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ON STRONG INTEGRAL SUMMABILITY

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Strong summability has been studied by many authors, including Borwein and Cass (1, 2) who have studied sequence-to-sequence transforms. Here we study integral transforms; and due to the lack of a limitation theorem for such transforms, some results do not follow directly as in the sequence cases. The strong methods defined here can be applied to construct known and new strong summability methods. We do not give details here, but refer the reader to (3) with the suggestion that the natural scale operator method be used for Q and the named method for P. For example, with the Cesàro methods, let $P = (C, \kappa, \delta)$ and $Q = (C, \delta)$ to obtain $[C, \kappa]_\lambda^\delta = [(C, \delta, \kappa), (C, \kappa)]_\lambda$.

Suppose throughout that $f(x)$ exists for $x \geqq 0$ and is integrable L over every finite interval. Suppose further that the transform

$$P(f; x) = \int_0^\infty p(x, t) f(t) dt$$

exists for all $x > 0$. If $P(f; x) \to \sigma$ as $x \to \infty$, then we shall say that f is limitable to σ by the method P and write $f(x) \to \sigma(P)$. This general method includes many well-known integral summability methods such as the Cesàro and Hausdorff methods.

We shall now define strong integral summability methods.

Definition. If $P(|Q(f; .) - \sigma|^\lambda; x)$ exists for $x > 0$ and tends to zero as $x \to \infty$, then we shall say that f is strongly limitable P, Q with index λ, to σ, and write $f(x) \to \sigma[P, Q]_\lambda$. Note that this is the same as

$$\int_0^\infty p(x, t) |Q(f; t) - \sigma|^\lambda dt \to 0 \quad \text{as} \quad x \to \infty.$$

Here, Q can be any summability method, either of sequence-to-function or of function-to-function type.

In what follows we shall only consider methods P in which $p(x, t) \geqq 0$ for all x and t concerned. When we require more than one such method, we shall write them as P_1 and P_2. We shall also suppose throughout that $\lambda > 0$. Given any two summability methods (of the same type) we write $P \supseteq Q$ whenever any sequence or function (as the case may be) summable by Q to σ is also summable by P to σ.

In the rest of this paper, we prove some inclusion relationships that involve variations in the parameters P, Q and λ. We prove a necessary and sufficient condition for strong functional summability in terms of ordinary summability plus a

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condition, in a way that reflects the usual results of this type. We also prove some consistency theorems and define a set of numbers which includes those values which may be attained as strong limits of functions. All these results are comparable with the results of Borwein and Cass (2) for sequence-to-sequence transformations.

Theorem 1. (i) *If P and Q are regular, then so is* $[P, Q]_\lambda$;
(ii) *If* $P_1 \supseteq P_2$, *then* $[P_1, Q]_\lambda \supseteq [P_2, Q]_\lambda$;
(iii) $[P, Q]_\lambda$ *is linear; that is, if* $f(x) \to \sigma[P, Q]_\lambda$, $g(x) \to \tau[P, Q]_\lambda$ *and α and β are constants, then* $\alpha f(x) + \beta g(x) \to (\alpha\sigma + \beta\tau)[P, Q]_\lambda$.

These all follow directly from the definitions.

Theorem 2. *If P_2 is regular, if P_1 is zero-preserving and $\lambda \geqq 1$, then* $[P_1, P_2Q]_\lambda \supseteq [P_1P_2, Q]_\lambda$.

Proof.

$$P_1(|P_2Q(f;.)-\sigma|^\lambda; y) = \int_0^\infty p_1(y,t)\left|\int_0^\infty p_2(t,x)Q(f;x)dx - \sigma\right|^\lambda dt$$
$$= \int_0^\infty p_1(y,t)\left|\int_0^\infty p_2(t,x)\{Q(f;x)-\sigma\}dx + h(t)\right|^\lambda dt$$

where, since P_2 is regular, $h(t) \to 0$ as $t \to \infty$,

$$\leqq 2^\lambda \int_0^\infty p_1(y,t)\left|\int_0^\infty p_2(t,x)\{Q(f;x)-\sigma\}dx\right|^\lambda dt$$
$$+ 2^\lambda \int_0^\infty p_1(y,t)|h(t)|^\lambda dt$$
$$= 2^\lambda(I_1 + I_2).$$

