MEAN-VALUE PRINCIPLE UNDER CUMULATIVE PROSPECT THEORY

BY

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ABSTRACT

In the paper we introduce a generalization of the mean-value principle under Cumulative Prospect Theory. This new method involves some well-known ways of pricing insurance contracts described in the actuarial literature. Properties of this premium principle, such as translation and scale invariance, additivity for independent risks, risk loading and others are studied.

1. INTRODUCTION

From the practical point of view, the mean-value principle is based on a belief that people use a utility function while making decisions under risk and uncertainty and they can properly evaluate the probabilities of gains and losses. Gerber (1979), Goovaerts et al. (1984) and Rolski et al. (1999) study the properties of this principle assuming that a value function is convex and twice differentiable. However, in reality, these assumptions are usually incorrect. On the one hand, numerous experiments carried out by Kahneman and Tversky (1979) confirm the fact that under risk and uncertainty people make decisions using a function which assigns virtual value to monetary outcomes. On the other hand, they notice that people decide which outcomes they see as basically identical and they set a reference point and consider lower outcomes as losses and larger as gains. They suggest replacing the utility function, which measures absolute wealth, with a value function that depends on relative payoff and measures gains and losses. According to them, such a function should be convex for negative and concave for positive arguments.

In addition to this observation, Kőszegi and Rabin (2007) notice that making decisions under uncertainty increases risk aversion if the risk is expected. Reference points, which influence decision maker to take certain action under uncertainty, are allocated on the basis of beliefs of a decision maker concerning a possible outcome and they can be determined in a stochastic way. Taking action relies on maximizing the functional $E_F \int v(w|r)\, dG(r)$, where v is the value function proposed by Kahneman and Tversky, w is the wealth with distribution F and G is a probability distribution function of a discrete random variable R with finite support. Under these assumptions we may deal with value function

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$v(w) = \int v(w|r)\,dG(r)$ with very irregular shapes, in particular they can be functions which are not differentiable at many points. Gillen and Markowitz (2010) suggest functions which are piecewise convex and concave, thus have some inflection points, but are differentiable at every point. The analysis of certain subclasses of these functions allows us to determine a decision maker's willingness to undertake risk. It still remains an issue for debate which type of a value function corresponds the most to human behavior while making decisions. Therefore, in order to obtain results for possibly large class of value functions, we should put possibly weak assumptions concerning the value function.

Kahneman and Tversky (1979) also discover that people distort probabilities while making decisions under risk and uncertainty. Rank-dependent utility model eliminates the problem of overweighting very small probabilities (e.g. Quiggin 1982). Formalization of this idea relies on distorting the cumulative distribution function, not the single probabilities. Tversky and Kahneman (1992) use the concept of rank-dependent utility model and create Cumulative Prospect Theory in which they assume that probabilities of gains and losses are distorted in a different way. This theory has already been widely applied and discussed in many papers (e.g. Schmidt et al., 2008, Teitelbaum, 2007). During last years some authors have adjusted classical theories of finance and insurance to Cumulative Prospect Theory (see Schmidt and Zank, 2007, De Giorgi et al., 2009, De Giorgi and Hens, 2006, Bernard and Ghossoub, 2010). In this paper we would like to study a new version of the mean-value principle under Cumulative Prospect Theory.

The paper is organized as follows. In Section 2 we review the mathematical foundations of Cumulative Prospect Theory and define a new version of the mean-value principle adapted to this theory. In Section 3 we analyze properties of this premium principle. Section 4 summarizes the whole article. In the Appendix one may find proofs of theorems and propositions.

2. Premium principle under Cumulative Prospect Theory

In rank-dependent utility model it is assumed that probabilities are distorted by some non-decreasing function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$, called probability distortion function (e.g. Segal, 1989). Let $\mathcal{G}$ denote the class of all probability distortion functions. For a fixed $g \in \mathcal{G}$ and random variable X, the Choquet integral is defined by

$$E_g X := \int_{-\infty}^{0} \left(g\left(P(X > t)\right) - 1\right) dt + \int_{0}^{\infty} g\left(P(X > t)\right) dt,$$

provided both integrals are finite. Further we assume that all random variables are defined on some probability space $(\Omega, \mathcal{A}, P)$. If X takes finite number of values $x_1 < x_2 < \ldots < x_n$ with probabilities $P(X = x_i) = p_i > 0$, then $E_g X = x_1 +$

$\sum_{i=1}^{n-1} g(q_i)\ (x_{i+1} - x_i)$, where $q_i = \sum_{k=i+1}^{n} p_k$; in particular for $n = 2$ we have $E_g X = x_1(1 - g(p_2)) + g(p_2)x_2$. Further we write $X \geq 0$ if $P(X \geq 0) = 1$ and $X \in \mathcal{X}_2^+$ if $P(X = s) = q = 1 - P(X = 0)$, where $q \in [0,1]$ and $s > 0$ are arbitrary. The Choquet integral is additive for comonotonic risks, positively homogenous, monotonic (i.e. $E_g X \geq E_g Y$ if $X \geq Y$ a.e.) and $E_g(c) = c$ for all $c \in \mathbb{R}$. Moreover $E_g(-X) = -E_{\bar{g}}X$, where $\bar{g}(x) = 1 - g(1 - x)$ is dual distortion function. In the literature we can find some classes of probability distortion functions, e.g. $g(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}$, $g(p) = \frac{p^\gamma}{p^\gamma + (1-p)^\gamma}$, $g(p) = \exp(-(-\ln p)^\gamma)$, $g(p) = p + \gamma(p - p^2)$ (see Tversky and Kahneman, 1992, Prelec, 1998, Goldstein and Einhorn, 1987, Sereda et al., 2010). For $g, h \in \mathcal{G}$ we define the generalized Choquet integral as

$$E_{gh}X = E_g X_+ - E_h(-X)_+.$$

It is introduced by Tversky and Kahneman (1992) for discrete random variables and is used to describe the mathematical foundations of Cumulative Prospect Theory. In numerous experiments Tversky and Kahneman notice that probabilities of losses are distorted in a different way than probabilities of gains. If $h(x) = \bar{g}(x) = 1 - g(1 - x)$, then $E_{g\bar{g}}X = E_g X$. Usually, formulas for h are similar to those for g but with different values of coefficient γ.

