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Effective Actions of the Unitary Group on Complex Manifolds

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Abstract. We classify all connected n-dimensional complex manifolds admitting effective actions of the unitary group U_n by biholomorphic transformations. One consequence of this classification is a characterization of $\mathbb{C}^n$ by its automorphism group.

0 Introduction

We are interested in classifying all connected complex manifolds M of dimension $n \geq 2$ admitting effective actions of the unitary group U_n by biholomorphic transformations. It is not hard to show that if $\dim M < n$, then an action of U_n by biholomorphic transformations cannot be effective on M, and therefore n is the smallest possible dimension of M for which one may try to obtain such a classification.

One motivation for our study was the following question that we learned from S. Krantz: assume that the group $\mathrm{Aut}(M)$ of all biholomorphic automorphisms of M and the group $\mathrm{Aut}(\mathbb{C}^n)$ of all biholomorphic automorphisms of $\mathbb{C}^n$ are isomorphic as topological groups equipped with the compact-open topology; does it imply that M is biholomorphically equivalent to $\mathbb{C}^n$? The group $\mathrm{Aut}(\mathbb{C}^n)$ is very large (see, *e.g.*, [AL]), and it is not clear from the start what automorphisms of $\mathbb{C}^n$ one can use to approach the problem. The isomorphism between $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\mathbb{C}^n)$ induces a continuous effective action on M of any subgroup $G \subset \mathrm{Aut}(\mathbb{C}^n)$. If G is a Lie group, then this action is in fact real-analytic. We consider $G = U_n$ which, as it turns out, results in a very short list of manifolds that can occur.

In Section 1 we find all possible dimensions of orbits of a U_n-action on M. It turns out (see Proposition 1.1) that an orbit is either a point (hence U_n has a fixed point in M), or a real hypersurface in M, or a complex hypersurface in M, or the whole of M (in which case M is homogeneous).

Manifolds admitting actions with fixed point were found in [K] (see Remark 1.2).

In Section 2 we classify manifolds with U_n-actions such that all orbits are real hypersurfaces. We show that such a manifold is either a spherical layer in $\mathbb{C}^n$, or a Hopf manifold, or the quotient of one of these manifolds by the action of a discrete subgroup of the center of U_n (Theorem 2.7).

In Section 3 we consider the situation when every orbit is a real or a complex hypersurface in M and show that there can exist at most two orbits that are complex hypersurfaces. Moreover, such orbits turn out to be biholomorphically equivalent to

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$\mathbb{CP}^{n-1}$ and can only arise either as a result of blowing up $\mathbb{C}^n$ or a ball in $\mathbb{C}^n$ at the origin, or adding the hyperplane $\infty \in \mathbb{CP}^n$ to the exterior of a ball in $\mathbb{C}^n$, or blowing up $\mathbb{CP}^n$ at one point, or taking the quotient of one of these examples by the action of a discrete subgroup of the center of U_n (Theorem 3.3).

In Section 4 we consider the homogeneous case. In this case the manifold in question must be equivalent to the quotient of a Hopf manifold by the action of a discrete central subgroup (Theorem 4.5).

Thus, Remark 1.2, Theorem 2.7, Theorem 3.3 and Theorem 4.5 provide a complete list of connected manifolds of dimension $n \geq 2$ admitting effective actions of U_n by biholomorphic transformations. An easy consequence of this classification is the following characterization of $\mathbb{C}^n$ by its automorphism group that we obtain in Section 5:

Theorem 5.1 *Let M be a connected complex manifold of dimension n. Assume that* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(\mathbb{C}^n)$ *are isomorphic as topological groups. Then M is biholomorphically equivalent to $\mathbb{C}^n$.*

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1 Dimensions of Orbits

In this section we obtain the following result, which is similar to Satz 1.2 in [K].

Proposition 1.1 *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of U_n by biholomorphic transformations. Let $p \in M$ and let $O(p)$ be the U_n-orbit of p. Then $O(p)$ is either*

(i) *the whole of M (hence M is compact), or*
(ii) *a single point, or*
(iii) *a complex compact hypersurface in M, or*
(iv) *a real compact hypersurface in M.*

Proof For $p \in M$ let I_p be the isotropy subgroup of U_n at p, *i.e.*, $I_p := \{g \in U_n : gp = p\}$. We denote by Ψ the continuous homomorphism of U_n into $\mathrm{Aut}(M)$ (the group of biholomorphic automorphisms of M) induced by the action of U_n on M. Let $L_p := \{d_p(\Psi(g)) : g \in I_p\}$ be the linear isotropy subgroup, where $d_p f$ is the differential of a map f at p. Clearly, L_p is a compact subgroup of $\mathrm{GL}\big(T_p(M), \mathbb{C}\big)$. Since the action of U_n is effective, L_p is isomorphic to I_p. Let $V \subset T_p(M)$ be the tangent space to $O(p)$ at p. Clearly, V is L_p-invariant. We assume now that $O(p) \neq M$ (and therefore $V \neq T_p(M)$) and consider the following three cases.

Case 1 $d := \dim_{\mathbb{C}}(V + iV) < n$.

Since L_p is compact, one can consider coordinates on $T_p(M)$ such that $L_p \subset U_n$. Further, the action of L_p on $T_p(M)$ is completely reducible and the subspace $V + iV$ is invariant under this action. Hence L_p can in fact be embedded in $U_d \times U_{n-d}$. Since

$\dim O(p) \le 2d$, it follows that

$$n^2 \le d^2 + (n-d)^2 + 2d,$$

and therefore either $d = 0$ or $d = n-1$. If $d = 0$, then we obtain (ii). If $d = n-1$, then we have

$$n^2 = \dim L_p + \dim O(p) \le n^2 - 2n + 2 + \dim O(p).$$

Hence $\dim O(p) \ge 2n-2$ which implies that $\dim O(p) = 2d = 2n-2$, and therefore $iV = V$. This yields (iii).

Case 2 $T_p(M) = V + iV$ and $r := \dim_{\mathbb{C}}(V \cap iV) > 0$.

As above, L_p can be embedded in $U_r \times U_{n-r}$ (clearly, we have $r < n$). Moreover, $V \cap iV \ne V$ and since L_p preserves V, it follows that $\dim L_p < r^2 + (n-r)^2$. We have $\dim O(p) \le 2n-1$, and therefore

$$n^2 < r^2 + (n-r)^2 + 2n - 1,$$

which shows that either $r = 1$, or $r = n-1$. It then follows that $\dim L_p < n^2 - 2n + 2$. Therefore, we have

$$n^2 = \dim L_p + \dim O(p) < n^2 - 2n + 2 + \dim O(p).$$

Hence $\dim O(p) > 2n-2$ and thus $\dim O(p) = 2n-1$. This yields (iv).

Case 3 $T_p(M) = V \oplus iV$.

In this case $\dim V = n$ and L_p can be embedded in the real orthogonal group $O_n(\mathbb{R})$, and therefore

$$\dim L_p + \dim O(p) \le \frac{n(n-1)}{2} + n < n^2,$$

which is a contradiction.

The proof of the proposition is complete. ■

Remark 1.2 It is shown in [K] (see Folgerung 1.10 there) that if U_n has a fixed point in M, then M is biholomorphically equivalent to either

(i) the unit ball $B^n \subset \mathbb{C}^n$, or
(ii) $\mathbb{C}^n$, or
(iii) $\mathbb{CP}^n$.

The biholomorphic equivalence f can be chosen to be an isomorphism of U_n-spaces, more precisely,

$$f(gq) = \gamma(g) f(q),$$

where either $\gamma(g) = g$ or $\gamma(g) = \bar{g}$ for all $g \in U_n$ and $q \in M$ (here B^n, $\mathbb{C}^n$ and $\mathbb{CP}^n$ are considered with the standard actions of U_n).

2 The Case of Real Hypersurface Orbits

We shall now consider orbits in M that are real hypersurfaces. We require the following algebraic result.

Lemma 2.1 *Let G be a connected closed subgroup of U_n of dimension $(n-1)^2$, $n \geq 2$. Then either G contains the center of U_n, or G is conjugate in U_n to the subgroup of all matrices*

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \tag{2.1}$$

where $\alpha \in U_1$ and $\beta \in \mathrm{SU}_{n-1}$, or for some $k_1, k_2 \in \mathbb{Z}$, $(k_1, k_2) = 1$, $k_2 \neq 0$, it is conjugate to the subgroup H_{k_1,k_2} of all matrices

$$\begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}, \tag{2.2}$$

where $B \in U_{n-1}$ and $a \in (\det B)^{\frac{k_1}{k_2}} := \exp\big(k_1/k_2 \operatorname{Ln}(\det B)\big)$.

Proof Since G is compact, it is completely reducible, *i.e.*, $\mathbb{C}^n$ splits into a sum of G-invariant pairwise orthogonal complex subspaces, $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_m$, such that the restriction G_j of G to each V_j is irreducible. Let $n_j := \dim_{\mathbb{C}} V_j$ (hence $n_1 + \cdots + n_m = n$) and let U_{n_j} be the group of unitary transformations of V_j. Clearly, $G_j \subset U_{n_j}$, and therefore $\dim G \leq n_1^2 + \cdots + n_m^2$. On the other hand $\dim G = (n-1)^2$, which shows that $m \leq 2$.

Let $m = 2$. Then there exists a unitary change of coordinates $\mathbb{C}^n$ such that in the new variables elements of G are of the form

$$\begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}, \tag{2.3}$$

where $a \in U_1$ and $B \in U_{n-1}$. We note that the scalars a and the matrices B in (2.3) corresponding to the elements of G form compact connected subgroups of U_1 and U_{n-1}, respectively; we shall denote them by G_1 and G_2 as above.

If $\dim G_1 = 0$, then $G_1 = \{1\}$, and therefore $G_2 = U_{n-1}$. Thus we get the form (2.2) with $k_1 = 0$.

Assume that $\dim G_1 = 1$, *i.e.*, $G_1 = U_1$. Then $(n-1)^2 - 1 \leq \dim G_2 \leq (n-1)^2$. Let $\dim G_2 = (n-1)^2 - 1$ first. The only connected subgroup of U_{n-1} of dimension $(n-1)^2 - 1$ is SU_{n-1}. Hence G is conjugate to the subgroup of matrices of the form (2.1). Now let $\dim G_2 = (n-1)^2$, *i.e.*, $G_2 = U_{n-1}$. Consider the Lie algebra $\mathfrak{g}$ of G. It consists of matrices of the following form:

$$\begin{pmatrix} l(b) & 0 \\ 0 & b \end{pmatrix}, \tag{2.4}$$

where b is an arbitrary matrix in $\mathfrak{u}_{n-1}$ and $l(b) \not\equiv 0$ is a linear function of the matrix elements of b ranging in $i\mathbb{R}$. Clearly, $l(b)$ must vanish on the commutant of $\mathfrak{u}_{n-1}$,

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which is $\mathfrak{su}_{n-1}$. Hence matrices (2.4) form a Lie algebra if and only if $l(b) = c\cdot\text{trace}\, b$, where $c \in \mathbb{R}\setminus\{0\}$. Such an algebra can be the Lie algebra of a subgroup of $U_1 \times U_{n-1}$ only if $c \in \mathbb{Q} \setminus \{0\}$. Hence G is conjugate to the group of matrices (2.2) with some $k_1, k_2 \in \mathbb{Z}$, $k_2 \neq 0$, and one can always assume that $(k_1, k_2) = 1$.

Now let $m = 1$. We shall proceed as in the proof of Lemma 2.1 in [IKra]. Let $\mathfrak{g} \subset \mathfrak{u}_n \subset \mathfrak{gl}_n$ be the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g}+i\mathfrak{g} \subset \mathfrak{gl}_n$ its complexification. Then $\mathfrak{g}^{\mathbb{C}}$ acts irreducibly on $\mathbb{C}^n$ and by a theorem of É. Cartan (see, *e.g.*, [GG]), $\mathfrak{g}^{\mathbb{C}}$ is either semisimple or the direct sum of a semisimple ideal $\mathfrak{h}$ and the center of $\mathfrak{gl}_n$ (which is isomorphic to $\mathbb{C}$). Clearly, the action of the ideal $\mathfrak{h}$ on $\mathbb{C}^n$ must be irreducible.

