1
Preliminaries

If you would understand anything, observe its beginning.
– Aristotle

Oceanographers think of ocean circulation in terms of the "global conveyer belt," in which cold polar waters sink and then circulate around the ocean basins, eventually being warmed in the tropics. But the truth is that this larger-scale circulation has a typical speed of only a few cm s^{-1}, and it is generally accompanied by variable currents many times faster (1 m s^{-1} is not uncommon), fluctuating on periods ranging from months down to seconds.

The largest such variations are the majestic mesoscale eddies that spin off strong currents like the Gulf Stream (Figure 1.1). Today, much research is focused on the

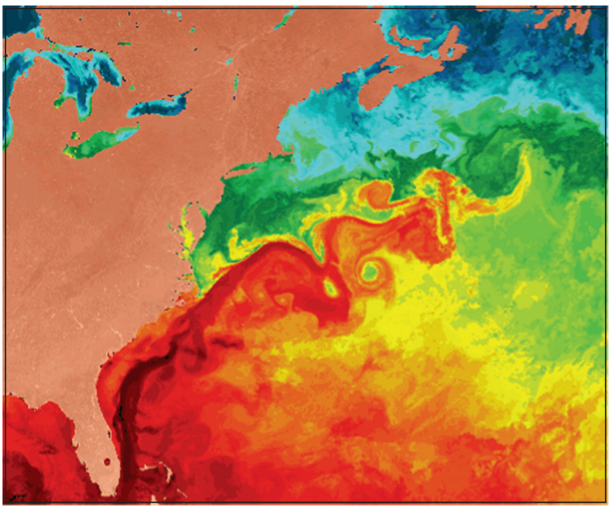

Figure 1.1 Instability of the Gulf Stream shown in a satellite image. Colors represent sea surface temperature. The darkest red represents warm water flowing northeast along the east coast of the United States. After departing from the coast at Cape Hatteras, the current becomes unstable and breaks down into turbulent eddies. (Image courtesy of the U.S. National Oceanic and Atmospheric Administration, hereafter NOAA.)

https://doi.org/10.1017/9781108640084.002 Published online by Cambridge University Press

next size smaller: the submesoscale eddies. Smaller than this are the gravity waves and, at the smallest scale, three-dimensional turbulence.

One must measure for a year or more in order to average out these various fluctuations and discern the mean "conveyer belt" current. But to think of the variations as something we can average away is to fool ourselves, for it is largely the oscillations that govern the conveyer belt. We can't understand one without the other.

One way to understand such a chaotic profusion of motions is to ask what would happen if, at some initial instant, the ocean was calm, with steady, orderly currents. Would the oscillations develop spontaneously? If so, how? This thought experiment is the essence of instability theory.

For example, Figure 1.1 shows eddies forming in the Gulf Stream. No human mathematician could solve the equations that describe these intricate motions; but, using the methods of linear perturbation theory, we can not only predict their length and time scales but also understand quite a lot about what causes them. The trick is to imagine a fictitious Gulf Stream that is straight and eddy-free, then study what happens in the very first few moments – after the current begins to buckle but before it grows so complex as to be mathematically intractable.

We can think of atmospheric motions in the same way. Imagine a fictitious atmosphere where the winds are purely zonal – mid-latitude westerlies, jet streams, and polar easterlies, all blowing straight in the east-west direction only. It turns out that those winds would be unstable, and as a result would break up into the large vortical structures we know as synoptic weather systems (Figure 1.2). To calculate the details requires a supercomputer, but we can understand the basic mechanics and predict the dominant length and time scales (a few thousand kilometers, a few days) quite easily.

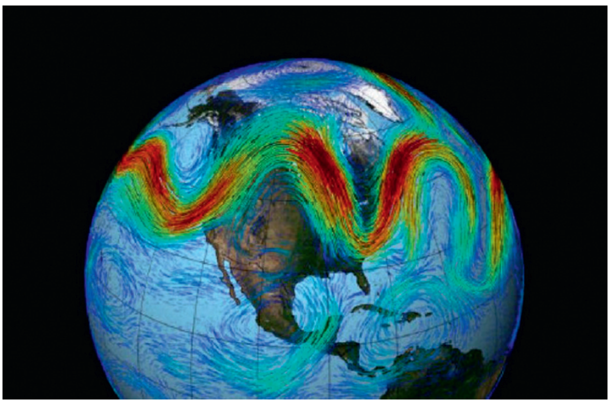

Figure 1.2 The atmospheric jet stream: speed (red = fast) and direction (streaks), showing baroclinic instability. (Visualization courtesy of the US National Aeronautics and Space Administration, hereafter NASA).

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The Earth's mantle provides a third example. Suppose the mantle were perfectly motionless. Heating from the radioactive core would lead to the growth of convection cells, exactly as we see reflected in the slow drift of the continents and the attendant seismic and volcanic activity.

In this book we will study instabilities on scales from centimeter to global, controlled mainly by gravity, sheared winds and currents, and the Earth's rotation. While our main focus is the Earth, analogous phenomena are found in atmospheres and magnetospheres of other planets, stellar interiors, and interstellar plasma flows.

1.1 What Is Instability?

Suppose that the emergency brake in your car doesn't work, and you have to park somewhere in hilly country. Where can you park so that your car doesn't roll away (Figure 1.3)? We hope you would park at point (a), the bottom of a valley. But what about point (b), the top of a hill? You could park there in theory, but you would have to park at *exactly* the right spot, and even then any little disturbance would cause your car to roll away.

