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PFA(S)[S]: More Mutually Consistent Topological Consequences of PFA and $V = L$

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Abstract. Extending the work of Larson and Todorcevic, we show that there is a model of set theory in which normal spaces are collectionwise Hausdorff if they are either first countable or locally compact, and yet there are no first countable L-spaces or compact S-spaces. The model is one of the form PFA(S)[S], where S is a coherent Souslin tree.

1 Introduction

Models we shall call "of form PFA(S)[S]" were introduced by S. Todorcevic in an unpublished note in 2001, where he used them to prove the consistency of the following statement.

> Every compact hereditarily normal space satisfying the countable chain condition is hereditarily separable and hereditarily Lindelöf.

These models are obtained by fixing a particular *coherent* Souslin tree S in a ground model (such trees are obtainable from $\diamondsuit$, for example), then iterating proper posets as in the consistency proof for PFA, but only those that preserve S, thus producing a model for $PFA(S)$, *i.e.,* PFA restricted to posets that preserve (the Souslinity of) S. That a countable support iteration of proper posets that preserve S preserves S is shown in [17]. Finally, one forces with S. A weaker technique, not requiring large cardinals, is to replace "proper" with "countable chain condition."

If all models formed by forcing with S over a model of PFA(S) satisfy φ, we say that PFA(S)[S] *implies* φ. If a particular ground model is used, we say φ *holds in a model of form* PFA(S)[S]. Which coherent S we use does not matter. The consistency of a supercompact cardinal is assumed.

Since we will be mainly dealing with locally compact spaces, for convenience we will assume all spaces are Hausdorff.

The solution by Larson and Todorcevic to Katětov's problem [15] depended on showing the remarkable fact that — using the weaker c.c.c. technique — some of the "Souslin-type" consequences [10] of MA_{ω_1}, namely that compact, first countable, hereditarily separable spaces are hereditarily Lindelöf, and that first countable,

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hereditarily Lindelöf spaces are hereditarily separable, are consistent with some of the "normal implies collectionwise Hausdorff" consequences of $V = L$, namely that separable normal first countable spaces are collectionwise Hausdorff. Since then the strength of both types of consequences has been increased. Larson and Tall [13] dropped the separability in the second type of consequence by starting with a particular ground model, while Todorcevic in 2001 and in a recent preprint improved [15] to get from PFA$(S)[S]$ that compact hereditarily separable spaces are hereditarily Lindelöf. Here we obtain another result in the $V = L$ column, getting that normal spaces that are either first countable or locally compact are collectionwise Hausdorff. The "locally compact" proof is considerably harder than the one for first countability, and unlike first countability, we do not know how to prove the locally compact result by just Souslin-tree forcing.

In [10] a model was constructed in which the "combinatorial" consequences of MA_{ω_1} held, but not the "Souslin-type" consequences. The current investigations of PFA$(S)[S]$ can be viewed as complementary: we construct a model in which the Souslin-type consequences of MA_{ω_1}, indeed of PFA, hold, but not the combinatorial ones.

Much of the set-theoretic portion of our proof follows a portion of Todorcevic's proof that PFA$(S)[S]$ implies that compact hereditarily separable spaces are hereditarily Lindelöf.

Definition 1.1 Let D be a discrete subspace of a topological space. An *expansion* of D is a collection of sets $\{A_d : d \in D\}$ such that $d \in A_d$ but not in A_e, $e \neq d$. A *separation* of D is an open disjoint expansion. A space is $(\lambda$-$)$*collectionwise Hausdorff* if every closed discrete subspace (of size $\leq \lambda$) is separated.

Main Theorem *Assuming the consistency of a supercompact cardinal, there is a model of form* PFA$(S)[S]$ *in which every locally compact normal space is collectionwise Hausdorff.*

The hard part of the proof is to show the following theorem.

Theorem 1.2 PFA$(S)[S]$ *implies that locally compact normal spaces are* $\aleph_1$*-collectionwise Hausdorff.*

In order to obtain full collectionwise Hausdorffness, we use a particular model of form PFA$(S)[S]$ (the one in [13]), so that GCH holds except at $\aleph_0$, and $\diamondsuit$ *for stationary systems* [7] holds for regular cardinals $\geq \aleph_2$. Following Watson [27], these will imply all locally compact normal spaces are collectionwise Hausdorff if all locally compact normal spaces are $\aleph_1$-collectionwise Hausdorff, which we will have.

To obtain the model of [13], start with a supercompact cardinal κ in the ground model. First make κ's supercompactness indestructible under κ-directed-closed forcing [16]. Then, via Easton forcing, add λ^+ Cohen subsets of size λ, for every regular $\lambda \geq \kappa$. This will establish GCH at κ and above. Then force $\diamondsuit$. This does not affect the supercompactness of κ. We then proceed as outlined in the first paragraph of this paper. That $\diamondsuit$ for stationary systems holds for all regular $\lambda \geq \aleph_2$ follows because the iteration satisfies the κ-chain condition and so preserves the $\diamondsuit$ for stationary systems created by the Cohen forcing [21].

Although the proof of our Main Theorem is given completely in this paper, subject to reliance on published results, some of its interesting consequences depend on unpublished work of Todorcevic, namely Theorem 1.3, Lemma 1.4, and Propositions 1.7 and 1.8. His work on forcing with a coherent Souslin tree is now available in preprints, however.

Remark 1 Since this paper is likely to be the first published paper containing details of the PFA(S)[S] methods, I also thought it useful to list here some of the principal consequences known to hold via these methods, even though none will be used in this article. The following consequences of PFA hold in this model:

(1) OCA [6].
(2) Every Aronszajn tree is special [6].
(3) $\mathfrak{b} = \aleph_2 = 2^{\aleph_0}$ [12].
(4) P-ideal dichotomy.
(5) Every compact, countably tight space is sequential.
(6) Every compact hereditarily separable space is hereditarily Lindelöf.
(7) Every first countable hereditarily Lindelöf space is hereditarily separable [15].
(8) In a compact, countably tight space, locally countable subspaces of size $\aleph_1$ are σ-discrete.
(9) Every compact hereditarily normal space satisfying the countable chain condition is hereditarily separable and hereditarily Lindelöf [19].

If PFA(S) is forced in the usual Laver-$\diamondsuit$ fashion [4, 16], we also obtain the following [14].

(10) Fleissner's Axiom R:

> ***Definition*** $\Gamma \subseteq [X]^{<\kappa}$ is *tight* if whenever $\{C_\alpha : \alpha < \delta\}$ is an increasing sequence from Γ and $\omega < cf(\delta) < \kappa$ then $\bigcup\{C_\alpha : \alpha < \delta\} \in \Gamma$.
>
> **Axiom R.** If $\Sigma \subseteq [X]^\omega$ is stationary and $\Gamma \subseteq [X]^{<\omega_2}$ is tight and unbounded, then there is $Y \in \Gamma$ such that $\mathcal{P}(Y) \cap \Sigma$ is stationary in $[Y]^\omega$.

Our main result has numerous easily established corollaries. We list one.

Theorem 1.3 ([23]) *There is a model of form* PFA(S)[S] *in which locally compact normal spaces are paracompact if they either have closed sets G_δ's or are metalindelöf.*

Theorem 1.3 follows immediately from Theorem 1.2 plus known results. In [13] the following lemma is shown.

Lemma 1.4 *There is a model of form* PFA(S)[S] *in which locally compact perfectly normal spaces are paracompact.*

In [8], the following lemma is proved.

Lemma 1.5 *Locally compact, normal, metalindelöf, $\aleph_1$-collectionwise Hausdorff spaces are paracompact.*

Thus Theorem 1.3 follows. It is interesting because Arhangel'skiĭ [2] proved the following proposition 40 years ago.

Proposition 1.6 *Locally compact perfectly normal metacompact spaces are paracompact.*

Let us also note that, using the Main Theorem, Larson and Tall [14] have established the following proposition.

Proposition 1.7 *There is a model of form* PFA$(S)[S]$ *in which locally compact hereditarily normal spaces are hereditarily paracompact if and only if they do not include a perfect pre-image of* ω_1.

In [19], the author has extended their methods to locally compact normal spaces, proving results such as the following proposition.

Proposition 1.8 *There is a model of form* PFA$(S)[S]$ *in which locally compact normal spaces are paracompact and countably tight if and only if they do not include a perfect pre-image of* ω_1, *and closures of their countable subspaces are Lindelöf.*

One can also obtain the following proposition.

Proposition 1.9 ([19]) PFA$(S)[S]$ *implies that every compact, homogeneous, hereditarily normal space is first countable.*

2 Preliminaries

We shall actually prove an apparently weaker form of Theorem 1.2, showing only that there is a closed unbounded $C \subseteq \omega_1$ such that $\{x_\alpha : \alpha \in C\}$ has a separation; however, by the following argument of Alan Taylor [24], that will suffice.