Assume first that $\mathfrak{g}^{\mathbb{C}}$ is semisimple, and let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be its decomposition into the direct sum of simple ideals. Then (see, *e.g.*, [GG]) the irreducible n-dimensional representation of $\mathfrak{g}^{\mathbb{C}}$ given by the embedding of $\mathfrak{g}^{\mathbb{C}}$ in $\mathfrak{gl}_n$ is the tensor product of some irreducible faithful representations of the $\mathfrak{g}_j$. Let n_j be the dimension of the corresponding representation of $\mathfrak{g}_j$, $j = 1, \ldots, k$. Then $n_j \geq 2$, $\dim_{\mathbb{C}} \mathfrak{g}_j \leq n_j^2 - 1$, and $n = n_1 \cdot \cdots \cdot n_k$. The following observation is simple.

Claim *If $n = n_1 \cdot \cdots \cdot n_k$, $k \geq 2$, $n_j \geq 2$ for $j = 1, \ldots, k$, then $\sum_{j=1}^{k} n_j^2 \leq n^2 - 2n$.*

Since $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} = (n-1)^2$, it follows from the above claim that $k = 1$, *i.e.*, $\mathfrak{g}^{\mathbb{C}}$ is simple. The minimal dimensions of irreducible faithful representations of complex simple Lie algebras are well-known (see, *e.g.*, [VO]). In the table below V denotes representations of minimal dimension.

$\mathfrak{g}$	$\dim V$	$\dim \mathfrak{g}$
$\mathfrak{sl}_k$, $k \geq 2$	k	$k^2 - 1$
$\mathfrak{o}_k$, $k \geq 7$	k	$\frac{k(k-1)}{2}$
$\mathfrak{sp}_{2k}$, $k \geq 2$	$2k$	$2k^2 + k$
$\mathfrak{e}_6$	27	78
$\mathfrak{e}_7$	56	133
$\mathfrak{e}_8$	248	248
$\mathfrak{f}_4$	26	52
$\mathfrak{g}_2$	7	14

Since $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} = (n-1)^2$, it follows that none of the above possibilities realize. Hence $\mathfrak{g}^{\mathbb{C}}$ contains the center of $\mathfrak{gl}_n$, and therefore $\mathfrak{g}$ contains the center of $\mathfrak{u}_n$. Thus G contains the center of U_n.

The proof of the lemma is complete. ■

We can now prove the following proposition.

Proposition 2.2 *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective action of U_n by biholomorphic transformations. Let $p \in M$ and let the orbit $O(p)$ be a real hypersurface in M. Then the isotropy subgroup I_p is isomorphic to U_{n-1}.*

Proof Since $O(p)$ is a real hypersurface in M, it arises in Case 2 in the proof of Proposition 1.1. We shall use the notation from that proof. Let W be the orthogonal complement to $V \cap iV$ in $T_p(M)$. Clearly, $\dim_{\mathbb{C}} V \cap iV = n - 1$ and $\dim_{\mathbb{C}} W = 1$. The group L_p is a subgroup of U_n and preserves V, $V \cap iV$, and W; hence it preserves the line $W \cap V$. Therefore, it can act only as $\pm\,\mathrm{id}$ on W. Since $\dim L_p = (n-1)^2$, the identity component L_p^c of L_p must in fact be the group of all unitary transformations preserving $V \cap iV$ and acting trivially on W. Thus, L_p^c is isomorphic to U_{n-1} and acts transitively on directions in $V \cap iV$. Hence $O(p)$ is either Levi-flat or strongly pseudoconvex.

We claim that $O(p)$ cannot be Levi-flat. For assume that $O(p)$ is Levi-flat. Then it is foliated by complex hypersurfaces in M. Let $\mathfrak{m}$ be the Lie algebra of all holomorphic vector fields on $O(p)$ corresponding to the automorphisms of $O(p)$ generated by the action of U_n. Clearly, $\mathfrak{m}$ is isomorphic to $\mathfrak{u}_n$. For $q \in O(p)$ we denote by M_q the leaf of the foliation passing through q and consider the subspace $\mathfrak{l}_q \subset \mathfrak{m}$ of all vector fields tangent to M_q at q. Since vector fields in $\mathfrak{l}_q$ remain tangent to M_q at each point in M_q, $\mathfrak{l}_q$ is in fact a Lie subalgebra of $\mathfrak{m}$. Clearly, $\dim \mathfrak{l}_q = n^2 - 1$, and therefore $\mathfrak{l}_q$ is isomorphic to $\mathfrak{su}_n$. Since there exists only one way to embed $\mathfrak{su}_n$ in $\mathfrak{u}_n$, we obtain that the action of $\mathrm{SU}_n \subset U_n$ preserves each leaf M_q for $q \in O(p)$. Hence each leaf M_q is a union of SU_n-orbits. But such an orbit must be open in M_q, and therefore the action of SU_n is transitive on each M_q.

Let $\tilde{I}_q$ be the isotropy subgroup of q in SU_n. Clearly, $\dim \tilde{I}_q = (n-1)^2$. It now follows from Lemma 2.1 that $\tilde{I}_q^c$, the connected identity component of $\tilde{I}_q$, is conjugate in U_n to the subgroup H_{k_1,k_2} (see (2.2)) with $k_1 = -k_2 = 1$. Hence $\tilde{I}_q$ contains the center of SU_n. The elements of the center act trivially on $\mathrm{SU}/\tilde{I}_q$ (which is equivariantly diffeomorphic to M_q). Thus, the central elements of SU_n act trivially on each M_q, and therefore on $O(p)$. Consequently, the action of U_n on the real hypersurface $O(p)$, and therefore on M, is not effective, which is a contradiction showing that $O(p)$ is strongly pseudoconvex.

Hence L_p can only act identically on W. Thus, L_p is isomorphic to U_{n-1} and so is I_p.

The proof is complete. ■

We now classify real hypersurface orbits up to equivariant diffeomorphisms.

Proposition 2.3 *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective action of U_n by biholomorphic transformations. Let $p \in M$ and assume that the orbit $O(p)$ is a real hypersurface in M. Then $O(p)$ is isomorphic as a homogeneous space to a lense manifold $\mathcal{L}_m^{2n-1} := S^{2n-1}/\mathbb{Z}_m$ obtained by identifying each point $x \in S^{2n-1}$ with $e^{\frac{2\pi i}{m}}x$, where $m = |nk+1|$, $k \in \mathbb{Z}$ (here $\mathcal{L}_m^{2n-1}$ is considered with the standard action of $U_n/\mathbb{Z}_m$).*

Proof By Proposition 2.2, I_p is isomorphic to U_{n-1}. Hence it follows from Lemma 2.1 that I_p either contains the center of U_n or is conjugate to some group H_{k_1,k_2} of matrices of the form (2.2) with $k_1, k_2 \in \mathbb{Z}$. The first possibility in fact cannot occur, since in that case the action of U_n on $O(p)$, and therefore on M, is not effective.

Assume that $K := k_1(n-1) - k_2 \neq \pm 1, 0$. Since $(k_1, k_2) = 1$, either k_1 or k_2 is not a multiple of K. We set $t := 2\pi k_1/K$ in the first case and $t := 2\pi k_2/K$ in the second case. Then $e^{it} \cdot \mathrm{id}$ is a nontrivial central element of U_n that belongs to H_{k_1,k_2}. Hence the action of U_n on $O(p)$ is not effective, which is a contradiction. Further, assuming

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that $K = 0$ we obtain $k_1 = \pm 1$ and $k_2 = \pm(n-1)$. But the center of U_n clearly lies in $H_{1,n-1}$, which yields that the action is not effective again. Hence $K = \pm 1$.

Now let $K = -1$. It is not difficult to show that each element of the corresponding group $H_{k_1,k_1(n-1)+1}$ can be expressed in the following form:

$$\begin{pmatrix} (\det B)^k & 0 \\ 0 & (\det B)^k B \end{pmatrix}, \tag{2.5}$$

where $B \in U_{n-1}$ and $k := k_1$. In a similar way, if $K = 1$, then each element of the corresponding group $H_{k_1,k_1(n-1)-1}$ can be expressed in the form (2.5) with $k := -k_1$.

Let $m := |nk + 1|$ and consider the lense manifold $\mathcal{L}_m^{2n-1}$. We claim that $O(p)$ is isomorphic to $\mathcal{L}_m^{2n-1}$. We identify $\mathbb{Z}_m$ with the subgroup of U_n consisting of the matrices $\sigma \cdot \mathrm{id}$ with $\sigma^m = 1$ and consider the standard action of $U_n/\mathbb{Z}_m$ on $\mathcal{L}_m^{2n-1}$. The isotropy subgroup S of the point in $\mathcal{L}_m^{2n-1}$ represented by the point $(1, 0, \dots, 0) \in S^{2n-1}$ is the standard embedding of U_{n-1} in $U_n/\mathbb{Z}_m$, namely, it consists of elements $C\mathbb{Z}_m$, where

$$C = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

and $B \in U_{n-1}$. The manifold $(U_n/\mathbb{Z}_m)/S$ is equivariantly diffeomorphic to $\mathcal{L}_m^{2n-1}$. We now show that it is also isomorphic to $O(p)$. Indeed, consider the Lie group isomorphism

$$\phi_{n,m} \colon U_n/\mathbb{Z}_m \to U_n, \quad \phi_{n,m}(A\mathbb{Z}_m) = (\det A)^k \cdot A, \tag{2.6}$$

where $A \in U_n$. Clearly, $\phi_{n,m}(S) \subset U_n$ is the subgroup of matrices of the form (2.5), that is, H_{k_1,k_2}. Thus, it is conjugate in U_n to I_p, and therefore $(U_n/\mathbb{Z}_m)/S$ is isomorphic to U_n/I_p and to $O(p)$. More precisely, the isomorphism $f \colon \mathcal{L}_m^{2n-1} \to O(p)$ is the following composition of maps:

$$f = f_1 \circ \phi^*_{n,m} \circ f_2, \tag{2.7}$$

where $f_1 \colon U_n/H_{k_1,k_2} \to O(p)$ and $f_2 \colon \mathcal{L}_m^{2n-1} \to (U_n/\mathbb{Z}_m)/S$ are the standard equivariant equivalences and the isomorphism $\phi^*_{n,m} \colon (U_n/\mathbb{Z}_m)/S \to U_n/H_{k_1,k_2}$ is induced by $\phi_{n,m}$ in the obvious way. Clearly, f satisfies

$$f(gq) = \phi_{n,m}(g) f(q), \tag{2.8}$$

for all $g \in U_n/\mathbb{Z}_m$ and $q \in \mathcal{L}_m^{2n-1}$.

Thus, f is an isomorphism between $\mathcal{L}_m^{2n-1}$ and $O(p)$ regarded as homogeneous spaces, as required. ■

The next result shows that isomorphism (2.7) in Proposition 2.3 is either a CR or an anti-CR diffeomorphism.

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Proposition 2.4 *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective action of U_n by biholomorphic transformations. For $p \in M$ suppose that $O(p)$ is a real hypersurface in M isomorphic as a homogeneous space to a lense manifold $\mathcal{L}_m^{2n-1}$. Then an isomorphism $\mathcal{F}\colon \mathcal{L}_m^{2n-1} \to O(p)$ can be chosen to be a* CR-*diffeomorphism that satisfies either the relation*

$$\mathcal{F}(gq) = \phi_{n,m}(g)\mathcal{F}(q), \tag{2.9}$$

or the relation

$$\mathcal{F}(gq) = \phi_{n,m}(\bar{g})\mathcal{F}(q), \tag{2.10}$$

for all $g \in U_n/\mathbb{Z}_m$ and $q \in \mathcal{L}_m^{2n-1}$ (here $\mathcal{L}_m^{2n-1}$ is considered with the CR-*structure inherited from S^{2n-1}).*

Proof Consider the standard covering map $\pi\colon S^{2n-1} \to \mathcal{L}_m^{2n-1}$ and the induced map $\tilde{\pi} := f \circ \pi\colon S^{2n-1} \to O(p)$, where f is defined in (2.7). It follows from (2.8) that the covering map $\tilde{\pi}$ satisfies

$$\tilde{\pi}(gq) = \tilde{\phi}_{n,m}(g)\tilde{\pi}(q), \tag{2.11}$$

for all $g \in U_n$ and $q \in S^{2n-1}$ where $\tilde{\phi}_{n,m} := \phi_{n,m} \circ \rho_{n,m}$ and $\rho_{n,m}\colon U_n \to U_n/\mathbb{Z}_m$ is the standard projection.