In mathematical terms, we say that both points (a) and (b) are equilibrium states,[1] i.e., states at which the system can remain steady in time. The difference between (a) and (b) is in what happens when the system is displaced slightly from equilibrium. If you park at the bottom of the valley (a), and if something then pushes the car slightly to the left or the right, gravity will pull it back toward its original location. The car will rock back and forth and eventually come to rest due to friction. In contrast, if you park at the top of the hill (b) and the car is moved even slightly, gravity pulls it further from the equilibrium point. The further the car travels, the steeper the slope and the stronger the pull of gravity. Goodbye car! We say that equilibrium (a) is stable, while (b) is unstable.

The equations that describe geophysical fluid systems are in general far too complicated to solve analytically. One way to get around this problem is to look for equilibria, i.e., solutions that are valid when all time derivatives are set to zero. Flows are often found to be close to such equilibria. For example, the surface of

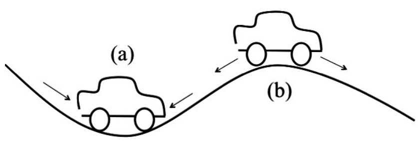

Figure 1.3 No brakes! Where would you park? Arrows show the gravitational force that acts when the car is displaced slightly from equilibrium.

[1] Highlighted text is used as an extra level of emphasis for important concepts.

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a lake is in equilibrium if it is horizontal. Although this is never exactly true, it is pretty close on average.

Once we have identified an equilibrium state, the next step is to determine its stability. If the equilibrium is stable, disturbances will often have the form of oscillations (e.g., the car in Figure 1.3a), or waves. If the equilibrium is unstable, then small disturbances grow exponentially. Instabilities will be our focus here, though we will find it useful to examine wave phenomena as well.

1.1.1 The Cycle of Instability

Because unstable systems are by their nature ephemeral, you might reasonably wonder why we ever observe them. It is much more usual to see systems close to stable equilibria. For example, the surface of a lake is never perfectly horizontal, but it's usually pretty close, because the horizontal equilibrium state is stable.

But a sufficiently strong wind destabilizes that horizontal equilibrium state, and waves grow as a result.[2] If the waves grow large enough, they fall prey to a second kind of instability as the crests roll over and break (related to convective instability; Chapter 2). The surface then relaxes toward the horizontal state until a new set of waves emerges. Eventually the wind dies down and the horizontal state is once again stable.

The oceans and atmosphere are almost always turbulent, and this cycle of instability is the reason. Forcing by wind, sun, gravity, and planetary rotation tends to push the system toward unstable states. Instability and turbulence then act to relax the system back toward stability. This cyclic instability regime is discussed further in section 12.3.

1.2 Goals

Our exploration of instability will have three main goals.

(i) **Mechanisms:** We aim to understand, on an intuitive level, the basic physical processes that generate instability. In the car example, we've seen how motion away from equilibrium alters the effect of gravity (arrows in Figure 1.3), resulting in oscillations or instability. Geophysical examples will take a bit more work to understand, but we'll do it.

(ii) **Rules of thumb:** We would like to be able to predict the stability or instability of a system quickly with minimal math. In the car example, we are able to predict whether an equilibrium point will be stable or unstable without knowing the details of the shape of the hill or valley. All we need to know

[2] The process is similar to shear instability, covered in Chapters 3–5.

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is whether the equilibrium is a maximum or a minimum of elevation, i.e., whether the curvature at that point is negative or positive.

We can invent similar rules for most types of geophysical flow instability. These allow us to estimate not only the likelihood of instability, but also the spatial and temporal scales on which it will grow. These can help us identify the particular mechanism through which a geophysical flow becomes unstable. For example, the Gulf Stream eddies shown in Figure 1.1 could be due to different instabilities (which you will learn about later). By comparing their observed length scales, and the time they take to grow, with rules of thumb based on various known instability types, we can take a first guess as to the mechanism.

(iii) **Numerical solution methods:** Sometimes a rule of thumb is not enough. We want to determine quantitative details of an instability, perhaps in a situation where many physical factors interact. In that case we may have to solve a nontrivial set of differential equations. Many advanced analytical methods are available, but in this book we will focus on numerical methods. Since the 1980s, computers have had the capacity to do something unprecedented: *to solve a differential equation whose coefficients are defined using observational measurements*. That capability is now in use in the analysis of oceanographic and atmospheric observation.

1.3 Tools

Below are three topics we'll expect you to have some familiarity with. Under each topic is listed one or more things that you should be able to do.

(i) **Calculus**:

- Solve this boundary value problem:

$$y'' = -y; \quad y(0) = y(\pi) = 0. \tag{1.1}$$

- Derive this Taylor series approximation:

$$\frac{1}{1+x} \approx 1 - x + x^2, \quad \text{for } |x| \ll 1.$$

- Understand the meaning of (though not necessarily solve) a partial differential equation, e.g.,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial \pi}{\partial x}.$$

- Define the Dirac delta function.

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(ii) **Linear algebra**:

- Compute the eigenvalues of a 2×2 matrix.

(iii) **Programming**: Homework will be done using the Matlab programming environment[3] or something equivalent.[4] You do not have to be an expert; you will learn as you go. But if you've never used the software at all it would be worth familiarizing yourself with the basic syntax. Try the following:

- Write a function, and a script that calls it.
- Define a matrix and compute its eigenvalues.
- Make a line plot and label the axes.
- Make an image plot.

1.4 Numerical Solution of a Boundary Value Problem

The basic geophysical flow instabilities are analyzed as solutions of two-point boundary value problems. In this section we'll define this class of problems and introduce a simple method for solving them.

1.4.1 Defining the Problem

Let $f(x)$ be the solution to a second-order ordinary differential equation with independent variable x. Complete specification of f requires two pieces of information in addition to the equation itself. These can be either

- values of f and its first-derivative at some initial point, which we'll label as zero, i.e., $f(0)$ and $f'(0)$, or
- values of f at two points, say $f(0)$ and $f(L)$.

The first case is called an initial value problem; the second is called a boundary value problem.

