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RESEARCH ARTICLE

Abelian actions on compact nonorientable Riemann surfaces

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Abstract
Given an integer $g > 2$, we state necessary and sufficient conditions for a finite Abelian group to act as a group of automorphisms of some compact nonorientable Riemann surface of genus g. This result provides a new method to obtain the symmetric cross-cap number of Abelian groups. We also compute the least symmetric cross-cap number of Abelian groups of a given order and solve the maximum order problem for Abelian groups acting on nonorientable Riemann surfaces.

1. Introduction

The study of groups of automorphisms of Riemann and Klein surfaces is a classical topic initiated by Schwartz, Hurwitz, Klein and Wiman, among others, at the end of the 19th century. Surfaces with a nontrivial finite group of automorphisms are of particular importance, since they correspond to the singular locus of the moduli space of such surfaces. By the uniformization theorem, compact Riemann and Klein surfaces of algebraic genus greater than one can be seen as the quotient of the hyperbolic plane under the action of a discrete subgroup of its isometries (a non-Euclidean crystallographic group, in general, or a Fuchsian group if it only contains orientation-preserving isometries). This approach gave rise to the use of combinatorial methods, which have proven the most fruitful in computing groups of automorphisms.

Here, we establish conditions on an Abelian group in order to act on nonorientable Riemann surfaces of a given genus in Theorem 4.3. Harvey [10] was the first in applying combinatorial methods to obtain such kind of results (for a cyclic group to act on a compact Riemann surface). For nonorientable surfaces, we will need the Abelianization of an NEC group, which is computed in Section 3. We also restrict Theorem 4.3 to cyclic groups, obtaining an extension of the results of Bujalance in [3].

Minimum genus and maximum order problems have been studied for a number of families of groups using diverse techniques. Some thorough surveys on these topics can be found in [5–7]. One of these techniques takes advantage of previously established conditions for the existence of surface-kernel epimorphisms onto a group of the family. This approach usually provides a shorter proof to the solution to the minimum genus and maximum order problems, as we will see in Sections 5 and 7. In Section 6, we obtain the least genus on which act some Abelian group of a given order.

2. Preliminaries

Klein surfaces constitute a generalization of Riemann surfaces that include bordered and nonorientable surfaces. They broaden the scope of Riemann surfaces by allowing transition functions that may include complex conjugation besides analytic functions and domains in the closed upper half-plane $\mathbb{C}^+$. This makes up what is called a dianalytic structure [1]. The topological genus g, the number k of boundary components, and the orientability are known as the *topological type* of a Klein surface, and the integer $p = \eta g + k - 1$ as its *algebraic genus*, where $\eta = 2$ if the surface is orientable and $\eta = 1$ otherwise. By a nonorientable Riemann surface, we mean a nonorientable unbordered Klein surface.

A *non-Euclidean crystallographic* (NEC) group Λ is a discrete subgroup of the group of isometries of the hyperbolic plane $\mathcal{H}$ for which $\mathcal{H}/\Lambda$ is compact. It is a *Fuchsian* group if it contains only orientation preserving isometries; otherwise, it is said to be a *proper* NEC group. An NEC group with no orientation preserving elements of finite order is called *surface* NEC group.

Every compact Klein surface with algebraic genus $p > 1$ can be represented by the orbit space $\mathcal{H}/\Gamma$ for some surface NEC group Γ, that is, surface NEC groups uniformize compact Klein surfaces. Furthermore, every group G of automorphisms of $\mathcal{H}/\Gamma$ is isomorphic to the factor group Λ/Γ for some NEC group Λ containing the surface NEC group Γ as a normal subgroup, that is, there is an epimorphism $\theta : \Lambda \to G$ for which $\ker\theta = \Gamma$ (we say that θ is a *surface-kernel epimorphism*).

It is well-known that every group of finite order acts on some compact Klein surface of algebraic genus greater than 1 [4]. A group may act on Klein surfaces of different genera. The *minimum genus problem* consists in finding the least genus on which a group acts. When dealing with nonorientable Riemann surfaces, such minimum (topological) genus is called *symmetric cross-cap number* of the group, and is denoted by $\tilde{\sigma}(G)$. Conversely, several groups may act on some Klein surface of a given algebraic genus. When the algebraic genus is greater than 1, there are only finitely many such groups. Computing the largest group order in a family of groups which act on a given genus is what we call the *maximum order problem* for that family.

Non-isomorphic NEC groups differ from one another in the *signature*, which is of the form

$$\left(g;\ \pm\,;[m_1,\dots,m_r];\ \left\{(n_{i1},\dots,n_{is_i}),\ i=1,\dots,k\right\}\right). \tag{2.1}$$

The signature of a Fuchsian group is usually denoted by $(g; m_1, \dots, m_r)$. For a surface NEC group, it is of the form $(g;\ \pm\,;\ [-]; \{(-), \overset{k}{\dots}, (-)\})$. The signature of an NEC group Λ determines both its algebraic structure and the topological structure of the orbit space $\mathcal{H}/\Lambda$.

The integers $m_i \geqslant 2$ are called *proper periods*, $n_{ij} \geqslant 2$ are the *link periods*, $(n_{i1}, \dots, n_{is_i})$ are the *period cycles* and g is the *orbit genus*. The orbit space $\mathcal{H}/\Lambda$ has topological genus g, k boundary components and is orientable if the sign of the signature is '$+$' and nonorientable otherwise. The covering map $\mathcal{H} \to \mathcal{H}/\Lambda$ ramifies over r interior points with ramification indices m_i and, on each boundary component, over s_i points with ramification indices n_{ij}. The number $\eta g + k - 1$ is the algebraic genus of $\mathcal{H}/\Lambda$, where $\eta = 2$ if the sign of the signature is '$+$' and $\eta = 1$ otherwise. An arbitrary set of such numbers and symbols define the signature of an NEC group if and only if

$$\eta g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right) > 0. \tag{2.2}$$

The expression in the left side is denoted by $\mu(\Lambda)$. The hyperbolic area of any fundamental region of $\mathcal{H}/\Lambda$ is $2\pi\mu(\Lambda)$. Also, if Λ' is a subgroup of Λ of finite index, then Λ' is an NEC group and

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda), \tag{2.3}$$

which is known as the Riemann–Hurwitz formula.

The signature provides a presentation of Λ with the following generators and relations depending on the sign of the signature:

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$x_1, \dots, x_r$ (hyperbolic rotations),
$c_{10}, \dots, c_{1s_1}, \dots, c_{k0}, \dots, c_{ks_k}$ (hyperbolic reflections),
$e_1, \dots, e_k$ (connecting generators),
$a_1, b_1, \dots, a_g, b_g$ if the sign is '+' (hyperbolic translations),
$d_1, \dots, d_g$ if the sign is '−' (hyperbolic glide reflections),

$$x_i^{m_i} = 1, \qquad c_{ij}^2 = 1, \qquad (c_{ij-1}c_{ij})^{n_{ij}} = 1, \qquad e_i^{-1}c_{i0}e_ic_{is_i} = 1,$$

$$x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \qquad \text{if the sign is '+' and}$$

$$x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2 = 1 \qquad \text{if the sign is '−'}.$$

The last one is called the *long relation*. (An abstract group with such a presentation is an NEC group with signature as above if and only if (2.2) is fulfilled.)

For further purposes, the following should be considered. For proper periods, we assume factorizations $m_i = p_1^{\mu_i(p_1)} \cdots p_s^{\mu_i(p_s)}$ with prime numbers $p_1 < \cdots < p_s$ and integers $\mu_i(p_j) \geqslant 0$ such that $\mu_1(p_j) + \cdots + \mu_r(p_j) > 0$. For each prime p_j, we rearrange the integers $\mu_1(p_j), \dots, \mu_r(p_j)$ to obtain increasing integers $\widehat{\mu}_1(p_j) \leqslant \widehat{\mu}_2(p_j) \leqslant \cdots \leqslant \widehat{\mu}_r(p_j)$ and define $\widehat{m}_i = p_1^{\widehat{\mu}_i(p_1)} \cdots p_s^{\widehat{\mu}_i(p_s)}$. Then, $\widehat{m}_i | \widehat{m}_{i+1}$ and there is an integer $\widehat{r}$ such that $\widehat{m}_i = 1$ for $i = 1, \dots, r - \widehat{r}$ and $\widehat{m}_i > 1$ for the $\widehat{r}$ integers $i = r - \widehat{r} + 1, \dots, r$. Moreover (see [13, Section 2]),

$$\sum_{i=1}^{r} \left(1 - \frac{1}{\widehat{m}_i}\right) \leqslant \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right). \tag{2.4}$$

Henceforth, we will deal with Abelian groups. The torsion subgroup of an Abelian group A is denoted by $\mathcal{T}(A)$. When A is a finitely generated Abelian group, its *invariant factor decomposition* is $A \approx \mathbb{Z}^n \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ for integers $n \geqslant 0$, the *torsion-free rank* of A, and $v_i > 1$, called *invariant factors* of A, with v_i dividing v_{i+1}, and *primary decomposition* $A \approx \mathbb{Z}^n \oplus A_{q_1} \oplus \cdots \oplus A_{q_\lambda}$, where $q_1 < \cdots < q_\lambda$ are the prime numbers dividing the order of A and $A_q = \{x \in A | q^m x = 0 \text{ for some } m \geqslant 0\}$ is the *q-primary component* of A —the q-Sylow subgroup $Syl_q(A)$. We also assume $v_i = q_1^{\alpha_i(q_1)} \cdots q_\lambda^{\alpha_i(q_\lambda)}$ for $i = 1, \dots, t$, so $0 \leqslant \alpha_1(q) \leqslant \cdots \leqslant \alpha_t(q)$ and $A_q \approx \mathbb{Z}_{q^{\alpha_1(q)}} \oplus \cdots \oplus \mathbb{Z}_{q^{\alpha_t(q)}}$. The integers $q_j^{\alpha_i(q_j)}$ are the *elementary divisors* of A.