Lemma 1. *The generalized Choquet integral has the following properties:*

W1 $E_{gh}\mathbf{1}_A = g(P(A))$;

W2 $E_{gh}(cX) = cE_{gh}X$ *for all* $c \geq 0$;

W3 $E_{gh}(-X) = -E_{hg}X$;

W4 if $X \leq Y$, *then* $E_{gh}X \leq E_{gh}Y$;

W5 if $g(x) \geq x$ *and* $h(x) \leq x$ *for* $x \in [0,1]$, *then* $E_{gh}X \geq EX$;

W5' if $g(x) \leq x$ *and* $h(x) \geq x$ *for* $x \in [0,1]$, *then* $E_{gh}X \leq EX$;

W6 if $g(x) = h(x) = x$, *then* $E_{gh}X = EX$;

W7 $E_{gh}c = c$ *for all* $c \in \mathbb{R}$;

W8 for all $c \in \mathbb{R}$ *we have*

$$E_{gh}(X + c) = E_{gh}X + c + \int_0^c [h(P(-X > s)) - \bar{g}(P(-X > s))]ds, \tag{1}$$

$$E_{gh}(X + c) = E_{gh}X + c + \int_0^{-c} [\bar{h}(P(X \geq s)) - g(P(X > s))]ds; \tag{2}$$

W9 Jensen's inequality: If $u : \mathbb{R} \to \mathbb{R}$ *is non-decreasing, concave and* $u(0) = 0$, *then for* $g, h \in \mathcal{G}$ *and arbitrary random variable* X *such that* $E_{gh}X$ *exists we have*

$$E_{gh}u(X) \leq u(m) + \int_0^{u'(m)m-u(m)} \left[\bar{h}\left(P\left(u'(m)X \geq s\right)\right) - g\left(P\left(u'(m)X \geq s\right)\right)\right] ds, \quad (3)$$

where $m = E_{gh}X$ and u' is the right-sided derivative of u. Moreover, if $\bar{h}(x) \geq g(x)$ or $X \geq 0$, then $E_{gh}u(X) \leq u(E_{gh}X)$.

We introduce a premium principle which is a modification of the mean-value principle adjusted to Cumulative Prospect Theory. Let X be an arbitrary random variable which does not have to be non-negative. Then X should be regarded as a total claim made by the insured, decreased by the possible gain earned from investition. This allows us to consider insurance products involving some investment options such as investment-linked life insurance or variable annuity. In case of non-life insurance it is plausible to study non-negative random variables.

Consider a decision-maker whose reference point is w and who wants to purchase an insurance policy paying out the monetary equivalent of the random outcome X. Further, we call $(X - w)_+$ losses and $(w - X)_+$ gains. If $X \geq 0$, then $(X - w)_+$ and $(w - X)_+$ denote catastrophic and non-catastrophic loss, respectively. In the latter case there is a direct analogy with stop-loss reinsurance. Assume that $u_1, u_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are some non-decreasing value functions, where u_1 measures gains and u_2 losses. Let g and h be probability distortion functions of gains and losses, respectively. We propose premium $H(X)$ for insuring X as the solution of

$$u_1\left(\left(w - H(X)\right)_+\right) - u_2\left(\left(H(X) - w\right)_+\right) = E_g u_1\left((w - X)_+\right) - E_h u_2\left((X - w)_+\right). \quad (4)$$

Notice that (4) can be rewritten as

$$u\left(w - H(X)\right) = E_{gh}u\left(w - X\right) \quad (5)$$

with non-decreasing function $u(x) = u_1(x_+) - u_2((-x)_+)$ for $x \in \mathbb{R}$. Gerber (1979) considers a similar equation for premium $H(X)$ under assumptions that the value function u is concave and probabilities are not distorted, i.e. $g(p) = h(p) = p$. In a more general model Luan (2001) assumes that $h = \bar{g}$, g is convex and the value function is concave. Van der Hoek and Sherris (2001) analyze a functional with different probability distortion functions for gains and losses. However, they study only the case when the value functions are linear.

Let us determine the minimum assumptions about u under which the premium defined by (5) exists and is determined uniquely. It is commonly accepted that u should be non-decreasing. However, if u was constant on some interval, then the premium could not be determined uniquely. Therefore we assume that u is increasing. It turns out that u should also be continuous. Otherwise, equation (5) may have no solutions. Without loss of generality we may assume that $u(0) = 0$. Further we consider increasing and continuous functions u such that $u(0) = 0$. To simplify the notation we write $u \in \mathcal{U}$ if u satisfies these three conditions. We also denote $u \in \mathcal{U}_0$ if $u(x) = cx$, $u(x) = (1 - e^{-cx})/a$ or $u(x) = (e^{cx} - 1)/a$ for $x \in \mathbb{R}$ and some $a, c > 0$.

In the following two examples we determine the premium $H(X)$ if $u \in \mathcal{U}_0$.

Example 1. *If $u(x) = cx$, then from W2, W3 and (1) we can rewrite (5) as*

$$\begin{aligned} c\big(w - H(X)\big) &= cE_{gh}(w - X) \\ &= c\Big(w - E_{hg}X + \int_0^w \big[h\big(P(X > s)\big) - \bar{g}\big(P(X > s)\big)\big]ds\Big). \end{aligned}$$

Finally we have

$$H(X) = E_{hg}X + \int_0^w \big[\bar{g}\big(P(X > s)\big) - h\big(P(X > s)\big)\big]ds. \tag{6}$$

Example 2. *For $u(x) = (1 - e^{-cx})/d$ from (1) and W3 we can rewrite (5) as*

$$\begin{aligned} 1 - e^{-c(w - H(X))} &= E_{gh}\big(1 - e^{-c(w - X)}\big) \\ &= 1 - E_h\big(e^{-cw}e^{cX}\big) + \int_0^1 \big[h\big(P\big(e^{-c(w-X)} > s\big)\big) - \bar{g}\big(P\big(e^{-c(w-X)} > s\big)\big)\big]ds. \end{aligned}$$

From the above and W2 we have

$$e^{cH(X)} + e^{cw}\int_0^1 \big[h\big(P\big(e^{cX} > se^{cw}\big)\big) - \bar{g}\big(P\big(e^{cX} > se^{cw}\big)\big)\big]ds = E_{hg}e^{cX}.$$

Thus

$$H(X) = \frac{1}{c}\ln\left[E_h e^{cX} + \int_0^{\exp(cw)} \big[\bar{g}\big(P\big(e^{cX} > t\big) - h\big(P\big(e^{cX} > t\big)\big)\big)\big]dt\right]. \tag{7}$$

In a similar way we can derive a formula for $H(X)$ if $u(x) = (e^{cx} - 1)/d$.