A critical difference between these two classes of problem is that the first generally has a solution while the second generally does not. Here's a simple example:

[3] Many universities make the Matlab software available free to students.

[4] Python is another programming environment that we recommend. It is freely available at www.python.org. The commands you need here are found in two packages that will be used over and over. Most of the numerical mathematics and matrix operations come from the `numpy` package, whereas the plotting commands are from the `matplotlib` package. You should start your Python scripts with the following lines:

```
import numpy as np
import matplotlib as plt
```

to load these packages and give them the short cuts `np` and `plt`, respectively. Plotting a line can then be done with the command `plt.plot`, and finding eigenvalues can be accomplished with `np.eig`.

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$$f'' = -k^2 f. \tag{1.2}$$

The general solution is

$$f = A\cos kx + B\sin kx, \tag{1.3}$$

where A and B are constants to be determined. Consider first the initial value problem. Suppose we have initial conditions $f(0) = 0$ and $f'(0) = 1$. The solution is then (1.3) with $A = 0$ and $B = 1/k$ (try it). Note that this solution works for *any* value of k.

Now, consider the boundary value problem with conditions

$$f(0) = 0; \quad f(L) = 0. \tag{1.4}$$

The first condition is satisfied if $A = 0$, but the second can then be satisfied only if $k = \pm n\pi/L$, where n is any integer. These special values of k are called the eigenvalues of the problem, and unless k is equal to one of those eigenvalues, the problem has no solution. We also call (1.2–1.4) a differential eigenvalue problem. It is analogous to the more familiar matrix eigenvalue problem, and can in fact be solved numerically using the same methods.

Here's how it works. Suppose that

- $\vec{x}$ is a list of possible values of x arranged as a vector;
- $\vec{f}$ and $\vec{f}^{(2)}$ are vectors composed of the corresponding values of f and its second-derivative, respectively;
- D is a matrix such that $\mathsf{D}\vec{f} = \vec{f}^{(2)}$.

We can now write (1.2) as

$$\mathsf{D}\vec{f} = -k^2\vec{f}, \tag{1.5}$$

which is a standard matrix eigenvalue problem with eigenvalue $-k^2$. Because the matrix eigenvalue problem can be easily solved using standard numerical routines (e.g., the Matlab function `eig`[5]), this approach suggests a convenient way to solve the differential eigenvalue problem. But first we have to identify this matrix D that transforms a vector into its second-derivative.

1.4.2 Discretization and the Derivative Matrix

We discretize the independent variable x by choosing a vector of values:

$$x_i = x_0 + i\Delta, \qquad \text{where } \ i = 0, 1, 2, \ldots, N+1.$$

[5] `Blue text` indicates Matlab syntax. We give coding examples in Matlab, assuming that readers preferring other software environments will substitute the equivalent expressions.

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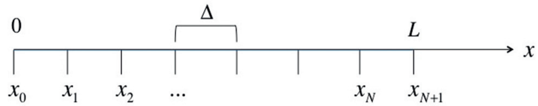

Figure 1.4 Discretizing the x axis.

The first and last values correspond to the boundaries, say $x_0 = 0$ and $x_{N+1} = L$ (Figure 1.4). This requires that

$$\Delta = L/(N+1).$$

Note that the x_i are evenly spaced. This restriction is not necessary, but it simplifies the math. We can now discretize the solution f,

$$f_i = f(x_i),$$

and the kth derivative

$$f_i^{(k)} = \left.\frac{d^k f}{dx^k}\right|_{x=x_i}.$$

The finite difference approximation to the derivative $f^{(k)}$ is a weighted sum of f_i values at neighboring points. A well-known example is:

$$f_i' = \frac{f_{i+1} - f_i}{\Delta}, \tag{1.6}$$

which approximates the first-derivative to arbitrary accuracy as $\Delta \to 0$. In general

$$\boxed{f_i^{(k)} = \sum_{j=j_1}^{j_2} A_j^{(k)} f_{i+j}.}$$

The range of the summation, $j = j_1, \ldots, j_2$, is called the stencil. For example, in (1.6), $k = 1$, $j_1 = 0$, and $j_2 = 1$, and the weights are $A_0^{(1)} = -1/\Delta$ and $A_1^{(1)} = 1/\Delta$.

The weights are computed by means of a Taylor series expansion of f about x_i:

$$\boxed{f_{i+j} = f(x_i + j\Delta) = f_i + j\Delta f_i^{(1)} + \frac{1}{2}(j\Delta)^2 f_i^{(2)} + \ldots + \frac{1}{k!}(j\Delta)^k f_i^{(k)}.} \tag{1.7}$$

For example, suppose we want to approximate the first-derivative using the three-point stencil $j = -1, 0, 1$:

$$\tilde{f}_i' = Af_{i-1} + Bf_i + Cf_{i+1},$$

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where the tilde identifies the approximation.[6] Substituting from (1.7) gives

$$
\begin{aligned}
Af_{i-1} + Bf_i + Cf_{i+1} &= A\left[f_i - \Delta f_i' + \frac{1}{2}\Delta^2 f_i'' - \frac{1}{6}\Delta^3 f_i''' + \ldots\right] + B\left[f_i\right] \\
&\quad + C\left[f_i + \Delta f_i' + \frac{1}{2}\Delta^2 f_i'' + \frac{1}{6}\Delta^3 f_i''' + \ldots\right] \\
&= (A+B+C)f_i + (-A+C)\Delta f_i' + (A+C)\frac{1}{2}\Delta^2 f_i'' \\
&\quad + (-A+C)\frac{1}{6}\Delta^3 f_i''' + \ldots = f_i'. \qquad (1.8)
\end{aligned}
$$

The final equality expresses our wish that the approximation $\tilde{f}_i'$ equal the true value f_i' (a wish that will not be granted). We try to find values for A, B, and C so that (1.8) is satisfied *for all functions* f, which requires that the final equality hold separately for the terms multiplying each derivative $f^{(k)}$. The terms multiplying f_i, f_i', and f_i'' (colored blue) give:

$$
\begin{aligned}
A + B + C &= 0 \\
(-A + C)\Delta &= 1 \\
(A + C) &= 0.
\end{aligned}
$$

We now have three equations for three unknowns. Since these are all the equations we can satisfy, we can equate only the blue terms in (1.8); the red term is neglected. The solution is:

$$A = -\frac{1}{2\Delta}, \quad B = 0, \quad C = \frac{1}{2\Delta}, \qquad (1.9)$$

or

$$\boxed{\tilde{f}_i' = \frac{f_{i+1} - f_{i-1}}{2\Delta}} \qquad (1.10)$$

This is called a centered difference owing to its symmetry.

