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GERSTENHABER BRACKET ON THE HOCHSCHILD COHOMOLOGY VIA AN ARBITRARY RESOLUTION

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Abstract We prove formulas of different types that allow us to calculate the Gerstenhaber bracket on the Hochschild cohomology of an algebra using some arbitrary projective bimodule resolution for it. Using one of these formulas, we give a new short proof of the derived invariance of the Gerstenhaber algebra structure on Hochschild cohomology. We also give some new formulas for the Connes differential on the Hochschild homology that lead to formulas for the Batalin–Vilkovisky (BV) differential on the Hochschild cohomology in the case of symmetric algebras. Finally, we use one of the obtained formulas to provide a full description of the BV structure and, correspondingly, the Gerstenhaber algebra structure on the Hochschild cohomology of a class of symmetric algebras.

Keywords: Hochschild cohomology; Gerstenhaber bracket; bimodule resolution; derived equivalence

2010 *Mathematics subject classification:* Primary 16E05; 16E35; 16E40

1. Introduction

Let A be an associative unital algebra over a field $\mathbf{k}$. The Hochschild cohomology $\mathrm{HH}^*(A)$ of A has a very rich structure. It is a graded commutative algebra via the cup product or the Yoneda product, and it has a graded Lie bracket of degree -1 so that it becomes a graded Lie algebra; these make $\mathrm{HH}^*(A)$ a Gerstenhaber algebra [**4**]. These structures have a good description in terms of the bar resolution of A, but this resolution is huge and so it is frequently useless for concrete computations.

The cup product has been well studied. There are various formulas for computing it using an arbitrary projective resolution, which have been used in many examples. The situation with the Lie bracket is more complicated. Almost all computations of it are based on the method of so-called comparison morphisms. This method allows elements of the Hochschild cohomology to be transferred from one resolution to another. For example, this method was used for the description of the Lie bracket on the Hochschild cohomology of the group algebra of a quaternion group of order 8 over a field of characteristic 2 in [**6**]. Later, it was applied to all local algebras of the generalized quaternion type over a

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field of characteristic 2 in [5]. Applications of the method of comparison morphisms can be found also in [1, 12, 14].

Recently, a formula for computing the bracket via a resolution which is not the bar resolution appeared in [11]. The proof given there is valid for a resolution that satisfies some conditions. Other formulas for the Lie bracket are proved in the current work. These formulas use chain maps from a resolution to its tensor powers and homotopies for some null homotopic maps defined by cocycles. Then the formula of [11] is slightly changed and proved for an arbitrary resolution. Note also that a nice formula for the bracket of a degree one element with an arbitrary element is given in [16].

It is well known that the Hochschild cohomology is a derived invariant. The proof of this fact can be found, for example, in [13]. The invariance of the cup product easily follows from this proof, while the derived invariance of the Gerstenhaber bracket was proved much later. In [8, 9], derived invariance of the Gerstenhaber bracket is proved using two different (relatively advanced) methods. In [9], Keller employs the derived Picard group, while [8] relies on the use of DG categories. Here, using our new formulation of the bracket and the approach to the Hochschild homology proposed in [18], we provide a direct proof of the derived invariance of the bracket which does not require any advanced technology.

Further, we give some formulas for the Lie bracket using so-called contracting homotopies. Then we discuss some formulas for the Connes differential on the Hochschild homology. One of these formulas is a slight modification of the formula from [7]. Also we give a formula using contracting homotopies for the Connes differential. Thus, in the case where the Connes differential induces a Batalin–Vilkovisky (BV) structure on Hochschild cohomology, we obtain an alternative way to compute the Lie bracket. We discuss this in the case where the algebra under consideration is symmetric.

Finally, we give an example of an application of the discussed formulas. We describe the BV structure and the Gerstenhaber bracket on the Hochschild cohomology of one family of symmetric local algebras of dihedral type. The Hochschild cohomology for these algebras was described in [2, 3]. Note also that the Hochschild cohomology groups and the Hochschild cohomology ring modulo nilpotent radical were described in [15] for a class of self-injective algebras, including the family of symmetric algebras considered in this work.

2. Hochschild cohomology via the bar resolution

In this paper, A always denotes some algebra over a field $\mathbf{k}$. We write simply $\otimes$ instead of $\otimes_{\mathbf{k}}$.

Let us recall how to define the Hochschild cohomology, the cup product and the Lie bracket in terms of the bar resolution. The Hochschild cohomology groups are defined as $\mathrm{HH}^n(A) \cong \mathrm{Ext}^n_{A^e}(A, A)$ for $n \geq 0$, where $A^e = A \otimes A^{\mathrm{op}}$ is the enveloping algebra of A.

Definition 2.1. An A^e*-complex* is a $\mathbb{Z}$-graded A-bimodule P with a differential of degree -1, i.e. an A-bimodule P with some fixed A-bimodule direct sum decomposition $P = \oplus_{n\in\mathbb{Z}} P_n$ and an A-bimodule homomorphism $\mathrm{d}_P : P \to P$ such that $\mathrm{d}_P(P_n) \subset P_{n-1}$ and $\mathrm{d}_P^2 = 0$. Let $\mathrm{d}_{P,n}$ denote $\mathrm{d}_P|_{P_n}$. The *nth homology of* P is the vector space $\mathrm{H}_n(P) = (\mathrm{Ker}\, \mathrm{d}_{P,n})/(\mathrm{Im}\, \mathrm{d}_{P,n+1})$. An A^e-complex P is called acyclic if $\mathrm{H}_n(P) = 0$ for all $n \in \mathbb{Z}$ and is called *bounded on the right* if $P_n = 0$ for small enough n. A *map of* A^e*-complexes*

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is a homomorphism of A-bimodules that respects the grading. If it also respects the differential, it is called a *chain map*. A complex is called positive if $P_n = 0$ for $n < 0$. A pair (P, μ_P) is called a *resolution* of the algebra A if P is a positive complex, $\mathrm{H}_n(P) = 0$ for $n > 0$ and $\mu_P : P_0 \to A$ is an A-bimodule homomorphism inducing an isomorphism $\mathrm{H}_0(P) \cong A$.

Given an A^e-complex P, (P, A) denotes the $\mathbf{k}$-complex $\oplus_{n\leqslant 0}\mathrm{Hom}_{A^e}(P_{-n}, A)$ with differential $\mathrm{d}_{(P,A),n} = \mathrm{Hom}_{A^e}(\mathrm{d}_{P,-1-n}, A)$. Let $\mu_A : A \otimes A \to A$ be the multiplication map.

Let $\mathrm{Bar}(A)$ be the positive A^e-complex with nth member $\mathrm{Bar}_n(A) = A^{\otimes(n+2)}$ for $n \geqslant 0$ and differential $\mathrm{d}_{\mathrm{Bar}(A)}$ defined by the equality

$$\mathrm{d}_{\mathrm{Bar}(A)}(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n}(-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}$$

for $n > 0$ and $a_i \in A$ $(0 \leqslant i \leqslant n+1)$. Then $(\mathrm{Bar}(A), \mu_A)$ is a projective A^e-resolution of A that is called the bar resolution.

The Hochschild cohomology of the algebra A is the homology of the complex $C(A) = (\mathrm{Bar}(A), A)$. We write $C^n(A)$ instead of $C_{-n}(A)$ and δ^n instead of $\mathrm{d}_{C(A),-1-n}$. Note that $C^0(A) \simeq A$ and $C^n(A) \simeq \mathrm{Hom}_{\mathbf{k}}(A^{\otimes n}, A)$. Given $f \in C^n(A)$, we introduce the notation

$$\delta_n^i(f)(a_1 \otimes \cdots \otimes a_{n+1}) := \begin{cases} a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) & \text{if } i = 0, \\ (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) & \text{if } 1 \leqslant i \leqslant n, \\ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1} & \text{if } i = n+1. \end{cases}$$

Then $\delta^n = \sum_{i=0}^{n+1} \delta_n^i$. We have $\mathrm{HH}^n(A) = (\mathrm{Ker}\,\delta^n)/(\mathrm{Im}\,\delta^{n-1})$.

The cup product $\alpha \smile \beta \in C^{n+m}(A) = \mathrm{Hom}_{\mathbf{k}}(A^{\otimes(n+m)}, A)$ of $\alpha \in C^n(A)$ and $\beta \in C^m(A)$ is given by

$$(\alpha \smile \beta)(a_1 \otimes \cdots \otimes a_{n+m}) := \alpha(a_1 \otimes \cdots \otimes a_n)\beta(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

This cup product induces a well-defined product in the Hochschild cohomology

$$\smile : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \longrightarrow \mathrm{HH}^{n+m}(A)$$

that turns the graded $\mathbf{k}$-vector space $\mathrm{HH}^*(A) = \bigoplus_{n\geq 0} \mathrm{HH}^n(A)$ into a graded commutative algebra [4, Corollary 1].

The Lie bracket is defined as follows. Let $\alpha \in C^n(A)$ and $\beta \in C^m(A)$. If $n, m \geq 1$, then, for $1 \leq i \leq n$, we define $\alpha \circ_i \beta \in C^{n+m-1}(A)$ by the equality

$$\begin{aligned}(\alpha \circ_i \beta)&(a_1 \otimes \cdots \otimes a_{n+m-1}) \\ &:= \alpha(a_1 \otimes \cdots \otimes a_{i-1} \otimes \beta(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1});\end{aligned}$$

if $n \geq 1$ and $m = 0$, then $\beta \in A$ and, for $1 \leq i \leq n$, we set

$$(\alpha \circ_i \beta)(a_1 \otimes \cdots \otimes a_{n-1}) := \alpha(a_1 \otimes \cdots \otimes a_{i-1} \otimes \beta \otimes a_i \otimes \cdots \otimes a_{n-1});$$

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and for any other case, we set $\alpha \circ_i \beta$ to be zero. Now we define

$$\alpha \circ \beta := \sum_{i=1}^{n} (-1)^{(m-1)(i-1)} \alpha \circ_i \beta \quad \text{and} \quad [\alpha, \beta] := \alpha \circ \beta - (-1)^{(n-1)(m-1)} \beta \circ \alpha.$$

Note that $[\alpha, \beta] \in C^{n+m-1}(A)$. The operation $[\ ,\ ]$ induces a well-defined Lie bracket on the Hochschild cohomology

$$[\ ,\ ]\colon \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \longrightarrow \mathrm{HH}^{n+m-1}(A)$$

such that $(\mathrm{HH}^*(A), \smile, [\ ,\ ])$ is a Gerstenhaber algebra (see [4]).

3. Comparison morphisms

Here we recall the method of comparison morphisms. First, we introduce some notation.