Since P_1 is zero-preserving, we have that $I_2 \to 0$ as $y \to \infty$. And by Hölder's inequality,

$$I_1 \leqq \int_0^\infty p_1(y,t)dt \int_0^\infty p_2(t,x)|Q(f;x)-\sigma|^\lambda dx \left(\int_0^\infty p_2(t,x)dx\right)^{\lambda-1}$$
$$= O\left(\int_0^\infty p_1(y,t)dt \int_0^\infty p_2(t,x)|Q(f;x)-\sigma|^\lambda dx\right)$$
$$= O(P_1P_2(|Q(f;.)-\sigma|^\lambda; y)) \to 0 \quad \text{as} \quad y \to \infty.$$

Note that $\int_0^\infty p_2(t,x)dx$ is uniformly bounded is a consequence of the regularity of P_2.

Theorem 3. *If $\int_0^\infty p(x,t)dt < M$ for all $x \geqq 0$ and $\lambda > \mu > 0$, then* $[P, Q]_\mu \supseteq [P, Q]_\lambda$.

Proof. This result follows since

$$P(|Q(f;.)-\sigma|^\mu; x) = \int_0^\infty p(x,t)|Q(f;t)-\sigma|^\mu dt$$
$$\leqq \left(\int_0^\infty p(x,t)|Q(f;t)-\sigma|^\lambda dt\right)^{\mu/\lambda}\left(\int_0^\infty p(x,t)dt\right)^{(\lambda-\mu)/\lambda}$$

Corollary. *If P is regular and $\lambda > \mu > 0$, then $[P, Q]_\mu \supseteq [P, Q]_\lambda$.*

Theorem 4. *If P is zero-preserving, then $[P, Q]_\lambda \supseteq Q$.*

Proof. We have that $Q(f; x) \to \sigma$. Thus $|Q(f; x) - \sigma|^\lambda \to 0$ and the result follows.

Theorem 5. *Suppose that P is regular and that $\lambda \geqq 1$. Then $f(x) \to \sigma[P, Q]_\lambda$ if and only if $f(x) \to \sigma(PQ)$ and $g(x) \to 0(P)$, where $g(x) = |Q(f; x) - PQ(f; x)|^\lambda$.*

Proof. Suppose that $f(x) \to \sigma[P, Q]_\lambda$. In view of Theorem 3, we may take $\lambda = 1$, and so we can easily deduce that $f(x) \to \sigma(PQ)$. Thus we get that $|PQ(f; x) - \sigma|^\lambda \to 0$ and *a fortiori* that $|PQ(f; x) - \sigma|^\lambda \to 0(P)$. Further, for $y > 0$, we have

$$\begin{aligned} P(g; y) &= \int_0^\infty p(y, t)|Q(f; t) - PQ(f; t)|^\lambda dt \\ &\leqq 2^\lambda \int_0^\infty p(y, t)|Q(f; t) - \sigma|^\lambda dt + 2^\lambda \int_0^\infty p(y, t)|PQ(f; t) - \sigma|^\lambda dt \\ &\to 0 \quad \text{as} \quad y \to \infty. \end{aligned}$$

For the converse, we note that the given conditions are equivalent to

$$\int_0^\infty p(y, t)|Q(f; t) - PQ(f; t)|^\lambda dt \to 0 \quad \text{as} \quad y \to \infty,$$

and

$$\int_0^\infty p(y, t)|PQ(f; t) - \sigma|^\lambda dt \to 0 \quad \text{as} \quad y \to \infty.$$

It easily follows that

$$\int_0^\infty p(y, t)|Q(f; t) - \sigma|^\lambda dt \to 0 \quad \text{as} \quad y \to \infty,$$

that is, that $f(x) \to \sigma[P, Q]_\lambda$.

We now prove some consistency theorems.

Theorem 6. *Suppose that* $\limsup_{x\to\infty} \int_0^\infty p(x, t)\,dt > 0$. *If $f(x) \to \sigma[P, Q]_\lambda$ and $f(x) \to \tau[P, Q]_\lambda$, then $\sigma = \tau$.*

Proof. This follows since

$$|\sigma - \tau|^\lambda \int_0^\infty p(x, t)\,dt \leqq 2^\lambda \int_0^\infty p(x, t)|Q(f; t) - \sigma|^\lambda dt + 2^\lambda \int_0^\infty p(x, t)|Q(f; t) - \tau|^\lambda dt.$$

We note the contrapositive, for it is of interest in its own right.