How accurate is this approximation? Recall that, to solve (1.8), we had to ignore the red term. That term is a measure of the error in our approximation. Substituting (1.9) for $A - C$ in (1.8), we have

$$\tilde{f}_i' = \frac{f_{i+1} - f_{i-1}}{2\Delta} = f_i' + \frac{1}{6}\Delta^2 f_i''' + \ldots$$

We can't tell the value of the error term in general because it depends on the function f. What we can do is recognize that, as we shrink the grid spacing Δ to zero for a given f, the error decreases in proportion to Δ^2. We therefore say that the approximation is accurate to second order in Δ.

[6] The constants A, B, and C are just simpler expressions for A_{-1}, A_0, and A_1.

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For comparison, you could derive (1.6) in the same way (try it!), and you'd find that the error is proportional to Δ, i.e., (1.6) is accurate only to first order. We conclude that (1.10) is more accurate than (1.6) in the sense that the error decreases more rapidly as $\Delta \to 0$.

We can represent (1.10) using a matrix:

$$f_i' = D_{ij}^{(1)} f_j.$$

For example, if $N = 4$, then

$$\mathsf{D} = \frac{1}{2\Delta}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Note, however, that the matrix is not square. It requires values of f at $x_0, x_1, \ldots, x_{N+1}$, but returns the derivative only at $x_1, \ldots, x_N$. This is unsatisfactory. In particular, for an eigenvalue problem such as (1.5), only a square matrix will do.

One solution is to replace the first and last equations (the top and bottom rows) with expressions that don't depend on f_0 and f_{N+1}. This requires the use of one-sided derivatives, which are derived in the same way as (1.10). The simplest choice is to use (1.6) for the top row and its counterpart for the bottom row:

$$f_1' = \frac{f_2 - f_1}{\Delta}; \quad f_N' = \frac{f_N - f_{N-1}}{\Delta} \tag{1.11}$$

We can represent the result using the matrix

$$\mathsf{D} = \frac{1}{2\Delta}\begin{bmatrix} -2 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \end{bmatrix},$$

which we call a derivative matrix. Recall that (1.6) is only accurate to first order. This is not ideal, since (1.10) is second order. In homework problem 1 in Appendix A, you will derive an alternative to (1.6) that is second order accurate. Another strategy is to use the top and bottom rows to incorporate boundary conditions, as we discuss in the next section.

1.4.3 Incorporating Boundary Conditions

If the derivative matrix is intended for use in solving a boundary value problem, then instead of one-sided derivatives we incorporate the boundary conditions into the top and bottom rows. For example, suppose we have the Dirichlet boundary

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conditions $f(0) = f_0 = 0$ and $f(L) = f_{N+1} = 0$.[7] Then (1.10) gives, for $i = 1$ and N,

$$f_1' = \frac{f_2}{2\Delta}; \qquad f_N' = \frac{-f_{N-1}}{2\Delta}, \tag{1.12}$$

and the derivative matrix for $N = 4$ becomes

$$\mathsf{D} = \frac{1}{2\Delta}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

For higher N, of course, the pattern is repeated through the interior of the matrix.

Note that we don't actually calculate the derivative at the boundary points x_0 and x_{N+1}. For that reason they are referred to as "ghost points"; their influence is felt, not seen.

Test your understanding: In homework exercise 2, you will derive the matrix representing the second-derivative, using either one-sided derivatives or boundary conditions $f_0 = f_{N+1} = 0$:

$$\mathsf{D}^{(2)} = \frac{1}{\Delta^2}\begin{bmatrix} -2 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ & & \ddots & & \\ \dots & 0 & 1 & -2 & 1 \\ \dots & & & 1 & -2 \end{bmatrix} \tag{1.13}$$

In exercise 3, you will verify that the eigenvalues and eigenvectors of $\mathsf{D}^{(2)}$ correspond to the analytical solution of the differential eigenvalue problem (1.2) with the same boundary conditions. Try it with different values of N and see how it affects the accuracy of the results. See if the error decreases in proportion to Δ^2.

1.5 The Equations of Motion

Assume that space is measured by a Cartesian coordinate system $\vec{x} = \{x, y, z\}$, with z directed opposite to gravity.[8] Corresponding unit vectors are $\hat{e}^{(x)}$, $\hat{e}^{(y)}$, and $\hat{e}^{(z)}$. The velocity is $\vec{u} = D\vec{x}/Dt = \{u, v, w\}$. Here D/Dt represents the material derivative, i.e., the time derivative as measured by an observer moving with the flow:

[7] In case you don't remember, Dirichlet boundary conditions specify the value of the solution, while Neumann conditions specify the derivative. We will use both kinds in this book.

[8] We'll generalize to other coordinate systems later.