If P is a complex, then we denote by $P[t]$ the complex that is equal to P as an A-bimodule, with grading $P[t]_n = P_{t+n}$ and differential defined as $\mathrm{d}_{P[t]} = (-1)^t \mathrm{d}_P$. Note that d_P defines a map from P to $P[-1]$. Let us now take some map of complexes $f : P \to Q$. For any $t \in \mathbb{Z}$, $f[t]$ denotes the map from $P[t]$ to $Q[t]$ induced by f, i.e. such a map that $f[t]|_{P[t]_i} = f|_{P_{i+t}}$. For simplicity we will write simply f instead of $f[t]$, since in each situation t can be easily recovered. Let $\mathbf{d}f$ denote the map $f\,\mathrm{d}_P - \mathrm{d}_Q f : P \to Q[-1]$. We will frequently use the equality $\mathbf{d}(fg) = (-1)^m(\mathbf{d}f)g + f\,\mathbf{d}g : N \to Q[m-1]$, which is valid for any $g : N \to P[m]$. For two maps of complexes $f, g : P \to Q$, we write $f \sim g$ if $f - g = \mathbf{d}s$ for some $s : U \to V[1]$. Note that if $f \sim 0$ and $\mathbf{d}g = 0$, then $fg \sim 0$ and $gf \sim 0$ (for a composition that makes sense). Also, we always identify an A-bimodule M with the complex $\tilde{M}$ such that $\tilde{M}_i = 0$ for $i \neq 0$ and $\tilde{M}_0 = M$. Note also that if $f \sim 0$, then $\mathbf{d}f = 0$. It is not hard to see that if P is a projective complex, Q is exact in Q_i for $i \geqslant n$, and $Q_i = 0$ for $i < n$, then for any $f : P \to Q$ the equality $\mathbf{d}f = 0$ holds if and only if $f \sim 0$. Moreover, we have the following fact.

Lemma 3.1. *Let P be a projective complex, let Q be exact in Q_i for $i > n$, and let $Q_i = 0$ for $i < n$. Let $\mu_Q : Q \to \mathrm{H}_n(Q)$ denote the canonical projection. If $f : P \to Q$ is such that $\mathbf{d}f = 0$ and $\mu_Q f \sim 0$, then $f \sim 0$.*

Proof. Assume that $\mu_Q f = \phi\,\mathrm{d}_P$. Since P_{n-1} is projective, there is some $\psi : P_{n-1} \to Q_n$ such that $\mu_Q \psi = \phi$. Then $f - \mathbf{d}\psi$ is a chain map such that $\mu_Q(f - \mathbf{d}\psi) = 0$. Then it is easy to see that $f \sim \mathbf{d}\psi \sim 0$. □

Now let (P, μ_P) and (Q, μ_Q) be two A^e-projective resolutions of A. The method of comparison morphisms is based on the following idea. Since P is positive projective and Q is exact in Q_i for $i > 0$, there is some chain map of complexes $\Phi_P^Q : P \to Q$ such that $\mu_Q \Phi_P^Q = \mu_P$. Analogously, there is a chain map $\Phi_Q^P : Q \to P$ such that $\mu_P \Phi_Q^P = \mu_Q$. Then Φ_P^Q and Φ_Q^P induce maps from (Q, A) to (P, A) and backwards. Thus, we also have the maps

$$(\Phi_P^Q)^* : \mathrm{H}_*(Q, A) \to \mathrm{H}_*(P, A) \quad \text{and} \quad (\Phi_Q^P)^* : \mathrm{H}_*(P, A) \to \mathrm{H}_*(Q, A).$$

Since $\mathbf{d}(1_P - \Phi_Q^P \Phi_P^Q) = 0$, we have $1_P \sim \Phi_Q^P \Phi_P^Q$ by the arguments above. Then it is easy to see that $(\Phi_P^Q)^*(\Phi_Q^P)^* = (\Phi_Q^P \Phi_P^Q)^* = 1_{\mathrm{H}_*(P,A)}$ and, analogously, $(\Phi_Q^P)^*(\Phi_P^Q)^* = 1_{\mathrm{H}_*(Q,A)}$.

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So we can define the Hochschild cohomology of A as the homology of (P, A), and this definition does not depend on the A^e-projective resolution (P, μ_P) of A. If we define some bilinear operation $*$ on (Q, A) that induces an operation on $\mathrm{HH}^*(A)$, then we can define the operation $*_\Phi$ on (P, A) by the formula $f *_\Phi g = (f\Phi_Q^P * g\Phi_Q^P)\Phi_P^Q$ for $f, g \in (P, A)$. It is easy to see that $*_\Phi$ induces an operation on $\mathrm{HH}^*(A)$ and that the induced operation coincides with $*$. Now we can take $Q = \mathrm{Bar}(A)$ and define the cup product and the Lie bracket on (P, A) by the equalities

$$f \smile_\Phi g = (f\Phi_{\mathrm{Bar}(A)}^P \smile g\Phi_{\mathrm{Bar}(A)}^P)\Phi_P^{\mathrm{Bar}(A)} \quad \text{and} \quad [f, g]_\Phi = [f\Phi_{\mathrm{Bar}(A)}^P, g\Phi_{\mathrm{Bar}(A)}^P]\Phi_P^{\mathrm{Bar}(A)}.$$

Thus, to apply the method of comparison morphism one has to describe the maps $\Phi_P^{\mathrm{Bar}(A)}$ and $\Phi_{\mathrm{Bar}(A)}^P$ and then use them to describe the bracket in terms of the resolution P. The problem is that for some $x \in P$ the formula $\Phi_P^{\mathrm{Bar}(A)}(x)$ is complicated, and to describe $\Phi_{\mathrm{Bar}(A)}^P$ one has to define it on a lot of elements.

Let now recall one formula for the cup product that uses an arbitrary A^e-projective resolution of A instead of the bar resolution. But first let us introduce some definitions and notation.

Definition 3.2. Given A^e-complexes P and Q, we define the *tensor product complex* $P \otimes_A Q$ by the equality $(P \otimes_A Q)_n = \sum_{i+j=n} P_i \otimes_A Q_j$. The differential $\mathrm{d}_{P\otimes_A Q}$ is defined by the equality $\mathrm{d}_{P\otimes_A Q}(x \otimes y) = \mathrm{d}_P(x) \otimes y + (-1)^i x \otimes \mathrm{d}_Q(y)$ for $x \in P_i$, $y \in Q_j$.

We always identify $P \otimes_A A$ and $A \otimes_A P$ with P by the obvious isomorphisms of complexes. For any $n \in \mathbb{Z}$ we also identify $P \otimes_A Q[n]$ and $P[n] \otimes_A Q$ with $(P \otimes_A Q)[n]$. Note that this identification uses isomorphisms $\alpha_{P,Q}^n : P \otimes_A Q[n] \to (P \otimes_A Q)[n]$ and $\beta_{P,Q}^n : P[n] \otimes_A Q \to (P \otimes_A Q)[n]$ defined by the equalities $\alpha_{P,Q}^n(x \otimes y) = (-1)^{in} x \otimes y$ and $\beta_{P,Q}(x \otimes y) = x \otimes y$ for $x \in P_i$ and $y \in Q$. In particular, we have two different isomorphisms $\beta_{P,Q}^n \alpha_{P[n],Q}^m$ and $\alpha_{P,Q}^m \beta_{P,Q[m]}^n$ from $P[n] \otimes_A Q[m]$ to $(P \otimes_A Q)[n + m]$. For convenience, we always identify $P[n] \otimes_A Q[m]$ with $(P \otimes_A Q)[n + m]$ using the isomorphism $\beta_{P,Q}^n \alpha_{P[n],Q}^m$, which sends $x \otimes y$ to $(-1)^{(i+n)m} x \otimes y$ for $x \in P_i$ and $y \in Q$. In particular, we identify $A[n] \otimes_A A[m]$ with $A[n + m]$ by the isomorphism $\beta_{A,A}^n \alpha_{A[n],A}^m$, which sends $a \otimes b$ to $(-1)^{mn} ab$ for $a, b \in A$.

Definition 3.3. Given an A^e-projective resolution (P, μ_P) of A, a chain map $\Delta_P : P \to P^{\otimes_A n}$ is called a *diagonal n-approximation* of P if $\mu_P^{\otimes n} \Delta_P = \mu_P$.

Let (P, μ_P) be an A^e-projective resolution of A. Suppose also that $\Delta_P : P \to P \otimes_A P$ is a diagonal 2-approximation of P. Then the operation $\smile_{\Delta_P}$ on (P, A) defined for $f : P \to A[-n]$ and $g : P \to A[-m]$ by the equality $f \smile_{\Delta_P} g = (-1)^{mn}(f \otimes g)\Delta_P$ induces the cup product on $\mathrm{HH}^*(A)$. Note also that if $f \in C^n(A)$ and $g \in C^m(A)$, then the equality $f \smile g = (-1)^{mn}(f \otimes g)\Delta$ holds for Δ defined by the equality

$$\Delta(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) = \sum_{i=0}^{n} (1 \otimes a_1 \otimes \cdots a_i \otimes 1) \otimes_A (1 \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes 1). \quad (3.1)$$

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4. Gerstenhaber bracket via an arbitrary resolution

In this section we prove some new formulas for the Gerstenhaber bracket. The existence of these formulas is based on the following lemma.

Lemma 4.1. *Let (P,μ_P) be an A^e-projective resolution of A and let $f: P\to A[-n]$ be such that $f\,\mathrm{d}_P=0$. Then $f\otimes 1_P-1_P\otimes f: P\otimes_A P\to P[-n]$ is homotopic to 0.*

Proof. It is easy to check that $\mathbf{d}(f\otimes 1_P-1_P\otimes f)=0$. Since $\mu_P(\mu_P\otimes 1_P-1_P\otimes \mu_P)=0$, there is some map $\phi: P\otimes_A P\to P[1]$ such that $\mu_P\otimes 1_P-1_P\otimes\mu_P=\mathbf{d}\phi$. Then $\mu_P(f\otimes 1_P-1_P\otimes f)=-f\mathbf{d}\phi\sim 0$ and so $f\otimes 1_P-1_P\otimes f\sim 0$ by Lemma 3.1. □

Corollary 4.2. *Let P, f be as above and Δ_P be some diagonal 2-approximation of P. Then $(f\otimes 1_P-1_P\otimes f)\Delta_P: P\to P[-n]$ is homotopic to 0.*

Proof. Since $\mathbf{d}\Delta_P=0$, everything follows directly from Lemma 4.1. □

Definition 4.3. Let P, f and Δ_P be as above. We call $\phi_f: P\to P[1-n]$ a *homotopy lifting* of (f,Δ_P) if $\mathbf{d}\phi_f=(f\otimes 1_P-1_P\otimes f)\Delta_P$ and $\mu_P\phi_f+f\phi\sim 0$ for some $\phi: P\to P[1]$ such that $\mathbf{d}\phi=(\mu_P\otimes 1_P-1_P\otimes\mu_P)\Delta_P$.