In the next example we will determine the premium if we assume that probability distortion functions are neo-additive weighting functions (see Wakker, 2010, p. 208). Further we denote $\sup X = \sup\{t : P(X \le t) < 1\}$ and $\inf X = -\sup(-X)$.

Example 3. *Let $u(x) = x$. Assume that $g(x) = c + dx$, $h(x) = a + bx$ for $x \in (0,1)$, where $b, d \ge 0$, $a, c \ge 0$, $a + b \le 1$ and $c + d \le 1$. If $\inf X \le 0 \le \sup X$, then from (6) we have*

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$$
\begin{aligned}
H(X) = & \int_0^{\sup X} [a + bP(X > s)]ds - \int_0^{-\inf X} [c + dP(-X > s)]ds \\
& + \int_0^{w(X)} [(1 - a - c - d) + (d - b)P(X > s)]ds \\
= & (1 - a - c - d)w(X) + (d - b)\int_0^{w(X)} P(X > s)ds \\
& + a \sup X + c \inf X + bEX_+ - dE(-X)_+,
\end{aligned}
\tag{8}
$$

where $w(X) = \min(\max(\inf X, w), \sup X)$. *If* $c = 0$, $0 \le d < 1$, $\inf X = 0$ *and* $\sup X = w$, *then* $H(X) = EX + (1 - d)(\sup X - EX)$. *This premium is analyzed by Kaluszka and Okolewski (2008). If* $w = 0$, $b = 1 - a$ *and* $d = 1 - c$, *then from (8) we get*

$$H(X) = EX + a(\sup X_+ - EX_+) - c(\sup(-X)_+ - E(-X)_+).$$

If $\inf X \le w \le \sup X$ *and* $b = d$, *then*

$$H(X) = (1 - a - c - d)w + a \sup X + c \inf X + dEX.$$

3. Properties of premium principle under Cumulative Prospect Theory

Further we assume that X is an arbitrary random variable, unless it is stated otherwise.

1. *Non-excessive loading:* $\inf X \le H(X) \le \sup X$.

This property holds for all $u \in \mathcal{U}$ and $g, h \in \mathcal{G}$, which is the consequence of W4 and W7.

2. *No unjustified risk loading:* $H(a) = a$ *for all* $a \in \mathbb{R}$.

From W7 it follows that this condition is satisfied for all $u \in \mathcal{U}$ and $g, h \in \mathcal{G}$.

3. *Translation invariance:* $H(X + b) = H(X) + b$.

Proposition 1. *Let* $u \in \mathcal{U}_0$ *and* $g, h \in \mathcal{G}$. *Then* $H(X)$ *is translation invariant for all* X *and* $b \in \mathbb{R}$ *if and only if* $h = \bar{g}$.

Proposition 2. *Let* $u \in \mathcal{U}_0$ *and* $g, h \in \mathcal{G}$. *Then* $H(X)$ *is translation invariant for all* $X \ge 0$ *and* $b \ge 0$ *if and only if* $h = \bar{g}$ *or* $w \le 0$.

Theorem 1. *Let $u \in \mathcal{U}$ and $g, h \in \mathcal{G}$ be continuous. If $H(X)$ is translation invariant for $w = 0$ and some $w > 0$, then $u \in \mathcal{U}_0$ and $h = \bar{g}$.*

4. *Scale invariance: $H(aX) = aH(X)$ for all $a > 0$.*

Theorem 2. Let *$u \in \mathcal{U}$ and $g, h \in \mathcal{G}$.*

(i) *Let $u(x) = cx$ for some $c > 0$. If $w = 0$ or $h = \bar{g}$, then $H(aX) = aH(X)$ for all $a > 0$.*
(ii) *Let h be continuous. If $X \geq 0$ and $H(X)$ is scale invariant for $w = 0$, then $u(x) = -c(-x)^d$ for $x \leq 0$ and some $c, d > 0$.*
(iii) *Let g, h be continuous. If $H(X)$ is scale invariant for $w = 0$ and all X, then $u(x) = -c(-x)^d$ for $x \leq 0$ and some $c, d > 0$ and $u(x) = ax^b$ for $x > 0$ and some $a, b > 0$.*
(iv) *If h is continuous and $H(X)$ is scale invariant for all $w \geq 0$ and all X, then $u(x) = cx$ for some $c > 0$ and all $x \in \mathbb{R}$ and $\bar{g} = h$.*

5. *Additivity for comonotonic risks.*

Theorem 3. *(i) Let $u(x) = cx$. If $h = \bar{g} \in G$, then $H(X)$ is additive for comonotonic risks.*
(ii) If $u \in \mathcal{U}$, $g, h \in \mathcal{G}$, h is continuous and $H(X)$, which is the solution of (5), is additive for comonotonic risks for all $w \geq 0$ and all X, then $u(x) = cx$ and $\bar{h} = g$.

6. *Additivity for independent risks.*

Theorem 4. *(i) If $g(p) = h(p) = p$ and $u \in \mathcal{U}$, then $H(X)$ is additive for independent risks if and only if $u \in \mathcal{U}_0$.*
(ii) Let $u \in \mathcal{U}_0$, $g, h \in \mathcal{G}$ be such that $h(0+) = 0$, $h(1-) = 1$ and there exists left-sided derivative of h at $x = 0$. If $H(X)$ is additive for independent risks for $w = 0$ and some $w > 0$, then $g(p) = h(p) = p$.

Notice that in Theorem 4 we do not put any additional requirements on function g. Moreover, from (ii) it follows that in practice it is enough to check the additivity for independent risks for two values of w in order to be certain that probabilities are not distorted.

7. *Subadditivity: $H(X + Y) \leq H(X) + H(Y)$.*

Theorem 5. *Let $u(x) = cx$ for some $c > 0$ and $h = \bar{g}$, where $g \in \mathcal{G}$. Then $H(X)$ is subadditive if and only if g is convex.*

8. *Stop-loss order preserving: $X \leq_{sl} Y$ implies $H(X) \leq H(Y)$.*

Theorem 6. *If $u \in \mathcal{U}$ is concave, $g, h \in \mathcal{G}$ are such that $\bar{g} = h$, g is convex and $X \leq_{sl} Y$, then $H(X) \leq H(Y)$.*

9. *Risk loading.*

One of the properties characterizing the premium is risk loading, i.e. $H(X) \geq E(X)$. If this condition is not satisfied, then obviously no insurance company would decide to sell the policy. The following propositions describe in terms of rank-dependent utility theory two groups of people: those who either can afford to buy an insurance or refuse to be insured. Generally, risk loading holds if and only if

$$E(X) \leq w - u^{-1}\big(E_{gh}u(w-X)\big). \tag{9}$$

Since it can be difficult to evaluate the right-hand side of (9), we give some sufficient conditions when risk loading is satisfied.