Below it will be helpful to express a finite Abelian group as follows:

$$A \approx \mathbb{Z}_2^n \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t},$$

(for readability, $\mathbb{Z}_v \oplus \overset{n}{\cdots} \oplus \mathbb{Z}_v$ will be denoted by $\mathbb{Z}_v^n$) where $v_i > 2$ and v_i divides v_{i+1}, so that $v_1, \dots, v_{t-m}$ are odd and the m integers $v_{t-m+1}, \dots, v_t$ are multiple of 4 for some integer $m \leqslant t$ —note that, though unique, this expression may not coincide with the invariant factor decomposition of A.

3. Abelianization of NEC groups

In this section, we lay out Breuer conditions for the existence of epimorphisms between Abelian groups and the abelianization of NEC groups.

We are mainly concerned with conditions for the existence of epimorphisms $\theta : \Lambda \to A$ from an NEC group onto a finite Abelian group. In this context, the Abelianization Λ_{ab} of Λ provides significant information. By the universal property of the quotient group, θ factors (uniquely) through the canonical homomorphism $\pi : \Lambda \to \Lambda_{ab}$. In other words, there is a (unique) epimorphism $\overline{\theta} : \Lambda_{ab} \to A$ such that $\theta = \overline{\theta} \circ \pi$. Conversely, given an epimorphism $\overline{\theta} : \Lambda_{ab} \to A$, the homomorphism $\theta = \overline{\theta} \circ \pi : \Lambda \to A$ is onto.

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Therefore, it is worth considering epimorphisms between Abelian groups. Breuer stated conditions for the existence of such epimorphisms as a set of inequations on the free-rank and the number of cyclic factors in the primary decomposition of the Abelian groups [2, lemmas A.1 and A.2]:

Lemma 3.1. *Let q be a prime number and $R, N_1, \ldots, N_s, T, n_1, \ldots, n_s$ be non-negative integers. There is an epimorphism*

$$\mathbb{Z}^R \oplus \mathbb{Z}_{q^1}^{N_1} \oplus \cdots \oplus \mathbb{Z}_{q^s}^{N_s} \quad \rightarrow \quad \mathbb{Z}^T \oplus \mathbb{Z}_{q^1}^{n_1} \oplus \cdots \oplus \mathbb{Z}_{q^s}^{n_s}$$

if and only if

$$R \geqslant T \quad \text{and} \quad R + N_j + N_{j+1} + \cdots + N_s \geqslant T + n_j + n_{j+1} + \cdots + n_s \quad \text{for} \quad j = 1, \ldots, s. \tag{3.1}$$

For arbitrary finite Abelian groups A and B, there is an epimorphism $\mathbb{Z}^R \oplus A \rightarrow \mathbb{Z}^T \oplus B$ if and only if there is an epimorphism $\mathbb{Z}^R \oplus A_q \rightarrow \mathbb{Z}^T \oplus B_q$ for each prime q dividing the order of B.

The requirements in (3.1) can be rewritten as follows.

Lemma 3.2. *Let q be a prime number and $R, \alpha_1, \ldots, \alpha_t, \beta_1, \ldots, \beta_r$ be non-negative integers with $\alpha_i \leqslant \alpha_{i+1}$ and $\beta_i \leqslant \beta_{i+1}$. There is an epimorphism*

$$\mathbb{Z}^R \oplus \mathbb{Z}_{q^{\beta_1}} \oplus \cdots \oplus \mathbb{Z}_{q^{\beta_r}} \quad \rightarrow \quad \mathbb{Z}^T \oplus \mathbb{Z}_{q^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{q^{\alpha_t}}$$

if and only if the following conditions hold: $R \geqslant T$ and if $R < T + t$, then q^{α_i} divides, at least, $T + t - R - i + 1$ elementary divisors q^{β_j} for $i = 1, \ldots, T + t - R$.

In order to study these conditions for epimorphisms $\Lambda_{ab} \rightarrow A$, we need to know the structure of Λ_{ab} in terms of the signature of Λ. Here a distinction is made between Fuchsian and proper NEC groups. For a Fuchsian group Λ, we find its Abelianization in [2, Lemma A.3].

Lemma 3.3. *The Abelianization of a Fuchsian group Λ with signature $(g; m_1, \ldots, m_r)$ is isomorphic to $\mathbb{Z}^{2g}$ if $r = 0$ or 1 and*

$$\Lambda_{ab} \approx \mathbb{Z}^{2g} \oplus \mathbb{Z}_{\widehat{m}_{r-\widehat{r}+1}} \oplus \cdots \oplus \mathbb{Z}_{\widehat{m}_{r-1}}$$

otherwise.

Now, we compute the Abelianization Λ_{ab} of a proper NEC group Λ. When the signature has some period cycle, Λ_{ab} is obtained by some considerations on the canonical presentation of Λ. Otherwise, it has no period cycle and we compute the *Smith normal form* ([16][11, Section 3.3][12, Chapter 2]) of the relation matrix of the Abelianized canonical presentation of Λ.

Lemma 3.4. *The Abelianization of a proper NEC group Λ with signature $(g; \pm; [m_1, \ldots, m_r]; \{(n_{i1}, \ldots, n_{is_i}),\ i = 1, \ldots, k\})$ is $\Lambda_{ab} \approx \mathbb{Z}^{\eta g + k - 1} \oplus \mathcal{T}(\Lambda_{ab})$, where*

$$\mathcal{T}(\Lambda_{ab}) \approx \begin{cases} \mathbb{Z}_2 & \text{if } k = r = 0, \\ \mathbb{Z}_{\widehat{m}_{r-\widehat{r}+1}} \oplus \cdots \oplus \mathbb{Z}_{\widehat{m}_{r-1}} \oplus \mathbb{Z}_{2\widehat{m}_r} & \text{if } k = 0,\ r > 0, \\ \mathbb{Z}_2^S \oplus \mathbb{Z}_{\widehat{m}_{r-\widehat{r}+1}} \oplus \cdots \oplus \mathbb{Z}_{\widehat{m}_r} & \text{otherwise} \end{cases} \tag{3.2}$$

is the torsion subgroup of Λ_{ab}, η equals 2 if the sign of the signature is '+' and 1 otherwise, and

$$S = \#\{\text{period cycles with no even link periods}\} + \#\{\text{even link periods}\}.$$

Proof. When $k > 0$, we remove one generator e_i by the long relation in the Abelianized presentation, and the other relations only contain generators of finite order. The remaining canonical generators e_i,

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a_i, b_i and d_i provide the factor $\mathbb{Z}^{\eta g+k-1}$, while the elliptic generators and their relations turn into $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r} \approx \mathbb{Z}_{\widehat{m}_{r-\widehat{r}+1}} \oplus \cdots \oplus \mathbb{Z}_{\widehat{m}_r}$.

The factor $\mathbb{Z}_2^S$ originates from the generators c_{ij}. There are $k+s_1+\cdots+s_k$ of such generators; we remove k of them (each relation $e_i^{-1}c_{i0}e_ic_{is_i}=1$ lets us remove c_{i0} or c_{is_i} when Abelianized) and also those generators c_{ij} for which n_{ij} is odd (when Abelianized, the relation $(c_{ij-1}c_{ij})^{n_{ij}}=1$ becomes either $c_{ij-1}c_{ij}=1$ or trivial for odd and even values of n_{ij}, respectively).