Using $\tilde{\pi}$ we can pull back the CR-structure from $O(p)$ to S^{2n-1}. We denote by $\tilde{S}^{2n-1}$ the sphere S^{2n-1} equipped with this new CR-structure. It follows from (2.11) that the CR-structure on $\tilde{S}^{2n-1}$ is invariant under the standard action of U_n on S^{2n-1}.

We now prove the following lemma.

Lemma 2.5 *There exist exactly two* CR-*structures on S^{2n-1} invariant under the standard action of U_n, namely, the standard* CR-*structure on S^{2n-1} and the structure obtained by conjugating the standard one.*

Proof of Lemma 2.5 For $q_0 := (1, 0, \dots, 0) \in S^{2n-1}$ let I_{q_0} be the isotropy subgroup of this point with respect to the standard action of U_n on S^{2n-1}. Clearly, $I_{q_0} = U_{n-1}$, where U_{n-1} is embedded in U_n in the standard way. Let L_{q_0} be the corresponding linear isotropy subgroup. Clearly, the only $(2n-2)$-dimensional subspace of $T_{q_0}(S^{2n-1})$ invariant under the action of L_{q_0} is $\{z_1 = 0\}$. Hence there exists a unique contact structure on S^{2n-1} invariant under the standard action of U_n.

On the other hand there exist exactly two ways to introduce in $\mathbb{R}^{2n-2}$ a U_{n-1}-invariant structure of complex linear space: the standard complex structure and its conjugation (this is obvious for $n = 2$, and easy to show for $n \geq 3$, and therefore we shall omit the proof). Let J_q be the operator of complex structure in the corresponding subspace of $T_q(S^{2n-1})$, $q \in S^{2n-1}$. Since there exist only two possibilities for J_q, and J_q depends smoothly on q, the lemma follows. ■

Proposition 2.4 easily follows from Lemma 2.5. Indeed, if the CR-structure of $\tilde{S}^{2n-1}$ is identical to that of S^{2n-1}, then we set $\mathcal{F} := f$. Clearly, $\mathcal{F}$ is a CR-diffeomorphism and satisfies (2.9). On the other hand, if the CR-structure of $\tilde{S}^{2n-1}$ is obtained

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from the structure of S^{2n-1} by conjugation, then we set $\mathcal{F}(t) := f(\bar{t})$ for $t \in \mathcal{L}_m^{2n-1}$. Clearly, $\mathcal{F}$ is a CR-diffeomorphism and satisfies (2.10).

The proof of the proposition is complete. ■

We introduce now additional notation.

Definition 2.6 Let $d \in \mathbb{C} \setminus \{0\}$, $|d| \neq 1$, let M_d^n be the Hopf manifold constructed by identifying $z \in \mathbb{C}^n \setminus \{0\}$ with $d \cdot z$, and let $[z]$ be the equivalence class of z. Then we denote by $M_d^n/\mathbb{Z}_m$, with $m \in \mathbb{N}$, the complex manifold obtained from M_d^n by identifying $[z]$ and $[e^{\frac{2\pi i}{m}} z]$.

We are now ready to prove the following theorem.

Theorem 2.7 *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of U_n by biholomorphic transformations. Suppose that all orbits of this action are real hypersurfaces. Then there exists $k \in \mathbb{Z}$ such that, for $m = |nk+1|$, M is biholomorphically equivalent to either*

(i) *$S_{r,R}^n/\mathbb{Z}_m$, where $S_{r,R}^n := \{z \in \mathbb{C}^n : r < |z| < R\}$, $0 \leq r < R \leq \infty$, is a spherical layer, or*
(ii) *$M_d^n/\mathbb{Z}_m$.*

The biholomorphic equivalence f can be chosen to satisfy either the relation

$$f(gq) = \phi_{n,m}^{-1}(g) f(q), \tag{2.12}$$

or the relation

$$f(gq) = \phi_{n,m}^{-1}(\bar{g}) f(q), \tag{2.13}$$

for all $g \in U_n$ and $q \in M$, where $\phi_{n,m}$ is defined in (2.6) (here $S_{r,R}^n/\mathbb{Z}_m$ and $M_d^n/\mathbb{Z}_m$ are equipped with the standard actions of $U_n/\mathbb{Z}_m$).

Proof Assume first that M is non-compact. Let $p \in M$. By Propositions 2.3 and 2.4, for some $m = |nk+1|$, $k \in \mathbb{Z}$, there exists a CR-diffeomorphism $f\colon O(p) \to \mathcal{L}_m^{2n-1}$ such that either (2.12) or (2.13) holds for all $q \in O(p)$. Assume first that (2.12) holds. The map f extends to a biholomorphic map of a neighborhood U of $O(p)$ onto a neighborhood of $\mathcal{L}_m^{2n-1}$ in $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_m$. We can take U to be a connected union of orbits. Then the extended map satisfies (2.12) on U, and therefore maps U biholomorphically onto the quotient of a spherical layer by the action of $\mathbb{Z}_m$.

Let D be a maximal domain in M such that there exists a biholomorphic map f from D onto the quotient of a spherical layer by the action of $\mathbb{Z}_m$ that satisfies a relation of the form (2.12) for all $g \in U_n$ and $q \in D$. As was shown above, such a domain D exists. Assume that $D \neq M$ and let x be a boundary point of D. Consider the orbit $O(x)$. Extending a map from $O(x)$ into a lense manifold to a neighborhood of $O(x)$ as above, we see that the orbits of all points close to x have the same type as

$O(x)$. Therefore, $O(x)$ is also equivalent to $\mathcal{L}_m^{2n-1}$. Let $h\colon O(x) \to \mathcal{L}_m^{2n-1}$ be a CR-isomorphism. It satisfies either relation (2.12) or relation (2.13) for all $g \in U_n$ and $q \in O(x)$.

Assume first that (2.12) holds for h. The map h extends to some neighborhood V of $O(x)$ that we can assume to be a connected union of orbits. The extended map satisfies (2.12) on V. For $s \in V \cap D$ we consider the orbit $O(s)$. The maps f and h take $O(s)$ into some surfaces $r_1 S^{2n-1}/\mathbb{Z}_m$ and $r_2 S^{2n-1}/\mathbb{Z}_m$, respectively, where $r_1, r_2 > 0$. Hence $F := h \circ f^{-1}$ maps $r_1 S^{2n-1}/\mathbb{Z}_m$ onto $r_2 S^{2n-1}/\mathbb{Z}_m$ and satisfies the relation

$$F(ut) = uF(t), \tag{2.14}$$

for all $u \in U_n/\mathbb{Z}_m$ and $t \in r_1 S^{2n-1}/\mathbb{Z}_m$. Let $\pi_1\colon r_1 S^{2n-1} \to r_1 S^{2n-1}/\mathbb{Z}_m$ and $\pi_2\colon r_2 S^{2n-1} \to r_2 S^{2n-1}/\mathbb{Z}_m$ be the standard projections. Clearly, F can be lifted to a map between $r_1 S^{2n-1}$ and $r_2 S^{2n-1}$, *i.e.*, there exists a CR-isomorphism $G\colon r_1 S^{2n-1} \to r_2 S^{2n-1}$ such that

$$F \circ \pi_1 = \pi_2 \circ G. \tag{2.15}$$

We see from (2.14) and (2.15) that, for all $g \in U_n$ and $y \in r_1 S^{2n-1}$,

$$\begin{aligned}\pi_2\big(G(gy)\big) &= F\big(\pi_1(gy)\big) = F\big(\rho_{n,m}(g)\pi_1(y)\big)\\ &= \rho_{n,m}(g)F\big(\pi_1(y)\big) = \rho_{n,m}(g)\pi_2\big(G(y)\big) = \pi_2\big(gG(y)\big),\end{aligned}$$

where $\rho_{n,m}\colon U_n \to U_n/\mathbb{Z}_m$ is the standard projection. Since the fibers of π_2 are discrete, this leads to the relation

$$G(gy) = gG(y), \tag{2.16}$$

for all $g \in U_n$ and $y \in r_1 S^{2n-1}$.

The map G extends to a biholomorphic map of the corresponding balls $r_1 B^n$, $r_2 B^n$, and the extended map satisfies (2.16) on $r_1 B^n$. Setting $y = 0$ in (2.16) we see that $G(0)$ is a fixed point of the standard action of U_n on $r_2 B^n$, and therefore $G(0) = 0$. Combined with (2.16) this shows that $G = d \cdot \mathrm{id}$, where $d \in \mathbb{C} \setminus \{0\}$. This means, in particular, that F is biholomorphic on $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_m$. Now,

$$H := \begin{cases} F \circ f & \text{on } D \\ h & \text{on } V \end{cases}$$

is a holomorphic map on $D \cup V$, provided that $D \cap V$ is connected.

We now claim that we can choose V such that $D \cap V$ is connected. We assume that V is small enough, hence the strictly pseudoconvex orbit $O(x)$ partitions V into two pieces. Namely, $V = V_1 \cup V_2 \cup O(x)$, where $V_1 \cap V_2 = \varnothing$ and each intersection $V_j \cap D$ is connected. Indeed, there exist holomorphic coordinates on D in which $V_j \cap D$ is a union of the quotients of spherical layers by the action of $\mathbb{Z}_m$. If there are several such "factorized" layers, then there exists a layer with closure disjoint from $O(x)$ and

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hence D is disconnected, which is impossible. Therefore, $V_j \cap D$ is connected and, if V is sufficiently small, then each V_j is either a subset of D or is disjoint from D. If $V_j \subset D$ for $j = 1, 2$, then $M = D \cup V$ is compact which contradicts our assumption. Thus, only one set of V_1, V_2 lies in D, and therefore $D \cap V$ is connected. Hence the map H is well-defined. Clearly, it satisfies (2.12) for all $g \in U_n$ and $q \in D \cup V$.

We will now show that H is one-to-one on $D \cup V$. Obviously, H is one-to-one on each of V and D. Assume that there exist points $p_1 \in D$ and $p_2 \in V$ such that $H(p_1) = H(p_2)$. Since H satisfies (2.12) for all $g \in U_n$ and $q \in D \cup V$, it follows that $H\big(O(p_1)\big) = H\big(O(p_2)\big)$. Let $\Gamma(\tau)$, $0 \le \tau \le 1$ be a continuous path in $D \cup V$ joining p_1 to p_2. For each $0 \le \tau \le 1$ we set $\rho(\tau)$ to be the radius of the sphere corresponding to the lense manifold $H\Big(O\big(\Gamma(\tau)\big)\Big)$. Since ρ is continuous and $\rho(0) = \rho(1)$, there exists a point $0 < \tau_0 < 1$ at which ρ attains either its maximum or its minimum on $[0, 1]$. Then H is not one-to-one in a neighborhood of $O\big(\Gamma(\tau_0)\big)$, which is a contradiction.

We have thus constructed a domain containing D as a proper subset that can be mapped onto the quotient of a spherical layer by the action of $\mathbb{Z}_m$ by means of a map satisfying (2.12). This is a contradiction showing that in fact $D = M$.