Proposition 3. *If $u \in \mathcal{U}$ is concave and $g, h \in \mathcal{G}$ are such that $g(x) \leq \bar{h}(x) \leq x$ for all $0 \leq x \leq 1$, then $H(X) \geq E(X)$.*

In the next two propositions we assume that $X \geq 0$.

Proposition 4. *Assume that $u \in \mathcal{U}$, $g, h \in \mathcal{G}$ and X is non-negative, bounded random variable such that $w < s = \sup X$. Then $H(X) \geq E(X)$ holds, if*

$$E(X) \leq w - u^{-1}\big[g\big(P(X < w)\big)\, u(w) + h\big(P(X = s)\big)\, u(w - s)\big]. \tag{10}$$

If X takes only the values from the set $\{0, w, s\}$, then (10) is equivalent to $H(X) \geq E(X)$.

Proposition 5. *Assume that $u \in \mathcal{U}$ and $g \in \mathcal{G}$. Then $H(X) \geq E(X)$, if*

$$E(X) \leq w - u^{-1}\big(u(w)g\big(P(X < w)\big)\big). \tag{11}$$

If $P(X = 0) + P(X = w) = 1$, then (11) is equivalent to $H(X) \geq E(X)$.

Example 4. *For some types of value functions and probability distortion functions we can check directly when risk loading holds. Let $u(x) = x$, $g(p) = p + \gamma_1(p - p^2)$ and $h(p) = p + \gamma_2(p - p^2)$, where $|\gamma_1|, |\gamma_2| \leq 1$. If $\gamma_1 \leq 0$, then g is convex and for $\gamma_1 \geq 0$ function g is concave. Moreover $\bar{g}(p) - h(p) = -p(1-p)(\gamma_1 + \gamma_2)$. From (6) we have*

$$\begin{aligned} H(X) = & \int_0^\infty [P(X > s) + \gamma_2 P(X > s)P(X \leq s)]ds - (\gamma_1 + \gamma_2)\int_0^w P(X > s)P(X \leq s)\, ds \\ & - \int_0^\infty \big[P(-X \geq s) + \gamma_1 P(-X \geq s)P(-X < s)\big]\, ds \\ = & \; EX + \gamma_2 \int_w^\infty P(X > s)P(X \leq s) ds - \gamma_1 \int_{-\infty}^w P(X > s)P(X \leq s)\, ds. \end{aligned}$$

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Thus $H(X) \geq EX$ if and only if

$$\gamma_2 \int_w^{\infty} P(X > s) P(X \leq s) ds \geq \gamma_1 \int_{-\infty}^{w} P(X > s) P(X \leq s) ds.$$

This condition is satisfied for example if $\gamma_2 \geq 0$ and $\gamma_1 \leq 0$. Let $\bar{X} = \max(X, w)$, $\underline{X} = \min(X, w)$. Notice that if X and X' are i.i.d. random variables, then

$$\begin{aligned} \int_w^{\infty} P(X > s)\, P(X \leq s)\, ds &= E \int_w^{\infty} \mathbf{1}(X' > s)\, \mathbf{1}(X \leq s)\, ds \\ &= E(\overline{X'} - \overline{X})_+ = \frac{1}{2} E|\overline{X'} - \overline{X}|, \end{aligned}$$

which is known as Gini coefficient.

In a smilar way we prove that $\int_{-\infty}^{w} P(X > s)\, P(X \leq s)\, ds = \frac{1}{2} E|\underline{X'} - \underline{X}|$. Since $|x - y| = 2\max(x, y) - x - y$, *we have*

$$H(X) = EX + \gamma_2 (E\overline{X}_{2:2} - E\overline{X}) - \gamma_1 (E\underline{X}_{2:2} - E\underline{X}),$$

where $\bar{X}_{2:2} = \max(\bar{X}, \bar{X}')$ *and* $\underline{X}_{2:2} = \max(\underline{X}, \underline{X}')$.

4. Concluding remarks

We have presented a new version of premium principle under Cumulative Prospect Theory. This premium principle satisfies non-excessive and no unjustified risk loading conditions for all types of value functions and probability distortion functions. It is translation invariant only if the value function is linear or exponential and $h = \bar{g}$. Scale invariance and additivity for comonotonic risks are satisfied if the value function is linear and $h = \bar{g}$. Additivity for independent risk holds if the value function is linear and exponential and probabilities are not distorted which corresponds to the case described by Gerber (1979). If the value function is linear and $h = \bar{g}$, then the principle is subadditive if and only if g is convex. In general this premium principle does not satisfy risk loading, but we give some conditions under which this property holds. This paper extends results by Gerber (1979), Van der Hoek and Sherris (2001) and Luan (2001) under mild assumptions on considered functions. The results are obtained by examining functional equations instead of differential equations.

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APPENDIX

Proof of Lemma 1. Proofs of W1, W3 and W6 are obvious. Proofs of W2, W5, W5' and W7 are the consequence of the definition of the generalized Choquet integral, the Choquet integral and properties of the Choquet integral.

Ad W4 If $X \leq Y$, then $P(X > t) \leq P(Y > t)$ for $t \in \mathbb{R}$. Thus

$$E_g X_+ = \int_0^\infty g(P(X > t))\,dt \leq \int_0^\infty g(P(Y > t))dt = E_g Y_+ .$$

Moreover, if $X \leq Y$, then $-Y \leq -X$ and $P(-Y > t) \leq P(-X > t)$ for $t \in \mathbb{R}$. Hence $E_h(-X)_+ \geq E_h(-Y)_+$.

Ad W8 Firstly, we will prove (1). We have

$$\begin{aligned} E_{gh}(X+c) &= \int_0^\infty g(P(X > t-c))\,dt - \int_0^\infty h(P(-X > t+c))\,dt \\ &= \int_{-c}^\infty g(P(X > t))\,dt - \int_c^\infty h(P(-X > t))\,dt \\ &= E_{gh}X + \int_{-c}^0 g(P(X > t))\,dt - \int_c^0 h(P(-X > t))\,dt \\ &= E_{gh}X + \int_0^c g(P(-X < s))\,ds + \int_0^c h(P(-X > s))\,ds \\ &= E_{gh}X + c + \int_0^c [h(P(-X > s)) - \bar{g}(P(-X > s))]\,ds, \end{aligned}$$

because the modification of values of the integrated function at a countable number of points yields $\int_0^c \bar{g}(P(-X > s))ds = \int_0^c \bar{g}(P(-X \leq s))ds$. Formula (2) is obtained from (1) after making some elementary calculations.