If $k=0$ (hence the sign of the signature is '–' since Λ is proper), a presentation of Λ_{ab} is

$$\langle x_1,\ldots,x_r,d_1,\ldots,d_g \mid x_i^{m_i},\ x_1\cdots x_r d_1^2\cdots d_g^2, [x_i,x_j], [x_i,d_j], [d_i,d_j]\rangle.$$

The relation matrix of this presentation is

$$\mathbf{R}=\begin{pmatrix} m_1 & & & & & & & \\ & m_2 & & & & 0 & & \\ & & \ddots & & & & & \\ & 0 & & & & & & \\ & & & m_r & & & & \\ 1 & \cdots & 1 & 1 & 2 & \overset{g}{\cdots} & 2 \end{pmatrix}.$$

The entries ϵ_i in the main diagonal of the Smith normal form of $\mathbf{R}$ can be computed as follows: $\epsilon_1=\rho_1$, $\epsilon_2=\rho_2/\rho_1$, $\epsilon_3=\rho_3/\rho_2$, etc., where ρ_i is the greatest common divisor of the minor determinants of order i of $\mathbf{R}$.

Clearly, $\rho_1=1$. Non-null 2×2 minors of $\mathbf{R}$ are m_i, $2m_i$ or $m_{i_1}m_{i_2}$ and thus $\rho_2=\gcd(m_1,\ldots,m_r)=\widehat{m}_1$. Likewise, non-null 3×3 minors take values $m_{i_1}m_{i_2}$, $2m_{i_1}m_{i_2}$ or $m_{i_1}m_{i_2}m_{i_3}$ so that $\rho_3=\gcd(m_1m_2,\ldots,m_1m_r,m_2m_3,\ldots,m_{r-1}m_r)=\widehat{m}_1\widehat{m}_2$. In general,

$$\rho_k=\gcd\{m_{i_1}\cdots m_{i_{k-1}}\}_{1\leqslant i_1<i_2<\cdots<i_{k-1}\leqslant r}=\widehat{m}_1\cdots\widehat{m}_{k-1},$$
$$\rho_{r+1}=2m_1\cdots m_r=2\widehat{m}_1\cdots\widehat{m}_r$$

for $k=2,\ldots,r$ and thus $\epsilon_1=1$, $\epsilon_2=\widehat{m}_1$, $\epsilon_3=\widehat{m}_2$, $\ldots$, $\epsilon_r=\widehat{m}_{r-1}$, $\epsilon_{r+1}=2\widehat{m}_r$. Therefore, $\Lambda_{ab}\approx\mathbb{Z}^{g-1}\oplus\mathbb{Z}_{\widehat{m}_{r-\widehat{r}+1}}\oplus\cdots\oplus\mathbb{Z}_{\widehat{m}_{r-1}}\oplus\mathbb{Z}_{2\widehat{m}_r}$ when $k=0$. □

4. Nonorientable unbordered surface-kernel epimorphisms

In this section, conditions for the existence of nonorientable unbordered surface-kernel epimorphisms onto an Abelian group are established in terms of its algebraic structure.

We say that a homomorphism θ of a proper NEC group Λ into a finite group G is *nonorientable unbordered surface-kernel* if $\ker\theta$ is a surface group with signature having sign '–' and no period cycles. The condition on the sign of the signature means that $\theta(\Lambda^+)=G$, as stated by Singerman [15, Theorem 1]. As a consequence, the following result establishes the type of signatures we will deal with.

Lemma 4.1. *[3, Corollary 3.3][9, Corollary 2.3] Let A be a finite Abelian group, Λ a proper NEC group. If there exists a nonorientable unbordered surface-kernel epimorphism $\Lambda\to A$, then every link period of the signature of Λ equals 2 and no period cycle has a single link period. If the order of A is odd, then the signature of Λ has no period cycle.*

Remark 4.2. *In order to prove Theorem 4.3, we need to make some considerations on the Smith normal form of (4.1), which is obtained as follows. There is some orientation-reversing element in the kernel of a nonorientable unbordered surface-kernel epimorphism $\theta:\Lambda\to A$ onto a finite Abelian group A. Since $\theta(c_{i0})=\theta(c_{is_i})$, we can reorder the product of canonical generators in the expression of such orientation-reversing element to obtain another orientation-reversing element h with the same image and an expression like the following:*

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$$h = \begin{cases} h' \cdot c_{10}^{\kappa_1} \cdots c_{1,s_1-1}^{\kappa_{s_1}} \cdot c_{2,0}^{\kappa_{s_1+1}} \cdots c_{k,s_k-1}^{\kappa_S} \\ h' \cdot d_1^{\zeta_1} \cdots d_g^{\zeta_g} \cdot c_{10}^{\kappa_1} \cdots c_{1,s_1-1}^{\kappa_{s_1}} \cdot c_{2,0}^{\kappa_{s_1+1}} \cdots c_{k,s_k-1}^{\kappa_S} \end{cases}$$

for signature sign '+' or '–', respectively, without any of the k canonical reflections c_{is_i}, *where h' is a product of powers of orientation-preserving canonical generators,* $\kappa_1 + \cdots + \kappa_S$ *or* $\zeta_1 + \cdots + \zeta_g + \kappa_1 + \cdots + \kappa_S$, *depending on the signature sign, is odd, and* $\kappa_i = 0$ *or 1 since* $c_{ij}^2 = 1$.

We can factor θ *through* $N = \langle h \rangle^{\Lambda}$, *the normal subgroup generated by h. By the universal property of the quotient group, there exists a unique homomorphism* $\phi : \Lambda/N \to A$ *such that* $\phi \circ \pi = \theta$, *where* $\pi : \Lambda \to \Lambda/N$ *is the canonical projection. Since* θ *and* π *are onto,* ϕ *is onto as well. Now, we factor* ϕ *through* $(\Lambda/N)_{ab}$ *to obtain another epimorphism* $\overline{\phi} : (\Lambda/N)_{ab} \to A$. *We will extract some information from this epimorphism between Abelian groups by applying lemmas 3.1 and 3.2. It is not mandatory to compute* $(\Lambda/N)_{ab}$; *the following considerations will be enough for our purposes.*

By Lemma 4.1, we will focus on proper NEC groups with signatures of type $(g; \pm; [m_1, \ldots, m_r]; \{(-)^{\varepsilon}, (2, \overset{s_{\varepsilon+1}}{\ldots}, 2), \ldots, (2, \overset{s_k}{\ldots}, 2)\})$. *In that case, a presentation of* $(\Lambda/N)_{ab}$ *is*

$$\langle x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g, e_1, \ldots, e_k, c_{10}, \ldots, c_{1,s_1-1}, c_{20}, \ldots, c_{k,s_k-1} \mid \\ x_i^{m_i},\ x_1 \cdots x_r e_1 \cdots e_k, c_{ij}^2, h, [\cdot,\cdot] \rangle$$

(by abuse of notation, we use the same symbols for the generators) if the signature sign is '+' and

$$\langle x_1, \ldots, x_r, d_1, \ldots, d_g, e_1, \ldots, e_k, c_{10}, \ldots, c_{1,s_1-1}, c_{20}, \ldots, c_{k,s_k-1} \mid \\ x_i^{m_i},\ x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2, c_{ij}^2, h, [\cdot,\cdot] \rangle$$

if the signature sign is '–', where $[\cdot,\cdot]$ *denotes all commutation relations of pairs of generators. We have removed the generators* c_{is_i} *(by means of the Abelianized relations* $e_i^{-1} c_{i0} e_i c_{is_i} = 1$*) and the relations* $(c_{ij-1}c_{ij})^2 = 1$ *(they become trivial when Abelianized). The relation matrix of this presentation is*

$$\begin{pmatrix} m_1 & & & & & & & & & & & & 0 \\ & \ddots & & & & & & & & & & & \\ & & m_r & & & & & & & & & & \\ 1 & \cdots & 1 & 1 & \overset{k}{\cdots} & 1 & 2 & \overset{g}{\cdots} & 2 & 0 & \overset{S}{\cdots} & 0 \\ & \cdots & & & \cdots & & \zeta_1 & \cdots & \zeta_g & \kappa_1 & \cdots & \kappa_S \\ & & & & & & & & & 2 & & \\ & & & & 0 & & & & & & \ddots & \\ & & & & & & & & & & & 2 \end{pmatrix} \qquad (4.1)$$

if the signature sign is '–'; if the signature sign is '+', remove the g columns containing the submatrix $\begin{pmatrix} 2 & \overset{g}{\cdots} & 2 \\ \zeta_1 & \cdots & \zeta_g \end{pmatrix}$ *(to lessen clutter, the 2g columns corresponding to the canonical generators* a_i *and* b_i *are omitted, since they consist of null entries, and also entries corresponding to orientation-preserving generators in the* $(r+2)$*nd row are not denoted).*