Assume now that h satisfies (2.13) (rather than (2.12)) for all $g \in U_n$ and $q \in O(x)$. Then h extends to a neighborhood V of $O(x)$ and satisfies (2.13) there. For a point $s \in V \cap D$ we consider its orbit $O(s)$. The maps f and h take $O(s)$ into some lense manifolds $r_1 S^{2n-1}/\mathbb{Z}_m$ and $r_2 S^{2n-1}/\mathbb{Z}_m$, respectively, where $r_1, r_2 > 0$. Hence $F := h \circ f^{-1}$ maps $r_1 S^{2n-1}/\mathbb{Z}_m$ onto $r_2 S^{2n-1}/\mathbb{Z}_m$ and satisfies the relation

$$F(ut) = \bar{u}F(t), \tag{2.17}$$

for all $u \in U_n/\mathbb{Z}_m$ and $t \in r_1 S^{2n-1}/\mathbb{Z}_m$. As above, F can be lifted to a map G from $r_1 S^{2n-1}$ into $r_2 S^{2n-1}$. By (2.17) and (2.15), for all $g \in U_n$ and $y \in r_1 S^{2n-1}$ we obtain

$$\begin{aligned}\pi_2\big(G(gy)\big) &= F\big(\pi_1(gy)\big) = F\big(\rho_{n,m}(g)\pi_1(y)\big)\\ &= \overline{\rho_{n,m}(g)}F\big(\pi_1(y)\big) = \rho_{n,m}(\bar{g})\pi_2\big(G(y)\big) = \pi_2\big(\bar{g}G(y)\big).\end{aligned}$$

As above, this shows that

$$G(gy) = \bar{g}G(y), \tag{2.18}$$

for all $g \in U_n$ and $y \in r_1 S^{2n-1}$.

The map G extends to a biholomorphic map between the corresponding balls $r_1 B^n$, $r_2 B^n$, and the extended map satisfies (2.18) on $r_1 B^n$. By setting $y = 0$ in (2.18) we see similarly to the above that $G(0)$ is a fixed point of the standard action of U_n on $r_1 B^n$, and thus $G(0) = 0$. Hence $G = d \cdot U$, where $d \in \mathbb{C} \setminus \{0\}$ and U is a unitary matrix. This, however, contradicts (2.18), and therefore h cannot satisfy (2.13) on $O(x)$.

The proof in the case when f satisfies (2.13) on $O(p)$ is analogous to the above. In this case we obtain an extension to the whole of M satisfying (2.13). This completes the proof in the case of non-compact M.

Assume now that M is compact. We consider a domain D as above and assume first that the corresponding map f satisfies (2.12). Since M is compact, $D \neq M$. Let x be a boundary point of D, and consider the orbit $O(x)$. We choose a connected neighborhood V of $O(x)$ as above, and let $V = V_1 \cup V_2 \cup O(x)$, where $V_1 \cap V_2 = \varnothing$ and each V_j is either a subset of D or is disjoint from D. If one domain of V_1, V_2 is disjoint from D, then, arguing as above, we arrive at a contradiction with the maximality of D. Hence $V_j \subset D$, $j = 1, 2$, and $M = D \cup O(x)$.

We can now extend $f|_{V_1}$ and $f|_{V_2}$ to biholomorphic maps f_1 and f_2, respectively, that are defined on V, map it onto spherical layers factorized by the action of $\mathbb{Z}_m$, and satisfy (2.12) on V. Then f_1 and f_2 map $O(x)$ onto $r_1 S^{2n-1}/\mathbb{Z}_m$ and $r_2 S^{2n-1}/\mathbb{Z}_m$, respectively, for some $r_1, r_2 > 0$. Clearly, $r_1 \neq r_2$. Hence $F := f_2 \circ f_1^{-1}$ maps $r_1 S^{2n-1}/\mathbb{Z}_m$ onto $r_2 S^{2n-1}/\mathbb{Z}_m$ and satisfies (2.14). This shows, similarly to the above, that $F(\langle t \rangle_1) = \langle d \cdot t \rangle_2$ for all $\langle t \rangle_1 \in r_1 S^{2n-1}/\mathbb{Z}_m$, where $d \in \mathbb{C} \setminus \{0\}$ and $\langle t \rangle_j \in r_j S^{2n-1}/\mathbb{Z}_m$ is the equivalence class of $t \in r_j S^{2n-1}$, $j = 1, 2$. Since $r_1 \neq r_2$, it follows that $|d| \neq 1$. Now, the map

$$H := \begin{cases} f & \text{on } D \\ f_1 & \text{on } O(x) \end{cases}$$

establishes a biholomorphic equivalence between M and $M^n_d/\mathbb{Z}_m$ and satisfies (2.12).

The proof in the case when f satisfies (2.13) on D is analogous to the above. In this case we obtain an extension H that satisfies (2.13).

The proof of the theorem is complete. ■

3 The Case of Complex Hypersurface Orbits

We now discuss orbits that are complex hypersurfaces. We start with several examples.

Example 3.1 Let B^n_R be the ball of radius $0 < R \leq \infty$ in $\mathbb{C}^n$ and let $\widehat{B^n_R}$ be its blow-up at the origin, *i.e.*,

$$\widehat{B^n_R} := \{(z, w) \in B^n_R \times \mathbb{CP}^{n-1} : z_i w_j = z_j w_i, \text{ for all } i, j\},$$

where $z = (z_1, \dots, z_n)$ are the standard coordinates in $\mathbb{C}^n$ and $w = (w_1 : \cdots : w_n)$ are the homogeneous coordinates in $\mathbb{CP}^{n-1}$. We define an action of U_n on $\widehat{B^n_R}$ as follows. For $(z, w) \in \widehat{B^n_R}$ and $g \in U_n$ we set

$$g(z, w) := (gz, gw),$$

where in the right-hand side we use the standard actions of U_n on $\mathbb{C}^n$ and $\mathbb{CP}^{n-1}$. The points $(0, w) \in \widehat{B^n_R}$ form an orbit O, which is a complex hypersurface biholomorphically equivalent to $\mathbb{CP}^{n-1}$. All other orbits are real hypersurfaces that are the boundaries of strongly pseudoconvex neighborhoods of O.

We fix $m \in \mathbb{N}$ and denote by $\widehat{B^n_R}/\mathbb{Z}_m$ the quotient of $\widehat{B^n_R}$ by the equivalence relation $(z, w) \sim e^{\frac{2\pi i}{m}}(z, w)$. Let $\{(z, w)\} \in \widehat{B^n_R}/\mathbb{Z}_m$ be the equivalence class of $(z, w) \in \widehat{B^n_R}$. We

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now define in a natural way an action of $U_n/\mathbb{Z}_m$ on $\widehat{B^n_R}/\mathbb{Z}_m$: for $\{(z,w)\} \in \widehat{B^n_R}/\mathbb{Z}_m$ and $g \in U_n$ we set

$$(g\mathbb{Z}_m)\{(z,w)\} := \{g(z,w)\}.$$

The points $\{(0,w)\}$ form the unique complex hypersurface orbit O, which is biholomorphically equivalent to $\mathbb{CP}^{n-1}$, and each real hypersurface orbit is the boundary of a strongly pseudoconvex neighborhood of O.

Now let $S^n_{r,\infty} = \{z \in \mathbb{C}^n : |z| > r\}$, $r > 0$, be a spherical layer with infinite outer radius and let $\widetilde{S^n_{r,\infty}}$ be the union of $S^n_{r,\infty}$ and the hypersurface at infinity in $\mathbb{CP}^n$, namely,

$$\widetilde{S^n_{r,\infty}} := \{(z_0 : z_1 : \cdots : z_n) \in \mathbb{CP}^n : (z_1, \ldots, z_n) \in S^n_{r,\infty}, z_0 = 0, 1\}.$$

We shall equip $\widetilde{S^n_{r,\infty}}$ with the standard action of U_n. For $(z_0 : z_1 : \cdots : z_n) \in \widetilde{S^n_{r,\infty}}$ and $g \in U_n$ we set

$$g(z_0 : z_1 : \cdots : z_n) := (z_0 : u_1 : \cdots : u_n),$$

where $(u_1, \ldots, u_n) := g(z_1, \ldots, z_n)$. The points $(0 : z_1 : \cdots : z_n)$ at infinity form an orbit O, which is a complex hypersurface biholomorphically equivalent to $\mathbb{CP}^{n-1}$. All other orbits are real hypersurfaces that are the boundaries of strongly pseudoconcave neighborhoods of O.

We fix $m \in \mathbb{N}$ and denote by $\widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$ the quotient of $\widetilde{S^n_{r,\infty}}$ by the equivalence relation $(z_0 : z_1 : \cdots : z_n) \sim e^{\frac{2\pi i}{m}}(z_0 : z_1 : \cdots : z_n)$. Let $\{(z_0 : z_1 : \cdots : z_n)\} \in \widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$ be the equivalence class of $(z_0 : z_1 : \cdots : z_n) \in \widetilde{S^n_{r,\infty}}$. We consider $\widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$ with the standard action of $U_n/\mathbb{Z}_m$, namely, for $\{(z_0 : z_1 : \cdots : z_n)\} \in \widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$ and $g \in U_n$ we set

$$(g\mathbb{Z}_m)\{(z_0 : z_1 : \cdots : z_n)\} := \{g(z_0 : z_1 : \cdots : z_n)\}.$$

The points $\{(0 : z_1 : \cdots : z_n)\}$ form a unique complex hypersurface orbit O which is biholomorphically equivalent to $\mathbb{CP}^{n-1}$, and each real hypersurface orbit is the boundary of a strongly pseudoconcave neighborhood of O.

Finally, let $\widehat{\mathbb{CP}^n}$ be the blow-up of $\mathbb{CP}^n$ at the point $(1 : 0 : \cdots : 0) \in \mathbb{CP}^n$:

$$\widehat{\mathbb{CP}^n} := \Big\{ \big((z_0 : z_1 : \cdots : z_n), w\big) \in \mathbb{CP}^n \times \mathbb{CP}^{n-1} : z_i w_j = z_j w_i \\ \text{for all } i, j \neq 0, z_0 = 0, 1 \Big\},$$

where $w = (w_1 : \cdots : w_n)$ are the homogeneous coordinates in $\mathbb{CP}^{n-1}$. We define an action of U_n in $\widehat{\mathbb{CP}^n}$ as follows. For $\big((z_0 : z_1 : \cdots : z_n), w\big) \in \widehat{\mathbb{CP}^n}$ and $g \in U_n$ we set

$$g\big((z_0 : z_1 : \cdots : z_n), w\big) := \big((z_0 : u_1 : \cdots : u_n), gw\big),$$

where $(u_1, \ldots, u_n) := g(z_1, \ldots, z_n)$. This action has exactly two orbits that are complex hypersurfaces: the orbit O_1 consisting of the points $\big((1 : 0 : \cdots : 0), w\big)$ and the orbit O_2 consisting of the points $\big((0 : z_1 : \cdots : z_n), w\big)$. Both O_1 and O_2 are biholomorphically equivalent to $\mathbb{CP}^{n-1}$. The real hypersurface orbits are the boundaries of strongly pseudoconvex neighborhoods of O_1 and strongly pseudoconcave neighborhoods of O_2.

We fix $m \in \mathbb{N}$ and denote by $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$ the quotient of $\widehat{\mathbb{CP}^n}$ by the equivalence relation $\big((z_0 : z_1 : \cdots : z_n), w\big) \sim e^{\frac{2\pi i}{m}}\big((z_0 : z_1 : \cdots : z_n), w\big)$. Let $\big\{\big((z_0 : z_1 : \cdots : z_n), w\big)\big\} \in \widehat{\mathbb{CP}^n}/\mathbb{Z}_m$ be the equivalence class of $\big((z_0 : z_1 : \cdots : z_n), w\big) \in \widehat{\mathbb{CP}^n}$. We shall consider $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$ with the standard action of $U_n/\mathbb{Z}_m$, namely, for $\Big\{\big((z_0 : z_1 : \cdots : z_n), w\big)\Big\} \in \widehat{\mathbb{CP}^n}/\mathbb{Z}_m$ and $g \in U_n$ we set:

$$(g\mathbb{Z}_m)\big\{\big((z_0 : z_1 : \cdots : z_n), w\big)\big\} := \big\{g\big((z_0 : z_1 : \cdots : z_n), w\big)\big\}.$$

As above, there exist exactly two orbits that are complex hypersurfaces: the orbit O_1 consisting of the points $\big\{\big((1 : 0 : \cdots : 0), w\big)\big\}$ and the orbit O_2 consisting of the points $\big\{\big((0 : z_1 : \cdots : z_n), w\big)\big\}$. Both O_1 and O_2 are biholomorphically equivalent to $\mathbb{CP}^{n-1}$. The real hypersurface orbits are the boundaries of strongly pseudoconvex neighborhoods of O_1 and strongly pseudoconcave neighborhoods of O_2.