Let $\{x_\alpha : \alpha < \omega_1\}$ be an enumeration of a closed discrete subspace of a normal space X. Let $\mathcal{I}$ be the ideal of *separated* subsequences of $\{x_\alpha : \alpha < \omega_1\}$, *i.e.,* those that have a separation. Then, by normality and the well-known Lemma 2.2, $\mathcal{I}$ is a countably complete ideal on ω_1 containing all singletons, and $\mathcal{I}$ is proper if and only if $\{x_\alpha : \alpha < \omega_1\}$ is not separated. Note that every bijection $f : \omega_1 \to \omega_1$ gives rise to a rearrangement of the sequence $\{x_\alpha : \alpha < \omega_1\}$, as well as to an isomorph $\mathcal{I}^*$ of $\mathcal{I}$. But the following lemma was sketched in [24].

Lemma 2.1 *For every countably complete proper ideal* $\mathcal{I}$ *on* ω_1, *some isomorph* $\mathcal{I}^*$ *of* $\mathcal{I}$ *contains no closed unbounded set.*

Thus, if every enumeration of $\{x_\alpha : \alpha < \omega_1\}$ includes a separated closed unbounded set, then the subspace $\{x_\alpha : \alpha < \omega_1\}$ is separated. ■

Taylor cites [3] as a source for Lemma 2.1. The statement and proof apparently appeared in a preprint of [3], but not in the published version. However, the proof can be reconstructed from the sketch Taylor gives.

We will need a well-known result.

Lemma 2.2 *In a normal space, every countable discrete collection of closed sets can be expanded to a discrete collection of open sets.*

Rather than forcing a separation (on a closed unbounded C) of our closed discrete set D, we will get away with forcing something prima facie much weaker. First of all, by Theorem 2.5, it will suffice to expand the points in C to a discrete collection of compact G_δ's.

Definition 2.3 **CKG** (Collectionwise Compact G_δ's) is the assertion that in a normal space, if $\mathcal{K} = \{K_\alpha\}_{\alpha<\omega_1}$ is a discrete collection of compact sets, each of which has *countable character*, *i.e.,* there exist open $\{U_{\alpha n}\}_{n<\omega}$, $U_{\alpha n} \supseteq K_\alpha$, such that each open set including K_α includes some $U_{\alpha n}$, then $\mathcal{K}$ can be separated by disjoint open sets.

Note that in a locally compact space, compact G_δ's have countable character. A slightly weaker assertion than **CKG** is that normal first countable spaces are $\aleph_1$-collectionwise Hausdorff. In [13] it was established that

Proposition 2.4 *Force with a Souslin tree. Then normal first countable spaces are $\aleph_1$-collectionwise Hausdorff.*

Exactly the same proof establishes the following theorem.

Theorem 2.5 *Force with a Souslin tree. Then* **CKG** *holds.*

For the convenience of the reader, I will present the proof of Theorem 2.5.

Proof Suppose $\{K_\alpha\}_{\alpha<\omega_1}$ is a discrete collection of closed sets in a normal space, and that for each α, $\{N(\alpha,i)\}_{i<\omega}$ is a descending neighbourhood base for K_α, in an extension obtained by forcing with a Souslin tree S. Let $\{\dot N(\alpha,i) : i<\omega,\ \alpha<\omega_1\}$ be S-names for the corresponding sets. For $s \in S$, let $\mathrm{ht}(s)$ be the height of s. Since S has countable levels and its corresponding forcing poset is ω-distributive, we can construct an increasing function $h\colon \omega_1 \to \omega_1$ such that the following holds.

> For all $\alpha < \omega_1$ and all $s \in S$ with $\mathrm{ht}(s) = h(\alpha)$, s decides all statements of the form "$\dot N(\beta, j) \cap \dot N(\alpha,i) = 0$", for all $i, j < \omega$ and $\beta < \alpha$.

Let $\dot A$ be an S-name for a subset of ω_1 such that for no $\alpha < \omega_1$ does any $s \in S$ with $\mathrm{ht}(s) = h(\alpha)$ decide whether $\alpha \in A$. To define such an $\dot A$, for each $\alpha < \omega_1$ pick two successors of each $s \in S$ with $\mathrm{ht}(s) = h(\alpha)$ and let one force $\alpha \in \dot A$ and let the other force $\alpha \notin \dot A$.

Let $\dot f$ be an S-name for a function $f\colon \omega_1 \to \omega$ such that

$$\bigcup\bigl\{N(\alpha, f(\alpha)) : \alpha \in A\bigr\} \cap \bigcup\bigl\{N(\alpha, f(\alpha)) : \alpha \in \omega_1 - A\bigr\} = \varnothing.$$

Let C be a closed unbounded subset of ω_1 in V such that for each $s \in S$ with $\mathrm{ht}(s) \in C$, s decides $f|\,\mathrm{ht}(s)$ and $A|\,\mathrm{ht}(s)$, and such that for all $\alpha < \beta < \omega_1$, if $\beta \in C$, then $h(\alpha) < \beta$. We will define an S-name $\dot g$ for a function from ω_1 to ω such that whenever $\alpha < \beta < \omega_1$, if $(\alpha, \beta] \cap C \neq \varnothing$, then $N(\alpha, g(\alpha)) \cap N(\beta, g(\beta)) = \varnothing$.

Let $c\colon \omega_1 \to \omega_1$ be defined by $c(\alpha) = \sup(C \cap (\alpha+1))$. Fix $\beta < \omega_1$. Each $s \in S$ with $\mathrm{ht}(s) = h(\beta)$ decides $f|c(\beta)$ and $A|c(\beta)$ and "$\dot N(\alpha, f(\alpha)) \cap \dot N(\beta, i) = \varnothing$" for all $i < \omega$, $\alpha < c(\beta)$, but not whether $\beta \in A$. Fix $s \in S$ with $\mathrm{ht}(s) = h(\beta)$.

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Since s does not decide whether $\beta \in A$, we claim that there is an $i_0 < \omega$ such that for all $\alpha < c(\beta)$ such that $s \Vdash \alpha \in \dot{A}$, $s \Vdash \dot{N}(\alpha, \dot{f}(\alpha)) \cap \dot{N}(\beta, i_0) = \varnothing$. To see this, extend s to $t \in S$ forcing that $\beta \notin A$ and deciding $f(\beta)$. Let i_0 be the value of $f(\beta)$ as decided by t. Then for each $\alpha < c(\beta)$ such that $s \Vdash \alpha \in \dot{A}$, t forces that $N(\alpha, f(\alpha)) \cap N(\beta, i_0) = \varnothing$, but these facts were already decided by s.

Similarly, there is an $i_1 < \omega$ such that for all $\alpha < c(\beta)$ such that $s \Vdash \alpha \notin \dot{A}$, $s \Vdash \dot{N}(\alpha, \dot{f}(\alpha)) \cap \dot{N}(\beta, \dot{i}_1) = \varnothing$. Since s decides $A|c(\beta)$, letting $\bar{i} = \max\{i_0, i_1\}$, for all $\alpha < c(\beta)$, $s \Vdash \dot{N}(\alpha, \dot{f}(\alpha)) \cap \dot{N}(\beta, \bar{i})) = \varnothing$. We have such an $\bar{i}_s$ for each s in the $h(\beta)$-th level of the tree, so we can construct a name $\dot{g}$ such that

$$s \Vdash \dot{g}(\beta) = \max\{\bar{i}_s, \dot{f}(\beta)\}$$

for each $s \in S$ with $\operatorname{ht}(s) = h(\beta)$. Then $\dot{g}$ is as required.

To finish the proof, define $c\colon \omega_1 \to \omega_1$ by letting $c(\alpha) = \sup((C - \{0\}) \cap \alpha)$, and let $\alpha \sim \beta$ if $c(\alpha) = c(\beta)$. The $\sim$-classes are countable and so, by Lemma 2.2, there is a $q\colon \omega_1 \to \omega$ such that $c(\alpha) = c(\beta)$ implies that $N(\alpha, q(\alpha)) \cap N(\beta, q(\beta)) = \varnothing$. Let $r(\alpha) = \max(g(\alpha),\ q(\alpha))$. Then $\{N(\alpha, r(\alpha))\}_{\alpha<\omega_1}$ is the required separation. ■

As a further reduction, the following lemma shows that it suffices to expand the points in C to a σ-relatively-discrete collection of compact G_δ's.

Lemma 2.6 *Suppose Y is closed discrete in a normal space X, $Y = \bigcup_{n<\omega} Y_n$. Suppose (as* e.g., *if the space is also locally compact) there exist open sets U_y, $y \in Y$ and compact G_δ's K_y, $y \in Y$, such that $y \in K_y \subseteq U_y$, and that $y' \neq y$ implies $y' \notin U_y$. Further suppose that for each n and each $y \in Y_n$, $U_y \cap \bigcup\{K_{y'} : y' \in Y_n,\ y' \neq y\} = 0$. Then Y has a discrete expansion by compact G_δ's.*

Proof Without loss of generality, we may assume the Y_n's are disjoint. By normality, take open F_σ's W_n, $n < \omega$, such that $Y_n \subseteq W_n$, and $W_n \cap W_l = 0$, for $l \neq n$. For $y \in Y_n$, let $U'_y = U_y \cap W_n$, and let V be an open set with $Y \subseteq V \subseteq \overline{V} \subseteq \bigcup\{U'_y : y \in Y\}$. Let K'_y be a compact G_δ about y, $K'_y \subseteq K_y \cap V \cap U'_y$. Then $U'_y \supseteq K'_y$ but meets no $K_{y'}$, $y' \neq y$. Then $\{U'_y : y \in Y\} \cup \{X - \overline{V}\}$ witness the discreteness of $\{K'_y : y \in Y\}$. ■

Next let us recall that a disjoint collection of sets is relatively discrete if it is both left- and right-separated. The following lemma shows that left-separation is the sticking point.