One can show following the proofs of Lemmas 3.1 and 4.1 that some homotopy lifting exists for any cocycle. Alternatively, the existence of some $\tilde\phi_f$ such that $\mathbf{d}\tilde\phi_f=(f\otimes 1_P-1_P\otimes f)\Delta_P$ follows from Corollary 4.2 and, in particular, there is some ϕ satisfying the equality from the definition of a homotopy lifting. Easy calculation shows that $\mu_P\tilde\phi_f+f\phi$ is a cocycle. Then there is $u: P\to P[1-n]$ such that $\mathbf{d}u=0$ and $\mu_P u=\mu_P\tilde\phi_f+f\phi$, and hence $\tilde\phi_f-u$ is a homotopy lifting. Now we are ready to prove our first formula.

Theorem 4.4. *Let (P,μ_P) be an A^e-projective resolution of A, and let $\Delta_P: P\to P\otimes_A P$ be a diagonal 2-approximation of P. Let $f: P\to A[-n]$ and $g: P\to A[-m]$ represent some cocycles. Suppose that ϕ_f and ϕ_g are homotopy liftings for (f,Δ_P) and (g,Δ_P), respectively. Then the Gerstenhaber bracket of the classes of f and g can be represented by the class of the element*

$$[f,g]_{\phi,\Delta}=(-1)^m f\phi_g+(-1)^{m(n-1)}g\phi_f. \tag{4.1}$$

Proof. We will prove the assertion of the theorem in three steps.

(1) Let us prove that the operation induced on the Hochschild cohomology by $[,]_{\phi,\Delta_P}$ does not depend on the choice of Δ_P and ϕ. We do this in two steps.

- If ϕ_g and ϕ_g' are two homotopy liftings for g, then $\mathbf{d}(\phi_g-\phi_g')=0$ and $\mu_P(\phi_g-\phi_g')\sim g\epsilon$ for some chain map $\epsilon: P\to P[1]$. Then $\epsilon\sim 0$ and $\mu_P(\phi_g-\phi_g')\sim 0$. Hence, $\phi_g-\phi_g'\sim 0$ and $f\phi_g'\sim f\phi_g$. Analogously, $g\phi_f'\sim g\phi_f$ and so $[f,g]_{\phi',\Delta}\sim[f,g]_{\phi,\Delta}$.
- Let Δ_P' and Δ_P be two diagonal 2-approximations of P, and let ϕ_f and ϕ_g be homotopy liftings for (f,Δ_P) and (g,Δ_P) correspondingly. Then $\Delta_P'=\Delta_P+\mathbf{d}u$ for some u. Note that if $\mathbf{d}\phi=(\mu_P\otimes 1_P-1_P\otimes\mu_P)\Delta_P$, then $\mathbf{d}(\phi+(\mu_P\otimes 1_P-1_P\otimes\mu_P)u)=$

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$(\mu_P\otimes 1_P-1_P\otimes\mu_P)\Delta'_P$, hence $\phi'_f=\phi_f+(f\otimes 1_P-1_P\otimes f)u$ and $\phi'_g=\phi_g+(g\otimes 1_P-1_P\otimes g)u$ are homotopy liftings for (f,Δ'_P) and (g,Δ'_P). Now we have

$$\begin{aligned}[f,g]_{\phi',\Delta'}-[f,g]_{\phi,\Delta}&=(-1)^m f(g\otimes 1_P-1_P\otimes g)u\\&\quad+(-1)^{m(n-1)}g(f\otimes 1_P-1_P\otimes f)u\\&=((-1)^{m+mn}g\otimes f-(-1)^m f\otimes g+(-1)^{m(n-1)+mn}f\\&\quad\otimes g-(-1)^{m(n-1)}g\otimes f)u=0.\end{aligned}$$

(2) Let us prove that the operation induced on the Hochschild cohomology does not depend on the choice of an A^e-projective resolution of A. Let (Q,μ_Q) be another A^e-projective resolution of A. Let $\Phi_P^Q:P\to Q$ and $\Phi_Q^P:Q\to P$ be comparison morphisms, and let $\phi_{f\Phi_Q^P\Phi_P^Q}$ and $\phi_{g\Phi_Q^P\Phi_P^Q}$ be homotopy liftings for $(f\Phi_Q^P\Phi_P^Q,\Delta_P)$ and $(g\Phi_Q^P\Phi_P^Q,\Delta_P)$ correspondingly. It is not difficult to check that $\phi_{f\Phi_Q^P}=\Phi_P^Q\phi_{f\Phi_Q^P\Phi_P^Q}\Phi_Q^P$ and $\phi_{g\Phi_Q^P}=\Phi_P^Q\phi_{g\Phi_Q^P\Phi_P^Q}\Phi_Q^P$ are homotopy liftings for $(f\Phi_Q^P,\Delta_Q)$ and $(g\Phi_Q^P,\Delta_Q)$ correspondingly in this case. Here, Δ_Q denotes the map $(\Phi_P^Q\otimes\Phi_P^Q)\Delta_P\Phi_Q^P$. Then

$$\begin{aligned}[f\Phi_Q^P,g\Phi_Q^P]_{\phi,\Delta}&=(-1)^m f\Phi_Q^P\Phi_P^Q\phi_{g\Phi_Q^P\Phi_P^Q}\Phi_Q^P+(-1)^{m(n-1)}g\Phi_Q^P\Phi_P^Q\phi_{f\Phi_Q^P\Phi_P^Q}\Phi_Q^P\\&=[f\Phi_Q^P\Phi_P^Q,g\Phi_Q^P\Phi_P^Q]_{\phi,\Delta_P}\Phi_Q^P=[f,g]_{\phi,\Delta}\Phi_Q^P.\end{aligned}$$

(3) Suppose now that $(P,\mu_P)=(\mathrm{Bar}(A),\mu_A)$ and $\Delta_P=\Delta$, where Δ is the map from (3.1). Let us define

$$\begin{aligned}\phi_g(1\otimes a_1\otimes\cdots a_{i+m-1}\otimes 1)&=\sum_{j=1}^{i}(-1)^{(m-1)j-1}\otimes a_1\otimes\cdots\otimes a_{j-1}\\&\quad\otimes g(a_j\otimes\cdots\otimes a_{j+m-1})\otimes a_{j+m}\otimes\cdots\otimes a_{i+m-1}\otimes 1\end{aligned}$$

and analogously for ϕ_f. Then we have $(-1)^m f\phi_g+(-1)^{m(n-1)}g\phi_f=[f,g]$ by definition. Direct calculations show that ϕ_f and ϕ_g are homotopy liftings for (f,Δ) and (g,Δ) (in fact, ϕ_g coincides with $(-1)^m G(1_B\otimes g\otimes 1_B)\Delta^{(2)}$ in [**11**, Notation 2.3], and the fact that ϕ_g is a homotopy lifting follows from [**11**, Proposition 2.4] and our discussion below). □

Let (P,μ_P) be an A^e-projective resolution for A, and let $\Delta_P^{(2)}:P\to P\otimes_A P\otimes_A P$ be a diagonal 3-approximation of P. There is some homotopy ϕ_P for $\mu_P\otimes 1_P-1_P\otimes\mu_P$. Since

$$(\mu_P\otimes\mu_P)(\mu_P\otimes 1_P\otimes 1_P-1_P\otimes 1_P\otimes\mu_P)\Delta_P^{(2)}=0,$$

there is some homotopy ϵ_P for $(\mu_P\otimes 1_P\otimes 1_P-1_P\otimes 1_P\otimes\mu_P)\Delta_P^{(2)}$. We define

$$f\circ_{\Delta_P^{(2)},\phi_P,\epsilon_P}g=f\phi_P(1_P\otimes g\otimes 1_P)\Delta_P^{(2)}-(-1)^m(f\otimes g)\epsilon_P:P\to A[1-n-m]\qquad(4.2)$$

and

$$[f,g]_{\Delta_P^{(2)},\phi_P,\epsilon_P}=f\circ_{\Delta_P^{(2)},\phi_P,\epsilon_P}g-(-1)^{(n-1)(m-1)}g\circ_{\Delta_P^{(2)},\phi_P,\epsilon_P}f.$$

This formula is a slightly corrected variant of the formula from [**11**].

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Corollary 4.5. *The operation* $[,]_{\Delta_P^{(2)},\phi_P,\epsilon_P}$ *induces an operation on* $\mathrm{HH}^*(A)$ *that coincides with the usual Lie bracket on the Hochschild cohomology.*

Proof. By Theorem 4.4 it is enough to check that $-(1_P\otimes g)\epsilon_P+(-1)^m\phi_P(1_P\otimes g\otimes 1_P)\Delta_P^{(2)}$ is a homotopy lifting for $(g,(\mu_P\otimes 1_P\otimes 1_P)\Delta_P^{(2)})$ if $g\,\mathrm{d}_P=0$. Let us verify the first condition:

$$\begin{aligned}&-\mathbf{d}((1_P\otimes g)\epsilon_P+(-1)^m\phi_P(1_P\otimes g\otimes 1_P)\Delta_P^{(2)})\\&=-(1_P\otimes g)\mathbf{d}\epsilon_P+\mathbf{d}\phi_P(1_P\otimes g\otimes 1_P)\Delta_P^{(2)}\\&=(1_P\otimes g)(1_P\otimes 1_P\otimes\mu_P-\mu_P\otimes 1_P\otimes 1_P)\Delta_P^{(2)}\\&\quad+(\mu_P\otimes 1_P-1_P\otimes\mu_P)(1_P\otimes g\otimes 1_P)\Delta_P^{(2)}\\&=(g\otimes 1_P-1_P\otimes g)(\mu_P\otimes 1_P\otimes 1_P)\Delta_P^{(2)}.\end{aligned}$$

The second condition can be easily verified after noting that $\operatorname{Im}\phi_P\subset\oplus_{i>0}P_i\subset\operatorname{Ker}\mu_P$. Indeed, we have

$$\begin{aligned}&\mu_P(-(1_P\otimes g)\epsilon_P+(-1)^m\phi_P(1_P\otimes g\otimes 1_P)\Delta_P^{(2)})+g\phi_P(\mu_P\otimes 1_P\otimes 1_P)\Delta_P^{(2)}\\&\quad=g(\phi_P(\mu_P\otimes 1_P\otimes 1_P)\Delta_P^{(2)}-(\mu_P\otimes 1_P)\epsilon_P)\sim 0\end{aligned}$$

because $\mathbf{d}(\phi_P(\mu_P\otimes 1_P\otimes 1_P)\Delta_P^{(2)}-(\mu_P\otimes 1_P)\epsilon_P)=0$. □

Remark 4.6. Usually, the diagonal 3-approximation $\Delta_P^{(2)}$ is constructed using some 2-approximation Δ_P by the rule $\Delta_P^{(2)}=(\Delta_P\otimes 1_P)\Delta_P$. It often occurs that the maps Δ_P and μ_P satisfy the equality

$$(\mu_P\otimes 1_P)\Delta_P=1_P=(1_P\otimes\mu_P)\Delta_P. \tag{4.3}$$

In this case, some things become easier. First, one can set $\phi=0$ in the definition of a homotopy lifting. Then the second condition simply means that $\mu_P\phi_f$ is a coboundary. In particular, one can simply set $\phi_f|_{P_{n-1}}=0$. Second, if (4.3) holds and the diagonal 3-approximation is defined as above, then one can set $\epsilon_P=0$ in equality (4.2). Thus, we get the formula from [**11**] in the case where (4.3) holds. Note that the condition (4.3) is weaker than the conditions proposed in [**11**].