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Table 1.1 *Typical terrestrial parameter values. Beware when using these "constants" for general purposes; some of them can vary significantly. Viscosity and diffusivity are assumed to be molecular in origin.*

name	symbol	unit	seawater	air
dynamic viscosity	μ	kg $m^{-2}s^{-3}$	10^{-3}	1.6×10^{-5}
gravitational acceleration	g	ms^{-2}	9.81	9.81
Coriolis parameter	f_0	s^{-1}	1.458×10^{-4}	1.458×10^{-4}
density	ρ_0	kg m^{-3}	1027	1.2
kinematic viscosity	ν	m^2s^{-1}	10^{-6}	1.4×10^{-5}
thermal density coefficient	α	K^{-1}	10^{-4}	3×10^{-3}
saline density coefficient	β	psu^{-1}	7×10^{-4}	
thermal diffusivity	κ_T	m^2s^{-1}	1.4×10^{-7}	1.9×10^{-5}
saline diffusivity	κ_S	m^2s^{-1}	10^{-9}	

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}. \tag{1.14}$$

The principle of mass conservation (valid provided that no nuclear reactions are present) requires that the density (mass per unit volume) ρ vary as the velocity field converges or diverges:

$$\frac{D\rho}{Dt} = -\rho \vec{\nabla} \cdot \vec{u}. \tag{1.15}$$

This is also called the continuity equation.

Conservation of momentum (Newton's second law) is represented by the Navier-Stokes equation for velocities measured in a rotating reference frame:

$$\rho \left[\frac{D\vec{u}}{Dt} - \vec{u} \times f\hat{e}^{(z)} \right] = -\vec{\nabla} p - \rho g \hat{e}^{(z)} + \mu \nabla^2 \vec{u}. \tag{1.16}$$

The second term in brackets is the acceleration due to the Coriolis force. On a spherical planet, $f = f_0 \sin\phi$, where ϕ is the latitude and f_0 is twice the planetary rotation rate (see Table 1.1 for values). In this planetary context, we assume that phenomena of interest are small compared with the size of the planet, so that the Cartesian geometry is valid and ϕ can be treated as a constant.

The three terms on the right-hand side represent forces (per unit volume): the pressure gradient force, gravity (with acceleration g), and friction due to viscosity (with dynamic viscosity μ).

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1.5.1 Approximate Forms

For most of the book we will use a simplified form of (1.16) based on two assumptions:

(i) The fluid is incompressible:

$$\boxed{\vec{\nabla} \cdot \vec{u} = 0.} \tag{1.17}$$

(ii) The density remains close to a uniform value ρ_0:

$$\rho = \rho_0 + \rho^*, \quad \text{where } |\rho^*| \ll \rho_0. \tag{1.18}$$

Expanding the pressure as

$$p = p_0 + p^*,$$

where p_0 is in hydrostatic balance with ρ_0 (i.e., $\vec{\nabla} p_0 = -\rho_0 g \hat{e}^{(z)}$; discussed further in section 2.1.1), gives

$$(\rho_0 + \rho^*) \left[\frac{D\vec{u}}{Dt} - \vec{u} \times f\hat{e}^{(z)} \right] = -\vec{\nabla}(\cancel{p_0} + p^*) - (\cancel{\rho_0} + \rho^*) g\hat{e}^{(z)} + \mu \nabla^2 \vec{u}$$

or

$$(\rho_0 + \rho^*) \left[\frac{D\vec{u}}{Dt} - \vec{u} \times f\hat{e}^{(z)} \right] = -\vec{\nabla} p^* - \rho^* g\hat{e}^{(z)} + \mu \nabla^2 \vec{u}.$$

Now, based on (1.18), we replace $(\rho_0 + \rho^*)$ on the left-hand side with ρ_0. Finally, we divide through by ρ_0, resulting in

$$\boxed{\frac{D\vec{u}}{Dt} = -\vec{\nabla}\pi + b\hat{e}^{(z)} + \nu \nabla^2 \vec{u} + \vec{u} \times f\hat{e}^{(z)}.} \tag{1.19}$$

The accelerations appearing on the right-hand side of (1.19) are

- the pressure gradient, with

$$\pi = \frac{p^*}{\rho_0}, \tag{1.20}$$

- the buoyancy, defined as

$$b = -g\frac{\rho^*}{\rho_0}, \tag{1.21}$$

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- the viscosity, with kinematic viscosity

$$\nu = \frac{\mu}{\rho_0}. \tag{1.22}$$

- the Coriolis acceleration.

Equation (1.19) is commonly known as the Boussinesq approximation to the momentum equations.

The density of seawater is governed by two separate properties: temperature T and salinity S. If the water is close to a uniform state with $T = T_0$ and $S = S_0$, we can use the linearized equation of state:

$$b = \alpha g(T - T_0) - \beta g(S - S_0). \tag{1.23}$$

Here, α and β are coefficients for thermal and saline buoyancy, taken to be constants. In the absence of sources, T and S obey Fickian diffusion equations with constant diffusivities κ_T and κ_S:

$$\frac{DT}{Dt} = \kappa_T \nabla^2 T; \qquad \frac{DS}{Dt} = \kappa_S \nabla^2 S. \tag{1.24}$$

In Chapter 9, we will discuss double diffusive instabilities, which depend on the separate effects of temperature and salinity as described by (1.23) and (1.24). In the atmosphere, salinity is obviously not a factor, but buoyancy is affected by humidity in a related way. For most applications, however, the fact that buoyancy has two components is not important, and we will use a single equation

$$\boxed{\frac{Db}{Dt} = \kappa \nabla^2 b.} \tag{1.25}$$

In these cases, you can think of b as proportional to temperature.

1.5.2 Viscosity and Turbulence

To the human senses, and in most measurements, a fluid appears as a continuous medium. Although we recognize that a fluid is really made of discrete molecules, the science of fluid mechanics is not concerned with such microscopic details. When we talk about, say, the velocity $\vec{u}$ at a point $\vec{x}$, we really mean the average molecular velocity in some volume of space, centered on $\vec{x}$, that is tiny but nonetheless large enough to encompass many molecules. With that understanding, we can think of $\vec{u}$ as a continuous function of $\vec{x}$, and therefore employ the powerful tools of calculus to understand the motion.

