [1, +\infty]$, or $m = 1$ and $p \in [2, +\infty]$, then*

$$\left(\frac{1}{T}\int_t^{t+T} \mathbb{E}|u(s)|_{m,p}^2\right)^{1/2} \geqslant C(m, p)\nu^{-\gamma}, \quad t \geqslant 1,\ T \geqslant N + 1,$$

where N is the same as in lemma 4.1.

Now it remains to deal with the case where $m = 1$ and $p \in (1, 2)$.

Lemma 6.6. *For $p \in (1, 2)$, we have*

$$\left(\frac{1}{T}\int_t^{t+T} \mathbb{E}|u(s)|_{1,p}^2\right)^{1/2} \geqslant C(p)\nu^{-\gamma}, \quad t \geqslant 2,\ T \geqslant N + 1,$$

where N is the same as in lemma 4.1. Here, note that $\gamma = 1 - 1/p$.

Proof. In the proof of this lemma, $C'(p)$ denotes various positive constants depending only on p. By Hölder's inequality in space, we have

$$\|u(s)\|_1^2 \leqslant |u(s)|_{1,p}^p |u(s)|_{1,\infty}^{(2-p)}.$$

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Therefore, using Hölder's inequality in time and in ensemble, as well as lemma 6.1, we obtain

$$\frac{1}{T}\int_t^{t+T}\mathbb{E}\|u(s)\|_1^2 \leqslant \left(\frac{1}{T}\int_t^{t+T}\mathbb{E}|u(s)|_{1,\infty}^2\right)^{(2-p)/2}\left(\frac{1}{T}\int_t^{t+T}\mathbb{E}|u(s)|_{1,p}^2\right)^{p/2}$$
$$\leqslant C'(p)\nu^{(p-2)}\left(\frac{1}{T}\int_t^{t+T}\mathbb{E}|u(s)|_{1,p}^2\right)^{p/2}.$$

Furthermore, lemma 4.1 implies that

$$\frac{1}{T}\int_t^{t+T}\mathbb{E}|u(s)|_{1,p}^2 \geqslant C'(p)\left(\nu^{(2-p)}\frac{1}{T}\int_t^{t+T}\mathbb{E}\|u(s)\|_1^2\right)^{2/p}$$
$$\geqslant C'(p)(\nu^{(2-p)}\nu^{-1})^{2/p}$$
$$\geqslant C'(p)\nu^{-(2p-2)/p}.$$

□

Remark 6.7. Upper estimates for

$$\left(\frac{1}{T}\int_t^{t+T}\mathbb{E}|u(s)|_{m,p}^n\right)^{1/n}, \quad n \geqslant 2,$$

follow from the lemmas above and Hölder's inequality.

7. Conclusion

Putting together the estimates that we have obtained, we formulate our main result.

Theorem 7.1. *For $m \in \{0,1\}$ and $p \in [1,+\infty]$, or, for $m \geqslant 2$ and $p \in (1,+\infty]$, we have*

$$\left(\mathbb{E}\sup_{t\in[k,k+1)}|u(t)|_{m,p}^n\right)^{1/n} \leqslant C(m,p,n)\nu^{-\gamma}, \quad n \geqslant 1,\ k \geqslant 2. \tag{7.1}$$

Moreover, there is an integer $N' \geqslant 1$ such that, for $m \geqslant 0$ and $p \in [1,+\infty]$, we have

$$\left(\frac{1}{T}\int_t^{t+T}\mathbb{E}|u(s)|_{m,p}^n\right)^{1/n} \geqslant C(m,p)\nu^{-\gamma}, \quad n \geqslant 2,\ t \geqslant 2,\ T \geqslant N'. \tag{7.2}$$

In both inequalities

$$\gamma = \max\left(0, m - \frac{1}{p}\right).$$

For a solution u of (3.4), we have obtained asymptotic estimates for expectations of a large class of Sobolev norms. The power of ν is clearly optimal except for $m \geqslant 2$ and $p = 1$, since it coincides for upper and lower estimates: we are in a *quasi-stationary regime*. Let us stress again that the upper bound $t = 2$ for the time needed for a quasi-stationary regime to be established has no dependence on u_0. The condition $t \geqslant T_0$ for some time $T_0 \geqslant 1$ is necessary. We need damping if u_0 is large and an injection of energy at a kick point if u_0 is small.