We show below that the complex hypersurface orbits in Example 3.1 are in fact the only ones that can occur.

Proposition 3.2 *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of U_n by biholomorphic transformations. Suppose that each orbit is a real or a complex hypersurface in M. Then there exist at most two complex hypersurface orbits.*

Proof We fix a smooth U_n-invariant distance function ρ on M. Let O be an orbit that is a complex hypersurface. Consider the ϵ-neighborhood of $U_\epsilon(O)$ of O in M:

$$U_\epsilon(O) := \big\{p \in M : \inf_{q\in O} \rho(p, q) < \epsilon\big\}.$$

If ϵ is sufficiently small, then the boundary of $U_\epsilon(O)$,

$$\partial U_\epsilon(O) = \big\{p \in M : \inf_{q\in O} \rho(p, q) = \epsilon\big\},$$

is a smooth connected real hypersurface in M. Clearly, ∂U_ϵ is U_n-invariant, and therefore it is a union of orbits. If $\partial U_\epsilon(O)$ contains an orbit that is a real hypersurface, then $\partial U_\epsilon(O)$ obviously coincides with that orbit.

Assume that $\partial U_\epsilon(O)$ contains an orbit that is a complex hypersurface. Then $\partial U_\epsilon(O)$ is a union of such orbits. It follows from the proof of Proposition 1.1 (see Case 1 there) that if an orbit $O(p)$ is a complex hypersurface, then I_p is isomorphic to $U_1 \times U_{n-1}$. By Lemma 2.1 of [IKra], I_p is in fact conjugate to $U_1 \times U_{n-1}$ embedded in U_n in the standard way. Hence the action of the center of U_n on $O(p)$ is trivial. Thus, the center of U_n acts trivially on each complex hypersurface orbit and hence on the entire $\partial U_\epsilon(O)$. Then its action on M is also trivial, which contradicts the assumption of the effectiveness of the action of U_n on M.

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Hence, if ϵ is sufficiently small, then $U_\epsilon(O)$ contains no complex hypersurface orbits other than O itself, and the boundary of $U_\epsilon(O)$ is a real hypersurface orbit. Let $\tilde{M}$ be the manifold obtained by removing all complex hypersurface orbits from M. Since such an orbit has a neighborhood containing no other complex hypersurface orbits, $\tilde{M}$ is connected. It is also clear that $\tilde{M}$ is non-compact. Hence, by Theorem 2.7, $\tilde{M}$ can be mapped onto $S^n_{r,R}/\mathbb{Z}_m$, for some $0 \le r < R \le \infty$, by a biholomorphic map f satisfying either (2.12) or (2.13). The manifold $S^n_{r,R}/\mathbb{Z}_m$ has two ends at infinity, and therefore the number of removed complex hypersurfaces is at most two, which completes the proof. ∎

We can now prove the following theorem.

Theorem 3.3 *Let M be a connected complex manifold of dimension $n \ge 2$ endowed with an effective action of U_n by biholomorphic transformations. Suppose that each orbit of this action is either a real or complex hypersurface and at least one orbit is a complex hypersurface. Then there exists $k \in \mathbb{Z}$ such that, for $m = |nk+1|$, M is biholomorphically equivalent to either*

(i) $\widehat{B^n_R}/\mathbb{Z}_m$, $0 < R \le \infty$, *or*
(ii) $\widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$, $0 \le r < \infty$, *or*
(iii) $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$.

The biholomorphic equivalence f can be chosen to satisfy either (2.12) or (2.13) for all $g \in U_n$ and $q \in M$.

Proof Assume first that only one orbit O is a complex hypersurface. Consider $\tilde{M} := M \setminus O$. Since $\tilde{M}$ is clearly non-compact, by Theorem 2.7 there exists $k \in \mathbb{Z}$ such that for $m = |nk+1|$ and some r and R, $0 \le r < R \le \infty$, the manifold $\tilde{M}$ is biholomorphically equivalent to $S^n_{r,R}/\mathbb{Z}_m$ by means of a map f satisfying either (2.12) or (2.13) for all $g \in U_n$ and $q \in \tilde{M}$. We shall assume that f satisfies (2.12) because the latter case can be dealt with in the same way.

Suppose first that $n \ge 3$. We fix $p \in O$ and consider I_p. We denote for the moment by $H \subset U_n$ the standard embedding of $U_1 \times U_{n-1}$ in U_n. As mentioned in the proof of Proposition 3.2, there exists $g \in U_n$ such that $I_p = g^{-1}Hg$. For an arbitrary real hypersurface orbit $O(q)$ we set

$$N_{p,q} := \{s \in O(q) : I_s \subset I_p\}.$$

Since I_s is conjugate in U_n to a subgroup H_{k_1,k_2}, where $k_1 := k$ and $k_2 = k(n-1)+1 \neq 0$ (see (2.5) in the proof of Proposition 2.3), it follows that

$$N_{p,q} = \{s \in O(q) : I_s = g^{-1}H_{k_1,k_2}g\}.$$

It is easy to show now that if we fix $t \in N_{p,q}$, then $N_{p,q} = \{ht\}$, where

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$$h = g^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \mathrm{id} \end{pmatrix} g, \quad \alpha \in U_1.$$

Let N_p be the union of the $N_{p,q}$'s over all real hypersurface orbits $O(q)$. Also let N'_p be the set of points in $S^n_{r,R}/\mathbb{Z}_m$ whose isotropy subgroup with respect to the standard action of $U_n/\mathbb{Z}_m$ is $\phi^{-1}_{n,m}(g^{-1}H_{k_1,k_2}g)$ (see (2.6) for the definition of $\phi_{n,m}$). It is easy to verify that N'_p is a complex curve in $S^n_{r,R}/\mathbb{Z}_m$ biholomorphically equivalent to either an annulus of modulus $(R/r)^m$ (if $0 < r < R < \infty$), or a punctured disk (if $r = 0$, $R < \infty$ or $r > 0$, $R = \infty$), or $\mathbb{C} \setminus 0$ (if $r = 0$ and $R = \infty$). Clearly, $f^{-1}(N'_p) = N_p$, and hence N_p is a complex curve in $\tilde{M}$.

Obviously, N_p is invariant under the action of I_p. By Bochner's theorem there exist local holomorphic coordinates in the neighborhood of p such that the action of I_p is linear in these coordinates and coincides with the action of the linear isotropy subgroup L_p introduced in the proof of Proposition 1.1 (upon the natural identification of the coordinate neighborhood in question and a neighborhood of the origin in $T_p(M)$). Recall that L_p has two invariant complex subspaces in $T_p(M)$: $T_p(O)$ and a one-dimensional subspace, which correspond in our coordinates to O and some holomorphic curve. It can be easily seen that $\overline{N_p}$ is precisely this curve. Hence $\overline{N_p}$ near p is an analytic disc with center at p, and therefore N'_p cannot in fact be equivalent to an annulus, and we have either $r = 0$ or $R = \infty$.

Assume first that $r = 0$ and $R < \infty$. We consider a holomorphic embedding $\nu\colon S^n_{0,R}/\mathbb{Z}_m \to \widehat{B^n_R}/\mathbb{Z}_m$ defined by the formula

$$\nu(\langle z \rangle) := \{(z, w)\},$$

where $w = (w_1 : \cdots : w_n)$ is uniquely determined by the conditions $z_i w_j = z_j w_i$ for all i, j, and $\langle z \rangle \in (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_m$ is the equivalence class of $z = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\}$. Clearly, ν is $U_n/\mathbb{Z}_m$-equivariant. Now let $f_\nu := \nu \circ f$. We claim that f_ν extends to O as a biholomorphic map of M onto $\widehat{B^n_R}/\mathbb{Z}_m$.

Let $\hat{O}$ be the orbit in $\widehat{B^n_R}/\mathbb{Z}_m$ that is a complex hypersurface and let $\hat{p} \in \hat{O}$ be the (unique) point such that its isotropy subgroup $I_{\hat{p}}$ (with respect to the action of $U_n/\mathbb{Z}_m$ on $\widehat{B^n_R}/\mathbb{Z}_m$ as described in Example 3.1) is $\phi^{-1}_{n,m}(I_p)$. Then $\{\hat{p}\} \cup \nu(N'_p)$ is a smooth complex curve. We define the extension F_ν of f_ν by setting $F_\nu(p) := \hat{p}$ for each $p \in O$.

We must show that F_ν is continuous at each point $p \in O$. Let $\{q_j\}$ be a sequence of points in M accumulating to p. Since all accumulation points of the sequence $\{F_\nu(q_j)\}$ lie in $\hat{O}$ and $\hat{O}$ is compact, it suffices to show that each convergent subsequence $\{F_\nu(q_{j_k})\}$ of $\{F_\nu(q_j)\}$ converges to $\hat{p}$. For every q_{j_k} there exists $g_{j_k} \in U_n$ such that $g^{-1}_{j_k} I_{q_{j_k}} g_{j_k} \subset I_p$, *i.e.*, $g^{-1}_{j_k} q_{j_k} \in \overline{N_p}$. We select a convergent subsequence $\{g_{j_{k_l}}\}$ and denote its limit by g. Then $\{g^{-1}_{j_{k_l}} q_{j_{k_l}}\}$ converges to $g^{-1}p$. Since $g^{-1}p \in O$ and $g^{-1}_{j_{k_l}} q_{j_{k_l}} \in \overline{N_p}$, it follows that $g^{-1}p = p$, *i.e.*, $g \in I_p$. The map F_ν satisfies (2.12) for all $g \in U_n$ and $q \in M$, hence $F_\nu(q_{j_{k_l}}) \in \overline{N_{\phi^{-1}_{n,m}(g_{j_{k_l}})\hat{p}}}$, where $N_{\phi^{-1}_{n,m}(g_{j_{k_l}})\hat{p}} \subset \widehat{B^n_R}/\mathbb{Z}_m$ is constructed similarly to $N_p \subset \tilde{M}$. Therefore the limit of $\{F_\nu(q_{j_{k_l}})\}$ (equal to the

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limit of $\{F_\nu(q_{j_k})\}$) is $\hat{p}$. Hence F_ν is continuous, and therefore holomorphic on M. It obviously maps M biholomorphically onto $\widehat{B^n_R}/\mathbb{Z}_m$.

The case when $r > 0$ and $R = \infty$ can be treated along the same lines, but one must consider the holomorphic embedding $\sigma\colon S^n_{r,\infty}/\mathbb{Z}_m \to \widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$ such that

$$\sigma(\langle z\rangle) := \{(1 : z_1 : \cdots : z_n)\},$$

the map $f_\sigma := \sigma \circ f$, and prove that f_σ extends to O as a biholomorphic map of M onto $\widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$.

If $r = 0$ and $R = \infty$, then precisely one of f_ν and f_σ extends to O, and the extension defines a biholomorphic map from M to either $\widehat{\mathbb{C}^n}/\mathbb{Z}_m$, or $\widetilde{S^n_{0,\infty}}/\mathbb{Z}_m$.