Ad W9 Obviously $u(x) \leq u(m) + u'(m)(x-m)$ for all x, where u' is the right-sided derivative of u. From this, W2, W4 and (2) we have

$$\begin{aligned} E_{gh}u(X) &\leq E_{gh}[u(m) - u'(m)m + u'(m)X] \\ &= u(m) + \int_0^{u'(m)m-u(m)} [\bar{h}(P(u'(m)X \geq s)) - g(P(u'(m)X \geq s))]\,ds \end{aligned}$$

and we get (3). Moreover $u'(m)m - u(m) \le 0$ and $u(0) = 0$, so if $\bar{h} \ge g$, then $E_{gh}u(X) \le u(E_{gh}X)$. If $X \ge 0$, then $P(u'(m)X \ge s) = 1$, hence $E_{gh}u(X) \le u(E_{gh}X)$. □

Proof of Proposition 1. Let $u(x) = cx$. From (6) and W8 we have

$$\begin{aligned} H(X+b) &= E_{hg}X + b + \int_0^b [h(P(-X>s)) - \bar{g}(P(-X>s))]\,ds \\ &\quad + \int_0^w [\bar{g}(P(X>s-b)) - h(P(X>s-b))]\,ds. \end{aligned}$$

Hence $H(X+b) = H(X) + b$ for all $b \in \mathbb{R}$ if and only if

$$\begin{aligned} &\int_0^b [h(P(-X>s)) - \bar{g}(P(-X>s))]\,ds \\ &= \int_0^w [\bar{g}(P(X>s)) - \bar{g}(P(X>s-b)) - (h(P(X>s)) - h(P(X>s-b)))]\,ds. \end{aligned} \tag{12}$$

Clearly, if $h = \bar{g}$, then $H(X)$ is translation invariant. Suppose now that $h(z) \ne \bar{g}(z)$ for some z. Let $b > w \ge 0$ and X be the random variable such that $P(X = -s_0) = z = 1 - P(X = 0)$, where $b - w < s_0 < b$. Then (12) can be rewritten as

$$\begin{aligned} [h(z) - \bar{g}(z)]s_0 &= \int_{-s_0}^{w-b} [h(1-z) - \bar{g}(1-z)]\,dt \\ &= (h(1-z) - \bar{g}(1-z))(w - b + s_0). \end{aligned}$$

Since $(h(z) - \bar{g}(z))s_0 \ne 0$, we get a contradiction that b is arbitrary such that $\max(s_0, w) < b < s_0 + w$. For $w < 0$ let $b < w$ and let X be such that $P(X = s_0) = z = 1 - P(X = 0)$, where $0 < w - b < s_0 < -b$. Then (12) can be rewritten as

$$s_0(\bar{g}(1-z) - h(1-z)) = (s_0 - w + b)(h(z) - \bar{g}(z)),$$

which contradicts that b is arbitrary such that $w - s_0 < b < \min(-s_0, w)$.

Let $u(x) = (1 - e^{-cx})/a$. From (7) under $b > w \ge 0$ we have

$$H(X+b) = \frac{1}{c}\ln\left[E_h e^{cX} + \int_0^{\exp(c(w-b))} [\bar{g}(P(e^{cX}>s)) - h(P(e^{cX}>s))]\,ds\right] + b. \tag{13}$$

If $h = \bar{g}$, then $H(X+b) = H(X) + b$ for all $b \in \mathbb{R}$. Suppose that $\bar{g}(z) \ne h(z)$ for some z. Let X be such that $P(X = s_0) = 1 - z = 1 - P(X = 0)$, where $s_0 < w - b < 0$. Then

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$$\int_0^{\exp(c(w-b))} \left[\bar{g}(P(e^{cX} > s)) - h\left(P(e^{cX} > s)\right)\right] ds = (\bar{g}(z) - h(z))\left(e^{cs_0} - e^{c(w-b)}\right) \neq 0$$

and from (13) and (7) it follows that $H(X)$ is not translation invariant. An analogous proof can be carried out for $u(x) = (e^{cx} - 1)/d$. □

Proof of Proposition 2. Since $P(X > t) = 1$ for $t < 0$ and $\bar{g}(1) = h(1) = 1$, then for $b \geq 0$ we have

$$\begin{aligned}\int_0^w [\bar{g}(P(X > s-b)) - h(P(X > s-b))]\, ds &= \int_{-b}^{w-b} [\bar{g}(P(X > t)) - h(P(X > t))]\, dt \\ &= \int_0^{w-b} [\bar{g}(P(X > t)) - h(P(X > t))]\, dt.\end{aligned}$$

Notice that $E_{hg}(X + b) = E_{hg}X + b$, because $X + b \geq 0$. Hence for $u(x) = cx$ from (6) it follows that

$$H(X+b) = H(X) + b + \int_{(w-b)_+}^{w} [h(P(X > s)) - \bar{g}(P(X > s))]\, ds. \tag{14}$$

From (14) it follows that if $w \leq 0$ or $\bar{g} = h$, then $H(X + b) = H(X) + b$. Suppose that $w > 0$ and $\bar{g}(z) \neq h(z)$ for some z. Let $b > w$ and X be such that $P(X = s_0) = z = 1 - P(X = 0)$, where $0 < s_0 < w$. Then

$$\int_{(w-b)_+}^{w} \left[h(P(X > s)) - \bar{g}(P(X > s))\right] ds = s_0(h(z) - \bar{g}(z)) \neq 0,$$

which means that $H(X)$ is not translation invariant.