Corollary. *If $f(x) \to \sigma[P, Q]^\lambda$ and $f(x) \to \tau[P, Q]_\lambda$ where $\sigma \neq \tau$, then* $\lim_{x\to\infty} \int_0^\infty p(x, t)\,dt = 0$.

Theorem 7. *Suppose that* $\lim_{x\to\infty}\int_0^\infty p(x,t)dt=0$ *and that* σ *is any real number. If* $f(x)$ *is bounded on* $[0,\infty)$, *then* $f(x)\to\sigma[P,I]_\lambda$. (*Here, I denotes the identity transform*).

Proof.

$$\int_0^\infty p(x,t)|f(t)-\sigma|^\lambda dt \leqq \sup_{t\geqq 0}\{|f(t)-\sigma|^\lambda\}\int_0^\infty p(x,t)dt\to 0 \quad \text{as} \quad x\to\infty.$$

This theorem is not totally satisfactory because of the restriction on f, and in the next theorem we shall consider unbounded functions.

Theorem 8. *Suppose that* $f(x)$ *is unbounded as* $x\to\infty$, *that* $\lim_{x\to\infty}\int_0^\infty p(x,t)dt=0$, *and that* $\int_0^\infty p(x,\cdot t)dt$ *converges uniformly in* x, *for* $x\geqq 0$. *Then there is a positive monotonic increasing unbounded step-function,* $v(x)$, *such that* $f(v(x))\to 0[P,I]_\lambda$.

For the proof of Theorem 8, we shall first prove a lemma.

Lemma. *Suppose that* $F(x)$ *is positive and unbounded as* $x\to\infty$, *and that* $\int_0^\infty p(x,t)dt$ *converges uniformly in* x *for* $x\geqq 0$. *Then there is a positive monotonic increasing unbounded sequence* $\{T_n\}$ *with* $T_0=0$, *and a positive monotonic increasing unbounded step-function,* $v(x)$, *with steps at* T_n *such that* $\tau_n(x)=\int_{T_n}^\infty p(x,t)dt\leqq 1/(2^nF(v(T_n)))$ *for all* $x\geq 0$.

Proof. First we note that for any such step-function $v(x)$, it follows that $\tau_n(x)\to 0$ as $n\to\infty$ uniformly in x for $x\geqq 0$. Suppose then that $v(x)$ has been chosen for $x<T_{n-1}$ with the required properties. Let $T=T_{n-1}+n$ and choose $v(x)$ to be constant on $[T_{n-1},T)$.

Let $\tau(x)=\int_T^\infty p(x,t)dt$. If $\tau(x)\leqq 1/(2^nF(2v(T_{n-1})))$ then we may choose $T_n=T$ and $v(T_n)=2v(T_{n-1})$. If $\tau(x)>1/(2^nF(2v(T_{n-1})))$ then we may choose $T^*>T$, such that $v(x)$ is constant on $[T_{n-1},T^*)$ and such that $\tau^*(x)=\int_{T^*}^\infty p(x,t)dt\leqq 1/2^nF(2v(T_{n-1})))$. This is possible since $\tau(x)\to 0$ as $T\to\infty$ uniformly in x. Now we may choose $T_n=T^*$ and $v(T_n)=2v(T_{n-1})$.

Proof of Theorem 8. By the lemma, we have that

$$\tau_n(x)=\int_{T_n}^\infty p(x,t)dt\leqq 1/(2^n|f(v(T_n))|^\lambda). \tag{1}$$

Also,

$$\begin{aligned}\int_0^\infty p(x,t)|f(v(t))|^\lambda dt &= \sum_{n=0}^\infty\int_{T_n}^{T_{n+1}} p(x,t)|f(v(t))|^\lambda dt\\ &\leqq \sum_{n=0}^\infty |f(v(T_n))|^\lambda\int_{T_n}^{T_{n+1}} p(x,t)dt\\ &\leqq \sum_{n=0}^\infty \tau_n(x)|f(v(T_n))|^\lambda.\end{aligned}$$

By (1), this sum is uniformly convergent. Since $\lim_{x\to\infty}\tau_n(x)=0$, the result follows.