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Although we are not interested in molecular motions *per se*, we must account for the effect they have on the motions that we *are* interested in. That is where viscosity comes in – it models the frictional effect that molecular interactions exert on the macroscopic motions that we can perceive and measure. The assumption that molecular effects can be represented in this way is called the continuum hypothesis.

In the study of geophysical fluids, the continuum hypothesis is extended to larger scales. We are not only not interested in the motions of individual molecules, but we are also not interested in macroscopic motions smaller than a certain scale. In the study of weather, for example, we do not try to understand every little gust of wind. When we talk about the wind speed at a certain time and place, we usually mean an average that encompasses many gusts.

As with molecular motions, though, we must account for the effect the gusts have on the larger-scale motions that we're trying to understand. Exploiting the obvious analogy, the effect of the gusts is usually represented as an "effective" viscosity, often called turbulent viscosity or eddy viscosity. While this analogy is imperfect,[9] the eddy viscosity concept is a useful first step toward understanding the effect of small-scale turbulence.

The assumption that eddy viscosity is uniform in space and time is, well, better than nothing. In this book, the quantity ν that we call "viscosity" can refer to either molecular or eddy viscosity. Similarly, the diffusivity κ can refer to diffusion either by molecular motions or by small-scale turbulence.

1.6 Further Reading

The derivation of (1.16) is something every student should see at least once. A detailed account is given in Smyth (2017), which has the added virtue of being free. Thorpe (2005) provides an excellent introduction to instability and turbulence in the oceanic context. A theoretical discussion of turbulence in general, with particular attention to eddy viscosity, may be found in chapters 4, 5, and 10 of Pope (2000).

1.7 Appendix: A Closer Look at Perturbation Theory

1.7.1 The Parking Problem Revisited

We now consider the car-parking example from a more rigorous perspective. This example will illustrate some of the fundamental ideas of perturbation theory. Let ℓ be the distance the car rolls along the hill, measured left-to-right from some

[9] Eddy viscosity is not a property of the fluid but of the flow, and it can vary greatly in space and time in ways we do not understand. Most properly, eddy viscosity is not even a scalar but is actually a second-rank tensor.

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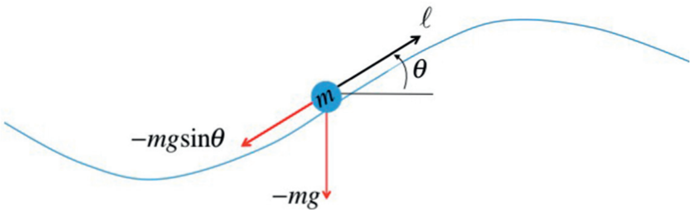

Figure 1.5 Force diagram for the car-parking example.

arbitrary origin (Figure 1.5). The force of gravity is $-mg$, where m is the mass of the car and $g = 9.81 \text{ m s}^{-2}$ at Earth's surface. The component of gravity in the direction of ℓ is $-mg \sin\theta$, where $\theta(\ell)$ is the angle of the road from the horizontal at any point ℓ. Newton's second law is therefore

$$m\frac{d^2\ell}{dt^2} = -mg\sin\theta. \tag{1.26}$$

Abbreviating $\sin\theta$ as $s(\ell)$ and removing the common factor m, we have

$$\frac{d^2\ell}{dt^2} + gs(\ell) = 0. \tag{1.27}$$

A general solution requires specifying the function $s(\ell)$, which, of course, is different for every road. An exact solution would be very difficult since $s(\ell)$ is in general nonlinear. Progress can be made if we identify equilibrium points: values of ℓ at which $s(\ell) = 0$, and concern ourselves only with the behavior close to those points. Equilibria exist when $\theta = 0$, i.e., at the bottom of a valley or the top of a hill (as can be seen by setting all time derivatives to zero in 1.27).

Let $\ell = \ell_0$ be an equilibrium point, and seek a solution

$$\ell(t) = \ell_0 + \epsilon\ell_1(t) + \epsilon^2\ell_2(t) + \ldots, \tag{1.28}$$

where ϵ is a parameter measuring the amplitude of the disturbance. Because we'll be assuming that ϵ is small, the early terms in the series are the most important. We now expand the unknown function $s(\ell)$ in a Taylor series about ℓ_0 and substitute (1.28) into the result:

$$\begin{aligned} s(\ell) &= s(\ell_0) + s'(\ell_0)(\ell - \ell_0) + \ldots \\ &= s(\ell_0) + \epsilon s'(\ell_0)\ell_1 + \epsilon^2 G + \ldots \end{aligned}$$

Here, G stands for some complicated terms whose details don't matter.[10] Now substitute into (1.27):

[10] Try it if you like. You should get $G = s'(\ell_0)\ell_2 + \frac{1}{2}s''(\ell_0)\ell_1^2$.