If $u(x) = (1 - e^{-cx})/d$, then from (7) and (13) for $b > 0$ we have

$$H(X+b) = \frac{1}{c}\ln\left[E_h e^{cX} + \int_0^{e^{c(w-b)}} \left[\bar{g}\left(P\left(e^{cX} > s\right)\right) - h\left(P\left(e^{cX} > s\right)\right)\right] ds\right] + b.$$

If $w \leq 0$ or $\bar{g} = h$, then $H(X)$ is translation invariant because $e^{c(w-b)} < 1$ for $b \geq 0$ and $e^{cX} \geq 1$. Let $w > 0$ and $\bar{g}(z) \neq h(z)$ for some z. For random variable X such that $P(X = s_0) = z = 1 - P(X = 0)$, where $0 < s_0 < w - b$ we get

$$\int_0^{e^{c(w-b)}} \left[\bar{g}\left(P\left(e^{cX} > s\right)\right) - h\left(P\left(e^{cX} > s\right)\right)\right] ds = \left(e^{cs_0} - 1\right)(\bar{g}(z) - h(z)) \neq 0,$$

which means that $H(X)$ is not translation invariant. An analogous proof can be carried out for $u(x) = (e^{cx} - 1)/d$. □

Proof of Theorem 1. Assume that $H(X + b) = H(X) + b$ for all $b \geq 0$. Consider $X \in \mathcal{X}_2^+$. Then from (5) under $w = 0$ we have

$$u(-H(X)) = E_{gh}u(-X) = -E_h(-u(-X))_+ = u(-s)h(q).$$

Thus

$$h(q) = \frac{u(-H(X))}{u(-s)}. \tag{15}$$

Since $H(X) = 0$ for $q = 0$ and $H(X) = s$ for $q = 1$, from the monotonicity and continuity of u and h it follows that $H(X)$ is continuous and non-decreasing function of variable q, thus it takes all values from $[0, s]$. From the translation invariance of $H(X)$, we can rewrite (5) for $X + b$ as

$$\begin{aligned} u(-H(X) - b) &= E_{gh}u(-X - b) = -E_h(-u(-X - b))_+ \\ &= u(-b)(1 - h(q)) + u(-s - b)h(q). \end{aligned} \tag{16}$$

Putting (15) into (16), denoting $x = -H(X)$, $y = -b$ and dividing both sides of obtained equation by $u(x)\,u(-s)$, we get

$$f(x, y) = f(-s, y) \tag{17}$$

for all $y \leq 0$, $s > 0$ and $-s \leq x \leq 0$, where $f(x, y) = (u(x + y) - u(y))/u(x)$. Setting $s = 1$ in (17) yields

$$f(x, y) = f(-1, y) \tag{18}$$

for $y \leq 0$, $-1 \leq x \leq 0$. Putting $x = -1$ in (17) gives

$$f(-s, y) = f(-1, y) \tag{19}$$

for $y \leq 0$ and $-s \leq -1$. From (18) and (19) we have $f(x, y) = f(-1, y)$ for all $x, y \leq 0$. Hence for a fixed y we have

$$u(x + y) = c(y)u(x) + u(y) \tag{20}$$

for all $x \leq 0$, where $c(y)$ is some function. By the symmetry we have

$$c(y)\,u(x) + u(y) = c(x)\,u(y) + u(x)$$

for $x, y \leq 0$, which is equivalent to

$$\big(c(y) - 1\big)\, u(x) \;=\; \big(c(x) - 1\big)\, u(y) \tag{21}$$

for $x, y \leq 0$. If $c(y) \neq 1$ for all $y < 0$, then

$$\frac{u(x)}{c(x) - 1} \;=\; \frac{u(y)}{c(y) - 1}$$

for $x, y \leq 0$. Hence $u(x) = d(c(x) - 1)$ for some $d > 0$ and from (20) we get

$$c(x + y) \;=\; c(x)\, c(y) \tag{22}$$

for $x, y \leq 0$. Since $u(x) = d(c(x) - 1)$ and $u \in \mathcal{U}$, it follows that c is continuous and increasing if $d > 0$. Thus the only solution of (22) is $c(x) = e^{ax}$ for $x \leq 0$ and some $a > 0$ (see Kuczma, 2009, p. 349). Hence the only solution of (20) is $u(x) = d(e^{ax} - 1)$ for $x \leq 0$. If $d < 0$, then using a similar reasoning we get $u(x) = (1 - e^{-ax})/d$, where $a > 0$. If $c(y) = 1$ for some y, then from (21) it follows that $c(x) = 1$ for $x \leq 0$. From (20) and continuity of u it follows that $u(x) = ax$ for $x \leq 0$.

We will show that u is linear or exponential on $\mathbb{R}$ and $h = \bar{g}$. Rewriting (5) for $X \in \mathcal{X}_2^+$, from the translation invariance of $H(X)$ we have

$$u\big(w - H(X) - b\big) \;=\; u(w - b)\, g(1 - q) + u(w - s - b)\, h(q) \tag{23}$$

if $b \leq w \leq b + s$ and

$$u\big(w - H(X) - b\big) \;=\; u(w - b)\,(1 - h(q)) + u(w - s - b)\, h(q) \tag{24}$$

for $b \geq w$. Let $u(x) = cx$ for $x < 0$. For $b > w$ from (24) we have $H(X) = sh(q)$. For $q_0 \in (0, 1)$ such that $h(q_0) = 1/2$, $s = 2(w - b)$ and $H(X) = sh(q_0)$ from (23) we get $u(x) = c_1 x$ for $x > 0$, where $c_1 = c/(2g(1 - q_0))$. Setting $s = w - b$ in (23) yields $\bar{g} = h$ and $c_1 = c$. Now, let $u(x) = (1 - e^{-cx})/a$ for $x < 0$. From (24) we have $H(X) = \frac{1}{c}\ln(e^{cs}h(q) + (1 - h(q)))$. Putting $s = \ln(2e^{c(w-b)} - 1)/c$ and the formula for $H(X)$ with q_0 into (23) gives $u(x) = (1 - e^{-cx})/a_1$ for $x > 0$, where $a_1 = 2ag(1 - q_0)$. In order to prove that $u(x) = (1 - e^{-cx})/a$ for $x \geq 0$ and $h = \bar{g}$ we use a similar reasoning as in the previous case. An analogous proof can be carried out for $u(x) = (e^{cx} - 1)/a$. □

Proof of Theorem 2. (i) If $u(x) = cx$, then from (6) for $a > 0$ we have

$$\begin{aligned} H(aX) \;&=\; E_{hg}(aX) + \int_0^w \big[\bar{g}\big(P(X > s/a)\big) - h\big(P(X > s/a)\big)\big]\, ds \\ &=\; aE_{hg}X + a\int_0^{w/a} \big[\bar{g}\big(P(X > s)\big) - h\big(P(X > s)\big)\big]\, ds. \end{aligned}$$

If $w = 0$ or $h = \bar{g}$, then $H(X)$ is scale invariant.