Lemma 2.7 *Let $D = \{x_\alpha : \alpha < \omega_1\}$ be a closed discrete subspace of a locally compact normal space. Then $\{x_\alpha : \alpha < \omega_1\}$ has a right-separated expansion by compact G_δ's.*

Proof By normality, we can find an expansion of D by open F_σ's $\{U_\alpha : \alpha < \omega_1\}$. By local compactness, we can find compact G_δ's $\{H_\alpha\}_{\alpha<\omega_1}$, with $x_\alpha \in H_\alpha \subseteq U_\alpha$. Let $K_\alpha = H_\alpha - \bigcup_{\beta<\alpha} U_\beta$. Then $\{K_\alpha : \alpha < \omega_1\}$ is the required expansion. ■

Remark 2 Thus it will suffice to expand the points in C to compact G_δ's that are σ-left-separated by the right-separating U's. We shall do this by simultaneously approximating a countable partition of ω_1 by finite partial functions from ω_1 into ω and approximating finitely many of the desired compact G_δ's by finite decreasing sequences of compact G_δ's. Forcing the left-separation is based on Todorcevic's proof that $PFA(S)[S]$ implies that there are no compact S-spaces; the idea of simultaneously approximating the partition and the compact G_δ's was inspired by an analogous approximation of infinite subsequences in an MA argument in [8].

3 The Proof

The proof of Theorem 1.2 is long and notationally dense. I have included as remarks considerable explanatory commentary to assist the reader.

From now on, we assume PFA(S). We have an S-name $\dot{Z}$, such that S forces $\dot{Z}$ is a locally compact normal space. It is convenient to assume that $\{\alpha : \alpha < \omega_1\}$ is a closed discrete subspace of Z. We shall usually omit the "ˇ" that should be placed over elements of the ground model. Let $\dot{\mathcal{E}}$ be a name such that S forces $\dot{\mathcal{E}}$ to be the collection of non-empty compact G_δ's of $\dot{Z}$. We shall assume that for each $\alpha < \omega_1$, we have S-names $\dot{U}_\alpha$, $\dot{K}_\alpha$, $\dot{K}_{\alpha,\beta}$, $\beta < \alpha$, such that S forces

(i) $\alpha \in \dot{U}_\alpha$;
(ii) $\dot{U}_\alpha$ is open; $\overline{\dot{U}}_\alpha$ is compact;
(iii) $\alpha \neq \beta$ implies $\alpha \notin \overline{\dot{U}}_\beta$;
(iv) $\alpha \in \dot{K}_\alpha \subseteq \dot{U}_\alpha$;
(v) $\dot{K}_\alpha \in \dot{\mathcal{E}}$, $\dot{K}_{\alpha,\beta} \in \dot{\mathcal{E}}$;
(vi) $\beta < \alpha$ implies $\dot{K}_\alpha \cap \overline{\dot{U}}_\beta = 0$
(vii) for each α, $\{\dot{K}_{\alpha,\beta} : \beta < \alpha\} \subseteq \dot{\mathcal{E}}$ is discrete, with $\beta \in \dot{K}_{\alpha,\beta} \subseteq \dot{K}_\beta$, and if $\alpha < \gamma$, then $\dot{K}_{\gamma,\beta} \subseteq \dot{K}_{\alpha,\beta}$.

Item (vii) is easy to accomplish: discretely separate $\{\beta : \beta < \alpha\}$, shrink the separating open sets to compact G_δ's, and then intersect with the corresponding K_β's. We then can recursively shrink the compact G_δ's to get $K_{\gamma,\beta} \subseteq K_{\alpha,\beta}$. That is, having obtained say the discrete collection $\{K'_{\gamma,\beta} : \beta < \gamma\}$, let $K_{\gamma,\beta} = K'_{\gamma,\beta} \cap \bigcap\{K_{\alpha,\beta} : \alpha < \gamma\}$.

Let C be a closed unbounded subset of ω_1 such that for each $\delta \in C$, every node of the δ-th level of S decides all statements of form $\dot{K}_{\gamma,\beta} \cap \overline{\dot{U}}_\alpha = 0$ for all $\beta < \gamma \leq \alpha < \delta$. To see that there is such a club, note that we may take a maximal antichain A deciding $\dot{K}_{\gamma,\beta} \cap \dot{U}_\alpha = \varnothing$. Since A is countable, we can choose $h(\gamma, \beta, \alpha) < \omega_1$ above $\sup\{\mathrm{ht}(a) : a \in A\}$. Let C be closed unbounded such that $h(\gamma, \beta, \alpha) < \delta$ whenever $\gamma, \beta, \alpha < \delta \in C$. Let $C^\circ = \{\delta \in C : \sup(C \cap \delta) < \delta\}$. For $\delta \in C^\circ$, let $\delta^- = \sup(C \cap \delta)$. Note that every member of C is a δ^- for some $\delta \in C^\circ$. For $\delta \in C$, let δ^+ be the least element of C greater than δ.

Let $\mathcal{P}$ be the collection of all triples $p = \langle f_p, \mathcal{E}_p, \mathfrak{N}_p \rangle$ where

(1) f_p is a finite partial function from $S \mid C^\circ$ to ω. Let $\mathrm{dom}_l\, f_p = \{s : f_p(s) = l\}$. We require that each non-empty $\mathrm{dom}_l\, f_p$ consists of nodes of different heights.
(2) $\mathfrak{N}_p$ is a finite $\in$-chain of countable elementary submodels of H_κ where κ is regular and sufficiently large, containing all relevant objects, such that $\mathfrak{N}_p$ separates each $\mathrm{dom}_l\, f_p$ in the sense that if $s, s' \in \mathrm{dom}_l\, f_p$ with $s \neq s'$, then there is an $N \in \mathfrak{N}_p$ such that $s \in N$ and $s' \notin N$.
(3) $\mathcal{E}_p$ is a finite partial function from $\omega \times S \mid C^\circ$ to ω_1 such that, letting π_2 be the projection map from $\omega \times S \mid C^\circ$ onto $S \mid C^\circ$,

(a) $\pi_2[\operatorname{dom} \mathcal{E}_p] = \operatorname{dom} f_p$,
(b) $\mathcal{E}_p(n, s) \geq \operatorname{ht}(s)$,
(c) whenever $s \in N \in \mathfrak{N}_p$, $\mathcal{E}_p(n, s) \in N$,
(d) if $s, s' \in \operatorname{dom}_l f_p$ and s' strictly extends s and $\operatorname{ht}(s') = \tau$, then

$$s' \Vdash \bigcap\{\dot{K}_{\mathcal{E}_p(n,s),\operatorname{ht}(s)^-} : \langle n, s\rangle \in \operatorname{dom} \mathcal{E}_p\} \cap \dot{\overline{U}}_{\tau^-} = \varnothing.$$

For $p, q \in \mathcal{P}$, we let $p \leq q$ if and only if
(4) $f_p \mid \operatorname{dom} f_q = f_q$;
(5) $\mathcal{E}_p \mid \operatorname{dom} \mathcal{E}_q = \mathcal{E}_q$;
(6) $\mathfrak{N}_p \supseteq \mathfrak{N}_q$.

Remark 3 The rationale for the "if s' extends s" clause in (3)(d) is that we are coding the discrete subspace by a generic branch, and do not care what happens off that branch. The superscript minuses are there because we only expect conditions of height α to know about things of smaller index. Clause (3)(c) will ensure that the restriction of a condition p to N will be a member of N.

That $\mathcal{P}$ is transitive is clear.

Lemma 3.1 *Let $D_s = \{p \in \mathcal{P} : s \in \operatorname{dom} f_p\}$. Let $D_{s,n} = \{p \in \mathcal{P} : \langle n, s\rangle \in \operatorname{dom} \mathcal{E}_p\}$. Then for each $s \in S \mid C^\circ$ and each $n < \omega$, D_s and $D_{s,n}$ are dense.*

Proof Given any $q \in \mathcal{P}$, if $q \notin D_s$, take $m > \max\{f_q(t) : t \in \operatorname{dom} f_q\}$. Then $\langle f_q \cup \{\langle s, m\rangle\}, (\mathcal{E}_q \cup \{\langle\langle 0, s\rangle, \operatorname{ht}(s)\rangle\}, \mathfrak{N}_q\rangle$ is the required extension of q in D_s. Given $q \in D_s - D_{s,n}$, suppose k is least such that $\langle k, s\rangle \in \operatorname{dom} \mathcal{E}_q$. Let

$$q' = \langle f_q, \mathcal{E}_q \cup \{\langle\langle n, s\rangle, \mathcal{E}_q(k, s)\rangle\}, \mathfrak{N}_q\rangle.$$

Then q' is $\leq q$ and is a member of $D_{s,n}$. ■

Once we show that $\mathcal{P}$ is proper and preserves S, we will be able to finish the proof of Theorem 1.2 by proving the following lemma.