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$$\frac{d^2\ell_0}{dt^2} + \epsilon \frac{d^2\ell_1}{dt^2} + \epsilon^2 \frac{d^2\ell_2}{dt^2} + \ldots + g\, \left[s(\ell_0) + \epsilon s'(\ell_0)\ell_1 + \epsilon^2 G + \ldots\right] = 0,$$

or, gathering powers of ϵ,

$$\frac{d^2\ell_0}{dt^2} + gs(\ell_0) + \epsilon\left[\frac{d^2\ell_1}{dt^2} + gs'(\ell_0)\ell_1\right] + \epsilon^2\left[\frac{d^2\ell_2}{dt^2} + gG\right] + \ldots = 0. \quad (1.29)$$

Now here is a subtle but important point. Regardless of values of the quantities in square brackets, we can always find one or more values of ϵ that satisfy (1.29). But that's not what we're looking for. What we want is to take the limit $\epsilon \to 0$, and have the equation be satisfied throughout that limiting process. In other words, the equation has to be true for every value of ϵ. That can only be true if the coefficients of the powers of ϵ all vanish individually.[11] That leaves us with an infinite sequence of equations whose solutions are the unknown functions ℓ_0, $\ell_1(t)$, $\ell_2(t)$, etc.:

$$\frac{d^2\ell_0}{dt^2} + gs(\ell_0) = 0, \quad (1.30)$$

$$\frac{d^2\ell_1}{dt^2} + gs'(\ell_0)\ell_1 = 0, \quad (1.31)$$

$$\frac{d^2\ell_2}{dt^2} + gG = 0, \quad (1.32)$$

$$\ldots \text{etc.}$$

Now, let's look again at our putative solution (1.28). We already know the first term, ℓ_0; it's just the equilibrium position (hill or valley). As a result, the first equation (1.30) is satisfied trivially; $s(\ell_0) = 0$ and $d^2\ell_0/dt^2 = 0$.

The next term in (1.28), $\epsilon\ell_1(t)$, is the only one that matters if ϵ is made sufficiently small. We therefore concern ourselves only with the second equation, (1.31), whose solution is ℓ_1. This is a linear ordinary differential equation, and very easy to solve because $s'(\ell_0)$ is a constant. For tidiness we'll abbreviate $s'(\ell_0)$ as s'_0. We'll consider two cases.

If $s'_0 > 0$, we define $gs'_0 = \omega^2$, and the general solution is

$$\ell_1 = A \sin \omega t + B \cos \omega t,$$

where A and B are constants to be determined by the initial conditions. This oscillatory solution describes the car rocking back and forth after being displaced from a stable equilibrium point (a valley).

[11] As an analogy, consider a quadratic equation $ax^2 + bx + c = 0$. For given values of a, b, and c, you can easily find solutions for x using the quadratic formula. But what if the equation has to be satisfied for *every* x? That's only possible if $a = b = c = 0$.

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If $s_0' < 0$, we define $-gs_0' = \sigma^2$, and the solution is

$$\ell_1 = Ae^{\sigma t} + Be^{-\sigma t}.$$

As long as $A \neq 0$, the first term grows exponentially and will eventually dominate the solution. This describes the unbounded motion of the car away from an unstable equilibrium, i.e., the top of a hill.

Here are some general features of stability analysis that the car-parking problem illustrates:

(i) The equation (1.31) is a linear, homogeneous ordinary differential equation. This is always true. In general, though, the coefficients will not be constant, and the solution will be much more difficult. For that reason we will often resort to numerical methods.
(ii) Solutions can be oscillatory or exponential. Our main interest is in exponential solutions with positive growth rate σ. Often, we can define a simple rule of thumb to tell us which type of solution will be found; here, we only need to know the sign of s_0'.
(iii) Having solved for ℓ_1, it is possible to substitute the result into (1.32) and solve for ℓ_2, and so on to even higher orders. Some brave souls do this, but we won't.
(iv) The solution is valid only if the neglected terms in (1.28) are indeed negligible. How can we tell if this is true? Most commonly, we regard the smallness of the neglected terms as a *hypothesis* whose validity we test by comparing the solution with reality.[12]

1.7.2 A Mechanical Spring-Mass System

Consider a fixed wire frame in the shape of a circle with a bead of mass m that is free to slide, without friction, along it. This is a direct analog of the car-parking problem we just saw. If we were to do a linear stability analysis of this system, we would find that the two equilibrium points correspond to the bottom (stable) and top (unstable) points of the frame. However, we will now modify the problem by adding a spring that connects point Q, at the top of the wire frame, with the mass at point P (as is depicted in Figure 1.6).[13] We wish to determine the stability of the bead in this new system.

As a first step we need to find an equation to describe its motion. To do this, we let the resting length of the spring be l, i.e., equal to the radius of the frame, and measure the position of the bead in terms of the angle, $\theta(t)$. The tension force

[12] We don't recommend this for the car-parking problem.
[13] This example comes from the excellent book by Acheson (1997).

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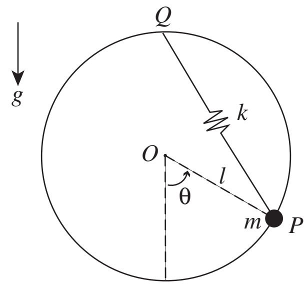

Figure 1.6 Pendulum system with a spring connecting the bead of mass m, at point P, to the top position of the frame at point Q.

exerted by the spring will be proportional to the displacement from the resting length, and given by

$$T = k \times \text{displacement} = k[2l\cos(\beta) - l], \tag{1.33}$$

where k is the spring coefficient with units of N m^{-1}, and β is the angle made by the spring to the vertical. Using some geometry we can express this in terms of θ as $\beta = \theta/2$. The component of this force that is along the wire is then $T\sin(\theta/2)$, and we can write our spring-pendulum equation as

$$ml\frac{d^2\theta}{dt^2} + mg\sin(\theta) - kl[2\cos(\theta/2) - 1]\sin(\theta/2) = 0, \tag{1.34}$$

given that the radial component of the acceleration is $ld^2\theta/dt^2$.

At this point it is helpful to stop and compare this equation with our original nonlinear equation from the car-parking problem (1.26).[14] Since the mass does not drop out of this equation, we have four parameters: m, l, g, and k. In addition we have the extra term that arises from the presence of the spring. We can simplify the problem by measuring time in the units of $(l/g)^{1/2}$, so that the dimensionless time variable becomes $t_\star \equiv t/(l/g)^{1/2}$ and

$$\frac{d^2\theta}{dt_\star^2} = \frac{d^2\theta}{dt^2}\frac{g}{l},$$

We can now write (1.34) as

$$\frac{d^2\theta}{dt_\star^2} + \sin(\theta) - S\sin(\theta/2)[2\cos(\theta/2) - 1] = 0, \tag{1.35}$$

where $S = kl/mg$. By representing our equation in dimensionless form we have achieved a great simplification – we have reduced the number of parameters from four to one.