Theorem 4.3. *Let* Λ *be a proper NEC group with signature* $(g; \pm; [m_1, \ldots, m_r]; \{(-)^{\varepsilon}, (2, \overset{s_{\varepsilon+1}}{\ldots}, 2), \ldots, (2, \overset{s_k}{\ldots}, 2)\})$, $s_i \neq 1$, *having* r_2 *even proper periods, and* $A \approx \mathbb{Z}_2^n \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ *a nontrivial Abelian group, where* $t \geqslant 0$, $v_i > 2$, $v_i | v_{i+1}$, v_i *is odd for* $i = 1, \ldots, t-m$ *and* $4 \mid v_i$ *if* $i > t-m$ *for some integer* $m \leqslant t$. *Let* $A_2 \approx \mathbb{Z}_{2^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{\alpha_{m+n}}}$ *be the Sylow 2-subgroup of A,* $\alpha_i \leqslant \alpha_{i+1}$, $w = \eta g + k - 1$, $S = \varepsilon + s_{\varepsilon+1} + \cdots + s_k$, $\eta = 2$ *if '+' is the signature sign of* Λ *and* $\eta = 1$ *otherwise. Then, there exists a nonorientable unbordered surface-kernel epimorphism* $\Lambda \to A$ *if and only if the following conditions hold:*

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(i) $m_i = 2$ *if* $t = 0$, $m_i|v_t$ *if* $n = 0$ *and* $m_i\,|lcm(2, v_t)$ *otherwise for all i.*

(ii) *If* $t > w$ *and* $i \in \{1, \ldots, t-w\}$, *then every elementary divisor of* $\mathbb{Z}_{v_i}$ *divides, at least,* $t - w + 1 - i$ *proper periods.*

(iii) *If* $k = 0$, $r_2 > 0$ *and* $m + n > g - 2$, *then* 2^{α_i} *divides, at least,* $m + n - g + 3 - i$ *proper periods for* $i = 1, \ldots, m + n - g + 1$*; if, in addition, g is even, then* $2^{\alpha_{m+n-g+2}-1}$ *divides some proper period, and there is an odd number of such proper periods if, in addition,* $g = 2$, $2^{\alpha_{m+n}}$ *divides no proper period and* $\alpha_{m+n-1} < \alpha_{m+n} - 1$.

(iv) *If* $k = 0$ *and* $m + n > 0$, *then* $2^{\alpha_{m+n}}$ *divides either no proper period or, at least, two proper periods; if, in addition,* $\alpha_{m+n-1} < \alpha_{m+n}$, *then* $2^{\alpha_{m+n}}$ *divides an even number of proper periods.*

(v) *If* $k = 0$ *and* $r_2 = 0$, *then* $g > m + n$, *and* $g > m + 1$ *if, in addition,* $n = 0$ *and m is odd.*

(vi) *If* $k > 0$, *then* $r_2 \geqslant m + n - w - S + 1$.

(vii) *If* $m + n = 0$, *then* $k = 0$.

(viii) *If* $m + n = 1$, *then* $k = \varepsilon$.

(ix) *If* $m + n = 2$, *then* s_i *is even for all i.*

Proof. Let $\theta : \Lambda \to A$ be a nonorientable unbordered surface-kernel epimorphism.

(i) The order of $\theta(x_i)$ is m_i and the order of every element of A divides the exponent of A ($\exp A = 2$ if $t = 0$, v_t if $n = 0$ and $lcm(2, v_t)$ otherwise).

(ii) This condition follows from Lemma 3.2 applied to the epimorphism $\Lambda_{ab} \to A$ (we note that v_i is either odd or multiple of 4, so that we do not have to consider the factors $\mathbb{Z}_2$ in (3.2)).

(iii) Let $k = 0$. If $g = 1$, then ζ_1 is odd in (4.1) and it follows that $(\Lambda/N)_{ab}$ has null free-rank and $Syl_2((\Lambda/N)_{ab}) \approx \mathbb{Z}_{2^{\hat{\mu}_1(2)}} \oplus \cdots \oplus \mathbb{Z}_{2^{\hat{\mu}_{r-1}(2)}}$. Now, assume that $g > 1$ and let 2^{δ} be the greatest power of 2 dividing any minor of order 2 of the submatrix $\left(\begin{smallmatrix} 2 & \overset{g}{\cdots} & 2 \\ \zeta_1 & \cdots & \zeta_g \end{smallmatrix}\right)$ of (4.1). These minors have the form $2(\zeta_i - \zeta_j)$. If g is even, some ζ_i is even and thus $2(\zeta_i - \zeta_j)$ is even but not multiple of 4 in some cases, since there is also some ζ_j that is odd; therefore, $\delta = 1$ if g is even. If g is odd, every ζ_i may be odd and δ may be greater than 1. If $k = 0$ and $r_2 > 0$, we obtain from (4.1) that the free-rank of $(\Lambda/N)_{ab}$ is $g - 2$ and

$$Syl_2(\mathcal{T}((\Lambda/N)_{ab})) \approx \mathbb{Z}_{2^{\hat{\mu}_1(2)}} \oplus \cdots \oplus \mathbb{Z}_{2^{\hat{\mu}_{r-1}(2)}} \oplus \mathbb{Z}_{2^{\hat{\mu}_r(2)+\delta}}$$

The claim for 2^{α_i} and $2^{\alpha_{m+n-g+2}-1}$ follows from Lemma 3.2 when applied to the epimorphism $Syl_2((\Lambda/N)_{ab}) \to A_2$ if $g < 3$ and $\mathbb{Z}^{g-2} \oplus Syl_2(\mathcal{T}((\Lambda/N)_{ab})) \to A_2$ otherwise.

Now, suppose also that $g = 2$, $2^{\alpha_{m+n}}$ divides no proper period and $\alpha_{m+n-1} < \alpha_{m+n} - 1$. If $2^{\alpha_{m+n}-1}$ divides an even number of proper periods, then the last component of $\theta_2(x_1 \cdots x_r)$ is doubly even. Also, the last component of either $\theta_2(d_1)$ or $\theta_2(d_2)$ is odd in order to generate $\mathbb{Z}_{\alpha_{m+n}}$, but both last components cannot be odd (otherwise, every element in $\ker\theta_2$ would contain an even number of glide reflections and would be orientation-preserving) and thus the last component of $\theta_2(d_1^2 d_2^2)$ is singly even. Therefore, the long relation would not be preserved, since the last component of $\theta_2(x_1 \cdots x_r d_1^2 d_2^2)$ would be singly even.

(iv) Otherwise, the component of $\theta_2(x_1 \cdots x_r \cdot d_1^2 \cdots d_g^2)$ corresponding to some factor of order $2^{\alpha_{m+n}}$ would be odd and the long relation would not be preserved.

(v) If $k = 0$ and $r_2 = 0$, the dimensions of the relation matrix (4.1) are $(r + 2) \times (r + g)$. Since ζ_i is odd for some $i \in \{1, \ldots, g\}$ and m_i is odd, it follows that ρ_i is odd for $i < r + 2$ and ρ_{r+2} is even, and thus the free-rank of $(\Lambda/N)_{ab}$ is $g - 2$ and its Sylow 2-subgroup is nontrivial cyclic. As an epimorphism $(\Lambda/N)_{ab} \to A$ exists, we have $g - 2 + 1 \geqslant m + n$ by choosing $q = 2$ and $j = 1$ in (3.1), hence $g > m + n$. So if, in addition, $n = 0$ then $g \geqslant m + 1$; but m odd and $g = m + 1$ (even) would be inconsistent with (3.1): g even (and $k = 0$) implies that some ζ_i is even ($\sum \zeta_i$ is odd) and thus ρ_{r+2} is even but not multiple of 4 (since $r_2 = 0$), so the Sylow 2-subgroup of $(\Lambda/N)_{ab}$ is $\mathbb{Z}_2$ (ρ_i is odd for $i < r + 2$ since $r_2 = 0$ and some ζ_i is odd) and we obtain $g - 2 \geqslant m$ for $q = 2$ and $j = 2$ in (3.1) and thus $g > m + 1$.

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(vi) Let **R** be the matrix (4.1), depending on the signature sign of Λ. If $k>0$, then either some κ_i equals 1 or some ζ_i is odd, say, ζ_1 (recall that $\sum \zeta_i + \sum \kappa_i$ is odd). Then, ρ_2 is odd since, in the $r+1$-th and $r+2$-th rows of **R**, there is a 2×2-submatrix like $\left(\begin{smallmatrix} 1 & 0 \\ \cdots & 1 \end{smallmatrix}\right)$ in the first case, and like $\left(\begin{smallmatrix} 2 & 1 \\ \zeta_1 & \cdots \end{smallmatrix}\right)$ in the second case. Also, $\rho_1 = 1$ since some entry equals 1. Therefore, $\epsilon_1 = \rho_1 = 1$ and $\epsilon_2 = \rho_2/\rho_1$ is odd in the Smith normal form of **R**, so the free-rank of $(\Lambda/N)_{ab}$ is $\eta g + k - 2 = w-1$ and its Sylow 2-subgroup has, at most, $r_2 + S$ factors. By (3.1) applied for $q=2$ and $j=1$ to the epimorphism $\mathbb{Z}^{w-1} \oplus Syl_2((\Lambda/N)_{ab}) \to A_2$, the free-rank of $(\Lambda/N)_{ab}$ and the number of factors of its Sylow 2-subgroup add up to $m+n$ (the number of factors of A_2) or more. Hence, $w-1+r_2+S \geqslant m+n$.