(ii) Assume that $H(aX) = aH(X)$. For $X \in \mathcal{X}_2^+$ from (5) under $w = 0$ we have

$$u(-ay) \;=\; h(q)\,u(-as) \tag{25}$$

for all $a > 0$, where $y = H(X)$. Setting $f(x) = -u(-x)$ and determining $h(q)$ from (25) with $a = 1$, we can rewrite (25) as

$$f(ay) \;=\; f(as)\,\frac{f(y)}{f(s)} \tag{26}$$

for all $s > 0$ and $0 \le y \le s$. If we put $s = 1$ in (26) and divide both sides of this equation by $u(-1)$, we get $z(ay) = z(a)z(y)$ for all $0 \le y \le 1$ and $a > 0$, where $z(x) = f(x)/(-u(-1))$. Setting $y = 1$ in (26) yields $z(a)z(s) = z(as)$ for $s \ge 1$ and $a > 0$. From the last two equations we have $z(ax) = z(a)z(x)$ for all $x > 0$ and $a > 0$. By the continuity of z we get $z(x) = x^d$ for all $x \ge 0$ and some $d > 0$ (see Kuczma, 2009, p. 349). Hence $u(x) = -c(-x)^d$ for all $x \le 0$, some $c = -u(-1) > 0$ and $d > 0$.

(iii) The formula for $u(x)$ if $x \le 0$ follows from (ii). Assume that $H(aX) = aH(X)$. Let X be such that $P(X = -s) = q = 1 - P(X = 0)$, where $s > 0$ and $q \in [0, 1]$ are arbitrary. From (5) under $w = 0$ we have

$$u(ay) \;=\; g(q)\,u(as) \tag{27}$$

for all $a > 0$, where $y = -H(X)$. If we determine $g(q)$ from (27) with $a = 1$ and put this expression into (27), we obtain again equation (26) with $f(x) = u(x)$. Hence $u(x) = ax^b$ for $x \ge 0$ and some $a, b > 0$.

(iv) From the scale invariance of $H(X)$ and (5) we have

$$u\big(w - aH(X)\big) = u(w)\,g(1-q) + u(w - as)\,h(q) \tag{28}$$

when $a \ge w/s$ and

$$u\big(w - aH(X)\big) = u(w)\,g(1-q) + u(w - as)\big(1 - g(1-q)\big) \tag{29}$$

if $0 \le a < w/s$. Setting $as = 2w$ in (28) and choosing $q_0 \in [0,1]$ such that $H(X) = s/2$, from (ii) it follows that $u(w) = cw^d h(q_0)/(1 - \bar{g}(q_0))$. Thus $u(x) = c_1 x^d$ for $x \ge 0$ and some $c_1 > 0$. Putting this into (29), differentiating both sides of (29) with respect to a and setting $a = 0$ we get $H(X) = s\bar{g}(q)$. If we put $H(X) = s\bar{g}(q)$, $a = 1$ and $w = s$ into (29), we obtain $(1 - \bar{g}(q))^d = (1 - \bar{g}(q))$. Thus $d = 1$. Putting $w = 0$ into (28) yields $H(X) = sh(q)$. Since $H(X) = s\bar{g}(q)$, we have $\bar{g} = h$. As $c_1 = ch(q_0)\,/\,(1 - \bar{g}(q_0))$ and $h(q_0) = 1/2$, we obtain $c = c_1$. □

Proof of Theorem 3. (i) Let $u(x) = cx$. If $h = \bar{g}$, then $H(X) = E_g X$ and the additivity of the Choquet integral for comonotonic risks ends the proof of (i). (ii) If $H(X)$ is additive for comonotonic risks, then it is scale invariant. From Theorem 2 it follows that $u(x) = cx$ and $\bar{h} = g$. □

Remark 1. *From the proofs of Theorem 2 (iv) and Theorem 3 (ii) it follows that it is enough to assume that* $X \in \mathcal{X}_2^+$.

Lemma 2. *If the domain of u is* $\left[0, \frac{1}{2}\right]$, *then the solution of*

$$u(2x) = 2u(x) \tag{30}$$

is $u(x) = xh(\ln x)$, *where h is a periodic function with period* $\ln 2$ *and* $0 \cdot h(-\infty) = 0$. *If we additionally assume that u has right-sided derivative at* $x = 0$ *(we allow* $u^+(0) = \infty$*), then the only solution is* $u(x) = cx$ *for some* $c > 0$.

Proof. Putting $x = 0$ in (30) implies $u(0) = 0$. Let u be a solution of (30). It is easy to check that $h(t) = e^{-t}u(e^t)$ is periodic with period $\ln 2$. Setting $x = e^t$ we have $u(x) = xh(\ln x)$ for $x > 0$, where h is an arbitrary periodic function with period $\ln 2$. Since u has the right-sided derivative at $x = 0$ and

$$u'(0) = \lim_{x \to 0^+} \frac{u(x)}{x} = \lim_{x \to 0^+} h(\ln x),$$

h is constant, as a periodic function which has the limit in $-\infty$. □

Proof of Theorem 4. (i) The proof will be carried out basing on the idea by Gerber (1979). Let X be an arbitrary risk and Y be constant, i.e. $P(Y = d) = 1$ for some $d > 0$. As $H(X)$ satisfies no unjustified risk loading, then from the additivity for independent risk it follows that $H(X)$ is translation invariant. From Theorem 1 we conclude that $u \in \mathcal{U}_0$.

(ii) Let $u(x) = cx$ and $w = 0$. Assume that $H(X)$ is additive for independent risks. Assume that $X, Y \in \mathcal{X}_2^+$ are independent random variables such that $P(X = 1) = p$, $P(Y = 1) = q$. Then

$$H(X) = h(p), H(Y) = h(q), \tag{31}$$

$$H(X + Y) = h(p + q - pq) + h(pq). \tag{32}$$

Since $H(X)$ is additive for independent risk, from (31) and (32) it follows that

$$h(p + q - pq) + h(pq) = h(p) + h(q) \tag{33}$$

for all $0 \le p, q \le 1$. Put $q = c - p$, where $0 \le c \le 1$. Let $(p_n)_{n \in \mathbb{N}}$ be the sequence such that $p_0 = \frac{c}{2}$ and $p_{n+1} = p_n(c - p_n)$. Then $(p_n)_{n \in \mathbb{N}}$ is generated by logistic difference equation (see Polyanin, Manzhirov, 2007, p. 875). From (33) we have