Let now $n = 2$. We fix $p \in O$ and consider I_p. There exists $g \in U_2$ such that $I_p = g^{-1}Hg$. As above, we introduce the sets $N_{p,q}$, *i.e.*, for an arbitrary real hypersurface orbit $O(q)$ we set

$$N_{p,q} := \{s \in O(q) : I_s \subset I_p\}.$$

Since I_s is conjugate in U_2 to a subgroup H_{k_1,k_2}, where $k_1 := k$ and $k_2 = k + 1 \neq 0$, it follows that

$$N_{p,q} = \{s \in O(q) : I_s = g^{-1}H_{k_1,k_2}g\} \cup \{s \in O(q) : I_s = g^{-1}h_0H_{k_1,k_2}h_0g\},$$

where

$$h_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

i.e., for $n = 2$, $N_{p,q}$ has two connected components. We denote them $N^1_{p,q}$ and $N^2_{p,q}$, respectively. It is easy to show now that if we fix $t \in N_{p,q}$, then $N^1_{p,q} = \{ht\}$ and $N^2_{p,q} = \{g^{-1}h_0ght\}$, where

$$h = g^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g, \quad \alpha \in U_1.$$

We now consider the corresponding sets N^1_p and N^2_p. The point p is the accumulation point in O for exactly one of these sets. As above, we obtain that either $r = 0$, or $R = \infty$. For example, assume that $r = 0$ and $R < \infty$. Let $\hat{O}$ be the orbit in $\widehat{B^2_R}/\mathbb{Z}_m$ that is a complex hypersurface. There are precisely two points in $\hat{O}$ whose isotropy subgroups in $U_2/\mathbb{Z}_m$ coincide with $\phi^{-1}_{2,m}(I_p)$. These points $\hat{p}_1$ and $\hat{p}_2$ are the accumulation points in $\hat{O}$ of $\nu(N'^1_p)$ and $\nu(N'^2_p)$, where $N'^1_p, N'^2_p \subset S^2_{0,R}/\mathbb{Z}_m$ are the sets of points with isotropy subgroups equal to $\phi^{-1}_{2,m}(g^{-1}H_{k_1,k_2}g)$ and $\phi^{-1}_{2,m}(g^{-1}h_0H_{k_1,k_2}h_0g)$ respectively. We then define the extension F_ν of f_ν by setting $F_\nu(p) = \hat{p}_1$ if N^1_p accumulates to p and $F_\nu(p) = \hat{p}_2$ if N^2_p accumulates to p. The proof of the continuity of F_ν proceeds as for $n \geq 3$. The arguments in the cases $r > 0$, $R = \infty$ and $r = 0$, $R = \infty$ are analogous to the above.

Assume now that two orbits O_1 and O_2 in M are complex hypersurfaces. As above, we consider the manifold $\tilde{M}$ obtained from M by removing O_1 and O_2. For

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some $k \in \mathbb{Z}$, $m = |nk + 1|$, and some r and R, $0 \leq r < R \leq \infty$, it is biholomorphically equivalent to $S^n_{r,R}/\mathbb{Z}_m$ by means of a map f satisfying either (2.12) or (2.13). Arguments very similar to the ones used above show that in this case $r = 0$, $R = \infty$, and $f_\tau := \tau \circ f$ extends to a biholomorphic map $M \to \widehat{\mathbb{CP}^n}/\mathbb{Z}_m$. Here $\tau\colon (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_m \to \widehat{\mathbb{CP}}^n/\mathbb{Z}_m$ is a $U_n/\mathbb{Z}_m$-equivariant map defined as

$$\tau(\langle z\rangle) := \Big\{ \big((1 : z_1 : \cdots : z_n), w\big) \Big\},$$

where $w = (w_1 : \cdots : w_n)$ is uniquely determined from the conditions $z_i w_j = z_j w_i$ for all i, j.

The proof is complete. ∎

4 The Homogeneous Case

We consider now the case when the action of U_n on M is transitive.

Example 4.1 Examples of manifolds on which U_n acts transitively and effectively are the Hopf manifolds M^n_d (see Definition 2.6). Let λ be a complex number such that $e^{\frac{2\pi(\lambda - i)}{nK}} = d$ for some $K \in \mathbb{Z} \setminus \{0\}$. We define an action of U_n on M^n_d as follows. Let $A \in U_n$. We can represent A in the form $A = e^{it} \cdot B$, where $t \in \mathbb{R}$ and $B \in \mathrm{SU}_n$. Then we set

$$A[z] := [e^{\lambda t} \cdot Bz]. \tag{4.1}$$

Of course, we must verify that this action is well-defined. Indeed, the same element $A \in U_n$ can be also represented in the form $A = e^{i(t + \frac{2\pi k}{n} + 2\pi l)} \cdot (e^{-\frac{2\pi i k}{n}} B)$, $0 \leq k \leq n-1$, $l \in \mathbb{Z}$. Then formula (4.1) yields

$$A[z] = [e^{\lambda(t + \frac{2\pi k}{n} + 2\pi l)} \cdot e^{-\frac{2\pi i k}{n}} Bz] = [d^{kK + nKl} e^{\lambda t} \cdot Bz] = [e^{\lambda t} \cdot Bz].$$

It is also clear that (4.1) does not depend on the choice of representative in the class $[z]$.

The action in question is obviously transitive. It is also effective. For let $e^{it} \cdot B[z] = [z]$ for some $t \in \mathbb{R}$, $B \in \mathrm{SU}_n$, and all $z \in \mathbb{C}^n \setminus \{0\}$. Then, for some $k \in \mathbb{Z}$, $B = e^{\frac{2\pi i k}{n}} \cdot \mathrm{id}$, and some $s \in \mathbb{Z}$ the following holds

$$e^{\lambda t} \cdot e^{\frac{2\pi i k}{n}} = d^s.$$

Using the definition of λ we obtain

$$t = \frac{2\pi s}{nK},$$
$$e^{\frac{2\pi i k}{n}} = e^{-\frac{2\pi i s}{nK}}.$$

Hence $e^{it} \cdot B = \mathrm{id}$, and thus the action is effective.

The isotropy subgroup of the point $[(1, 0, \dots, 0)]$ is $G_{K,1} \cdot \mathrm{SU}_{n-1}$, where SU_{n-1} is embedded in U_n in the standard way and $G_{K,1}$ consists of all matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \beta \cdot \mathrm{id} \end{pmatrix},$$

where $\beta^{(n-1)K} = 1$.

Another example is provided by the manifolds $M_d^n/\mathbb{Z}_m$ (see Definition 2.6). Let $\{[z]\} \in M_d^n/\mathbb{Z}_m$ be the equivalence class of $[z]$. We define an action of U_n on $M_d^n/\mathbb{Z}_m$ by the formula $g\{[z]\} := \{g[z]\}$ for $g \in U_n$. This action is clearly transitive; it is also effective if, *e.g.*, $(n, m) = 1$ and $(K, m) = 1$.

The isotropy subgroup of the point $\{[(1, 0, \dots, 0)]\}$ is $G_{K,m} \cdot \mathrm{SU}_{n-1}$, where $G_{K,m}$ consists of all matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \cdot \mathrm{id} \end{pmatrix}, \tag{4.2}$$

with $\alpha^m = 1$ and $\alpha^K \beta^{K(n-1)} = 1$. Note that in this case every orbit of the induced action of SU_n is equivariantly diffeomorphic to the lense manifold $\mathcal{L}_m^{2n-1}$.

One can consider more general actions by choosing λ such that $e^{\frac{2\pi(\lambda - i)}{n}} = d^K$, but not all such actions are effective.

We shall now describe complex manifolds admitting effective transitive actions of U_n. It turns out that such a manifold is always biholomorphically equivalent to one of the manifolds $M_d^n/\mathbb{Z}_m$. To prove this we shall look at orbits of the induced action of SU_n. We require the following algebraic lemma first.

Lemma 4.2 *Let G be a connected closed subgroup of U_n of dimension $n^2 - 2n$, $n \geq 2$. Then either*

(i) *G is irreducible as a subgroup of $\mathrm{GL}_n(\mathbb{C})$, or*
(ii) *G is conjugate to SU_{n-1} embedded in U_n in the standard way, or*
(iii) *for $n = 3$, G is conjugate to $U_1 \times U_1 \times U_1$ embedded in U_3 in the standard way, or*
(iv) *for $n = 4$, G is conjugate to $U_2 \times U_2$ embedded in U_4 in the standard way.*

Proof We start as in the proof of Lemma 2.1. Since G is compact, it is completely reducible, *i.e.*, $\mathbb{C}^n$ splits into a sum of G-invariant pairwise orthogonal complex subspaces, $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_m$, such that the restriction G_j of G to every V_j is irreducible. Let $n_j := \dim_{\mathbb{C}} V_j$ (hence $n_1 + \cdots + n_m = n$) and let U_{n_j} be the unitary transformation group of V_j. Clearly, $G_j \subset U_{n_j}$, and therefore $\dim G \leq n_1^2 + \cdots + n_m^2$. On the other hand $\dim G = n^2 - 2n$, which shows that $m \leq 2$ for $n \neq 3$. If $n = 3$, then it is also possible that $m = 3$, which means that G is conjugate to $U_1 \times U_1 \times U_1$ embedded in U_3 in the standard way.

Now let $m = 2$. Then either there exists a unitary transformation of $\mathbb{C}^n$ such that each element of G has in the new coordinates the form (2.3) with $a \in U_1$ and $B \in U_{n-1}$ or, for $n = 4$, G is conjugate to $U_2 \times U_2$. We note that, in the first case, the scalars a and the matrices B, that arise from elements of G in (2.3) form compact connected subgroups of U_1 and U_{n-1} respectively; we shall denote them by G_1 and G_2 as above.

If $\dim G_1 = 0$, then $G_1 = \{1\}$, and therefore $G_2 = \mathrm{SU}_{n-1}$.

Assume that $\dim G_1 = 1$, *i.e.*, $G_1 = U_1$. Therefore, $n \geq 3$. Then $(n-1)^2 - 2 \leq \dim G_2 \leq (n-1)^2 - 1$. It follows from Lemma 2.1 of [IKra] that, for $n \neq 3$, we have $G_2 = \mathrm{SU}_{n-1}$. For $n = 3$ it is also possible that $G_2 = U_1 \times U_1$, and therefore G is conjugate to $U_1 \times U_1 \times U_1$ embedded in U_3 in the standard way. Assume that $G_2 = \mathrm{SU}_{n-1}$ and consider the Lie algebra $\mathfrak{g}$ of G. It consists of all matrices of the form (2.4) with b an arbitrary matrix in $\mathfrak{su}_{n-1}$ and $l(b)$ a linear function of the matrix elements of b ranging in $i\mathbb{R}$. However, $l(b)$ must vanish on the commutant of $\mathfrak{su}_{n-1}$ which is $\mathfrak{su}_{n-1}$ itself. Consequently, $l(b) \equiv 0$, which contradicts our assumption that $G_1 = U_1$.

The proof is complete. ∎

We can now prove the following proposition.

Proposition 4.3 *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective transitive action of U_n by biholomorphic transformations. Then there exists $m \in \mathbb{N}$, $(n, m) = 1$, such that for each $p \in M$ the orbit $\tilde{O}(p)$ of the induced action of SU_n is a real hypersurface in M that is SU_n-equivariantly diffeomorphic to the lense manifold $\mathcal{L}_m^{2n-1}$ endowed with the standard action of $\mathrm{SU}_n \subset U_n/\mathbb{Z}_m$.*

Proof Since M is homogeneous under the action of U_n, for every $p \in M$ we have $\dim I_p = n^2 - 2n$. We now apply Lemma 4.2 to the identity component I_p^c. Clearly, if I_p^c contains the center of U_n, then the action of U_n on M is not effective, and therefore cases (iii) and (iv) cannot occur. We claim that case (i) does not occur either.

Since M is compact, the group $\mathrm{Aut}(M)$ of all biholomorphic automorphisms of M is a complex Lie group. Hence we can extend the action of U_n to a holomorphic transitive action of $\mathrm{GL}_n(\mathbb{C})$ on M (see [H], pp. 204–207). Let J_p be the isotropy subgroup of p with respect to this action. Clearly, $\dim_{\mathbb{C}} J_p = n^2 - n$. Consider the normalizer $N(J_p^c)$ of J_p^c in $\mathrm{GL}_n(\mathbb{C})$. It is known from results of Borel-Remmert and Tits (see Theorem 4.2 in [A2]) that $N(J_p^c)$ is a parabolic subgroup of $\mathrm{GL}_n(\mathbb{C})$. We note that $N(J_p^c) \neq \mathrm{GL}_n(\mathbb{C})$. For otherwise J_p^c would be a normal subgroup of $\mathrm{GL}_n(\mathbb{C})$. But $\mathrm{GL}_n(\mathbb{C})$ contains no normal subgroup of dimension $n^2 - n$. Indeed, considering the intersection of such a subgroup with $\mathrm{SL}_n(\mathbb{C})$, we would obtain a normal subgroup of $\mathrm{SL}_n(\mathbb{C})$ of positive dimension thus arriving at a contradiction.