(vii) This condition follows from Lemma 4.1.

(viii) If $Syl_2(A) \approx \mathbb{Z}_{2^\alpha}$ and $(n_{i1}, \ldots, n_{is_i}) = (2, 2, \ldots, 2)$ is a period cycle of Λ, then $\theta(c_{ij-1}) = \theta(c_{ij}) = 2^{\alpha-1}$, since both $\theta(c_{ij-1})$ and $\theta(c_{ij})$ are elements of order 2 in $Syl_2(A)$. But then $\theta(c_{ij-1}c_{ij}) = 0$ and $\theta(c_{ij-1}c_{ij})$ cannot have order $n_{ij} = 2$, in contradiction with Theorem [3, Proposition 3.2].

(ix) Assume that $Syl_2(A) \approx \mathbb{Z}_{2^{\alpha_1}} \oplus \mathbb{Z}_{2^{\alpha_2}}$. By the relation $e_i^{-1} c_{i0} e_i c_{is_i} = 1$, $\theta(c_{i0}) = \theta(c_{is_i})$, and thus s_i is even, since $\theta(c_{ij-1}) \neq \theta(c_{ij})$ (otherwise $\theta(c_{ij-1}c_{ij}) = (0, 0)$ and $\theta(c_{ij-1}c_{ij})$ would not have order $n_{ij} = 2$).

We prove the sufficiency of the conditions by defining epimorphisms $\theta_q : \Lambda \to A_q$ for each prime q in the set $\{q_1, \ldots, q_\lambda\}$ of prime numbers dividing the order of A, and a surface-kernel epimorphism $\theta : \Lambda \to A$ as the direct product epimorphism

$$\theta : \Lambda \to A : g \mapsto \theta(g) = (\theta_{q_1}(g), \ldots, \theta_{q_\lambda}(g)).$$

For readability, we let $\mu_i = \mu_i(q)$ (see Section 3) in the definition of each homomorphism θ_q. Also, we assume that $\mu_i \leqslant \mu_{i+1}$; otherwise, there is a permutation (in general, different for each value of q) such that $\widehat{\mu}_i = \mu_{\tau(i)}$ and we replace x_i by $x_{\tau(i)}$ and μ_i by $\widehat{\mu}_i$ in the definition of $\theta_q(x_i)$ below, so that the order of $\theta(x_i)$ is m_i.

First, we define θ_2 (whenever $m+n>0$) for $k=0$ (the cases not showed here are defined alike) and $k>0$, and then we define θ_q for $q \neq 2$. Let $\Sigma = \sum_i 2^{\alpha_{m+n}-\mu_i}$ (note that, if Σ is odd, then $2^{\alpha_{m+n}}$ divides an odd number of proper periods; recall condition *(iv)* in that case).

a) $k = r_2 = 0$ and $n > 0$ (so $g > m+n$).

$$\theta_2(d_1) = (1, -1, \ldots, -1), \theta_2(d_i) = (0, \overset{i-1}{\ldots}, 0, 1, 0, \ldots, 0), \quad i = 2, \ldots, m+n,$$
$$\theta_2(d_i) = (0, \ldots, 0), \quad i > m+n, \quad d_g \in \ker\theta_2 \cap (\Lambda - \Lambda^+).$$

b) $k = r_2 = n = 0$ (so $g > m+1$ or $g = m+1$ is odd).

$$\theta_2(d_1) = (-1, \ldots, -1), \theta_2(d_i) = (0, \overset{i-2}{\ldots}, 0, 1, 0, \ldots, 0), \quad i = 2, \ldots, m+1,$$
$$\theta_2(d_i) = (0, \ldots, 0), \quad i > m+1, \text{ either } d_g \text{ or } d_1 \cdots d_g \in \ker\theta_2 \cap (\Lambda - \Lambda^+).$$

c) $k = 0$, $r_2 > 0$ and $m+n \leqslant g-2$.

$$\theta_2(x_i) = (0, \ldots, 0, 2^{\alpha_{m+n}-\mu_i}), \quad i = 1, \ldots, r,$$
$$\theta_2(d_i) = (0, \overset{i-1}{\ldots}, 0, 1, 0, \ldots, 0), \quad i = 1, \ldots, m+n,$$
$$\theta_2(d_{m+n+1}) = (-1, \ldots, -1, -1 - \Sigma/2),$$
$$\theta_2(d_i) = (0, \ldots, 0), \quad i \geqslant m+n+2,$$
$$d_g \in \ker\theta_2 \cap (\Lambda - \Lambda^+).$$

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d) $k=0$, $r_2>0$, $m+n>g-2$ and $g>1$ is odd.

$$\theta_2(x_i)=(0,\dots,0,2^{\alpha_{m+n}-\mu_i}),\quad i=1,\dots,r-m-n+g-2,$$
$$\theta_2(x_i)=(0,\dots,0,1,0,\overset{g+r-i-3}{\dots},0,2^{\alpha_{m+n}-\mu_i}),$$
$$i=r-m-n+g-1,\dots,r-2,$$
$$\theta_2(d_i)=(0,\dots,0,1,0,\overset{g-i-1}{\dots},0),\quad i=1,\dots,g-2.$$

We consider the following cases:

d.1) Σ is singly even.

$$\theta_2(x_{r-1})=(0,\dots,0,1,0,\overset{g-2}{\dots},0,2^{\alpha_{m+n}-\mu_{r-1}}),$$
$$\theta_2(x_r)=(-1,\dots,-1,0,\overset{g-2}{\dots},0,2^{\alpha_{m+n}-\mu_r}),$$
$$\theta_2(d_{g-1})=(0,\dots,0,-1,\overset{g-2}{\dots},-1,-\Sigma/2),$$
$$\theta_2(d_g)=(0,\dots,0),$$
$$d_g\in\ker\theta_2\cap(\Lambda-\Lambda^+).$$

d.2) Σ is doubly even.

$$\theta_2(x_{r-1})=(0,\dots,0,1,0,\overset{g-2}{\dots},0,2^{\alpha_{m+n}-\mu_{r-1}}),$$
$$\theta_2(x_r)=(-1,\dots,-1,0,\overset{g-2}{\dots},0,2^{\alpha_{m+n}-\mu_r}),$$
$$\theta_2(d_{g-1})=(0,\dots,0,1),$$
$$\theta_2(d_g)=(0,\dots,0,-1,\overset{g-2}{\dots},-1,-1-\Sigma/2),$$
$$d_1\cdots d_{g-2}\cdot d_{g-1}^{1+\Sigma/2}\cdot d_g\in\ker\theta_2\cap(\Lambda-\Lambda^+).$$

d.3) Σ is odd and $\Sigma-1$ is singly even.

$$\theta_2(x_{r-1})=(0,\dots,0,1,0,\overset{g-3}{\dots},0,1,1),$$
$$\theta_2(x_r)=(-1,\dots,-1,0,\overset{g-3}{\dots},0,-1,0),$$
$$\theta_2(d_{g-1})=(0,\dots,0,-1,\overset{g-2}{\dots},-1,-(\Sigma-1)/2),$$
$$\theta_2(d_g)=(0,\dots,0),$$
$$d_g\in\ker\theta_2\cap(\Lambda-\Lambda^+)$$

d.4) Σ is odd and $\Sigma-1$ is doubly even.

$$\theta_2(x_{r-1})=(0,\dots,0,1,0,\overset{g-3}{\dots},0,1,1),$$
$$\theta_2(x_r)=(-1,\dots,-1,0,\overset{g-3}{\dots},0,-1,0),$$
$$\theta_2(d_{g-1})=(0,\dots,0,1),$$
$$\theta_2(d_g)=(0,\dots,0,-1,\overset{g-2}{\dots},-1,-1-(\Sigma-1)/2),$$
$$d_1\cdots d_{g-2}\cdot d_{g-1}^{1+(\Sigma-1)/2}\cdot d_g\in\ker\theta_2\cap(\Lambda-\Lambda^+).$$

When $k>0$, we define $\theta_2(c_{10})=(0,\dots,0,2^{\alpha_{m+n}-1})$ and consider the sequence

$$(c_{20},\dots,c_{\varepsilon 0},c_{\varepsilon+1,0},\dots,c_{\varepsilon+1,s_{\varepsilon+1}-1},\dots,c_{k0},\dots,c_{k,s_k-1})$$

containing $S-1$ elements—we rule out the elements c_{10} and c_{is_i} for $i>\varepsilon$. We subsequently assign $(2^{\alpha_1-1},0,\dots,0)$, $(0,2^{\alpha_2-1},0,\dots,0)$, etc—beginning again with $(2^{\alpha_1-1},0,\dots,0)$ if there are more than $m+n$ elements in the sequence. Also, we define $\theta_2(c_{is_i})=\theta_2(c_{i0})$ for $i>\varepsilon$.