Corollary. *Suppose that $f(x)$ is unbounded as $x \to \infty$, that* $\lim_{x\to\infty} \int_0^\infty p(x,t)dt = 0$, *and that $\int_0^\infty p(x,t)dt$ is continuous in x for all positive x. Then there is a positive increasing unbounded step-function, $v(x)$, such that $f(v(x)) \to 0[P,I]_\lambda$.*

Proof. It is sufficient to show that if there is a positive monotonic increasing unbounded sequence $\{T_n\}$, such that $\tau_n(x) = \int_{T_n}^\infty p(x,t)dt$ is continuous in x in $[0,\infty)$ and for all n, and if $\lim_{x\to\infty} \int_0^\infty p(x,t)dt = 0$, then $\int_0^\infty p(x,t)dt$ converges uniformly in x for $x \geqq 0$.

Suppose then that $\epsilon > 0$ is given. Then there exists $X_0 = X_0(\epsilon)$ such that

$$\int_0^\infty p(x,t)dt < \epsilon \quad \text{for all } x \geqq X_0.$$

If we can now show that

$$\int_{T_0}^\infty p(x,t)dt < \epsilon \quad \text{for all } x \text{ in} \quad [0, X_0] \quad \text{and some} \quad T_0 = T_0(\epsilon) > 0 \tag{2}$$

then we would have that $\int_T^\infty p(x,t)dt < \epsilon$ for all $x \geqq 0$ and $T \geqq T_0$, and *a fortiori* that $\int_0^\infty p(x,t)dt$ is uniformly convergent in x for $x \geqq 0$.

Assume then that (2) is false: thus there exists sequences $\{x_n\} \subset [0, X_0]$ and $\{t_n\}$ such that $t_n \to \infty$, and such that

$$\int_{t_n}^\infty p(x,t)dt \geqq \epsilon \quad \text{for all } n. \tag{3}$$

We may also assume that $x_n \to x_0 \in [0, X_0]$.

Since $\int_0^\infty P(x_0,t)dt < \infty$, we have that $\int_{T_{n_0}}^\infty p(x_0,t)dt < \epsilon/2$ for some $n_0 = n_0(\epsilon)$, and since $\tau_{n_0}(x)$ is continuous, we further obtain that

$$\left|\int_{T_{n_0}}^\infty \{p(x_0,t) - p(x_n,t)\}dt\right| < \epsilon/2 \quad \text{for all } n \geqq n_0.$$

Hence

$$\begin{aligned}\left|\int_{T_{n_0}}^\infty p(x_n,t)dt\right| &\leqq \left|\int_{T_{n_0}}^\infty p(x_0,t)dt\right| + \left|\int_{T_{n_0}}^\infty \{p(x_n,t) - p(x_0,t)\}dt\right| \\ &< \epsilon \quad \text{for all } n \geqq n_0,\end{aligned}$$

which contradicts (3).

Finally, we prove a theorem which determines which numbers can occur as the generalised strong limits of functions.

Definition. We call σ a *limit value of $f(x)$ at infinity* (abbreviated LVI) if for every positive number ϵ and for every real y, there exists a real x, greater than y, such that $|f(x) - \sigma| < \epsilon$.

Theorem 9. *Suppose that $\int_0^Y p(x,t)dt \to 0$ as $x \to \infty$ for every finite Y, and that* $\limsup_{x\to\infty} \int_0^\infty p(x,t)dt = M > 0$.

(i) *If $f(x) \to \sigma[P,I]_\lambda$, then σ is an LVI of $f(x)$;*

(ii) *If $f(x)$ is bounded $[P,I]_\lambda$, then $f(x)$ has an LVI.*

Proof. (i) Suppose that σ is not an LVI of $f(x)$. Then there is a positive number ϵ

and a real y such that for all $t > y$, we have $|f(t)-\sigma| \geqq \epsilon$, Thus

$$\int_0^\infty p(x, t)|f(t)-\sigma|^\lambda dt \geqq \epsilon^\lambda \int_y^\infty p(x, t)dt$$

so that $\limsup_{x\to\infty} \int_0^\infty p(x, t)|f(t)-\sigma|^\lambda dt \geqq \epsilon^\lambda M > 0$, and this is a contradiction.

(ii) Suppose that $f(x)$ had no LVI. It follows that $|f(x)| \to \infty$ as $x \to \infty$, and further that $\int_0^\infty p(x, t)|f(t)|^\lambda dt$ is unbounded as $x \to \infty$, which again is a contradiction.

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