On the other hand, we always can set $\epsilon_P=(\phi_P\otimes 1_P+1_P\otimes\phi_P)\Delta_P^{(2)}$ and obtain the following formula for the bracket:

$$\begin{aligned}[f,g]_{\Delta_P^{(2)},\phi_P,\epsilon_P}&=-f\phi_P(g\otimes 1_P\otimes 1_P-1_P\otimes g\otimes 1_P+1_P\otimes 1_P\otimes g)\Delta_P^{(2)}\\&\quad+(-1)^{(n-1)(m-1)}g\phi_P(f\otimes 1_P\otimes 1_P-1_P\otimes f\otimes 1_P+1_P\otimes 1_P\otimes f)\Delta_P^{(2)}.\end{aligned}\tag{4.4}$$

Remark 4.7. In fact, Corollary 4.2 can be proved directly without Lemma 4.1. Then one can show that homotopy liftings exist using only the projectivity of P and not of its tensor powers over A. This allows us to define the Gerstenhaber bracket on $\mathrm{Ext}^*_{A^e}(A,A)$ for any associative ring A, even in the case where one cannot use the bar resolution for this.

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5. Derived invariance of the Gerstenhaber bracket

Let D^-A and K_p^-A denote the derived category of bounded on the right complexes of A-modules and the homotopy category of bounded on the right complexes of A-projective modules, respectively. Note that the construction of a projective resolution for a complex induces an equivalence between D^-A and K_p^-A. In this section, (P, μ_P) is called a projective bimodule resolution of A if $P \in \mathrm{K}_p^-A^e$ and the morphism of A^e-complexes $\mu_P : P \to A$ induces an isomorphism in homology, i.e. P does not have to be concentrated only in non-negative degrees. Then the chain map $\Delta_P : P \to P \otimes_A P$ is called a diagonal 2-approximation of P if $(\mu_P \otimes_A \mu_P)\Delta_P \sim \mu_P$.

One can easily check that all the arguments of the previous sections are valid for the settings of this section. In particular, for any map $f : P \to A[-n]$ there exists a homotopy lifting for (f, Δ_P) and the statement of Theorem 4.4 holds.

We will say that A is *standardly derived equivalent* to B if there exist $U \in \mathrm{D}^-(A \otimes B^{\mathrm{op}})$ and $V \in \mathrm{D}^-(B \otimes A^{\mathrm{op}})$ such that $U \otimes_B^L V \cong A$ in D^-A^e and $V \otimes_A^L U \cong B$ in D^-B^e. We will assume without loss of generality that $U \in \mathrm{K}_p^-(A \otimes B^{\mathrm{op}})$ and $V \in \mathrm{K}_p^-(B \otimes A^{\mathrm{op}})$. The paper [**13**] guarantees that if A and B are algebras over a field, then they are standardly derived equivalent if and only if they are derived equivalent. Since $U \in \mathrm{K}_p^-(A \otimes B^{\mathrm{op}})$, $V \in \mathrm{K}_p^-(B \otimes A^{\mathrm{op}})$, $U \otimes_B^L V \cong A$ in D^-A^e and $V \otimes_A^L U \cong B$ in D^-B^e, there are chain maps $\alpha : U \otimes_B V \to A$ and $\beta : V \otimes_A U \to B$ that induce isomorphisms in homology. We will need the following technical lemmas.

Lemma 5.1. *The maps α and β above can be chosen in such a way that*

$$\alpha \otimes 1_U \sim 1_U \otimes \beta : U \otimes_B V \otimes_A U \to U \quad \textit{and} \quad 1_V \otimes \alpha \sim \beta \otimes 1_V : V \otimes_A U \otimes_B V \to V.$$

Proof. Let $\tilde{\beta} : V \otimes_A U \to B$ be some chain map inducing isomorphism in homology. Note that $\alpha(\alpha \otimes 1_{U \otimes_B V} - 1_{U \otimes_B V} \otimes \alpha) = 0$. Since α is a quasi-isomorphism, we have

$$\alpha \otimes 1_{U \otimes_B V} \sim 1_{U \otimes_B V} \otimes \alpha : U \otimes_B V \otimes_A U \otimes_B V \to U \otimes_B V.$$

Analogously, $\tilde{\beta} \otimes 1_{V \otimes_A U} \sim 1_{V \otimes_A U} \otimes \tilde{\beta}$. Let β be a chain map that equals

$$\tilde{\beta}(1_V \otimes \alpha \otimes 1_U)(1_{V \otimes_A U} \otimes \tilde{\beta}^{-1})$$

in $\mathrm{Hom}_{\mathrm{D}^-B^e}(V \otimes_A^L U, B)$. In the derived category of $A \otimes B^{\mathrm{op}}$-modules we have

$$\begin{aligned} 1_U \otimes \beta &= (1_U \otimes \tilde{\beta})(1_{U \otimes_B V} \otimes \alpha \otimes 1_U)(1_{U \otimes_B V \otimes_A U} \otimes \tilde{\beta}^{-1}) \\ &= (1_U \otimes \tilde{\beta})(\alpha \otimes 1_{U \otimes_B V \otimes_A U})(1_{U \otimes_B V \otimes_A U} \otimes \tilde{\beta}^{-1}) \\ &= (1_U \otimes \tilde{\beta})(\alpha \otimes 1_U \otimes \tilde{\beta}^{-1}) = \alpha \otimes 1_U. \end{aligned}$$

Since $U \otimes_B V \otimes_A U$ is $A \otimes B^{\mathrm{op}}$-projective, we have $\alpha \otimes 1_U \sim 1_U \otimes \beta$. Analogously, $1_V \otimes \alpha \sim \beta \otimes 1_V$. □

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Lemma 5.2. *Suppose that α and β satisfy the compatibility conditions of Lemma 5.1 and the maps*

$$\varphi_{\alpha\beta} : U \otimes_B V \otimes_A U \to U[1] \quad \text{and} \quad \varphi_{\beta\alpha} : V \otimes_A U \otimes_B V \to V[1]$$

are such that $\mathbf{d}\varphi_{\alpha\beta} = \alpha \otimes 1_U - 1_U \otimes \beta$ and $\mathbf{d}\varphi_{\beta\alpha} = \beta \otimes 1_V - 1_V \otimes \alpha$. Then

$$\beta(\varphi_{\beta\alpha} \otimes 1_U + 1_V \otimes \varphi_{\alpha\beta}) : V \otimes_A U \otimes_B V \otimes_A U \to B[1]$$

is a null-homotopic chain map.

Proof. Let us set $\psi = \beta(\varphi_{\beta\alpha} \otimes 1_U + 1_V \otimes \varphi_{\alpha\beta})$. Since

$$\psi \mathrm{d}_{V\otimes_A U \otimes_B V \otimes_A U} = \beta(\beta \otimes 1_{V\otimes_A U} - 1_V \otimes \alpha \otimes 1_U + 1_V \otimes \alpha \otimes 1_U - 1_{V\otimes_A U} \otimes \beta) = 0,$$

ψ is a chain map. Note that $\psi(\beta\otimes\beta)^{-1} \in \mathrm{Hom}_{\mathrm{D}^-B^e}(B, B[1]) = 0$. Consequently, ψ equals 0 in the derived category of B^e-modules. Since $V \otimes_A U \otimes_B V \otimes_A U$ is projective, we have $\psi \sim 0$. □

Suppose that A and B are derived equivalent algebras, U and V are as above, and α, β, $\varphi_{\alpha\beta}$ and $\varphi_{\beta\alpha}$ are as in Lemma 5.2. If (P, μ_P) is a projective bimodule resolution of A, then it is easy to see that $(V \otimes_A P \otimes_A U, \beta(1_V \otimes_A \mu_P \otimes_A 1_U)) = (\tilde{P}, \mu_{\tilde{P}})$ is a projective bimodule resolution of B. For $f : P \to A[-n]$, we will denote by $\tilde{f}$ the map $1_V \otimes_A f \otimes_A 1_U : \tilde{P} \to B[-n]$. There is an isomorphism $\chi : \mathrm{HH}^*(A) \to \mathrm{HH}^*(B)$ that sends the element corresponding to $f : P \to A[-n]$ to the element corresponding to $\chi(f) = \beta\tilde{f} : \tilde{P} \to B[-n]$. Note that $\mu_{\tilde{P}} = \chi(\mu_P)$. Now let $\Delta_P : P \to P \otimes_A P$ be a diagonal 2-approximation for (P, μ_P). Since the map $1_P \otimes_A \alpha \otimes_A 1_P : P \otimes_A U \otimes_B V \otimes_A P \to P \otimes_A P$ is a quasi-isomorphism and all the complexes under consideration are projective, there exists a chain map $\gamma : P \otimes_A P \to P \otimes_A U \otimes_B V \otimes_A P$ such that $\gamma(1_P \otimes_A \alpha \otimes_A 1_P) \sim 1_{P\otimes_A U\otimes_B V\otimes_A P}$ and $(1_P \otimes_A \alpha \otimes_A 1_P)\gamma \sim 1_{P\otimes_A P}$. Then it is easy to check that the map $\Delta_{\tilde{P}} = 1_V \otimes_A \gamma\Delta \otimes_A 1_U$ is a diagonal approximation for $(\tilde{P}, \mu_{\tilde{P}})$. Note also that $(1_P \otimes_A \alpha \otimes_A 1_P)\gamma\Delta_P$ is a diagonal approximation for (P, μ_P). We have the following lemma.

Lemma 5.3. *Let $f : P \to A[-n]$ be a map of A^e-complexes and $\phi_f : P \to P[1-n]$ be a homotopy lifting for $(f, (1_P \otimes_A \alpha \otimes_A 1_P)\gamma\Delta_P)$. Then*

$$\begin{aligned}\psi_f = 1_V \otimes_A \phi_f \otimes_A 1_U + (-1)^n\big(\varphi_{\beta\alpha}(\tilde{f} \otimes_B 1_V) \otimes_A 1_{P\otimes_A U} \\ + 1_{V\otimes_A P} \otimes_A \varphi_{\alpha\beta}(1_U \otimes_B \tilde{f})\big)\Delta_{\tilde{P}}\end{aligned}$$

is a homotopy lifting for $(\chi(f), \Delta_{\tilde{P}})$.