We consider the sequence $a_g,b_g\dots,a_1,b_1,e_{k-1},\dots,e_1,x_r,\dots,x_1$ or $d_g,\dots,d_1,e_{k-1},\dots,e_1,$ $x_r,\dots,x_1$ according to the signature of Λ and subsequently assign $(0,\dots,0,1),(0,\dots,0,1,0),\dots,$ $(0,\overset{\min(n,S-1)}{\dots\dots},0,1,0,\dots,0)$ to its first $m+n-\min(n,S-1)$ elements and $(0,\dots,0)$ to the rest, but we

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let the last component be $(\ldots, 2^{\alpha_{m+n}-\mu_i})$ in case of an elliptic generator x_i; also, let $\theta_2(d_g) = (0, \ldots, 0)$ if $m = 0$. Finally, let

$$\theta_2(e_k) = (0, \overset{\min(n,S-1)}{\ldots\ldots}, 0, -1, \ldots, -1, -2, \overset{\eta g-1}{\ldots\ldots}, -2, \delta - \Sigma),$$

where $\delta = -1$ if $g = 0$, $\delta = 0$ if $g > 0$ and sign(Λ) is '+', and $\delta = -2$ otherwise. We observe that d_g (if $m = 0$), $c_{10}d_g^p$ (for some even integer p) or $c_{10}a_g^p$ or $c_{10}e_{k-1}^p$ (for some integer p) belong to $\ker\theta_2 \cap (\Lambda - \Lambda^+)$.

With the above definition for θ_2, we can easily find, for each component of A_2, an element in Λ whose image is a generator (an odd number) of that component and null for the other components, so that θ_2 is onto.

Now, let $q \neq 2$ be a prime number dividing $|A|$ and $A_q \approx \mathbb{Z}_{q^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{q^{\alpha_t}}$ be the q-Sylow subgroup of A (some factors of A_q may be trivial, i.e., $\alpha_1 = \cdots = \alpha_{t'} = 0$ for some $t' < t$). Let

$$\gamma_1 = e_1, \ \ldots, \ \gamma_{k-1} = e_{k-1}, \ \gamma_k = a_1, \ \gamma_{k+1} = b_1, \ \ldots, \ \gamma_{w-1} = a_g, \ \gamma_w = b_g,$$

or

$$\gamma_1 = e_1, \ldots, \gamma_{k-1} = e_{k-1}, \gamma_k = d_1, \ldots, \gamma_w = d_g,$$

according to the sign of the signature of Λ. We define θ_q as follows (note that $r + w \geqslant t$ by condition *(ii)*):

$$\theta_q(c_{i0}) = (0, \ldots, 0), \qquad i = 1, \ldots, k,$$

$$\theta_q(x_i) = (0, \ldots, 0, q^{\alpha_t-\mu_i}), \qquad i = \begin{cases} 1, \ldots, r-t+w & \text{if } t > w, \\ 1, \ldots, r & \text{if } t \leqslant w, \end{cases}$$

$$\theta_q(x_i) = (0, \overset{t-r-w+i-1}{\ldots\ldots\ldots}, 0, 1, 0, \overset{r+w-i-1}{\ldots\ldots\ldots}, 0, q^{\alpha_t-\mu_i}),$$
$$i = r-t+w+1, \ldots, r \quad \text{if } t > w,$$

$$\theta_q(\gamma_i) = (0, \ldots, 0), \qquad i = 1, \ldots, w-t \quad \text{if } t < w,$$

$$\theta_q(\gamma_i) = (0, \overset{t-w+i-1}{\ldots\ldots}, 0, 1, 0, \overset{w-i}{\ldots}, 0), \quad i = \begin{cases} 1, \ldots, w & \text{if } t \geqslant w, \\ w-t+1, \ldots, w & \text{if } t < w, \end{cases}$$

$$\theta_q(e_k) = \begin{cases} (-1, \ldots, -1, \delta, \overset{\eta g-1}{\ldots\ldots}, \delta, -u+\delta) & \text{if } t > \eta g > 0, \\ (\delta, \ldots, \delta, -u+\delta) & \text{if } t \leqslant \eta g \text{ or } g = 0, \end{cases}$$

where $u = \sum_{i=1}^{r} q^{\alpha_t-\mu_i}$. The homomorphism θ_q is onto by condition *(ii)*.

The homomorphism θ is also onto. For, consider an elementary divisor $q^{\alpha_i(q)}$ of A and the generator $h = (0, \ldots, 0, 1, 0, \ldots, 0)$ of some cyclic factor

$$H = \{0\} \oplus \cdots \oplus \{0\} \oplus \mathbb{Z}_{q^{\alpha_i(q)}} \oplus \{0\} \oplus \cdots \oplus \{0\}$$

of A_q. Then, $h = \theta_q(g)$ for some $g \in \Lambda$. Obviously, $\theta(g)$ may have nontrivial components in some other primary component $A_{q'}$ for a prime $q' \neq q$, but not the element $\frac{\exp A}{q^{\alpha_t(q)}}\,\theta(g)$ since $\frac{\exp A}{q^{\alpha_t(q)}}\,\theta_{q'}(g)$ is trivial whenever $q' \neq q$. Moreover, $\frac{\exp A}{q^{\alpha_t(q)}}\,\theta(g)$ has order $q^{\alpha_i(q)}$ since $\gcd(q, \exp A/q^{\alpha_t(q)}) = 1$. Hence, $\langle\theta(g^{\exp A/q^{\alpha_t(q)}})\rangle = H$.

It also preserves the long relation and the order of x_i, c_{ij} and $c_{ij-1}c_{ij}$. We have emphasized an element in $\ker\theta_2 \cap (\Lambda - \Lambda^+)$, say h, for each case in the definition of θ_2. Since v_t is odd (let $v_t = 1$ if $t = 0$), the element h^{v_t} belongs to $\ker\theta \cap (\Lambda - \Lambda^+)$ and thus $\theta(\Lambda^+) = A$. □

When A is cyclic, Theorem 4.3 reads as follows.

Theorem 4.4. *Let Λ be a proper NEC group with signature $(g; \pm; [m_1, \ldots, m_r]; \{(-)^k\})$ and $N = 2^\alpha q > 1$ be an integer, q odd. Let $w = \eta g + k - 1$, $\eta = 2$ if '+' is the signature sign of Λ and $\eta = 1$ otherwise. Then, there exists a nonorientable unbordered surface-kernel epimorphism $\Lambda \to \mathbb{Z}_N$ if and only if the following conditions hold:*

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i. $m_i|N$ *for all i.*
ii. *If* $N > 2$ *and* $w = 0$*, then every elementary divisor of* $\mathbb{Z}_N$ *divides some proper period.*
iii. *If* $k = 0$*,* $g = 2$ *and some proper period is even, then* 2^α *divides some proper period.*
iv. *If* $k = 0$ *and N is even, then* 2^α *divides an even number of proper periods.*
v. *If* $k = 0$ *and no proper period is even, then* $g > 1$ *if N is singly even and* $g > 2$ *if N is doubly even*
vi. *If N is odd, then* $k = 0$*.*

This result gathers theorems 3.5, 3.6 and 3.7 in [3], yet it rounds out Theorem 3.6 with a complete set of necessary and sufficient conditions.

5. Symmetric cross-cap number of an Abelian group

In this section, we develop a new way of achieving the expression for the symmetric cross-cap number of an Abelian group by means of Theorem 4.3.

The symmetric cross-cap number of a cyclic group was obtained by Bujalance in [3] from the above mentioned theorems 3.5, 3.6 and 3.7, so that, by the same token, it can also be obtained from Theorem 4.4 herein:

Theorem 5.1.

$$\tilde{\sigma}(\mathbb{Z}_N) = \begin{cases} 3 & \text{if } N = 2, \\ (q-1)(N/q-1)+1 & \text{if } N \text{ is not prime and } q^2 \nmid N, \\ (q-1)(N/q-1)+q & \text{otherwise,} \end{cases} \tag{5.1}$$

where q is the smallest prime divisor of N.