[14] Note that since the path of motion of the mass is circular we can write the left-hand side of (1.26) as $d^2\ell/dt^2 = ld^2\theta/dt^2$.

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The parameter S describes the strength of the spring relative to the weight of the bob. Consider two extreme cases. If S is very small, the spring is weak and has little effect, so the bob oscillates about $\theta = 0$ just as in a normal pendulum. If S is large, the spring is very tight, and we expect the bob to be pulled strongly away from the bottom of the hoop. In the latter extreme, is the equilibrium state $\theta = 0$ stable or unstable? Imagine that you started with a loose spring and the bob at $\theta = 0$, then gradually tightened the spring. What would happen? Can you picture, at a critical value of S, the bob suddenly snapping away from $\theta = 0$? Next we will examine that process mathematically.

First, it is necessary to search for equilibrium points of the system by setting the time derivatives to zero in (1.35) and solving for the position of the bead. This leads to

$$\sin(\theta) - S\sin(\theta/2)[2\cos(\theta/2) - 1] = 0,$$

which can be simplified using the identity $\sin(2x) = 2\cos(x)\sin(x)$ with $x = \theta/2$:

$$\sin(\theta/2)[2(S-1)\cos(\theta/2) - S] = 0.$$

The system is therefore in equilibrium if one of two criteria is satisfied:

$$\sin(\theta/2) = 0, \tag{1.36}$$

or

$$\cos(\theta/2) = S/[2(S-1)]. \tag{1.37}$$

The former criterion is satisfied if $\theta = 0$ or π. To perform a linear stability analysis about an equilibrium point θ_0 we substitute $\theta(t) = \theta_0 + \epsilon\theta'(t)$, and expand the nonlinear functions in their Taylor series. For the case $\theta_0 = 0$,

$$\sin(\epsilon\theta') = \epsilon\theta' + O(\epsilon^3)$$
$$\cos(\epsilon\theta') = 1 + O(\epsilon^2).$$

Collecting the terms of order ϵ, we have

$$\frac{d^2\theta'}{dt^2} + \left(1 - \frac{S}{2}\right)\theta' = 0.$$

This is analogous to equation (1.31) of the parking problem. As in that case, the result is a linear, homogeneous ordinary differential equation whose solutions are either oscillatory or exponential. Substituting the test solution $\theta(t) \propto e^{\sigma t}$ gives

$$\sigma = \pm\left(\frac{S}{2} - 1\right)^{1/2}.$$

When $S < 2$, σ is imaginary, and the mass oscillates about $\theta = 0$. When $S > 2$, however, σ is real (with one positive and one negative value), and the equilibrium

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point $\theta = 0$ is therefore *unstable*. The value $S = 2$ represents a critical state. This can be understood physically by interpreting S as the ratio of the destabilizing tension kl, which pulls the mass away from the equilibrium position, to the gravitational force mg that acts to restore the mass to equilibrium.

Something else very interesting happens at the critical value $S = 2$: the system acquires two new equilibrium points. Since $-\pi \leq \theta \leq \pi$, we know that $0 \leq \cos(\theta/2) \leq 1$, and therefore the second criterion for equilibrium, (1.37), can be satisfied only when $S \geq 2$.

Let's recall what the equilibrium points represent. When the time derivatives are set to zero in our equation of motion of the bead, we are essentially saying that the sum of the forces is equal to zero. In other words, the equilibrium points are positions of the mass for which all forces balance. In this example of the spring-pendulum system, we have just shown that the tension from the spring can balance the gravitational force for $\theta \neq 0, \pi$ when the spring force is strong enough, i.e., for $S > 2$. The locations (θ_e) at which this balance occurs are shown in Figure 1.7 for each value of S. We leave it as an exercise for the student to show that these equilibrium points are stable (Figure 1.7).

We have shown that this spring-mass system has either one or two equilibrium states depending on the value of the dimensionless parameter, S. When S exceeds the critical value 2, the bottom position loses stability and the two new stable equilibria appear. The determination of simple "rules of thumb" like this is one of the primary goals of linear stability analysis.

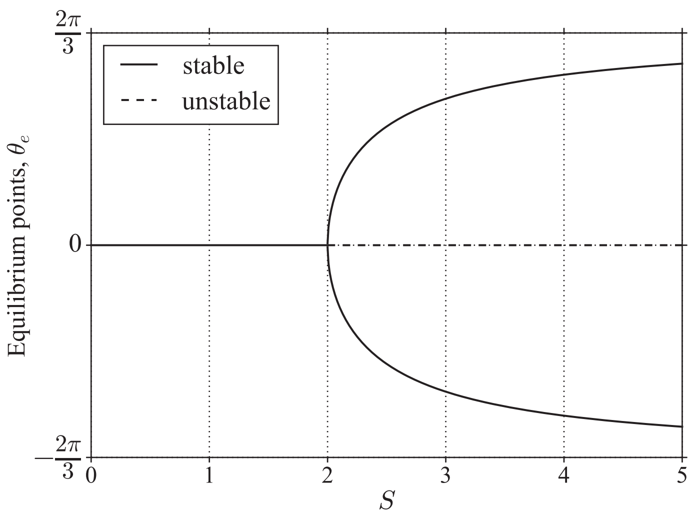

Figure 1.7 Equilibrium points of the spring-pendulum system and their stability. The asymptotic value $2\pi/3$ is reached for large S.

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