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Proof. Direct calculations show that

$$\begin{aligned}\mathbf{d}\psi_f &= (1_V \otimes_A f \otimes_A \alpha \otimes_A 1_{P\otimes_A U} - 1_{V\otimes_A P} \otimes_A \alpha \otimes_A f \otimes_A 1_U)\Delta_{\tilde{P}} + (\beta\tilde{f} \otimes_B 1_{\tilde{P}} \\ &\quad - 1_V \otimes_A f \otimes_A \alpha \otimes_A 1_{P\otimes_A U} + 1_{V\otimes_A P} \otimes_A \alpha \otimes_A f \otimes_A 1_U - 1_{\tilde{P}} \otimes_B \beta\tilde{f})\Delta_{\tilde{P}} \\ &= (\chi(f) \otimes_B 1_{\tilde{P}} - 1_{\tilde{P}} \otimes_B \chi(f))\Delta_{\tilde{P}}.\end{aligned}$$

In particular, $\mathbf{d}\psi_{\mu_P} = (\mu_{\tilde{P}} \otimes_B 1_{\tilde{P}} - 1_{\tilde{P}} \otimes_B \mu_{\tilde{P}})\Delta_{\tilde{P}}$. By the definition of the homotopy lifting, we have $\mu_P\phi_f + f\phi_{\mu_P} \sim 0$, and hence

$$\begin{aligned}\mu_{\tilde{P}}\psi_f + \chi(f)\psi_{\mu_P} &\sim (-1)^n\beta(\varphi_{\beta\alpha}(\tilde{f} \otimes_B 1_V) \otimes_A \mu_P \otimes_A 1_U + 1_V \otimes_A \mu_P \otimes_A \varphi_{\alpha\beta} \\ &\quad \times (1_U \otimes_B \tilde{f}))\Delta_{\tilde{P}} + \beta(1_V \otimes_A f \otimes_A 1_U)(\varphi_{\beta\alpha}(\tilde{\mu}_P \otimes_B 1_V) \otimes_A 1_{P\otimes_A U} \\ &\quad + 1_{V\otimes_A P} \otimes_A \varphi_{\alpha\beta}(1_U \otimes_B \tilde{\mu}_P))\Delta_{\tilde{P}} \\ &= (-1)^n\beta(\varphi_{\beta\alpha} \otimes_A 1_U + 1_V \otimes_A \varphi_{\alpha\beta})(\tilde{f} \otimes_B \tilde{\mu}_P + \tilde{\mu}_P \otimes_B \tilde{f})\Delta_{\tilde{P}} \sim 0\end{aligned}$$

by Lemma 5.2. Thus, ψ_f is a homotopy lifting for $(\chi(f), \Delta_{\tilde{P}})$. □

Now we are ready to prove the following theorem.

Theorem 5.4. *Suppose that A and B are* **k***-algebras. If A is derived equivalent to B, then $\mathrm{HH}^*(A) \cong \mathrm{HH}^*(B)$ as Gerstenhaber algebras.*

Proof. It is well known that the isomorphism χ defined above preserves the cup product. In fact, it coincides with the isomorphism from [**13**]. Thus, it remains to prove that it preserves the Gerstenhaber bracket.

By Lemma 5.3 and Theorem 4.4, it is enough to show that

$$(-1)^m\chi(f)\psi_g + (-1)^{m(n-1)}\chi(g)\psi_f \sim \chi((-1)^m f\phi_g + (-1)^{m(n-1)}g\phi_f)$$

for any two maps $f : P \to A[-n]$ and $g : P \to A[-m]$ of A^e-complexes. We have

$$\begin{aligned}&(-1)^m\chi(f)\psi_g + (-1)^{m(n-1)}\chi(g)\psi_f - \chi((-1)^m f\phi_g + (-1)^{m(n-1)}g\phi_f) \\ &\quad = \beta(1_V \otimes_A ((-1)^m f\phi_g + (-1)^{m(n-1)}g\phi_f) \otimes_A 1_U) - \chi((-1)^m f\phi_g + (-1)^{m(n-1)}g\phi_f) \\ &\qquad + \beta(1_V \otimes_A f \otimes_A 1_U)(\varphi_{\beta\alpha}(\tilde{g} \otimes_B 1_V) \otimes_A 1_{P\otimes_A U} + 1_{V\otimes_A P} \otimes_A \varphi_{\alpha\beta}(1_U \otimes_B \tilde{g}))\Delta_{\tilde{P}} \\ &\qquad - (-1)^{(m-1)(n-1)}\beta(1_V \otimes_A g \otimes_A 1_U)(\varphi_{\beta\alpha}(\tilde{f} \otimes_B 1_V) \otimes_A 1_{P\otimes_A U} \\ &\qquad + 1_{V\otimes_A P} \otimes_A \varphi_{\alpha\beta}(1_U \otimes_B \tilde{f}))\Delta_{\tilde{P}} \\ &\quad = (-1)^{(m-1)n}\beta(\varphi_{\beta\alpha} \otimes_A 1_U)(\tilde{g} \otimes_B \tilde{f})\Delta_{\tilde{P}} + (-1)^n\beta(1_V \otimes_A \varphi_{\alpha\beta})(\tilde{f} \otimes_B \tilde{g})\Delta_{\tilde{P}} \\ &\qquad + (-1)^n\beta(\varphi_{\beta\alpha} \otimes_A 1_U)(\tilde{f} \otimes_B \tilde{g})\Delta_{\tilde{P}} + (-1)^{(m-1)n}\beta(\varphi_{\beta\alpha} \otimes_A 1_U)(\tilde{f} \otimes_B \tilde{g})\Delta_{\tilde{P}} \\ &\quad = (-1)^n\beta(\varphi_{\beta\alpha} \otimes_A 1_U + 1_V \otimes_A \varphi_{\alpha\beta})(\tilde{f} \otimes_B \tilde{g} + (-1)^{mn}\tilde{g} \otimes_B \tilde{f})\Delta_{\tilde{P}} \sim 0\end{aligned}$$

by Lemma 5.2. Thus, the theorem is proved. □

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6. A formula via contracting homotopy

In this section, we present a formula that expresses the Lie bracket on the Hochschild cohomology in terms of an arbitrary resolution and a left contracting homotopy for it. Note that contracting homotopies can be used to construct comparison maps between resolutions; this method was applied to compute the bracket, for example, in [6].

Definition 6.1. Let (P, μ_P) be a projective A^e-resolution of A. Let $t_P : P \to P$ and $\eta_P : A \to P$ be homomorphisms of left modules such that $t_P(P_i) \subset P_{i+1}$ and $\eta_P(A) \subset P_0$. The pair (t_P, η_P) is called a *left contracting homotopy* for (P, μ_P) if $\mathrm{d}_P t_P + t_P \mathrm{d}_P + \eta_P\mu_P = 1_P$ and $t_P(t_P + \eta_P) = 0$.

Since A is projective as a left A-module, any A^e-projective resolution of A splits as a complex of left A-modules. Hence, a left contracting homotopy exists for any A^e-projective resolution of A (see [6, Lemma 2.3] and the remark after it for details).

Let us fix an A^e-projective resolution (P, μ_P) of A and a left contracting homotopy (t_P, η_P) for it.

For any $n \geqslant 0$, the map $\pi_n : A \otimes P_n \to P_n$ defined by the equality $\pi_n(a \otimes x) = ax$ for $a \in A$, $x \in P_n$ is an epimorphism of A-bimodules. Since P_n is projective, there is $\iota_n \in \mathrm{Hom}_{A^e}(P_n, A \otimes P_n)$ such that $\pi_n\iota_n = 1_{P_n}$. Let us fix such ι_n for each $n \geqslant 0$. Then π_n and ι_n $(n \geqslant 0)$ determine homomorphisms of graded A-bimodules $\pi : A \otimes P \to P$ and $\iota : P \to A \otimes P$.

Let us define

$$\begin{aligned}
t_L &:= (1_P \otimes \pi)(t_P \otimes 1_P)(1_P \otimes \iota) : P \otimes_A P \to (P \otimes_A P)[1],\\
\eta_L &:= (1_P \otimes \pi)(\eta_P \otimes 1_P)\iota : P \to P \otimes_A P,\\
\mathrm{d}_L &:= \mathrm{d}_P \otimes 1_P, \mathrm{d}_R := 1_P \otimes \mathrm{d}_P : P \otimes_A P \to (P \otimes_A P)[-1],\\
\mu_L &:= \mu_P \otimes 1_P, \mu_R := 1_P \otimes \mu_P : P \otimes_A P \to P.
\end{aligned}$$

Note that all the defined maps are homomorphisms of A-bimodules. Note also that we omit isomorphisms $\alpha^1_{P,P}$ and $\beta^{\pm 1}_{P,P}$ in our definitions according to our agreement. It is easy to see that the map $t_L\,\mathrm{d}_R : P \otimes_A P \to P \otimes_A P$ is locally nilpotent in the sense that for any $x \in P \otimes_A P$ there is an integer l such that $(t_L\,\mathrm{d}_R)^l(x) = 0$. This follows from the fact that $t_L\,\mathrm{d}_R(P \otimes_A P_j) \subset P \otimes_A P_{j-1}$ if $j > 0$ and $t_L\,\mathrm{d}_R(P \otimes_A P_0) = 0$. Hence, the map $1_{P \otimes_A P} + t_L\,\mathrm{d}_R$ is invertible.

Now let $f : P \to A[-n]$ and $g : P \to A[-m]$ be maps of complexes. Let us define

$$f \circ g = -f\mu_R S t_L(1_P \otimes g \otimes 1_P)(1_P \otimes S\eta_L)S\eta_L,$$

where $S = (1_{P \otimes_A P} + t_L\,\mathrm{d}_R)^{-1}$.

Theorem 6.2. *In the notation above, the operation defined by the equality* $[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f$ *induces the usual Lie bracket on the Hochschild cohomology.*

We divide the proof into several lemmas. First of all, note that

$$\mathrm{d}_L t_L + t_L\,\mathrm{d}_L + \eta_L\mu_L = 1_{P \otimes_A P}, \mu_L\eta_L = 1_P, (\mathrm{d}_R)^2 = (\mathrm{d}_L)^2 = 0 \text{ and } \mathrm{d}_L\,\mathrm{d}_R + \mathrm{d}_R\,\mathrm{d}_L = 0. \tag{6.1}$$

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Lemma 6.3. $(\mathrm{d}_L + \mathrm{d}_R)S = S(\mathrm{d}_L + \eta_L\mu_L\,\mathrm{d}_R)$.