Remark 5.2. *The symmetric cross-cap number of the groups* $\mathbb{Z}_2^2$*,* $\mathbb{Z}_2^3$ *and* $\mathbb{Z}_2 \oplus \mathbb{Z}_{2u}$ $(u > 1)$ *was obtained in [9, Proposition 6.4]:*

$$\tilde{\sigma}(\mathbb{Z}_2^2) = 3, \quad \tilde{\sigma}(\mathbb{Z}_2^3) = 4, \quad \tilde{\sigma}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2u}) = 2u, \tag{5.2}$$

By Theorem 4.3, we can obtain signatures of NEC groups attaining such topological genera: $(0; +; [-]; \{(2, 2, 2)\})$, $(0; +; [-]; \{(2, 2, 2, 2)\})$ *and* $(0; +; [2, 2u]; \{(-)\})$*, respectively, fulfill conditions of Theorem 4.3 and it can be proved that any other signature fulfilling such conditions yields a greater or equal topological genus.*

Theorem 5.3. *[9, propositions 6.1, 6.2 and 6.3] The symmetric cross-cap number of a noncyclic Abelian group A different to* $\mathbb{Z}_2^2$*,* $\mathbb{Z}_2^3$ *and* $\mathbb{Z}_2 \oplus \mathbb{Z}_{2u}$ $(u \geqslant 2)$ *is* $\tilde{\sigma}(A) = 2 + |A|\,\mu^*$*, where* μ^* *is, with the notation of Theorem 4.3,*

$$\begin{array}{ll} a)\ t - 1 - \dfrac{1}{v_1} - \cdots - \dfrac{1}{v_{t-n}} & \text{if } n \leqslant m, \\ b)\ t - 1 - \dfrac{1}{v_1} - \cdots - \dfrac{1}{v_{t-\epsilon}} + \dfrac{\delta}{2v_{t-\epsilon}} & \text{if } 2m < m + n < 2t, \\ c)\ t - 1 & \text{if } 2m < m + n = 2t, \\ d)\ t - 1 + \dfrac{m + n - 2t + 1}{4} & \text{if } 2t < m + n, \end{array}$$

$\epsilon = (m + n - \delta)/2$*,* $\delta = 1$ *if* $m + n$ *is odd and* $\delta = 0$ *otherwise.*

Proof. We first tackle odd order Abelian groups, so that $n = m = 0$ and we only consider case a), for which $\mu^* = -1 + \sum_{i=1}^{t} \left(1 - \frac{1}{v_i}\right)$. For, note that the signature $(1; - ; [v_1, \ldots, v_t]; \{-\})$ defines an NEC group Λ^* and fulfills conditions of Theorem 4.3. Now we prove that, if Λ is another NEC group with signature $(g; - ; [m_1, \ldots, m_r]; \{-\})$ fulfilling conditions of Theorem 4.3, then $\mu(\Lambda^*) \leqslant \mu(\Lambda) = g - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)$.

If $t \leqslant g-1$, then $\mu(\Lambda^*) < -1 + t < -1 + t + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) \leqslant g - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) = \mu(\Lambda)$. Suppose now that $t > g-1$. By (2.4), we may assume that $m_1 | \cdots | m_r$, since $\mu(\widehat{\Lambda}) \leqslant \mu(\Lambda)$ if $\widehat{\Lambda}$ is an NEC group with signature $(g; - ;[\widehat{m}_{r-\widehat{r}+1}, \ldots, \widehat{m}_r]; \{-\})$. By condition *(ii)* of Theorem 4.3, $v_1 \mid m_{r+g-t}$, …, $v_{t-g+1} \mid m_r$, hence $\sum_{i=1}^{t-g+1} \left(1 - \frac{1}{v_i}\right) \leqslant \sum_{i=r-t+g}^{r} \left(1 - \frac{1}{m_i}\right) \leqslant \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)$ (recall that $r - t + g \geqslant 1$ by condition *(ii)* of Theorem 4.3). It follows that $\mu(\Lambda^*) \leqslant \mu(\Lambda)$ if $g = 1$, and also if $g > 1$ since $\sum_{i=t-g+2}^{t} \left(1 - \frac{1}{v_i}\right) < \sum_{i=t-g+2}^{t} 1 = g - 1$.

Now, we consider Abelian groups of even order. Let Λ^* be an NEC group with signature

a) $(0; + ; [v_1, \ldots, v_{t-n}]; \{(-)^{n+1}\})$

b) $(0; + ; [v_1, \ldots, v_{t-\frac{m+n-\delta}{2}-1}, (\delta+1)v_{t-\frac{m+n-\delta}{2}}]; \{(-)^{\frac{m+n-\delta}{2}+1}\})$

c) $(0; + ; [-]; \{(-)^{t+1}\})$

d) $(0; + ; [-]; \{(-)^t, (2, \overset{m+n-2t+1}{\ldots}, 2)\})$

respectively. This NEC group fulfills conditions of Theorem 4.3 and $\mu(\Lambda^*) = \mu^*$. Now, we prove that it follows from Theorem 4.3 that $\mu^* \leqslant \mu(\Lambda)$ for any other NEC group Λ fulfilling such conditions.

We can assume that Λ has signature

$$(0; + ; [m_1, \ldots, m_r]; \{(-)^{k-1}, (2, \overset{s}{\ldots}, 2)\}), \tag{5.3}$$

where $m_i | m_{i+1}$, $k > 0$, $s \neq 1$, s is even if $m + n = 1$ and $s = 0$ if $m + n = 0$. For, the signature $(g; \pm ;[\widehat{m}_{r-\widehat{r}+1}, \ldots, \widehat{m}_r]; \{(-)^{\varepsilon}, (2, \overset{s_{\varepsilon+1}}{\ldots}, 2), \ldots, (2, \overset{s_k}{\ldots}, 2)\})$ defines an NEC group $\widehat{\Lambda}$ that fulfills conditions of Theorem 4.3 and, by (2.4), $\mu(\widehat{\Lambda}) \leqslant \mu(\Lambda)$ if the signature of Λ is $(g; \pm ; [m_1, \ldots, m_r]; \{(-)^{\varepsilon}, (2, \overset{s_{\varepsilon+1}}{\ldots}, 2), \ldots, (2, \overset{s_k}{\ldots}, 2)\})$ also fulfilling such conditions (we can easily see that also $\mu(\widehat{\Lambda}) > 0$ in that case). Therefore, we can assume that $m_1 | \cdots | m_r$. Moreover, consider an NEC group Λ_o with signature $(0; + ; [m_1, \ldots, m_r]; \{(-)^{\eta g+k-1}, (2, \overset{s}{\ldots}, 2)\})$, where $s = \sum_{i=1}^{k} s_i$ and η, g and k are the parameters of Λ (note that Λ_o has $\eta g + k$ period-cycles, and thus it is a proper NEC group: $\eta g + k - 1 \geqslant 0$ since Λ is proper). It is straightforward to check that $\mu(\Lambda_o) = \mu(\Lambda)$ and Λ_o fulfills the conditions of Theorem 4.3 if Λ does.

So assume that Λ has signature (5.3) and let $w = k - 1$, $\mu = \mu(\Lambda) = w - 1 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{s}{4}$ and $w^* = n$, $(m + n - \delta)/2$, t and t for cases a), b), c) and d) in the definition of μ^* in Theorem 5.3, respectively.

If $t \leqslant w$, then $\mu^* \leqslant t - 1 \leqslant w - 1 \leqslant \mu$ for cases a), b) and c). For case d), if $m + n - 2w - s + 1 \leqslant 0$, then $\mu \geqslant w - 1 + \frac{s}{4} \geqslant w - 1 + \frac{m+n-2w+1}{4} = \mu^* + \frac{w-t}{2} \geqslant \mu^*$, and, if $m + n - 2w - s + 1 > 0$, then $\sum (1 - 1/m_i) \geqslant (m + n - 2w - s + 1)/2$ by condition *(vi)* and $\mu \geqslant w - 1 + \frac{m+n-2w-s+1}{2} + \frac{s}{4} = \mu^* + \frac{m+n-2t-s+1}{4} \geqslant \mu^*$.

Otherwise, $t > w$. If $t > w > w^*$ (that discards cases c) and d)), then $\mu^* < w^* - 1 + \sum_{i=1}^{t-w} \left(1 - \frac{1}{v_i}\right) + \sum_{i=t-w+1}^{t-w^*} 1 = w - 1 + \sum_{i=1}^{t-w} \left(1 - \frac{1}{v_i}\right) \leqslant \mu$ since $v_1 | m_{r-t+w+1}, \ldots, v_{t-w} | m_r$ by condition *(ii)*.