Lemma 3.2 *PFA$(S)[S]$ implies that C has a σ-left-separated, right- separated expansion by compact G_δ's, and hence a discrete expansion by compact G_δ's.*

Proof Let G be $\mathcal{P}$-generic for the D_s's and the $D_{s,n}$'s. Let $f = \bigcup\{f_p : p \in G\}$. Let $e = \bigcup\{\mathcal{E}_p : p \in G\}$. Then $e\colon \omega \times S \mid C^\circ \to \omega_1$. For $\gamma = \operatorname{ht}(s)^-$, $s \in B \mid C^\circ$, where B is the generic branch of S, let $E_\gamma = \bigcap\{K_{e(n,s),\gamma} : n < \omega\}$. Then S forces that $\{E_\gamma : \gamma \in C\}^{\cdot}$ is the required right-separated, σ-left-separated expansion of C by compact G_δ's. ■

We shall now start the proof that $\mathcal{P}$ is a proper poset that preserves S. The following lemma is due to Miyamoto [17]. Since the lemma is not quite stated there in this form, and the proof is short, we give the proof here.

Lemma 3.3 *$\mathcal{P}$ is proper and preserves S if for all sufficiently large regular θ and for a closed unbounded family $\mathcal{C}$ (in $[H_\theta]^{\aleph_0}$) of countable elementary submodels M of H_θ with $\mathcal{P}, S \in M$, letting $\delta = M \cap \omega_1$, for every $p \in \mathcal{P} \cap M$, there is a $q \leq p$ such that for all $s \in S$ of height δ, $\langle q, s\rangle$ is $(\mathcal{P} \times S, M)$-generic.*

Proof First of all, for any $\langle q, s\rangle \in \mathcal{P} \times S$, if $\langle q, s\rangle$ is $(\mathcal{P} \times S, M)$-generic, then q is $(\mathcal{P}, M)$-generic, so $\mathcal{P}$ is proper. Suppose $\mathcal{P}$ forces $\dot{A}$ to be a maximal antichain of S. Let $A' = \{\langle r, s\rangle \in \mathcal{P} \times S : r \Vdash s \in \dot{A}\}$. Let $p \in \mathcal{P}$. Take θ regular and sufficiently large, and let $M \in \mathcal{C}$ be a countable elementary submodel of H_θ containing $p, A', \mathcal{P}$, and S. A' is predense in $\mathcal{P} \times S$, and by assumption, there is a $q \leq p$ such that for all s of height δ, $\langle q, s\rangle$ is $(\mathcal{P} \times S, M)$-generic. Thus $A' \cap M$ is predense below $\langle q, s\rangle$ for all s of height δ. Therefore $q \Vdash$ "for all s of height δ, there is a $t \in \dot{A}$ such that s extends t." But then $q \Vdash$ "$\dot{A} \subseteq S \restriction \delta$." ■

Remark 4 The overall strategy for using Miyamoto's lemma is the same as in the proof that PFA implies that there are no S-spaces, and many other proofs as well: "copy" the "growth" of a condition into an elementary submodel by a finite induction, using elementarity at each step. The fact that we want a σ-left-separated collection rather than just an uncountable left-separated subcollection will add an extra layer of complexity.

Todorcevic's proof that PFA$(S)[S]$ implies that there are no compact S-spaces depends on showing that such spaces are sequential. This allows him to reduce an uncountable amount of information down to a countable amount, which Souslin tree forcing can handle. Our proof is along the same lines: we in effect use the fact that any countably infinite subset of an uncountable closed discrete subspace in a locally compact normal space has a discrete expansion by compact G_δ's which converges to the point at infinity in the one-point compactification of the space.

Most of our proof is independent of the particular problem we are working on, but instead involves general properties of Souslin trees, in particular, coherent ones. To emphasize this and to render the technology more accessible to subsequent researchers, we have organized much of the proof as a sequence of lemmas and notation having nothing to do with topology. I am indebted to Arthur Fischer for the first lemma.

Lemma 3.4 *Let S be a Souslin tree and N a countable elementary submodel of some H_θ containing S. Suppose $A \subseteq S$, $A \in N$, $t \in A - N$. Suppose there is an $s \in S \cap N$, s below t. Then there is a $u \in [s, t) \cap N$ such that A is dense above u.*

Proof If s itself is not the desired u, let

$$\begin{aligned} E = \{x \in S : {} & s \text{ is below } x, \text{ the cone above } x \text{ does not contain a member of } A, \\ & \text{and } x \text{ is minimal among elements of } [s, x] \text{ with that property }\}. \end{aligned}$$

Then $E \in N$ and E is an antichain of S, so E is countable. Therefore $E \subseteq N$. Let $\eta = \sup\{\mathrm{ht}(x) : x \in E\}$. Then $\eta \in N$. Let u be the predecessor of t on the $(\eta + 1)$th level of S. Then $u \in N$ and $u \in [s, t)$. If A were not dense above u, there would be a y above u such that the cone above y would not include a member of A. The height of the least such y would be greater than η, a contradiction. ■

Remark 5 It is considerably easier to prove that PFA$(S)[S]$ implies that locally compact normal spaces are weakly $\aleph_1$-collectionwise Hausdorff, *i.e.*, each closed discrete subspace of size $\aleph_1$ includes a separated subspace of size $\aleph_1$. The key to removing

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"weakly" is to generalize the machinery developed by Todorcevic for proving that PFA$(S)[S]$ *implies that compact hereditarily separable spaces are hereditarily Lindelöf*, which works with subsets of S, to instead work with families of finite chains of S. There are several plausible attempts at doing this. Todorcevic did so in order to prove item (8) in the list of consequences above. I have not seen this part of his proof. The approach taken here seems appropriate for our situation. It is convenient to make the following definitions.

Definition 3.5 An *m-chain with possible repetitions* is an m-tuple $\langle a_1, \dots, a_m \rangle$, each $a_i \in S$, such that a_{i+1} extends a_i. We admit the possibility that $a_{i+1} = a_i$.

Definition 3.6 Let $\mathcal{A}$ be a family of chains with possible repetitions of a Souslin tree S. $\mathcal{A}$ is **dense above** $s \in S$ if for each s' extending s, there is an $A \in \mathcal{A}$ such that $\min A$ extends s'. We shall use "s' above s" and "s' extends s" synonymously, and admit the possibility that $s' = s$.

Corollary 3.7 *Let S be a Souslin tree and N a countable elementary submodel of some H_θ containing S. Suppose $\mathcal{A}$ is a family of chains with possible repetitions of S, $\mathcal{A} \in N$, and suppose there is an $A_0 \in \mathcal{A}$, $\min A_0 \notin N$. Suppose $s \in S \cap N$, s below $t = \min A_0$. Then there is a $u \in S \cap N$, $u \in [s, t)$, such that $\mathcal{A}$ is dense above u.*

Proof Let $\mathcal{A}^* = \{\min A : A \in \mathcal{A}\}$. Apply Lemma 3.4. ∎

Before proceeding further, let us define "coherent". We quote from [15]; see also the references listed there, as well as in [11].

Definition 3.8 A *coherent* tree is a downward closed subtree S of ${}^{<\omega_1}\omega$ with the property that $\{\vec{\xi} \in \operatorname{dom} s \cap \operatorname{dom} t : s(\vec{\xi}) \neq t(\vec{\xi})\}$ is finite for all $s, t \in S$. A *coherent Souslin tree* is a Souslin tree given by a coherent family of functions in ${}^{<\omega_1}\omega$ closed under finite modifications.

As noted in [15], for S a coherent (König calls these *uniformly coherent*) Souslin tree, and s, t on the same (η-th) level of S, there is a canonical isomorphism σ^S_{st} between the cones above (we think of our trees as growing upwards) s and t, defined by letting $\sigma^S_{st}(s')(\alpha)$ be $t(\alpha)$ if $\alpha < \eta$ and $s'(\alpha)$ otherwise, for each s' extending s. These isomorphisms are such that $\sigma^S_{su} = \sigma^S_{tu} \circ \sigma^S_{st}$ and $\sigma^S_{st} = (\sigma^S_{ts})^{-1}$. See [12] for a construction of a coherent Souslin tree from $\diamondsuit$.