All parabolic subgroups of $\mathrm{GL}_n(\mathbb{C})$ are well-known. Let $n = n_1 + \cdots + n_r$, $n_j \geq 1$, and let $P(n_1, \ldots, n_r)$ be the group of all matrices that have blocks of sizes $n_1, \ldots, n_r$ on the diagonal, arbitrary entries above the blocks, and zeros below. Then an arbitrary parabolic subgroup of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to some subgroup $P(n_1, \ldots, n_r)$.

Since the normalizer $N(J_p^c)$ does not coincide with $\mathrm{GL}_n(\mathbb{C})$, it is conjugate to a subgroup $P(n_1, \ldots, n_r)$ with $r \geq 2$. Hence there exists a proper subspace of $\mathbb{C}^n$ that is invariant under the action of $N(J_p^c)$, and therefore under the action of I_p^c. Thus, I_p^c cannot be irreducible.

Hence there exists $g \in U_n$ such that $gI_p^c g^{-1} = \mathrm{SU}_{n-1}$, where SU_{n-1} is embedded in U_n in the standard way. Clearly, the element g can be chosen from SU_n, and hence I_p^c is contained in SU_n and is conjugate in SU_n to SU_{n-1}.

Consider now the orbit $\tilde{O}(p)$ of a point $p \in M$ under the induced action of SU_n, and let $\tilde{I}_p \subset \mathrm{SU}_n$ be the isotropy subgroup of p with respect to this action. Clearly, $\tilde{I}_p = I_p \cap \mathrm{SU}_n$. Since I_p^c lies in SU_n, it follows that $\tilde{I}_p^c = I_p^c$. In particular, $\dim \tilde{I}_p = n^2 - 2n$, and therefore $\tilde{O}(p)$ is a real hypersurface in M.

Assume now that $n \geq 3$. We require the following lemma.

Lemma 4.4 *Let G be a closed subgroup of SU_n, $n \geq 3$, such that $G^c = \mathrm{SU}_{n-1}$, where SU_{n-1} is embedded in SU_n in the standard way. Let m be the number of connected components of G. Then $G = G_{1,m} \cdot \mathrm{SU}_{n-1}$, where the group $G_{1,m}$ is defined in (4.2).*

Proof of Lemma 4.4 Let $C_1, \dots, C_m$ be the connected components of G with $C_1 = \mathrm{SU}_{n-1}$. Clearly, there exist $g_1 = \mathrm{id}, g_2, \dots, g_m$ in SU_n such that $C_j = g_j \, \mathrm{SU}_{n-1}$, $j = 1, \dots, m$. Moreover, for each pair of indices i, j there exists k such that $g_i \, \mathrm{SU}_{n-1} \cdot g_j \, \mathrm{SU}_{n-1} = g_k \, \mathrm{SU}_{n-1}$, and therefore

$$g_k^{-1} g_i \, \mathrm{SU}_{n-1} \, g_j = \mathrm{SU}_{n-1} \,. \tag{4.3}$$

Applying (4.3) to the vector $v := (1, 0, \dots, 0)$, which is preserved by the standard embedding of SU_{n-1} in SU_n, we obtain

$$g_k^{-1} g_i \, \mathrm{SU}_{n-1} \, g_j v = v,$$

i.e.,

$$\mathrm{SU}_{n-1} \, g_j v = g_i^{-1} g_k v,$$

which implies that $g_j v = (\alpha_j, 0, \dots, 0)$, $|\alpha_j| = 1$, $j = 1, \dots, m$. Hence g_j has the form

$$g_j = \begin{pmatrix} \alpha_j & 0 \\ 0 & A_j \end{pmatrix},$$

where $A_j \in U_{n-1}$ and $\det A_j = 1/\alpha_j$. Since A_j can be written in the form $A_j = \beta_j \cdot B_j$ with $B_j \in \mathrm{SU}_{n-1}$, we can assume without loss of generality that $A_j = \beta_j \cdot \mathrm{id}$. Clearly, each matrix

$$g_j \cdot \begin{pmatrix} 1 & 0 \\ 0 & \sigma \cdot \mathrm{id} \end{pmatrix}$$

where j is arbitrary and $\sigma^{n-1} = 1$, also belongs to G. Further, it is clear that the parameters α_j, $j = 1, \dots, m$, are all distinct and form a finite subgroup of U_1, which is therefore the group of m-th roots of unity.

Thus, $G = G_{1,m} \cdot \mathrm{SU}_{n-1}$, as required. ■

It now follows from Lemma 4.4 that if $n \geq 3$, then for each $p \in M$, $\tilde{I}_p$ is conjugate in SU_n to one of the groups $G_{1,m} \cdot \mathrm{SU}_{n-1}$ with $m \in \mathbb{N}$. Hence $\tilde{O}(p)$ is SU_n-equivariantly diffeomorphic to $\mathcal{L}_m^{2n-1}$. Clearly, the SU_n-action is effective on $\tilde{O}(p)$

only if $(n, m) = 1$. The integer m does not depend on p since all isotropy subgroups I_p are conjugate in U_n. This proves Proposition 4.3 for $n \geq 3$.

Now let $n = 2$. Since $\tilde{O}(p)$ is a homogeneous real hypersurface, it is either strongly pseudoconvex or Levi-flat. Assume that $\tilde{O}(p)$ is Levi-flat. Then it is foliated by complex curves. Let $\mathfrak{m}$ be the Lie algebra of all holomorphic vector fields on $\tilde{O}(p)$ corresponding to the automorphisms of $\tilde{O}(p)$ generated by the action of SU_2. Clearly, $\mathfrak{m}$ is isomorphic to $\mathfrak{su}_2$. Let M_p be the leaf of the foliation passing through p, and consider the subspace $\mathfrak{l} \subset \mathfrak{m}$ of vector fields tangent to M_p at p. The vector fields in $\mathfrak{l}$ remain tangent to M_p at each point $q \in M_p$, and therefore $\mathfrak{l}$ is in fact a Lie subalgebra of $\mathfrak{m}$. However, $\dim \mathfrak{l} = 2$ and $\mathfrak{su}_2$ has no 2-dimensional subalgebras. Hence $\tilde{O}(p)$ must be strongly pseudoconvex.

Similarly to the proof of Proposition 2.2, we can now show that $\tilde{I}_p$ is isomorphic to a subgroup of U_1. This means that $\tilde{I}_p$ is a finite cyclic group, *i.e.*, $\tilde{I}_p = \{A^l, 0 \leq l < m\}$ for some $A \in \mathrm{SU}_2$ and $m \in \mathbb{N}$ such that $A^m = \mathrm{id}$. Choosing new coordinates in which A is in the diagonal form, we see that $\tilde{I}_p$ is conjugate in SU_2 to the group of matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \alpha^m = 1.$$

Hence $\tilde{O}(p)$ is SU_2-equivariantly diffeomorphic to the lense manifold $\mathcal{L}^3_m$. Clearly, the action of SU_2 is effective on $\tilde{O}(p)$ only if m is odd. The integer m does not depend on p since all isotropy subgroups I_p are conjugate in U_2. This proves Proposition 4.3 for $n = 2$ and completes the proof in general. ■

We can now establish the following result.

Theorem 4.5 *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective transitive action of U_n by biholomorphic transformations. Then M is biholomorphically equivalent to some manifold $M^n_d/\mathbb{Z}_m$, where $m \in \mathbb{N}$ and $(n, m) = 1$. The equivalence $f: M \to M^n_d/\mathbb{Z}_m$ can be chosen to satisfy either the relation*

$$f(gq) = gf(q), \tag{4.4}$$

or, for $n \geq 3$, the relation

$$f(gq) = \bar{g}f(q), \tag{4.5}$$

for all $g \in \mathrm{SU}_n$ and $q \in M$ (here $M^n_d/\mathbb{Z}_m$ is considered with the standard action of SU_n).

Proof We claim first that M is biholomorphically equivalent to some manifold $M^n_d/\mathbb{Z}_m$. For a proof we only need to show that M is diffeomorphic to $S^1 \times \mathcal{L}^{2n-1}_m$ for some $m \in \mathbb{N}$ such that $(n, m) = 1$. Then biholomorphic equivalence will follow from Theorem 3.1 of [A1].

Choose m provided by Proposition 4.3. For $p \in M$ we consider the SU_n-orbit $\tilde{O}(p)$. Let $t_0 := \min\{t > 0 : e^{it}p \in \tilde{O}(p)\}$. Clearly, $t_0 > 0$. For each point $q \in \tilde{O}(p)$ there exists $B \in \mathrm{SU}_n$ such that $q = Bp$. Hence

$$e^{it_0}q = e^{it_0}(Bp) = (e^{it_0}B)p = (Be^{it_0})p = B(e^{it_0}p), \tag{4.6}$$

and $e^{it_0}\tilde{O}(p) = \tilde{O}(p)$. This shows that $M' := \cup_{0\le t<t_0} e^{it}\tilde{O}(p)$ is a closed submanifold of M of dimension n. Since M is connected, it follows that $M' = M$.

Let $p_t := e^{it}p$, $0 \le t \le t_0$. We consider a curve $\gamma\colon [0,t_0] \to M$ such that $\gamma(0) = \gamma(t_0) = p$, $\gamma(t) \in \tilde{O}(p_t)$ for each t, and $\gamma([0,t_0])$ is diffeomorphic to S^1. We can assume that $\tilde{I}_p = G_{1,m}\cdot \mathrm{SU}_{n-1}$, which is also the isotropy subgroup, with respect to the standard action of SU_n on $\mathcal{L}_m^{2n-1}$, of the point $q \in \mathcal{L}_m^{2n-1}$ represented by the point $(1,0,\dots,0) \in S^{2n-1}$. Further, for each $0 < t < t_0$, there exists $g_t \in \mathrm{SU}_n$ such that $\tilde{I}_{\gamma(t)} = g_t \tilde{I}_p g_t^{-1}$. Clearly, $\tilde{I}_{\gamma(t)}$ is the isotropy subgroup of the point $q_t := g_t q$ in $\mathcal{L}_m^{2n-1}$. Hence the map

$$\phi_t\big(h\gamma(t)\big) = hq_t,$$

where $h \in \mathrm{SU}_n$, maps the orbit $\tilde{O}(p_t)$ diffeomorphically (and SU_n-equivariantly) onto $\mathcal{L}_m^{2n-1}$, $0 \le t \le t_0$ (here we set $g_0 := g_{t_0} := \mathrm{id}$, $q_0 := q_{t_0} := q$).

We define now a map $\Phi\colon M \to S^1 \times \mathcal{L}_m^{2n-1}$. For each $x \in M$ there exists a unique $0 \le t < t_0$, such that $x \in \tilde{O}(p_t)$. We set

$$\Phi(x) = \Big(e^{\frac{2\pi i t}{t_0}}, \phi_t(x)\Big).$$

It is clear that g_t, and therefore q_t can be chosen so that Φ is a diffeomorphism. Hence M is biholomorphically equivalent to one of the manifolds $M_d^n/\mathbb{Z}_m$.