Proof. Let us multiply the desired equality by $1_{P\otimes_A P} + t_L\,\mathrm{d}_R$ on the left and on the right at the same time. We obtain that we have to prove that

$$\mathrm{d}_L + \mathrm{d}_R + t_L\,\mathrm{d}_R\,\mathrm{d}_L + t_L(\mathrm{d}_R)^2 = \mathrm{d}_L + \eta_L\mu_L\,\mathrm{d}_R + \mathrm{d}_L t_L\,\mathrm{d}_R + \eta_L\mu_L\,\mathrm{d}_R t_L\,\mathrm{d}_R.$$

Using (6.1) one can see that it is enough to show that $\eta_L\mu_L\,\mathrm{d}_R t_L\,\mathrm{d}_R = 0$. But the last equality follows from the fact that the image of $\mathrm{d}_R t_L\,\mathrm{d}_R$ lies in $\oplus_{n>0} P_n \otimes_A P \subset \operatorname{Ker}\mu_L$. □

Lemma 6.4. *$S\eta_L$ is a diagonal 2-approximation of P.*

Proof. By Lemma 6.3 we have

$$\mathbf{d}(S\eta_L) = (\mathrm{d}_L + \mathrm{d}_R)S\eta_L - S\eta_L\,\mathrm{d}_P = S(\mathrm{d}_L + \eta_L\mu_L\,\mathrm{d}_R)\eta_L - S\eta_L\,\mathrm{d}_P.$$

Since $\operatorname{Im}\eta_L \subset \operatorname{Ker}\mathrm{d}_L$, it is enough to prove that $\eta_L\mu_L\,\mathrm{d}_R\eta_L = \eta_L\,\mathrm{d}_P$. It is easy to see that $\mu_L\,\mathrm{d}_R = \mathrm{d}_P\mu_L$. Hence, $\eta_L\mu_L\,\mathrm{d}_R\eta_L = \eta_L\,\mathrm{d}_P\mu_L\eta_L = \eta_L\,\mathrm{d}_P$ by (6.1). □

Proof of Theorem 6.2. It follows from Lemma 6.4 that $\Delta_P = (1_P \otimes \mu_R S\eta_L)S\eta_L$ is a diagonal 2-approximation of P.

It is enough to show that $\phi_g = (-1)^{m-1}\mu_R S t_L(1_P \otimes g \otimes 1_P)(1_P \otimes S\eta_L)S\eta_L$ is a homotopy lifting for (g, Δ_P). Using Lemma 6.3, we get $\mu_R\,\mathbf{d}(S)t_L = \mu_R(1_P - \eta_L\mu_L)\,\mathrm{d}_R t_L = \mu_R\,\mathrm{d}_R t_L - \mu_R\eta_L\,\mathrm{d}_P\mu_L t_L = 0$. Since $S\,\mathrm{d}_R = \mathrm{d}_R$ and $\mu_L S\eta_L = 1_P$, we now get

$$\begin{aligned}
\mathbf{d}\phi_g &= -\mu_R S((\mathrm{d}_L + \mathrm{d}_R)t_L + t_L(\mathrm{d}_L + \mathrm{d}_R))(1_P \otimes g \otimes 1_P)(1_P \otimes S\eta_L)S\eta_L\\
&= \mu_R S(\eta_L\mu_L - 1_{P\otimes_A P} - t_L\,\mathrm{d}_R)(1_P \otimes g \otimes 1_P)(1_P \otimes S\eta_L)S\eta_L\\
&\quad - \mu_R\,\mathrm{d}_R t_L(1_P \otimes g \otimes 1_P)(1_P \otimes S\eta_L)S\eta_L\\
&= (\mu_P \otimes g \otimes \mu_R S\eta_L)(1_P \otimes S\eta_L)S\eta_L - (1_P \otimes g \otimes \mu_P)(1_P \otimes S\eta_L)S\eta_L\\
&= (g \otimes 1_P)(1_P \otimes \mu_R S\eta_L)S\eta_L\mu_L S\eta_L - (1_P \otimes g)(1_P \otimes \mu_R S\eta_L)S\eta_L\\
&= (g \otimes 1_P - 1_P \otimes g)\Delta_P.
\end{aligned}$$

Note also that $\mu_L\Delta_P = \mu_R S\eta_L = \mu_R\Delta_P$ and $\mu_P\phi_g = 0$. Hence, ϕ_g is a homotopy lifting for (g, Δ_P) and the theorem is proved. □

7. Formulas for the Connes differential

In this section we discuss some formulas for the Connes differential. These formulas are based on the formula from [7]. In the case of a symmetric algebra, a formula for the Connes differential gives a formula for a BV differential. Thus, we obtain in this section an alternative way of computing the Lie bracket on the Hochschild cohomology of a symmetric algebra.

Let Tr denote the functor $A \otimes_{A^e} -$ from the category of A-bimodules to the category of $\mathbf{k}$-linear spaces. If M and N are A-bimodules, then there is an isomorphism $\sigma_{M,N} : Tr(M \otimes_A N) \to Tr(N \otimes_A M)$ defined by the equality $\sigma_{M,N}(1 \otimes x \otimes y) = 1 \otimes y \otimes x$ for

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$x \in M$ and $y \in N$. Moreover, for $f \in \mathrm{Hom}_{A^e}(M_1, M_2)$ and $g \in \mathrm{Hom}_{A^e}(N_1, N_2)$, one has $\sigma_{M_2,N_2} Tr(f \otimes g) = Tr(g \otimes f)\sigma_{M_1,N_1}$. It is easy to see also that Tr induces a functor from the category of A^e-complexes to the category of $\mathbf{k}$-complexes. In this case, $\sigma_{P,Q}$ is defined by the equality $\sigma_{P,Q}(1 \otimes x \otimes y) = (-1)^{ij} \otimes y \otimes x$ for $x \in P_i$ and $y \in Q_j$ and satisfies the property $\sigma_{P_2,Q_2} Tr(f \otimes g) = Tr(g \otimes f)\sigma_{P_1,Q_1}$ for $f : P_1 \to P_2$ and $g : Q_1 \to Q_2$.

The Hochschild homology $\mathrm{HH}_*(A)$ of the algebra A is simply the homology of the complex $Tr(\mathrm{Bar}(A))$. As in the case of cohomology, any comparison morphism $\Phi_P^Q : P \to Q$ between resolutions (P, μ_P) and (Q, μ_Q) of the algebra A induces an isomorphism $Tr(\Phi_P^Q) : \mathrm{H}_* Tr(P) \to \mathrm{H}_* Tr(Q)$. Thus, the Hochschild homology of A is isomorphic to the homology of $Tr(P)$ for any projective bimodule resolution (P, μ_P) of A.

Note that $Tr(\mathrm{Bar}_n(A)) \cong A^{\otimes(n+1)}$. Connes differential $\mathcal{B} : \mathrm{HH}_n(A) \to \mathrm{HH}_{n+1}(A)$ is the map induced by the map from $Tr(\mathrm{Bar}_n(A))$ to $Tr(\mathrm{Bar}_{n+1}(A))$ that sends $a_0 \otimes a_1 \otimes \ldots \otimes a_n \in A^{\otimes(n+1)}$ to

$$\sum_{i=0}^{n}(-1)^{in} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}$$
$$+ \sum_{i=0}^{n}(-1)^{in} a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

In fact, it follows from some standard arguments that the homological class of the second summand is zero. The following result is essentially stated in [7] (see Equation (4.8) of the cited paper and the explanations before and after it).

Proposition 7.1 (D. Kaledin). *Let (P, μ_P) be a projective bimodule resolution of A, let Δ_P be a diagonal 2-approximation for P, and let $\phi_P : P \otimes_A P \to P[1]$ be such that $\mu_P \otimes 1_P - 1_P \otimes \mu_P = \mathbf{d}\phi_P$. Then the map*

$$Tr(\phi_P)(1_{P \otimes_A P} + \sigma_{P,P}) Tr(\Delta_P) : Tr(P) \to Tr(P[1])$$

induces the Connes differential on the Hochschild homology.

This result can be written in a slightly different form.

Corollary 7.2. *Let (P, μ_P), Δ_P and ϕ_P be as in Proposition 7.1, and let $\epsilon : P \to P[1]$ be such that $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P = \mathbf{d}\epsilon$. Then the map*

$$Tr(\phi_P)\sigma_{P,P} Tr(\Delta_P) + Tr(\epsilon) : Tr(P) \to Tr(P[1])$$

induces the Connes differential on the Hochschild homology.

Proof. Since $\mathbf{d}(\phi_P \Delta_P) = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$, it is enough to note that the map $H_*(Tr(\phi_P)\sigma_{P,P} Tr(\Delta_P) + Tr(\epsilon)) : \mathrm{HH}_*(A) \to \mathrm{HH}_*(A)$ does not depend on the choice of ϵ. □

Now it is not difficult to express the Connes differential in terms of a contracting homotopy.

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Corollary 7.3. *Let S, t_L and η_L be as in the previous section. Then the map $-Tr(\mu_R St_L)\sigma_{P,P}Tr((1_P \otimes (\mu_R S\eta_L)^2)S\eta_L)$ induces the Connes differential on the Hochschild homology.*

Proof. It follows from Lemma 6.4 that $\mu_R S\eta_L : P \to P$ is a comparison morphism, i.e. there is some $u : P \to P[1]$ such that $1 - \mu_R S\eta_L = \mathbf{d}u$. It is not hard to show using Lemma 6.3 (see also the proof of Theorem 6.2) that $\mathbf{d}\phi_P = \mu_L - \mu_R$ for $\phi_P = u(\mu_L - \mu_R) - \mu_R St_L(\mu_R S\eta_L \otimes 1_P)$. Note also that $(\mu_L - \mu_R)\Delta_P = 0$ for $\Delta_P = (1_P \otimes \mu_R S\eta_L)S\eta_L$. Then the Connes differential is induced by the map

$$Tr(\phi_P)\sigma_{P,P}Tr(\Delta_P) = -Tr(\mu_R St_L)\sigma_{P,P}Tr((1_P \otimes (\mu_R S\eta_L)^2)S\eta_L). \qquad \square$$

Now we explain how one can obtain a formula for a BV differential on the Hochschild cohomology of a symmetric algebra in terms of an arbitrary resolution.