Remark 6 Intuitively, what coherence does for us is it deals with the following problem: in trying to go from a PFA proof to a PFA$(S)[S]$ proof, we have much less control over what the $\mathcal{P}$-generic S-name becomes when we force with S than we would have over simply an object — rather than a name — we construct with PFA. A coherent Souslin tree, however, has — up to automorphism — only one generic branch. Therefore the possible interpretations of a name will be "isomorphic," *i.e.,* although there are many possible objects to deal with, they are all essentially the same. We do not yet, however, have a clear understanding of under which circumstances this intuition leads to a PFA$(S)[S]$ proof from a PFA proof. Moreover, the collectionwise Hausdorff conclusion we are proving here by PFA$(S)[S]$ methods does not follow from PFA, since MA_{ω_1} implies that there is a locally compact normal space which is not $\aleph_1$-collectionwise Hausdorff [22]. However, a simplified version of our proof does establish the following:

> PFA *implies that in a locally compact normal space, every closed discrete subspace of size* $\aleph_1$ *has a discrete expansion by compact* G_δ*'s.*

It may interest the reader to see how such σ_{st}'s (we suppress the S) interact with the forcing. Let

$$S^\eta = \{s \in S : \mathrm{ht}(s) \geq \eta\}.$$

Then σ_{st} extends to an automorphism of S^η, by defining $\sigma_{st}(u) = u$, for any $u \in S^\eta$ incomparable with s or t. Let us now suppress mention of st unless necessary. σ can be extended to S^η-names by recursively defining

$$\sigma(\dot{x}) = \{\langle \sigma(\dot{y}), \sigma(u)\rangle : \dot{y} \text{ is an } S^\eta\text{-name and } u \in S^\eta \text{ and } \langle \dot{y}, u\rangle \in \dot{x}\}.$$

Since S^η is dense in the forcing poset S, it follows that if the only parameters in ϕ are S^η-names $\dot{x}_1, \dots, \dot{x}_n$, and if $v \in S^\eta$, then

$$v \Vdash_S \phi(\dot{x}_1, \dots, \dot{x}_n) \text{ if and only if } \sigma(v) \Vdash_{S^\eta} \phi\big(\sigma(\dot{x}_1), \dots, \sigma(\dot{x}_n)\big).$$

As usual, we may adjoin a greatest element $\mathbb{1}$ to our partial orders. Consider this done. We will abuse notation by continuing to use S and S^η to refer to these augmented partial orders. Note then that the canonical S-names for elements of V are also S^η-names for the same elements of V. If $\check{\alpha} = \{\langle \check{\beta}, \mathbb{1}\rangle : \beta \in \alpha\}$ is a canonical name for an ordinal, since $\sigma(0) = 0$ and $\sigma(\mathbb{1}) = \mathbb{1}$, by induction we see that $\sigma(\check{\alpha}) = \check{\alpha}$. In fact, $\sigma(\check{x}) = \check{x}$, for any $x \in V$. Also note that if $\dot{B}^\eta$ is the canonical S^η-name for the generic branch, then $\mathbb{1} \Vdash_S \dot{B}^\eta = \dot{B}|S^\eta$.

Let θ be a sufficiently large regular cardinal bigger than κ, and let M be a countable elementary submodel of H_θ containing everything relevant. (There will be a club of such M's.) In particular, let M contain the function $\mathcal{E} \subseteq \mathcal{P} \times \omega \times S \times \omega_1$ defined by $\mathcal{E}(p, n, s) = \alpha$ if and only if $\mathcal{E}_p(n, s)$ is defined and $= \alpha$. Then if p and s are in M, so is $\mathcal{E}_p(n, s)$, if that is defined. Let $\delta = M \cap \omega_1$. Let $p \in \mathcal{P} \cap M$. Let $p^M = \langle f_p, \mathcal{E}_p, \mathfrak{N}_p \cup \{M \cap H_\kappa\}\rangle$. Then, by a standard argument, $p^M \in \mathcal{P}$.

Remark 7 We now embark on a series of reductions (that we may assume without loss of generality) which will ease the proof that $\mathcal{P}$ is proper and preserves S. These reductions were set out in notes by Todorcevic in 2001; they depend solely on the coherence of S and the fact that the first coordinates of our conditions are finite sets. Much of what is written below is taken from or based on those notes. Any errors are of course my responsibility. I have added proofs of the many details that were not obvious to me.

Let t_M be an arbitrary node at the δ-th level of S. We will show that $\langle p^M, t_M\rangle$ is generic. Let $\mathcal{D} \in M$ be a given dense open subset of $\mathcal{P} \times S$ and let $\langle q, t\rangle$ be a given extension of $\langle p^M, t_M\rangle$. We need to show that $\langle q, t\rangle$ is compatible with some member of $\mathcal{D} \cap M$. Extending $\langle q, t\rangle$, we may assume that $\langle q, t\rangle \in \mathcal{D}$. Moreover, by extending further (since $\mathcal{D}$ is open), we may assume that t is not in the largest model of $\mathfrak{N}_q$, and that this model contains all the members of dom f_q.

Let

$$q_M = q \mid M = \left\langle f_q \cap M, \mathcal{E}_q \cap M, \mathfrak{N}_q \cap M \right\rangle.$$

Note that $q_M \in \mathcal{P} \cap M$ and that $q \le q_M$. That q_M is in M is clear. To see that it is in $\mathcal{P}$, the only point at issue is clause 2) — could the N's separating s and $s' \in \mathrm{dom}_n\, f_q \cap M$ all have been in $\mathfrak{N}_q - M$? Consider an $N \in \mathfrak{N}_q$ such that $\mathrm{ht}(s) \in N$ and $\mathrm{ht}(s') \notin N$. Since $\mathrm{ht}(s') \in M \cap H_\kappa \in \mathfrak{N}_q$, N must be a member of $M \cap H_\kappa$, so $N \in \mathfrak{N}_{q_M}$.

Remark 8 All we shall use about $\mathcal{P}$ until the paragraph with a (†) several pages hence is that the $\mathrm{f_p}$'s are finite, $\mathrm{q} \mid \mathrm{M} \in \mathrm{M}$, and $\mathrm{q} \le \mathrm{q} \mid \mathrm{M}$.

We then may assume that the maximal model of $\mathfrak{N}_{q_M}$ contains all the members of dom f_{q_M} ($=$ dom $f_q \cap M$), else we could have extended $\mathfrak{N}_q$ to ensure this. Let N^* be that maximal model. Since $N^* \in M \cap H_\kappa$, it is not the maximal model of $\mathfrak{N}_q$, so $N^* \in N'$, where N' is the minimal model of $\mathfrak{N}_q$, which is not in M. Then $N' = M \cap H_\kappa$. Then we can adjoin to $\mathfrak{N}_q$ a countable elementary submodel of H_κ in M containing N^* and dom $f_q \cap M$.

Let δ_M be the intersection of ω_1 with the maximal model of $\mathfrak{N}_{q_M}$. By taking the maximal model large enough, we may ensure that the projection of $(\mathrm{dom}\, f_q \cup \{t\}) - M$ on the δ-th level of S has the same size as its projection on the δ_M-th level. To see this, note that there is a $\delta^* < \delta$ such that the projection of $(\mathrm{dom}\, f_q \cup \{t\}) - M$ on the δ^*th level has the same size as its projection on the δth level, since δ is a limit ordinal and S is a normal tree. Then add to $\mathfrak{N}_q$ a countable elementary submodel N of H_κ, $N \in M$, with δ^* and the maximal model of $\mathfrak{N}_{q_M}$ as members.

Let $\{u_1, \dots, u_n\}$, $\{v_1, \dots, v_n\}$ respectively enumerate these projections on the δ_Mth and δth levels, such that $u_i = v_i \mid \delta_M$, $i \le n$, and such that $u_1 = t \mid \delta_M$ and $v_1 = t \mid \delta$. For $1 \le i, j \le n$, let σ_{ij} be the canonical isomorphism that moves u_i to u_j. Note that $\sigma_{ij}^{-1} = \sigma_{ji}$, and σ_{ii} is the identity isomorphism.

Let N_{q_M} be the maximal model of $\mathcal{N}_{q_M}$. For any $\langle r, t_r\rangle \in \mathcal{P} \times S$ that is $\le \langle q_M, u_1\rangle$, define:

$$F_r = \left\{ x \in (\mathrm{dom}\, f_r \cup \{t_r\}) - N_{q_M} : x \mid \delta_M = \text{ some } u_{i_x} \text{ and some } \sigma_{1 i_x}(t) \text{ extends } x \right\}.$$

Then, considering t as t_q, claim

$$F_q = \left\{ x \in \left(\mathrm{dom}\, f_q \cup \{t\}\right) - M : x \mid \delta = \text{ some } v_{i_x} \text{ and } \sigma_{1 i_x}(t) \text{ extends } x \right\}.$$

Clearly F_q includes the right-hand side. On the other hand, if $x \in \mathrm{dom}\, f_q - N_{q_M}$, then $x \notin M$, so $\mathrm{ht}(x) \ge \delta$. No two v_i's project onto the same u_j, so if $x \mid \delta_M = u_{i_x}$, then $x \mid \delta = v_{i_x}$. ■

We claim that if v_i and v_j are projections of elements of F_q, then $\sigma_{ij}(v_i) = v_j$. To see this, first note that if $x \in F_q$ extends v_i, then $\sigma_{i1}(v_i) \le \sigma_{i1}(x) \le t$. Hence $\sigma_{i1}(v_i) = v_1$, since both are of height δ below t. It follows that $\sigma_{ij}(v_i) = \sigma_{1j} \circ \sigma_{i1}(v_i) = \sigma_{1j}(v_1) = v_j$. We then have that for such v_i and v_j, $v_i \mid [\delta_M, \delta) = v_j \mid [\delta_M, \delta)$.