Let $F\colon M \to M_d^n/\mathbb{Z}_m$ be a biholomorphic equivalence. Using F, the action of SU_n on M can be pushed to an action of SU_n by biholomorphic transformations on $M_d^n/\mathbb{Z}_m$. The group $\mathrm{Aut}(M_d^n/\mathbb{Z}_m)$ of all biholomorphic automorphisms of $M_d^n/\mathbb{Z}_m$ is isomorphic to $Q_{d,m}^n := (\mathrm{GL}_n(\mathbb{C})/\{d^k\cdot \mathrm{id}, k\in\mathbb{Z}\})/\mathbb{Z}_m$ (this can be seen, for example, by lifting automorphisms of $M_d^n/\mathbb{Z}_m$ to its universal cover $\mathbb{C}^n \setminus \{0\}$). Each maximal compact subgroup of this group is conjugate to a subgroup of the form $(U_n/\mathbb{Z}_m)\times K$, where $U_n/\mathbb{Z}_m$ is embedded in $Q_{d,m}^n$ in the standard way, and K is isomorphic to S^1. The action of SU_n on $M_d^n/\mathbb{Z}_m$ induces an embedding $\tau\colon \mathrm{SU}_n \to Q_{d,m}^n$. Since SU_n is compact, there exists $s \in Q_{d,m}^n$ such that $\tau(\mathrm{SU}_n)$ is contained in $s\big((U_n/\mathbb{Z}_m)\times K\big)s^{-1}$. However, there exists no nontrivial homomorphism from SU_n into S^1, and therefore $\tau(\mathrm{SU}_n) \subset s(U_n/\mathbb{Z}_m)s^{-1}$. Since $(n,m) = 1$, it follows that $\tau(\mathrm{SU}_n) = s\,\mathrm{SU}_n\,s^{-1}$, where SU_n in the right-hand side is embedded in $Q_{d,m}^n$ in the standard way.

We now set $f := \hat{s}^{-1}\circ F$, where $\hat{s}$ is the automorphism of $M_d^n/\mathbb{Z}_m$ corresponding to $s \in Q_{d,m}^n$. Pushing now the action of SU_n on M to an action of SU_n on $M_d^n/\mathbb{Z}_m$ by means of f in place of F, for the corresponding embedding $\tau_s\colon \mathrm{SU}_n \to Q_{d,m}^n$ we obtain the equality $\tau_s(\mathrm{SU}_n) = \mathrm{SU}_n$, where SU_n in the right-hand side is embedded in $Q_{d,m}^n$ in the standard way. Thus, there exists an automorphism γ of SU_n such that

$$f(gq) = \gamma(g)f(q),$$

for all $g \in \mathrm{SU}_n$ and $q \in M$.

Assume first that $n \ge 3$. Then each automorphism of SU_n has either the form

$$g \mapsto h_0 g h_0^{-1}, \tag{4.7}$$

or the form

$$g \mapsto h_0 \bar{g} h_0^{-1}, \tag{4.8}$$

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for some fixed $h_0 \in \mathrm{SU}_n$ (see, *e.g.*, [VO]). If γ has the form (4.7), then considering in place of f the map $q \mapsto h_0^{-1} f(q)$ we obtain a biholomorphic map satisfying (4.4). If γ has the form (4.8), then considering in place of f the map $q \mapsto h_0^{-1} f(q)$ we obtain a biholomorphic map satisfying (4.5).

Let $n = 2$. Each automorphism of SU_2 has the form (4.7) and arguing as above we obtain a biholomorphic map satisfying (4.4).

The proof is complete. ∎

Remark 4.6 For $n \geq 3$ Theorem 4.5 can be proved without referring to the results in [A1]. We note first that the SU_n-equivariant diffeomorphism between $\mathcal{L}_m^{2n-1}$ and $\tilde{O}(p)$ constructed in Proposition 4.3 is either a CR or an anti-CR map (here we consider $\mathcal{L}_m^{2n-1}$ is with the CR-structure inherited from S^{2n-1}). The corresponding proof is similar to the proof of Proposition 2.4. We must only replace U_n and $U_n/\mathbb{Z}_m$ by SU_n and $\phi_{n,m}$ by the identity map. Further we argue as in the second part of the proof of Theorem 2.7 for compact M, replacing there U_n by SU_n.

Remark 4.7 Ideally, one would like the biholomorphic equivalence in Theorem 4.5 to be U_n-equivariant, rather than just SU_n-equivariant. However, as Example 4.1 shows, there is no canonical transitive action of U_n on $M_d^n/\mathbb{Z}_m$. It is not hard, however, to write a general formula for such actions, but we do not do it here.

5 A Characterization of $\mathbb{C}^n$

In this section we apply the results obtained above to prove the following theorem.

Theorem 5.1 *Let M be a connected complex manifold of dimension n. Assume that* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(\mathbb{C}^n)$ *are isomorphic as topological groups. Then M is biholomorphically equivalent to* $\mathbb{C}^n$.

Proof The theorem is trivial for $n = 1$, so we assume that $n \geq 2$. Since M admits an effective action of U_n by biholomorphic transformations, M is biholomorphically equivalent to one of the manifolds listed in Remark 1.2, Theorem 2.7, Theorem 3.3 and Theorem 4.5. The automorphism groups of the following manifolds are clearly Lie groups: B^n, $\mathbb{CP}^n$, $S_{r,R}^n/\mathbb{Z}_m$ for $r > 0$ or $R < \infty$, $M_d^n/\mathbb{Z}_m$, $\widehat{B_R^n}/\mathbb{Z}_m$, $\widetilde{S_{r,\infty}^n}/\mathbb{Z}_m$, $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$. Since $\mathrm{Aut}(M)$ is isomorphic to $\mathrm{Aut}(\mathbb{C}^n)$ and $\mathrm{Aut}(\mathbb{C}^n)$ is not locally compact, $\mathrm{Aut}(M)$ cannot be isomorphic to a Lie group and hence M is not biholomorphically equivalent to any of the above manifolds.

Therefore, M is biholomorphically equivalent to either $\mathbb{C}^n$, or $\mathbb{C}^{n*}/\mathbb{Z}_m$, where $\mathbb{C}^{n*} := \mathbb{C}^n \setminus \{0\}$ and $m = |nk + 1|$ for some $k \in \mathbb{Z}$. We will now show that the groups $\mathrm{Aut}(\mathbb{C}^n)$ and $\mathrm{Aut}(\mathbb{C}^{n*}/\mathbb{Z}_m)$ are not isomorphic.

Let first $m = 1$. The group $\mathrm{Aut}(\mathbb{C}^{n*})$ consists of exactly those elements of $\mathrm{Aut}(\mathbb{C}^n)$ that fix the origin. Suppose that $\mathrm{Aut}(\mathbb{C}^n)$ and $\mathrm{Aut}(\mathbb{C}^{n*})$ are isomorphic and let ψ: $\mathrm{Aut}(\mathbb{C}^n) \to \mathrm{Aut}(\mathbb{C}^{n*})$ denote an isomorphism. Clearly, $\psi(U_n)$ induces an action of U_n on $\mathbb{C}^{n*}$, and therefore, by our results above, there is $F \in \mathrm{Aut}(\mathbb{C}^{n*})$ such that for the isomorphism $\psi_F\colon \mathrm{Aut}(\mathbb{C}^n) \to \mathrm{Aut}(\mathbb{C}^{n*})$, $\psi_F(g) := F \circ \psi(g) \circ F^{-1}$, we have: either $\psi_F(g) = g$, or $\psi_F(g) = \bar{g}$ for all $g \in U_n$.

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Consider U_{n-1} embedded in U_n in the standard way, and consider its centralizer C in $\mathrm{Aut}(\mathbb{C}^n)$, *i.e.*,

$$C := \{f \in \mathrm{Aut}(\mathbb{C}^n) : f \circ g = g \circ f \text{ for all } g \in U_{n-1}\}.$$

It is easy to show that C consists of maps $f = (f_1, \dots, f_n)$ such that

$$\begin{aligned} f_1 &= az_1 + b, \\ f' &= h(z_1)z', \end{aligned} \tag{5.1}$$

where $z' := (z_2, \dots, z_n)$, $f' := (f_2, \dots, f_n)$, $a, b \in \mathbb{C}$, $a \neq 0$, $h(z_1)$ is a nowhere vanishing entire function. Similarly, let C^* be the centralizer of U_{n-1} in $\mathrm{Aut}(\mathbb{C}^{n*})$. It consists of maps $f = (f_1, \dots, f_n)$ such that

$$\begin{aligned} f_1 &= az_1, \\ f' &= h(z_1)z', \end{aligned} \tag{5.2}$$

where $a \in \mathbb{C}$, $a \neq 0$, $h(z_1)$ is entire and nowhere vanishing. Clearly, $\psi_F(C) = C^*$.

Let C' and $C^{*'}$ denote the commutants of C and C^* respectively. Clearly, $\psi_F(C') = C^{*'}$. It is easy to check that $C^{*'}$ consists exactly of all maps of the form (5.2) where $a = 1$ and $h(0) = 1$. In particular, $C^{*'}$ is Abelian. We will now show that C' is not Abelian. Indeed, consider the following elements of C (see (5.1)):

$$\begin{aligned} f(z_1, z') &:= (z_1 + 1, z'), \\ g(z_1, z') &:= (2z_1, z'), \\ u(z_1, z') &:= (z_1 + 1, e^{z_1} z'). \end{aligned}$$

We now see that

$$\begin{aligned} F(z_1, z') &:= f \circ g \circ f^{-1} \circ g^{-1} = (z_1 - 1, z'), \\ G(z_1, z') &:= u \circ g \circ u^{-1} \circ g^{-1} = (z_1 - 1, e^{\frac{z_1 - 2}{2}} z'). \end{aligned}$$

Clearly, $F, G \in C'$, and we have

$$\begin{aligned} F \circ G &= (z_1 - 2, e^{\frac{z_1 - 2}{2}} z'), \\ G \circ F &= (z_1 - 2, e^{\frac{z_1 - 3}{2}} z'). \end{aligned}$$

Hence $F \circ G \neq G \circ F$, and thus C' is not Abelian. Therefore, C' and $C^{*'}$ are not isomorphic. This contradiction shows that $\mathrm{Aut}(\mathbb{C}^n)$ and $\mathrm{Aut}(\mathbb{C}^{n*})$ are not isomorphic.

Let now $m > 1$. For $z \in \mathbb{C}^{n*}$ denote as before by $\langle z \rangle \in \mathbb{C}^{n*}/\mathbb{Z}_m$ its equivalence class. Let

$$H_m^n := \{f \in \mathrm{Aut}(\mathbb{C}^{n*}) : \langle f(z) \rangle = \langle f(\tilde{z}) \rangle, \text{if } \langle z \rangle = \langle \tilde{z} \rangle\}.$$

The group $\mathrm{Aut}(\mathbb{C}^{n*}/\mathbb{Z}_m)$ is isomorphic in the obvious way to $H^n_m/\mathbb{Z}_m$. Suppose that $\mathrm{Aut}(\mathbb{C}^n)$ and $\mathrm{Aut}(\mathbb{C}^{n*}/\mathbb{Z}_m)$ are isomorphic and let $\psi\colon \mathrm{Aut}(\mathbb{C}^n) \to \mathrm{Aut}(\mathbb{C}^{n*}/\mathbb{Z}_m)$ denote an isomorphism. Clearly, $\psi(U_n)$ induces an action of U_n on $\mathbb{C}^{n*}/\mathbb{Z}_m$, and therefore there is $F \in \mathrm{Aut}(\mathbb{C}^{n*}/\mathbb{Z}_m)$ such that for the isomorphism $\psi_F\colon \mathrm{Aut}(\mathbb{C}^n) \to \mathrm{Aut}(\mathbb{C}^{n*})$, $\psi_F(g) := F \circ \psi(g) \circ F^{-1}$, we have: either $\psi_F(g) = \phi^{-1}_{n,m}(g)$, or $\psi_F(g) = \phi^{-1}_{n,m}(\bar{g})$ for all $g \in U_n$, where we consider $U_n/\mathbb{Z}_m$ embedded in $H^n_m/\mathbb{Z}_m$.

The rest of the proof proceeds as for the case $m = 1$ above with obvious modifications. We consider the centralizer C^*_m of $\phi^{-1}_{n,m}(U_{n-1}) = \phi^{-1}_{n,m}(\overline{U_{n-1}}) \subset H^n_m/\mathbb{Z}_m$. Clearly, $\psi_F(C) = C^*_m$. Then we find the commutant ${C^*_m}'$ of C^*_m, and we have $\psi_F(C') = {C^*_m}'$. As above, it turns out that ${C^*_m}'$ is Abelian. Therefore, $\mathrm{Aut}(\mathbb{C}^n)$ and $\mathrm{Aut}(\mathbb{C}^{n*}/\mathbb{Z}_m)$ cannot be isomorphic.

The proof is complete. ■

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