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Now set $\hat{u}^k = a_k(u) + ib_k(u)$ (see (3.3)). For sufficiently large $t \geqslant 2$ and T (see theorem 7.1), consider the averaged quantities

$$F_{s,\theta} = \frac{1}{T}\int_t^{t+T} \frac{\sum_{k\in I(s,\theta)} \mathbb{E}|\hat{u}^k|^2(\tau)}{\sum_{k\in I(s,\theta)} 1}, \quad s,\theta > 0,$$

where $I(s,\theta) = [\nu^{-s+\theta}, \nu^{-s-\theta})$. In the same way as in [2, formulae (1.6)–(1.8)], the inequalities (7.1) and (7.2) yield

$$F_{s,\theta} \leqslant C\nu^{2s}, \tag{7.3}$$

$$F_{s,\theta} \leqslant C(m)\nu^{2+2m(s-1-\theta)}, \quad m > 0,\ s > 1+\theta, \tag{7.4}$$

$$F_{1,\theta} > C\nu^{2+2\theta}, \tag{7.5}$$

for $\nu \leqslant \nu(\theta)$ with some $\nu(\theta) > 0$. These results have some consequences for the energy spectrum of u.

Indeed, relation (7.4) implies that the energy of the kth Fourier mode,

$$E_k = \frac{1}{2T}\int_t^{t+T} \mathbb{E}|\hat{u}^k|^2,$$

averaged around $k = l$, where $l \gg \nu^{-1}$, decays faster than any negative degree of l. On the other hand, by (7.3) and (7.5), the energy E_k, averaged around $k = \nu^{-1}$, behaves as k^{-2}. That is, the interval $k \in (\nu^{-1}, +\infty)$ is the *dissipation range*, where the energy E_k decays fast.

As the force η is smooth in x, the energy is injected at frequencies $k \sim 1$. The estimate (7.3) readily implies that the energy $E = \sum E_k$ of a solution u is supported, when $\nu \to 0$, by any interval $(0, \nu^{-\gamma})$, $\gamma > 0$. That is, the *energy range* of the solution u is the interval $(0, \nu^0]$ (see [5]).

The *inertial range* (ν^0, ν^{-1}) is a complement to the energy and dissipation ranges. At $k \sim \nu^{-1}$ we have $E_k \sim k^{-2}$. It is plausible that, in this range, E_k decays algebraically, possibly $E_k \sim k^{-2}$. The study of the energy spectrum of solutions u in the inertial range is one of the objectives of our future research.

We recall that the behaviour of the energy spectrum E_k of turbulent fluid of the form 'some negative degree of k in the inertial range, followed by fast decay in the dissipation range' is suggested by the Kolmogorov theory of turbulence (see [5]). Our results (following those of [2]) show that, for the 'burgulence' (described by the Burgers equation, see [1]), the dissipation range is $(\nu^{-1}, +\infty)$ and suggest that the power law in the inertial range is $E_k \sim k^{-2}$.

We also see that, for $\nu \to 0^+$, solutions u display intermittency-type behaviour (see [5, chapter 8]). Indeed, in the quasi-stationary regime, up to averaging in time and in ensemble, $\max_{x\in S^1} u_x \sim 1$, whereas $\int_{S^1} u_x^2 \sim \nu^{-1}$. Thus, typically u has large negative gradients on a small subset of S^1, and small positive gradients on a large subset of S^1.

In a future paper we will look at the same problem with the kick force replaced by a spatially smooth white noise in time (see [4] for a possible definition). This problem is, heuristically, the limit case of the kick-forced problem with increasingly frequent appropriately scaled kicks.

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