For $x \in F_q$, let $\widehat{x} = \sigma^{-1}_{1 i_x}(x)$. For an m-tuple $\vec{x} = \langle x_1, \dots, x_m \rangle$, if $\widehat{x}_j$ is defined for $1 \leq j \leq m$, let $\widehat{\vec{x}} = \langle \widehat{x}_1, \dots, \widehat{x}_m \rangle$. In particular, let $\widehat{F}_q$ be the chain with possible repetitions of length $|F_q| : \langle \widehat{x} : x \in F_q \rangle$. Similarly define F^l_q and $\widehat{F}^l_q$ for $l \in L = \{l : \mathrm{dom}_l\, f_q \neq 0\}$. We can make analogous definitions of $\widehat{F}_r$ etc. for an arbitrary $\langle r, t_r \rangle$ extending $\langle q_M, u_1 \rangle$. Let $c = \text{length } \widehat{F}_q = |F_q|$ and $c_l = \text{length } \widehat{F}^l_q = |F^l_q|$.

Remark 9 We use sequence notation and chains with possible repetitions to avoid losing information when we pass from F_q to $\widehat{F}_q$. The next three paragraphs are a working out of Todorcevic's ideas due to Arthur Fischer. The intent is to define *in M* the set of all conditions in $\mathcal{D}$ that "look just like $\langle q, t \rangle$".

Let $\mathcal{D}_0 = \{\langle r, t_r \rangle \in \mathcal{D} : \langle r, t_r \rangle \leq \langle q_M, u_1 \rangle$ and

(i) q_M is an *initial part* of r, *i.e.,* for each l, $\mathrm{dom}_l\, f_{q_M}$ is an initial segment of $\mathrm{dom}_l\, f_r$, $\mathcal{E}_r \mid \mathrm{dom}\, \mathcal{E}_{q_M} = \mathcal{E}_{q_M}$, and $\mathfrak{N}_{q_M}$ is an initial segment of $\mathfrak{N}_r$,
(ii) the height of each node in $F_r - F_q$ is $> \delta_M$,
(iii) $L_r = L$, each $|F^l_r| = c_l$, $|F_r| = c$,
(iv) $f_r(\text{the } j\text{-th element of } F_r) = f_q(\text{the } j\text{-th element of } F_q)$,
(v) the height of t_r is greater than the height of any of the nodes in $\mathrm{dom}\, f_r$.

The above requirements will ensure that the natural correspondence between r and q induces a natural correspondence of F_r and $\widehat{F}_r$ to F_q and $\widehat{F}_q$ respectively.

Notice that the u_i's and hence the σ_{ij}'s are in M, and so $\mathcal{D}_0 \in M$ by definability. Clauses (iii) and (iv) do not violate definability, since c and the c_l's are just natural numbers and so are in M. Similarly, the range of f_q is just a finite subset of ω, so we could rewrite (iv) using specific natural numbers. Then

$$\mathcal{F} = \big\{ F \in S^c : F = \widehat{F}_r \text{ for some } \langle r, t_r \rangle \in \mathcal{D}_0 \big\},$$

and

$$\mathcal{F}_l = \big\{ F \in S^{c_l} : F = \widehat{F}^l_r \text{ for some } \langle r, t_r \rangle \in \mathcal{D}_0 \big\}$$

are also in M and in H_κ as well.

Since $M \cap H_\kappa \in N$ for each $N \in \mathfrak{N}_q - M$, it follows that $\mathcal{F}$ and $\mathcal{F}_l \in N$, for all such N. Note that $\widehat{F}_q \in \mathcal{F}$ and $\widehat{F}^l_q \in \mathcal{F}_l$, since $\langle q, t \rangle \in \mathcal{D}_0$. Note also that the terms of $\widehat{F}^l_q$ are separated by models of $\mathfrak{N}_q$. To see this, recall t is not in the largest model of $\mathfrak{N}_q$, which does contain all the members of $\mathrm{dom}\, f_q$. If $\widehat{x}, \widehat{x}'$ are terms of $\widehat{F}^l_q$, then there is an $N \in \mathfrak{N}_q$ such that $x \in N$ and $x' \notin N$. Then $\widehat{x} \in N$, and $\widehat{x}' \notin N$, else $x' \in N$. $N \notin M$, so the σ_{ij}'s $\in N$.

Our plan is to reflect $\langle q, t \rangle$ to an $\langle r, t_r \rangle \in \mathcal{D}_0 \cap M$ by using elementarity to systematically reflect the members of F_q down into M. Our topological hypotheses will be used to obtain such a reflection that is also compatible with $\langle q, t \rangle$. Let $\mathcal{N}'_q$ be a minimal subchain of $\mathfrak{N}_q$ containing $M \cap H_\kappa$ at its bottom and separating $\widehat{F}^l_q$ for each l. Let $\mathcal{N}'_q = \{N_a\}_{a \leq m-1}$ ordered by inclusion, with $N_0 = M \cap H_\kappa$. $\widehat{F}_q$ is a chain with possible repetitions; let us write it as

$$\langle \widehat{\vec{x}}_1, \dots, \widehat{\vec{x}}_{m-1}, t \rangle,$$

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where $\hat{\hat{x}}_a = \langle \hat{x}_{a,1}, \dots, \hat{x}_{a,d_a} \rangle$ enumerates in increasing order $\widehat{F}_q \cap (N_a - N_{a-1})$, $a \geq 1$. Thus the length of the vector $\vec{x}_a$ is equal to the size of $F_q \cap (N_a - N_{a-1})$. Since $\mathcal{F} \in N_{m-1}$,

$$\mathcal{F}(\vec{x}_1, \dots, \vec{x}_{m-1}) = \left\{ x \in S : \langle \hat{\hat{x}}_1, \dots, \hat{\hat{x}}_{m-1}, x \rangle \in \mathcal{F} \right\} \in N_{m-1}.$$

By Lemma 3.4, there is a $y_m \in N_{m-1} \cap S$, $y_m \in [\max \hat{\hat{x}}_{m-1}, t)$, such that $\mathcal{F}(\vec{x}_1, \dots, \vec{x}_{m-1})$ is dense above y_m. Next, consider

$$\mathcal{F}(\vec{x}_1, \dots, \vec{x}_{m-2}) = \left\{ \langle \vec{x}, y \rangle \in S^{d_{m-1}+1} : \langle \vec{x}, y \rangle \text{ is a chain with possible} \right. \\ \left. \text{repetitions and } \mathcal{F}(\vec{x}_1, \dots, \vec{x}_{m-2}, \vec{x}) \text{ is dense above } y \right\}.$$

Then $\mathcal{F}(\vec{x}_1, \dots, \vec{x}_{m-2}) \in N_{m-2}$ and $\langle \hat{\hat{x}}_{m-1}, y_m \rangle \in \mathcal{F}(\vec{x}_1, \dots, \vec{x}_{m-2})$. As before, this time by Corollary 3.7 with $\mathcal{F}(\vec{x}_1, \dots, \vec{x}_{m-2})$, $\langle \hat{\hat{x}}_{m-1}, y_m \rangle$, N_{m-2} playing the roles of $\mathcal{A}, A_0, N$ respectively, we can find $y_{m-1} \in N_{m-2} \cap S$, $y_{m-1} \in [\max \hat{\hat{x}}_{m-2}, \min \hat{\hat{x}}_{m-1})$, such that $\mathcal{F}(\vec{x}_1, \dots, \vec{x}_{m-2})$ is dense above y_{m-1}. Continuing, in m steps we find a $y_1 \in N_0$, $y_1 \in [u_1, v_1)$, such that

$$\mathcal{F}(\varnothing) = \left\{ \langle \vec{x}, y \rangle \in S^{d_1+1} : \langle \vec{x}, y \rangle \text{ is a chain with possible} \right. \\ \left. \text{repetitions and } \mathcal{F}(\vec{x}) \text{ is dense above } y \right\}$$

is $\in N_0$ and dense above y_1.

Let $\dot{\mathcal{X}}_1$ be a name for

$$\left\{ \langle \alpha_1, \dots, \alpha_{d_1} \rangle \in (C^\circ)^{d_1} : \text{for some } \langle \vec{z}, w \rangle \in \mathcal{F}(\varnothing), \right. \\ \left. \{\vec{z}, w\} \subseteq B \text{ and for each } i, 1 \leq i \leq d_1, \mathrm{ht}(z_i)^- = \alpha_i \right\}.$$

Then $\dot{\mathcal{X}}_1 \in M$. Let $\dot{\mathcal{X}}_1'$ be a name for $\{\vec{\xi} \in \mathcal{X}_1 : \min \vec{\xi} > \delta_M\}$.
Claim: $y_1 \Vdash \dot{\mathcal{X}}_1' \neq \varnothing$.