If $t > w = w^*$ (we discard cases c) and d) as well), then, in cases a) and b) with $m + n$ even ($\delta = 0$), $\mu^* = w - 1 + \sum_{i=1}^{t-w} \left(1 - \frac{1}{v_i}\right) \leqslant \mu$ by condition *(ii)* as above. In case b) with $m + n$ odd ($\delta = 1$), $1 - \frac{1}{2v_{t-w}} = 1 - \frac{1}{v_{t-w}} + \frac{1}{2v_{t-w}} \leqslant 1 - \frac{1}{m_r} + \frac{s}{4}$ provided that $s \geqslant 2$, and, if $s = 0$, then there is, at least, $m + n - 2w = m + n - (m + n - 1) = 1$ even proper period by condition *(vi)* (note that $S - 1 = w$ if $s = 0$) and thus $2v_{t-w} | m_r$ since v_{t-w} is odd (note that $t - w \leqslant t - m$ since, in case b), $w^* \geqslant m$). Therefore, $\mu^* \leqslant \mu$ either if $s > 0$ or $s = 0$.

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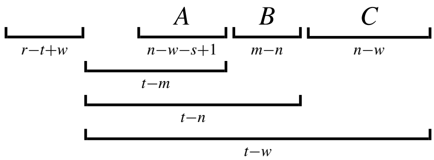
Finally, suppose that $t > w$ and $w^* > w$. We proof that $\mu^* < \mu$ for case a) (cases b), c) and d) are established similarly). To lessen clutter, we rename the first $t - w$ integers v_i by defining $v'_1 = \cdots = v'_{r-t+w} = 1$, $v'_{r-t+w+1} = v_1, \ldots, v'_r = v_{t-w}$. Hence, $v'_i | m_i$ for all i by condition *(ii)*, and thus $1 - \frac{1}{v'_i} \leqslant 1 - \frac{1}{m_i}$.

For case a) we have $t \geqslant m \geqslant n = w^* > w$. We consider the following partition of $\{1, \ldots, r\}$:



($\#A < t - m$ in the figure, but $\#A$ may be greater than $t - m$). Let $A = \varnothing$ if $n - w - s + 1 \leqslant 0$. In case that $s = 0$, let $\#A = n - w$.

Note that $4 | v'_i$ if $i \in B \cup C$ and v'_i is odd otherwise, and $\mu^* = n - 1 + \sum_{i \notin C} (1 - \frac{1}{v'_i})$.

If $w + s > n$ (hence $s \geqslant 2$ since $w < n$), then $\sum_C \left(1 - \frac{1}{m_i}\right) + \frac{s}{4} > \frac{3(n-w)}{4} + \frac{n-w}{4} = n - w$ since $4 | m_i$ for $i \in C$, and thus $\mu > w - 1 + \sum_{i \notin C} (1 - \frac{1}{m_i}) + n - w \geqslant \mu$.

If $w + s - 1 < n$ and $s \geqslant 2$, then $m + n - 2w - s + 1 > 0$ since $m \geqslant n > w$, and, by condition *(vi)*, m_i is even if $i \in A \cup B \cup C$ and thus $r \geqslant m + n - 2w - s + 1 = \#\{A \cup B \cup C\}$. Let $C = C_1 \cup C_2$, with $C_1 = \{r - n + w + 1, \ldots, r - s + 1\}$ and $C_2 = \{r - s + 2, \ldots, r\}$, $\#C_1 = \#A$, $\#C_2 = s - 1$. For $i \in A$, v'_i is odd and m_i is even, hence $2v'_i | m_i$. Also, $4 | m_j$ for $j \in C_1$, hence $4v'_i | m_j$ if $i \in A$. Then $\sum_A \left(1 - \frac{1}{m_i}\right) + \sum_{C_1} \left(1 - \frac{1}{m_i}\right) \geqslant \sum_A \left(1 - \frac{1}{2v'_i}\right) + \sum_A \left(1 - \frac{1}{4v'_i}\right) = n - w - s + 1 + \sum_A \left(1 - \frac{1}{2v'_i} - \frac{1}{4v'_i}\right) > n - w - s + 1 + \sum_A \left(1 - \frac{1}{v'_i}\right)$. As $4 | m_i$ for $i \in C_2$ and $\#C_2 = s - 1$, $\sum_{C_2} \left(1 - \frac{1}{m_i}\right) + \frac{s}{4} \geqslant (s-1)\left(1 - \frac{1}{4}\right) + \frac{s}{4} = s - 1 + \frac{1}{4} \geqslant s - 1$.

If $w + s - 1 < n$ and $s = 0$, then $C_2 = \varnothing$, $\#A = \#C = n - w$ and $\sum_{A \cup C} \left(1 - \frac{1}{m_i}\right) > n - w + \sum_A \left(1 - \frac{1}{v'_i}\right)$. It follows that $\mu^* \leqslant \mu$ if either $s \geqslant 2$ or $s = 0$. □

6. Least symmetric cross-cap number of Abelian groups of a given order

An easy consequence of the results of the previous section is the following: for a given integer $N > 1$, we find the least topological genus of any nonorientable Riemann surface of topological genus $g > 2$ on which some Abelian group of order N acts. For ease and by abuse of notation, we denote it by $\widetilde{\sigma}(N)$ (it is not the symmetric cross-cap number of a group but the least symmetric cross-cap number attained in a family of groups).

Theorem 6.1. *The least symmetric cross-cap number of Abelian groups of order $N > 1$ is*

$$\widetilde{\sigma}(N) = \begin{cases} 3 & \text{if } N \leqslant 4, \\ 6 & \text{if } N = 16, \\ N & \text{if } N > 4 \text{ is prime, and} \\ (q-1)(N/q - 1) + 1 & \text{otherwise,} \end{cases}$$

where q is the smallest prime divisor of N.

Remark 6.2. *For a proof of Theorem 6.1, we refer the reader to that of Theorem 2 in [14], both proofs are exactly alike since the cross-cap number of an Abelian group A relates with its real genus straightforwardly: $\widetilde{\sigma}(A) = \rho(A) + 1$.* □

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Remark 6.3. *The Abelian group of order N acting on genus $\widetilde{\sigma}(N)$ is unique ($\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $N = 16$ and either $\mathbb{Z}_N$ or $\mathbb{Z}_q \oplus \mathbb{Z}_{N/q}$ otherwise) unless $N = 4$ or 8.* □

7. Maximum order problem

The maximum order problem for Abelian groups acting on Riemann surfaces of genus $g > 1$ was solved in [2, Corollary 9.6], and in [8, Section 4.5] for Abelian groups acting on compact bordered Klein surfaces of algebraic genus $p > 1$. We now obtain the corresponding result for compact nonorientable Riemann surfaces, which expands that of Bujalance for cyclic groups [3, Corollary 4.4]. It follows easily from theorems 4.3 and 6.1.

Corollary 7.1. *The largest order of an Abelian group acting on a compact nonorientable Riemann surface of topological genus $g > 2$ is 16 if $g = 6$ and $2g$ otherwise.*

Proof. If $g = 6$, then the largest order is 16 since, by Theorem 6.1, $\widetilde{\sigma}(16) = 6$ and $\widetilde{\sigma}(N) \geqslant N/2 > 6$ if $N > 16$ (note that $(q-1)(N/q-1)+1 = N/2 + (q-2)(N-2q)/2q \geqslant N/2$ if $q \geqslant 2$).

If $g = 8$, then the largest order is 16 since $\widetilde{\sigma}(\mathbb{Z}_2 \oplus \mathbb{Z}_8) = 8$ and $\widetilde{\sigma}(N) \geqslant N/2 > 8$ if $N > 16$.

Otherwise, $g \neq 6$ or 8. In that case, $\widetilde{\sigma}(2g) = g$ and thus the largest order is, at least, $2g$. But no Abelian group of order greater than $2g$ acts on compact nonorientable Riemann surfaces of topological genus g. For, consider an Abelian group A of order N that acts on genus $g \neq 6$ or 8, so $g \geqslant \widetilde{\sigma}(N)$. If $N \neq 16$, then $\widetilde{\sigma}(N) \geqslant N/2$ by Theorem 6.1 and thus $N \leqslant 2g$. Now, suppose that $N = 16$, hence $g \geqslant \widetilde{\sigma}(16) = 6$. If $g > 8$, then $16 < 2g$. Finally, no Abelian group of order 16 acts on genus 7; indeed, $\widetilde{\sigma}(A) > 7$ for any Abelian group A of order 16 other than $\mathbb{Z}_2^4$ and, if $A \approx \mathbb{Z}_2^4$ acts on genus 7, then $5/4 = 4w - 4 + 2r + s$ for some nonnegative integers w, r, s by Theorem 4.3 and the Riemann-Hurwitz formula (2.3), which is not possible since $4w - 4 + 2r + s \in \mathbb{Z}$. □

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