First of all, let us recall that there are well-known maps $\mathbf{i}_f : \mathrm{HH}_*(A) \to \mathrm{HH}_*(A)$ for $f \in \mathrm{HH}^*(A)$, whose definition can be found, for example, in [10]. These maps satisfy the condition $\mathbf{i}_f\mathbf{i}_g = \mathbf{i}_{f\smile g}$. We also need the facts that $\mathbf{i}_f|_{\mathrm{HH}_n(A)} = 0$ for $n < |f|$ and $\mathbf{i}_f|_{\mathrm{HH}_{|f|}(A)}$ is the map induced by $Tr(\tilde{f}) : Tr(P_n) \to Tr(A) \cong \mathrm{HH}_0(A)$, where $\tilde{f} \in \mathrm{Hom}_{A^e}(P_n, A)$ represents f. After the correction of signs one obtains by [10, Lemma 15] that

$$\begin{aligned}\mathbf{i}_{[f,g]}(x) = (-1)^{(|f|+1)|g|}(&-(-1)^{(|f|+|g|)}\mathcal{B}\mathbf{i}_{f\smile g}(x)\\ &+ \mathbf{i}_f\mathcal{B}\mathbf{i}_g(x) - (-1)^{|f||g|}\mathbf{i}_g\mathcal{B}\mathbf{i}_f(x) - \mathbf{i}_{f\smile g}\mathcal{B}(x))\end{aligned}$$

for all $f, g \in \mathrm{HH}^*(A)$, $x \in \mathrm{HH}_*(A)$. Considering $x \in \mathrm{HH}_{|f|+|g|-1}(A)$, we get

$$Tr([f,g]) = -(-1)^{(|f|+1)|g|}(Tr(f \smile g)\mathcal{B} - Tr(f)\mathcal{B}\mathbf{i}_g - (-1)^{|f||g|}Tr(g)\mathcal{B}\mathbf{i}_f). \qquad (7.1)$$

Definition 7.4. A *BV algebra* is a Gerstenhaber algebra $(R^\bullet, \smile, [\ ,\ ])$ with an operator $\mathcal{D}\colon R^\bullet \to R^{\bullet-1}$ of degree -1 such that $\mathcal{D} \circ \mathcal{D} = 0$ and

$$[a,\, b] = -(-1)^{(|a|+1)|b|}(\mathcal{D}(a \smile b) - \mathcal{D}(a) \smile b - (-1)^{|a|}a \smile \mathcal{D}(b))$$

for homogeneous elements $a, b \in R^\bullet$.

Definition 7.5. A finite-dimensional algebra A is called *symmetric* if $A \cong \mathrm{Hom}_{\mathbf{k}}(A, k)$ as an A-bimodule.

Let A be symmetric. Let $\theta : A \to \mathbf{k}$ be an image of 1 under some bimodule isomorphism from A to $\mathrm{Hom}_{\mathbf{k}}(A, \mathbf{k})$. Then it is easy to see that θ induces a map from $Tr(A)$ to $\mathbf{k}$. We also denote this map by θ. Note also that if $f \in \mathrm{Hom}_{A^e}(M, A)$, then $\theta Tr(f) = 0$ if and only if $f = 0$.

Let $\mathcal{B}_P : Tr(P) \to Tr(P[1])$ be a map inducing the Connes differential on the Hochschild homology. Then we can define $\mathcal{D}_P(f) : P \to A[1-n]$ for $f : P \to A[-n]$ as the unique map such that $\theta Tr(\mathcal{D}_P(f)) = \theta Tr(f)\mathcal{B}_P$.

Proposition 7.6 (see [17]). *$\mathcal{D}_P$ induces a BV differential on the Hochschild cohomology.*

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Proposition 7.6 is the remark after [17, Theorem 1]. To see that it is valid, one can apply θ to the equality (7.1) with $\mathcal{B}=\mathcal{B}_P$ and get

$$\begin{aligned}\theta Tr([f,g]) &= -(-1)^{(|f|+1)|g|}(\theta Tr\mathcal{D}_P(f\smile g)-\theta Tr\mathcal{D}_P(f)\mathbf{i}_g-(-1)^{|f||g|}\theta Tr\mathcal{D}_P(g)\mathbf{i}_f)\\ &= -(-1)^{(|f|+1)|g|}\theta Tr(\mathcal{D}_P(f\smile g)-\mathcal{D}_P(f)\smile g-(-1)^{|f|}f\smile\mathcal{D}_P(g)).\end{aligned}$$

Note also that if one knows the BV differential and the cup product, then it is easy to compute the Gerstenhaber bracket.

8. Example of an application

Here, we apply the results of the previous sections to describe the BV structure on the Hochschild cohomology of the family of algebras considered in [2, 3]. For this section, we fix some integer $k>1$ and set $A=\mathbf{k}\langle x_0,x_1\rangle/\langle x_0^2,x_1^2,(x_0x_1)^k-(x_1x_0)^k\rangle$. The index α in the notation x_α is always specified modulo 2. If a is an element of $\mathbf{k}\langle x_0,x_1\rangle$, then we also denote by a its class in A.

Let G be a subset of $\mathbf{k}\langle x_0,x_1\rangle$ formed by the elements $(x_0x_1)^{i+1}$, $x_1(x_0x_1)^i$, $(x_1x_0)^i$ and $x_0(x_1x_0)^i$ for $0\leqslant i\leqslant k-1$. Note that the classes of the elements from G form a basis of A. Let G denote this basis too. Let l_v denote the length of $v\in G$. Note that the algebra A is symmetric with θ defined by the equalities $\theta((x_0x_1)^k)=1$ and $\theta(v)=0$ for $v\in G\setminus\{(x_0x_1)^k\}$. For $v\in G$, we introduce $v^*\in G$ as the unique element such that $\theta(vv^*)=1$. Note that $\theta(vw)=0$ for $w\in G\setminus\{v^*\}$. For $a=\sum_{v\in G}a_vv\in A$, where $a_v\in\mathbf{k}$ for $v\in G$, we define $a^*:=\sum_{v\in G}a_vv^*\in A$. It is clear that $(a^*)^*=a$ for any $a\in A$. If $v,w\in G$, then v/w denotes $(v^*w)^*$. If there is such $u\in G$ that $wu=v$, then this u is unique and $v/w=u$. If there is no such u, then $v/w=0$. Note that $(v/x_\alpha x_\beta)/x_\alpha$ is equal to v/x_α if $\alpha=\beta$ and $v\in\{x_\alpha,1^*\}$, and is equal to 0 in all remaining cases. For $a=\sum_{v\in G}a_vv\in A$ and $b=\sum_{v\in G}b_vv\in A$, where $a_v,b_v\in\mathbf{k}$ for $v\in G$, we define $a/b:=\sum_{v,w\in G}a_vb_wv/w\in A$.

In this section, we will use the bimodule resolution of A described in [2]. Here we present it in a slightly different form, but one can easily check that it is the same resolution. Let us introduce the algebra $B=\mathbf{k}[x_0,x_1,z]/\langle x_0x_1\rangle$. We introduce the grading on B by the equalities $|x_0|=|x_1|=1$ and $|z|=2$. Let us define the A^e-complex P. We set $P=A\otimes B\otimes A$ as an A-bimodule. The grading on P comes from the grading on B and the trivial grading on A. Let $\overline{a}$ $(a\in B)$ denote $1\otimes a\otimes 1$. For convenience we set $\overline{a}=0$ if $a=x_\alpha^iz^j$, where $\alpha\in\{0,1\}$ and i or j is less than 0. We define the differential d_P by the equality

$$\mathrm{d}_P(\overline{x_\alpha^iz^j})=\begin{cases}0 & \text{if } i=j=0,\\ x_\alpha\overline{x_\alpha^{i-1}}+(-1)^i\overline{x_\alpha^{i-1}}x_\alpha & \text{if } j=0, i>0,\\ \displaystyle\sum_{v\in G,\beta\in\{0,1\}}(-1)^{jl_v+\beta}v^*\overline{x_\beta z^{j-1}}\frac{v}{x_\beta} & \text{if } i=0, j>0,\\ x_\alpha\overline{x_\alpha^{i-1}z^j}+(-1)^{i+j}\overline{x_\alpha^{i-1}z^j}x_\alpha & \\ +(-1)^{i+\alpha}((-1)^jx_\alpha^*\overline{x_\alpha^{i+1}z^{j-1}}+\overline{x_\alpha^{i+1}z^{j-1}}x_\alpha^*) & \text{if } i,j>0,\end{cases}$$

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for $\alpha \in \{0,1\}$, $i,j \geqslant 0$. We define $\mu_P : P_0 \to A$ by the equality $\mu_P(\bar{1}) = 1$. Then one can check that (P, μ_P) is an A^e-projective resolution of A isomorphic to the resolution from [2]. Let us define the left contracting homotopy (t_P, η_P) for (P, μ_P). We define η_P by the equality $\eta_P(1) = \bar{1}$. Now, for $v \in G$, $\alpha \in \{0,1\}$ and $i,j \geqslant 0$, we define

$$t_P(\overline{x_\alpha^i z^j} v) = \begin{cases} \displaystyle\sum_{w\in G,\beta\in\{0,1\}} (-1)^{j(l_w+l_v+1)+1} \frac{w^*}{v^*} \overline{x_\beta z^j} \frac{w}{x_\beta} & \text{if } i=0,\ v \neq 1^*, \\ \displaystyle\sum_{w\in G} (-1)^{j(l_w+1)+1} w^* \overline{x_{l_w} z^j} \frac{w}{x_{l_w}} & \text{if } i=0 \text{ and } v=1^*, \\ (-1)^{i+j+1} \overline{x_\alpha^{i+1} z^j} \dfrac{v}{x_\alpha} + (-1)^{jl_v+j+l_v} \dfrac{v}{x_1^*} \overline{z^{j+1}} & \text{if } i=1 \text{ and } \alpha=1, \\ (-1)^{i+j+1} \overline{x_\alpha^{i+1} z^j} \dfrac{v}{x_\alpha} & \text{otherwise.} \end{cases}$$

In this section, we will use the notation of § 6. Our aim is to describe the BV structure on the Hochschild cohomology of A. As was explained in the previous section, it is enough to describe the Connes differential. By Corollary 7.3, we have to describe the map $-Tr(St_L)\sigma_{P,P}Tr((1_P \otimes (\mu_R S\eta_L)^2)S\eta_L)$. We will give here only the descriptions of the diagonal 2-approximation and the BV differential on the Hochschild cohomology, omitting the details of computations that can be found in the arXiv version. Let us start with the map $S\eta_L : P \to P \otimes_A P$.