Proof Given any y_1' extending y_1, since $\mathcal{F}(\varnothing)$ is dense above y_1, we can find a $\langle \vec{z}, w \rangle \in \mathcal{F}(\varnothing)$ with minimal element of height greater than δ_M extending y_1'. Take y_1'' above $\langle \vec{z}, w \rangle$. Then $y_1'' \Vdash \langle \mathrm{ht}(z_1)^-, \dots, \mathrm{ht}(z_{d_1})^- \rangle \in \dot{\mathcal{X}}_1'$. ■

There is a level of height greater than δ_M at which all extensions of y_1 at that level decide a $\dot{\vec{\xi}}$ that y_1 forces to be a member of $\mathcal{X}_1'$ to be some $\vec{\xi}$ and also decide a corresponding $\langle \vec{z}, w \rangle(\vec{\xi})$. Let μ_1 be the sup of the components of these countably many $\vec{\xi}$'s and repeat the process, extending each of the aforementioned extensions of y_1 to a level of height greater than μ_1, deciding $\vec{\xi}$ as before, but with the minimal component of such $\vec{\xi}$'s greater than μ_1. Continuing, we form a subtree of height ω of the cone above y_1 such that each element of each level of the subtree decides $\dot{\vec{\xi}} \in \dot{\mathcal{X}}_1$ and a corresponding $\langle \vec{z}, w \rangle(\vec{\xi})$, and such that the $\vec{\xi}$'s of one level all have minimal components greater than the maximal components of the $\vec{\xi}$'s of the previous level.

By elementarity, there is such a subtree in M. Therefore the sup ζ of the heights in S of the elements of the subtree is less than δ. We can thus take $\{y_{1,j} : j < \omega\}$

strictly ascending below v_1, all of height less than ζ, and associated strictly increasing $\vec{\xi}^1_j$'s and their corresponding $\langle \vec{z}^1_j, w^1_j\rangle$'s, with

$$\langle \vec{z}^1_j, w^1_j\rangle \in \mathcal{F}(\varnothing) \cap M \text{ and } y_{1,j} \Vdash \langle \vec{z}^1_j, w^1_j\rangle \subseteq \dot{B},$$

and $\pi_d(\vec{\xi}^1_j) = \operatorname{ht}(\pi_d(\vec{z}^1_j))^-$, where $\pi_d(\vec{\xi}^1_j)$ (respectively, $\pi_d(\vec{z}^1_j)$) is its d-th component.

Now, finally, we apply the general machinery to our specific situation. Since for $x \in \operatorname{dom} f_q - M$, $\operatorname{ht}(x) \geq \delta$, such x decides whether or not $\dot{K}_{\zeta,\pi_d(\vec{\xi}^1_j)^-}$ meets $\dot{\overline{U}}_{\operatorname{ht}(x)^-}$. Since $\overline{U}_{\operatorname{ht}(x)^-}$ is compact and for fixed d the $\pi_d(\vec{\xi}^1_j)^-$'s are distinct, there is a $j_x \in \omega$ such that for each $d \leq d_1$, x forces

$$\bigcup \left\{ \dot{K}_{\zeta,\pi_d(\vec{\xi}^1_j)^-} : j \geq j_x \right\} \cap \dot{\overline{U}}_{\operatorname{ht}(x)^-} = \varnothing. \tag{†}$$

To see this, note that x certainly forces that there is such a j_x. Then for some j_x, some extension of x forces (†). But then x must have already forced this, since it had decided whether $\dot{K}_{\zeta,\pi_d(\vec{\xi}^1_j)^-}$ met $\dot{\overline{U}}_{\operatorname{ht}(x)^-}$.

Let $j_1 = \max\{ j_x : x \in \operatorname{dom} f_q - M\}$. Let $\mathbf{z}_{1,d}$ be the element of height $\pi_d(\vec{\xi}^1_{j_1})^+$ below x_d, for $x_d \in F_q \cap (N_1 - N_0)$. Let $\vec{\mathbf{z}}_1 = \langle \mathbf{z}_{1,1}, \dots, \mathbf{z}_{1,d_1}\rangle$. Let $\mathbf{w}_1 = w^1_{j_1}$. Then $\langle \widehat{\vec{\mathbf{z}}}_1, \mathbf{w}_1\rangle = \langle \vec{z}^1_{j_1}, w^1_{j_1}\rangle \in \mathcal{F}(\varnothing)$, and for all $x \in \operatorname{dom} f_q - M$, x forces $\dot{K}_{\zeta,\operatorname{ht}(\mathbf{z}_{1,d})^-} \cap \dot{\overline{U}}_{\operatorname{ht}(x)^-} = \varnothing$, for all $d \leq d_1$. Notice that $\langle \vec{\mathbf{z}}_1, \mathbf{w}_1\rangle \in M$.

We now need to iteratively peel off the remaining "layers" of F_q. Let $\dot{\mathcal{X}}_2$ be a name for

$$\left\{ \langle \alpha_1, \dots, \alpha_{d_2}\rangle \in (C^\circ)^{d_2} : \text{ for some } \langle \vec{z}, w\rangle, \langle \widehat{\vec{z}}, w\rangle \in \mathcal{F}(\vec{\mathbf{z}}_1), \\ \{\vec{z}, w\} \subseteq B \text{ and for each } i, 1 \leq i \leq d_2, \operatorname{ht}(z_i)^- = \alpha_i \right\}.$$

We now carry out the same argument as before, with an infinite strictly ascending sequence of $y_{2,j}$'s below v_1 extending y_{1,j_1} and deciding $\vec{\xi} \in \mathcal{X}_2$, where $\min \vec{\xi} > \max \vec{\xi}^1_{j_1}$. As before, we obtain a $\vec{\mathbf{z}}_2 \in M$, each $\mathbf{z}_{2,d}$ below $x_d \in F_q \cap (N_2 - N_1)$, and with each $\operatorname{ht}(\mathbf{z}_{2,d}) > \operatorname{ht}(\mathbf{z}_{1,d_1})$, such that for each $x \in \operatorname{dom} f_q - M$, x forces $\dot{K}_{\zeta,\operatorname{ht}(\mathbf{z}_{2,d})^-} \cap \overline{U}_{\operatorname{ht}(x)^-} = \varnothing$, for all $d \leq d_2$.

Continuing, after m steps we will find $\langle \widehat{\vec{\mathbf{z}}}_1, \dots, \widehat{\vec{\mathbf{z}}}_m, \mathbf{w}_1\rangle \in \mathcal{F}$, each component of each $\vec{\mathbf{z}}_a$ below some v_i, and hence in M. Since $\langle \widehat{\vec{\mathbf{z}}}_1, \dots, \widehat{\vec{\mathbf{z}}}_m, \mathbf{w}_1\rangle \in \mathcal{F}$, there is an $\langle r, t_r\rangle \in \mathcal{D}_0 \cap M$ such that $\widehat{F}_r = \langle \widehat{\vec{\mathbf{z}}}_1, \dots, \widehat{\vec{\mathbf{z}}}_m, \mathbf{w}_1\rangle$. Then $\mathbf{w}_1 = t_r$. Now $\mathbf{w}_1 = w^1_{j_1}$ is below y_{1,j_1}, since otherwise y_{1,j_1} could not force it to be in B. Therefore it is below v_1 and so $t_r \leq t$. We claim that $\langle r, t_r\rangle$ is compatible with $\langle q, t\rangle$, which will finish the proof.

Since $r \leq q_M$, it follows that $f_r \cup f_q$ is a function. Let

$$\mathcal{E}_{r,q} = \mathcal{E}_r \cup \mathcal{E}_q \mid (\omega \times (\operatorname{dom} f_q - \operatorname{dom} f_r)) \cup \{\langle\langle n_{i,d}+1, \mathbf{z}_{i,d}\rangle, \zeta\rangle : \mathbf{z}_{i,d} \in \operatorname{dom} f_r\},$$

where $n_{i,d}$ is the maximal integer such that $\langle n_{i,d}, \mathbf{z}_{i,d}\rangle \in \operatorname{dom} \mathcal{E}_r$. Then $\mathcal{E}_{r,q}$ satisfies 3(c) in the definition of $\mathcal{P}$.

We next note that $\mathfrak{N}_r \cup \mathfrak{N}_q$ is an $\in$-chain, for by construction, $\mathfrak{N}_r \in M$, so $\mathfrak{N}_r \cup \{M \cap H_\kappa\}$ is an $\in$-chain. Now $\mathfrak{N}_q = \mathfrak{N}_{q_M} \cup (\mathfrak{N}_q - \mathfrak{N}_{q_M})$; the elements N of $\mathfrak{N}_q - \mathfrak{N}_{q_M}$ all have $M \cap H_\kappa$ in them, for if not, such an N would be in M. $\mathfrak{N}_r \cup \mathfrak{N}_q$ is thus the $\in$-chain $\mathfrak{N}_r \cup \{M \cap H_\kappa\} \cup (\mathfrak{N}_q - \mathfrak{N}_{q_M})$.

Let $\mathcal{R} = \langle\langle f_r \cup f_q, \mathcal{E}_{r,q}, \mathfrak{N}_r \cup \mathfrak{N}_q\rangle, t\rangle$.