$$h(c-p_{n+1})+h(p_{n+1}) = h(c-p_n)+h(p_n) = \ldots = 2h(c/2). \tag{34}$$

As $p_{n+1}/c = c\cdot p_n/c\cdot(1-p_n/c)$, where $c \le 1$, thus $\lim_{n\to\infty}\frac{p_n}{c}=0$. Hence $\lim_{n\to\infty}p_n=0$. Function h is continuous at 0 and at 1, thus letting $n\to\infty$ in (34) yields $h(c) = 2h(c/2)$ for all $0\le c\le 1$. Since h has right-sided derivative at 0 (we allow $h'(0)=\infty$), from Lemma 2 it follows that $h(p)=p$. We will prove that $g(p)=p$. Let $w>0$ be such that $H(X)$ is additive for independent risks. Let $X, Y\in \mathcal{X}_2^+$ be independent random variables such that $P(X=2w/3)=1$, $P(Y=2w/3)=q$. Then for $h(x)=x$ from (6) we have

$$H(X) = \frac{2}{3}w,\ H(Y) = \frac{2}{3}qw + \int_0^w[\bar g(q)-q]\mathbf{1}_{[0,2w/3]}(s)\,ds, \tag{35}$$

$$H(X+Y) = \frac{2}{3}w(1+q) + \int_0^w[\bar g(q)-q]\mathbf{1}_{[2w/3,\,4w/3]}(s)\,ds. \tag{36}$$

From (35), (36) and the additivity for independent risks we get

$$\int_0^w[\bar g(q)-q]\mathbf{1}_{[0,2w/3]}(s)\,ds = \int_0^w[\bar g(q)-q]\mathbf{1}_{[2w/3,\,4w/3]}(s)\,ds.$$

Thus $2w(\bar g(q)-q) = w(\bar g(q)-q)$. Hence $\bar g(q)=q$ and finally $g(q)=q$.

Now, let $u(x)=(1-e^{-cx})/d$ and $X, Y\in\mathcal{X}_2^+$ be independent random variables such that $P(X=s)=p$, $P(Y=s)=q$. From (7) under $w=0$ we get

$$H(X) = \frac{1}{c}\ln[1-h(p)+h(p)e^{cs}],\ H(Y) = \frac{1}{c}\ln[1-h(q)+h(q)e^{cs}],$$

$$H(X+Y) = \frac{1}{c}\ln[1-h(p+q-pq)+e^{cs}(h(p+q-pq)-h(pq))+h(pq)e^{2cs}].$$

Additivity for independent risks yields

$$(1-h(p))(1-h(q))+e^{cs}\big(h(p)+h(q)-2h(p)h(q)\big)+e^{2cs}h(p)h(q)$$
$$= 1-h(p+q-pq)+e^{cs}\big(h(p+q-pq)-h(pq)\big)+h(pq)e^{2cs}.$$

We obtained equality of two polynomials of variable e^{cs}. If we compare coefficients of these polynomials, we will get (33). Thus $h(x)=x$. Let $w>0$ be such that $H(X)$ is additive for independent risks. For $p=1$ we have

$$H(X)=s,\ H(Y) = \frac{1}{c}\ln\left[1-q+qe^{cs}-\int_0^{e^{cw}}[q-\bar g(q)]\mathbf{1}_{[0,e^{cs}]}(t)\,dt\right], \tag{37}$$

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$$H(X+Y) = \frac{1}{c}\ln\left[e^{cs}(1-q) + e^{2cs}q - \int_0^{e^{cw}} [q - \bar{g}(\bar{q})]\mathbf{1}_{[e^{cs},\, e^{2cs}]}(t)\,dt\right]. \quad (38)$$

From (37), (38) and the additivity for independent risks it follows that

$$\int_0^{e^{cw}} [q - \bar{g}(q)]\mathbf{1}_{[0,\, e^{cs}]}(t)\,dt = \int_0^{e^{cw}} [q - \bar{g}(q)]\mathbf{1}_{[e^{cs},\, e^{2cs}]}(t)\,dt.$$

Thus $\bar{g}(q) = q$ and finally $g(q) = q$. An analogous proof can be carried out for $u(x) = (e^{cx} - 1)/d$. □

Proof of Theorem 5. Let $u(x) = cx$. Then from (6) we have $H(X) = E_{\bar{g}}X$. Assume that g is convex. It is known that $E_g(X + Y) \leq E_g X + E_g Y$ if and only if g is concave (see Denneberg, 1994). Thus for $\bar{g}$, which is concave, we have

$$H(X+Y) = E_{\bar{g}}(X+Y) \leq E_{\bar{g}}X + E_{\bar{g}}Y = H(X) + H(Y). \quad (39)$$

Assume now that $H(X)$ is subadditive. Then (39) holds, hence $\bar{g}$ is concave. □

Proof of Theorem 6. Let $\bar{g} = h$ and $X \leq_{sl} Y$. Then $E_{\bar{g}}[-u(w - X)] \leq E_{\bar{g}}[-u(w - Y)]$. Hence $E_g[u(w - X)] \geq E_g[u(w - Y)]$, because $E_{\bar{g}}[-u(X)] = -E_g[u(X)]$. By the definition of $H(X)$ we have

$$u(w - H(Y)) = E_g[u(w - Y)] \leq E_g[u(w - X)] = u(w - H(X)).$$

From the monotonicity of u we get $H(Y) \geq H(X)$. □

Proof of Proposition 3. From W9 we have

$$u(w - H(X)) = E_{gh}(u(w - X)) \leq u(E_{gh}(w - X)),$$

Thus from W3 and (1) we get

$$H(X) \geq E_{hg}X + \int_0^{w} [\bar{g}(P(X > s)) - h(P(X > s))]\,ds.$$

Since $\bar{g}(x) \geq h(x) \geq x$, from W5 it follows that $H(X) \geq E_{hg}X \geq E(X)$. □

Proof of Proposition 4. Put $Y = 0$ if $X < w$, $Y = w$ when $w \leq X < s$ and $Y = s$ if $X = s$. Since $Y \leq X$, from the monotonicity of the generalized Choquet integral we obtain

$$u(w-H(X)) = E_{gh}\, u(w-X) \le E_{gh}\, u(w-Y)$$
$$= g(P(X<w))\, u(w) + h(P(X=s))\, u(w-s).$$

From the above and (10) it follows that $H(X) \ge E(X)$. □

Proof of Proposition 5. Put $Y = 0$ for $X < w$ and $Y = w$ for $X \ge w$. Then $X = Y$ if and only if $P(X = 0) + P(X = w) = 1$. Since $Y \le X$, then

$$u(w-H(X)) = E_{gh} u(w-X) \le E_g u(w-Y)$$
$$= g(P(X<w))\, u(w).$$

From the above and (11) we have $H(X) \ge E(X)$. □

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