First, let us introduce the following notation:

$$A_{t,j} = \sum_{\substack{v,\, w\, \in\, G \\ \alpha,\, \beta\, \in\, \{0,1\}}} (-1)^{jl_v+t(l_w+l_v+1)+\beta} \frac{w^*}{v} \overline{x_\alpha z^t} \frac{w}{x_\alpha} \otimes \overline{x_\beta z^{j-1}} \frac{v}{x_\beta},$$

$$B_{t,j} = \sum_{v\in G,\beta\in\{0,1\}} (-1)^{(j+t)(l_v+1)} \frac{x_{\beta+1}^*}{v} \overline{x_{\beta+1}^2 z^t} (x_\beta x_{\beta+1})^{k-1} \otimes \overline{x_\beta^2 z^{j-2}} \frac{v}{x_\beta},$$

$$C_{t,i,j,\alpha} = (-1)^{i+j+\alpha} \sum_{w\in G,\beta\in\{0,1\}} (-1)^{tl_w} \frac{w^*}{x_\alpha} \overline{x_\beta z^t} \frac{w}{x_\beta} \otimes \overline{x_\alpha^{i+1} z^{j-1}},$$

$$D_{t,i,j,\alpha} = (-1)^{(i+1)t} \sum_{v\in G,\beta\in\{0,1\}} (-1)^{jl_v+\beta} \overline{x_\alpha^{i+1} z^t} \frac{v^*}{x_\alpha} \otimes \overline{x_\beta z^{j-1}} \frac{v}{x_\beta},$$

$$\begin{aligned} E_{t,i,j,\alpha} &= (x_{\alpha+1}x_\alpha)^{k-1} \overline{x_{\alpha+1}^2 z^t} (x_\alpha x_{\alpha+1})^{k-1} \otimes \overline{x_\alpha^{i+2} z^{j-2}} \\ &\quad + (-1)^{it} \overline{x_\alpha^{i+2} z^t} (x_{\alpha+1}x_\alpha)^{k-1} \otimes \overline{x_{\alpha+1}^2 z^{j-2}} (x_\alpha x_{\alpha+1})^{k-1}. \end{aligned}$$

Lemma 8.1. *If $j \geqslant 0$ is some integer, then* $S\eta_L(\overline{z^j}) = \sum_{t=0}^{j} (-1)^{(j+1)t} (\overline{z^t} \otimes \overline{z^{j-t}} + A_{t,j-t} + B_{t,j-t})$.

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Lemma 8.2. *If $\alpha \in \{0,1\}$, and $j \geqslant 0$ and $i > 0$ are some integers, then*

$$S\eta_L(\overline{x_\alpha^i z^j}) = \sum_{t=0}^{j}(-1)^{(i+j+1)t} \times \left(\sum_{r=0}^{i}(-1)^{rt}\overline{x_\alpha^r z^t} \otimes \overline{x_\alpha^{i-r} z^{j-t}} + C_{t,i,j-t,\alpha} + D_{t,i,j-t,\alpha} + E_{t,i,j-t,\alpha}\right).$$

In particular, it follows from Lemmas 8.1 and 8.2 that $\mu_R S\eta_L = 1_P$. This effect occurs frequently and significantly simplifies the subsequent calculations.

Let us now recall the description of $\mathrm{HH}^*(A)$ given in [2, 3]. Note that $\mathrm{Hom}_{A^e}(P_n, A) \cong \mathrm{Hom}_{\mathbf{k}}(B_n, A) \cong A^{\dim_{\mathbf{k}} B_n} = A^{n+1}$. We choose this isomorphism in the following way. We send $f \in \mathrm{Hom}_{A^e}(P_n, A)$ to

$$\begin{cases} \sum\limits_{\substack{p+2j=n,\\ p>0,\alpha\in\{0,1\}}} f(\overline{x_\alpha^p z^j})e_{p+\alpha}^n & \text{if } 2 \nmid n,\\ f(\overline{z^{\frac{n}{2}}})e_1^n + \sum\limits_{\substack{p+2j=n,\\ p>0,\alpha\in\{0,1\}}} f(\overline{x_\alpha^p z^j})e_{p+\alpha}^n & \text{if } 2 \mid n. \end{cases}$$

Here, $e_i^n \in A^{n+1}$ is such an element that $\pi_j^n(e_i^n) = 0$ for $j \neq i$ and $\pi_i^n(e_i^n) = 1$, where $\pi_j^n : A^{n+1} \to A$ $(1 \leqslant j \leqslant n+1)$ is the canonical projection on the jth component of the direct sum. We identify $\mathrm{Hom}_{A^e}(P_n, A)$ and $A^{\dim_{\mathbf{k}} B_n}$ by the isomorphism just defined.

Let us introduce some elements of $\mathrm{Hom}_{A^e}(P, A) = \bigoplus_{n\geqslant 0} A^{\dim_{\mathbf{k}} B_n}$.

- $p_1 = x_0x_1 + x_1x_0$, $p_2 = x_1^*$, $p_2' = x_0^*$ and $p_3 = 1^*$ are elements of $\mathrm{Hom}_{A^e}(P_0, A) = A$;
- $u_1 = (x_0, 0)$, $u_1' = (0, x_1)$, $u_2 = (1, 0)$ and $u_2' = (0, 1)$ are elements of $\mathrm{Hom}_{A^e}(P_1, A) = A^2$;
- $v = (1,0,0)$, $v_1 = (x_0x_1 - x_1x_0, 0, 0)$, $v_2 = (0,1,0)$, $v_2' = (0,0,1)$ and $v_3 = (1^*, 0, 0)$ are elements of $\mathrm{Hom}_{A^e}(P_2, A) = A^3$;
- $w_1 = (x_0, 0, 0, 0)$, $w_2 = (x_0^*, 0, 0, 0)$ and $w_2' = (0, x_1^*, 0, 0)$ are elements of $\mathrm{Hom}_{A^e}(P_3, A) = A^4$;
- $t = (1,0,0,0,0)$ is an element of $\mathrm{Hom}_{A^e}(P_4, A) = A^5$.

It is proved in [2, 3] that the algebra $\mathrm{HH}^*(A)$ is generated by the cohomological classes of the elements from $\mathcal{X}$, where

$$\mathcal{X} = \begin{cases} \{p_1, p_2, p_2', p_3, u_1, u_1', u_2, u_2', v\} & \text{if } \mathrm{char}\,\mathbf{k} = 2,\\ \{p_1, p_2, p_2', u_1, u_1', v_1, v_2, v_2', v_3, w_1, w_2, w_2', t\} & \text{if } \mathrm{char}\,\mathbf{k} \neq 2, \mathrm{char}\,\mathbf{k} \mid k,\\ \{p_1, p_2, p_2', u_1, u_1', v_1, v_2, v_2', t\} & \text{if } \mathrm{char}\,\mathbf{k} \neq 2, \mathrm{char}\,\mathbf{k} \nmid k. \end{cases}$$

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Note that our notation is essentially the same as that of [2], but slightly differs from the notation of [3]. For simplicity, we denote the cohomological class of $a \in \mathrm{Hom}_{A^e}(P_n, A)$ by a too.

Obtaining the description of $Tr(\mu_R St_L)\sigma_{P,P}$ on the image of $Tr(S\eta_L)$, we get the map $\mathcal{B}_P : Tr(P) \to Tr(P[1])$. It follows from the previous section that we can define the BV differential $\mathcal{D}_P : \mathrm{HH}^*(A) \to \mathrm{HH}^*(A)$ by the formula

$$\mathcal{D}_P(f)(a) = \sum_{v \in G} \theta Tr(f)\mathcal{B}_P(v \otimes a)v^*$$

for $a \in P$. Finally, we get

$$\begin{aligned}
\mathcal{D}_P(u_2) &= \mathcal{D}_P(p_2'u_2) = \mathcal{D}_P(u_2^2) = \mathcal{D}_P(u_2') = \mathcal{D}_P(p_2u_2') = \mathcal{D}_P((u_2')^2)\\
&= \mathcal{D}_P(v) = \mathcal{D}_P(p_1v)\\
&= \mathcal{D}_P(u_1v) = \mathcal{D}_P(u_1'v) = \mathcal{D}_P(u_2v) = \mathcal{D}_P(u_2'v) = \mathcal{D}_P(v^2) = \mathcal{D}_P(v_1)\\
&= \mathcal{D}_P(p_1v_1) = \mathcal{D}_P(v_2)\\
&= \mathcal{D}_P(p_2'v_2) = \mathcal{D}_P(v_2^2) = \mathcal{D}_P(v_2') = \mathcal{D}_P(p_2v_2') = \mathcal{D}_P((v_2')^2)\\
&= \mathcal{D}_P(w_1) = \mathcal{D}_P(t) = \mathcal{D}_P(p_1t)\\
&= \mathcal{D}_P(v_1t) = \mathcal{D}_P(v_2t) = \mathcal{D}_P(v_2't) = \mathcal{D}_P(w_1t) = \mathcal{D}_P(t^2) = 0,\\
\mathcal{D}_P(u_1) &= \mathcal{D}_P(u_1') = k,\ \mathcal{D}_P(p_1u_1) = (k-1)p_1,\ \mathcal{D}_P(p_2u_1') = p_2,\ \mathcal{D}_P(p_2'u_1) = p_2',\\
\mathcal{D}_P(v_3) &= u_1' - u_1\,, \mathcal{D}_P(p_2v) = u_2',\ \mathcal{D}_P(p_2'v) = u_2,\ \mathcal{D}_P(p_3v) = u_1 + u_1',\\
\mathcal{D}_P(u_1u_1') &= k(u_1' - u_1),\ \mathcal{D}_P(u_1u_2) = ku_2,\ \mathcal{D}_P(u_1'u_2') = ku_2',\ \mathcal{D}_P(w_2) = v_2,\\
\mathcal{D}_P(w_2') &= -v_2',\ \mathcal{D}_P(u_1v_1) = (2k-1)v_1,\ \mathcal{D}_P(u_1v_2) = (k+2)v_2,\ \mathcal{D}_P(u_1v_2') = kv_2',\\
\mathcal{D}_P(u_1'v_2) &= kv_2,\ \mathcal{D}_P(u_1'v_2') = (k+2)v_2',\ \mathcal{D}_P(v_2v_3) = 3w_2,\ \mathcal{D}_P(v_2'v_3) = 3w_2',\\
\mathcal{D}_P(u_1t) &= \mathcal{D}_P(u_1't) = 3kt,\ \mathcal{D}_P(v_2w_2) = v_2^2,\ \mathcal{D}_P(v_2'w_2') = -(v_2')^2,\\
\mathcal{D}_P(v_3t) &= 3(u_1' - u_1)t,\ \mathcal{D}_P(w_2t) = 3v_2t,\ \mathcal{D}_P(w_2't) = 3v_2't.
\end{aligned}$$

During our calculations, the results of [2, 3] were used. In particular, we used the formulas for some products in $\mathrm{HH}^*(A)$ and the description of some coboundaries. Alternatively, one can use the formula $f \smile g = (f \otimes g)S\eta_L$ and Lemmas 8.1 and 8.2 to compute products in $\mathrm{HH}^*(A)$. Note also that in each of the formulas above we assume that all the elements included in the formula lie in $\mathcal{X}$. For example, if v appears in some equality, then this equality holds for $\mathrm{char}\,\mathbf{k} = 2$, but it does not have to hold for $\mathrm{char}\,\mathbf{k} \neq 2$. We also have $\mathcal{D}_P(a) = 0$ for all $a \in \mathrm{HH}^0(A)$. Now it is not hard to recover the Gerstenhaber bracket and the rest of the BV differential on the Hochschild cohomology of A using relations between the generators of $\mathrm{HH}^*(A)$ described in [2, 3] and the graded Leibniz rule for the Gerstenhaber bracket and the cup product.

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