Since $\operatorname{dom} f_r \subseteq M$ and $r \leq q_M$, each $\operatorname{dom}_l(f_r \cup f_q)$ consists of nodes of different heights. Suppose $b, c \in \operatorname{dom}_l(f_r \cup f_q)$. The only case of interest is when $b \in \operatorname{dom}_l f_r$ and $c \in \operatorname{dom}_l f_q$. If $c \in M$, then $c \in \operatorname{dom}_l f_{q_M}$ and the members of $\mathfrak{N}_r$ separate b and c since $r \leq q_M$. If $c \notin M$, then an $N \in \mathfrak{N}_r$ containing b will not contain c, since $N \subseteq M$. To finish showing that the first component of $\mathcal{R}$ is a condition, suppose $s' \in \operatorname{dom}_l f_q$, $s \in \operatorname{dom}_l f_r$, and s' extends s. If $s \in \operatorname{dom}_l f_{q_M}$, this is trivial, so suppose $s \in \operatorname{dom}_l f_r - \operatorname{dom}_l f_{q_M}$. Since s' extends s and also extends some v_i, it follows that v_i extends s, which then extends u_i, since $\operatorname{ht}(s) > \delta_M$. Then u_1 is below $\sigma_{i1}(s)$ which is below v_1 which is below t. Then s is below $\sigma_{1i}(t)$. But then $s \in F_r$. By construction then, 3d) of the definition of $\mathcal{P}$ is satisfied, so indeed $\langle f_r \cup f_q, \mathcal{E}_{r,q}, \mathfrak{N}_r \cup \mathfrak{N}_q\rangle \in \mathcal{P}$ and is below both r and q. But then $\mathcal{R} \in \mathcal{P} \times S$ is below both $\langle r, t_r\rangle$ and $\langle q, t\rangle$ as required. ∎

4 Conclusion

In conclusion, there are several questions we have been unable to answer.

Problem 4.1 Assuming $V = L$ and normal spaces which have character $\leq \aleph_1$ are collectionwise Hausdorff; *does this hold in a model of form* PFA$(S)[S]$*?*

Recall a space X is of *pointwise countable type* if for each $x \in X$ and each open set U containing X, there is a compact K containing x, $K \subseteq U$, and open $\{W_n(K)\}_{n<\omega}$, each $W_n(K) \supseteq K$, such that every open set including K includes some $W_n(K)$. Spaces of pointwise countable type include both first countable and locally compact spaces.

Problem 4.2 Assuming $V = L$ and normal spaces of pointwise countable type are collectionwise Hausdorff; *does this hold in a model of form PFA$(S)[S]$?*

Problem 4.3 *Is forcing with a Souslin tree sufficient to get that locally compact normal spaces are $\aleph_1$-collectionwise Hausdorff?*

Problem 4.4 *Does* MA$_{\omega_1}$ *imply that in a locally compact normal space, every closed discrete subspace of size $\aleph_1$ has a discrete expansion by compact G_δ's? Indeed, does* ZFC *imply this?*

If ZFC sufficed, we could get an affirmative answer to Problem 4.3 by the following result:

Theorem 4.5 *Suppose X is a locally compact normal space in which every closed discrete subspace of size $\aleph_1$ has a discrete expansion by compact G_δ's. Assume every normal first countable space is $\aleph_1$-collectionwise Hausdorff. Then X is $\aleph_1$-collectionwise Hausdorff.*

Proof Suppose X has a closed discrete subspace Y of size $\aleph_1$. Expand Y to a discrete collection of compact G_δ's $\{K_\alpha : \alpha < \omega_1\}$. Let X' be the space resulting from identifying each K_α to a point. The identification is a perfect map, so X' is locally compact normal. Let X'' be the result of isolating all points in X' outside $\{K_\alpha/\sim : \alpha \in \omega_1\}$. Then X'' is normal and first countable. To see this, note in X' that $K_\alpha/\sim$ is a G_δ and so has countable character. We can now take a separation of $\{K_\alpha/\sim : \alpha < \omega_1\}$ in X''. By removing some isolated points, we can take the sets in the separation to be open in X'. But then $\{K_\alpha/\sim : \alpha < \omega_1\}$ is separated in X', and thence $\{K_\alpha : \alpha < \omega_1\}$ is separated in X. ■

I rather doubt that ZFC suffices; a counterexample would be of interest in connection with a question of S. Watson [26]: *if every first countable normal space is collectionwise Hausdorff, is every locally compact normal space collectionwise Hausdorff?* There is no example in ZFC of a locally compact normal space and a closed discrete subspace without such an expansion, since such an expansion trivially exists if locally compact normal spaces are $\aleph_1$-collectionwise Hausdorff.

The use of a supercompact cardinal in our proof can almost certainly be avoided, since the part of PFA we used applied to objects of size $\aleph_1$. This will presumably require modifying the posets to get ones satisfying Shelah's $\aleph_2$-p.i.c., and then following a well-trodden path, as in *e.g.*, [5,25]. However, the addition of the Souslin-tree forcing complicates things.

In [20], in a sketch of a plan to prove that PFA(S)[S] implies that locally compact, hereditarily normal spaces not including a perfect pre-image of ω_1 are paracompact (which result we accomplished in a model of PFA(S)[S] in [14]), the author introduced the condition

$\sum^+$: *Suppose X is a countably tight compact space, $\mathcal{L} = \{L_\alpha\}_{\alpha<\omega_1}$ a collection of disjoint compact sets such that each L_α has a neighbourhood that meets only countably many L_β's, and $\mathcal{V}$ is a family of $\leq \aleph_1$ open sets such that*

(a) $\bigcup \mathcal{L} \subseteq \bigcup \mathcal{V}$;
(b) *For every $V \in \mathcal{V}$ there is an open set U_V such that $\overline{V} \subseteq U_V$ and U_V meets only countably many members of $\mathcal{L}$.*

Then $\mathcal{L} = \bigcup_{n<\omega} \mathcal{L}_n$, where each $\mathcal{L}_n$ is a discrete collection in $\bigcup \mathcal{V}$.

and conjectured that PFA(S)[S] implied it. MA_{ω_1} does, but it is not clear whether PFA(S)[S] does. Thus,

Problem 4.6 *Does* PFA(S)[S] *imply* $\sum^+$?

Finally, let us note that a minor variation of the proof we have given here yields a weak version of an important result of Todorcevic.

Theorem 4.7 PFA(S)[S] *implies that if $\{x_\alpha\}_{\alpha<\omega_1}$ is a locally countable subspace of a compact space Z with finite products Fréchet–Urysohn, and T is a stationary subset of ω_1, then there is a a stationary $T' \subseteq T$ such that $\{x_\alpha : \alpha \in T'\}$ is discrete.*

Remark 10 Todorcevic announced (in a seminar in Toronto in 2002) the stronger conclusion that $\{x_\alpha\}_{\alpha<\omega_1}$ is σ-discrete, from the weaker hypothesis that the space is compact and countably tight. The details of the proof of that stronger result will appear in a paper (in preparation) by Fischer, Tall, and Todorcevic.

It is considerably easier to prove Theorem 4.7 for the special case when $\{x_\alpha\}_{\alpha<\omega_1}$ is right-separated.

Definition 4.8 We say that $\{x_\alpha\}_{\alpha<\omega_1}$ is right-separated if there exist open U_α, $\alpha < \omega_1$, such that $x_\alpha \in U_\alpha$ and, if $\alpha < \beta$, then $x_\beta \notin U_\alpha$.

That will suffice, since a simple closing-off argument establishes that if $\{x_\alpha\}_{\alpha<\omega_1}$ is locally countable, there is a closed unbounded $C \subseteq \omega_1$ such that $\{x_\alpha : \alpha \in C\}$ is right-separated.

To prove Theorem 4.7, we use a version of Todorcevic's partial order. We have S-names $\dot{Z}$, $\dot{U}_\alpha$, $\alpha < \omega_1$, such that S forces $\dot{Z}$ is such a space and

(i) $\alpha \in \dot{U}_\alpha$, which is open,

(ii) $\beta < \alpha$ implies $\alpha \notin \bigcup\{\overline{U}_\beta : \beta < \alpha\}$.

$\mathcal{P}$ is similar to our partial order, except there is no need for $\dot{\mathcal{E}}$ and (2) is replaced by

- If $s, s' \in \mathrm{dom}_l\, f_p$ and s' strictly extends s and $\mathrm{ht}(s') = \tau$ and $\mathrm{ht}(s) = \sigma$, then $s' \Vdash \sigma^- \notin \overline{U}_{\tau^-}$.

Showing that the partial order is proper and preserves S is accomplished by an easier version of what we did here: by compactness, $\mathcal{X}_1$, as a subspace of a finite power of Z, has a complete accumulation point x; by right-separation, x does not project to any of the x_α's. By Fréchet–Urysohn, there is a sequence $\{x_{\alpha_n}\}_{n<\omega}$ from $\mathcal{X}_1$ which converges to x. Since the projections of x are not in any of the $\overline{U}_\alpha$'s for s's of height α in the condition we are trying to get away from, for n sufficiently large, x_{α_n} will not be in them either. Thus we find that $\{x_\alpha : \alpha \in C\}$ is σ-discrete, so there is a stationary T' as required. ■

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