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THE MAXIMAL IDEAL IN THE SPACE OF OPERATORS ON $(\sum \ell_q)_{c_0}$

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Abstract

We study the isomorphic structure of $(\sum \ell_q)_{c_0}$ $(1 < q < \infty)$ and prove that these spaces are complementably homogeneous. We also show that for any operator T from $(\sum \ell_q)_{c_0}$ into ℓ_q, there is a subspace X of $(\sum \ell_q)_{c_0}$ that is isometric to $(\sum \ell_q)_{c_0}$ and the restriction of T on X has small norm. If T is a bounded linear operator on $(\sum \ell_q)_{c_0}$ which is $(\sum \ell_q)_{c_0}$-strictly singular, then for any $\epsilon > 0$, there is a subspace X of $(\sum \ell_q)_{c_0}$ which is isometric to $(\sum \ell_q)_{c_0}$ with $\|T|_X\| < \epsilon$. As an application, we show that the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ forms the unique maximal ideal of $\mathcal{L}((\sum \ell_q)_{c_0})$.

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1. Introduction

Let X be a Banach space and $\mathcal{L}(X)$ be the space of bounded linear operators on X. The question of determining maximal ideals of $\mathcal{L}(X)$ has been studied intensively in the past twenty years. It is well known that the set of compact operators is the unique maximal ideal of $\mathcal{L}(X)$ when $X = c_0$ or ℓ_p $(1 \le p < \infty)$ [5]. In these cases, the set of compact operators coincides with the set

$$\mathcal{M}_X = \{T \in \mathcal{L}(X) : I_X \text{ does not factor through } T\}.$$

There are many other Banach spaces X for which $\mathcal{M}_X$ is the unique maximal ideal of $\mathcal{L}(X)$, including $L_p(0,1)$ $(1 \le p < \infty)$ [4], ℓ_∞ [3], $(\sum_{n=1}^\infty \ell_2^n)_{c_0}, (\sum_{n=1}^\infty \ell_2^n)_{\ell_1}, (\sum_{n=1}^\infty \ell_1^n)_{c_0}$, $(\sum_{n=1}^\infty \ell_\infty^n)_{\ell_1}, (\sum_{n=1}^\infty \ell_\infty^n)_{\ell_p}$ $(1<p<\infty)$ [13, 8, 10–12], $(\sum \ell_q)_{\ell_p}$ $(1 \le q < p < \infty)$ [2], $(\sum \ell_q)_{\ell_1}$ $(1 < q < \infty)$ [16], $d_{w,p}$ [7] and an Orlicz sequence space which is close to ℓ_p [14].

The main purpose of this paper is to show that $\mathcal{M}_X$ is also the unique maximal ideal in $\mathcal{L}(X)$ when $X = (\sum \ell_q)_{c_0}$ $(1 < q < \infty)$. A key step is to prove that $(\sum \ell_q)_{c_0}$ $(1 < q < \infty)$ is complementably homogeneous. Recall that a Banach space X is called complementably homogeneous [2] if every subspace Y of X that is isomorphic to X contains a further subspace isomorphic to X and complemented in X.

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THEOREM 1.1. *Let $1 < q < \infty$ and let X be a subspace of $(\sum \ell_q)_{c_0}$ which is C-isomorphic to $(\sum \ell_q)_{c_0}$. Then for any $\epsilon > 0$, there is a subspace Y of X which is $(C+\epsilon)$-isomorphic to $(\sum \ell_q)_{c_0}$ and $(C+\epsilon)$-complemented in $(\sum \ell_q)_{c_0}$.*

Our second result is that, for any operator from $(\sum \ell_q)_{c_0}$ into ℓ_q, there is a subspace of $(\sum \ell_q)_{c_0}$ that is isometric to $(\sum \ell_q)_{c_0}$ and the restriction of the operator on this subspace has small norm.

THEOREM 1.2. *Let $1 < q < \infty$ and let $T : (\sum \ell_q)_{c_0} \to \ell_q$ be a bounded linear operator. Then for any $\epsilon > 0$, there exists a subspace X of $(\sum \ell_q)_{c_0}$ such that X is isometric to $(\sum \ell_q)_{c_0}$ with $\|T|_X\| < \epsilon$.*

A further result follows from Theorems 1.1 and 1.2.

THEOREM 1.3. *Let $1 < q < \infty$ and let T be a bounded linear operator on $(\sum \ell_q)_{c_0}$ which is $(\sum \ell_q)_{c_0}$-strictly singular. Then for any $\epsilon > 0$, there is a subspace X of $(\sum \ell_q)_{c_0}$ which is isometric to $(\sum \ell_q)_{c_0}$ and $\|T|_X\| < \epsilon$.*

As an application, we derive the following corollary.

COROLLARY 1.4. *For $1 < q < \infty$, the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is the unique maximal ideal in the space $\mathcal{L}((\sum \ell_q)_{c_0})$.*

2. Operators on $(\sum \ell_q)_{c_0}$

Let X be a Banach space with a Schauder basis (e_i) and let S_X denote the unit sphere of X. A sequence (x_i) of nonzero vectors in X is a *block basic sequence* of (e_i) if there exists a sequence of strictly increasing integers (N_i) with $N_0 = 0$ and a sequence of real numbers (a_i) so that $x_i = \sum_{j=N_{i-1}+1}^{N_i} a_j e_j$ for every $i \in \mathbb{N}$. A *block subspace* of X is the closed linear span of a block basic sequence in X. A bounded linear operator between two Banach spaces X and Y is an *isomorphism* if there exists a $\delta > 0$ such that $\|Tx\| > \delta$ whenever $x \in X$ and $\|x\| = 1$. For $C \geq 1$, X and Y are *C-isomorphic* if there exists an isomorphism T from X onto Y so that $\|T\|\,\|T^{-1}\| \leq C$. When the isomorphic constant C is not relevant, we simply say X and Y are *isomorphic*. Two sequences $(x_i) \subset X$ and $(y_i) \subset Y$ are *equivalent* if there exists a constant $C \geq 1$ such that for all sequences of real numbers (a_i),

$$C^{-1}\Big\|\sum a_i y_i\Big\| \leq \Big\|\sum a_i x_i\Big\| \leq C\Big\|\sum a_i y_i\Big\|.$$

If $T : X \to Y$ is an operator between Banach spaces and Z is a subspace of X, define

$$f(T,Z) = \inf\{\|Tz\| : \|z\| = 1, z \in Z\} \quad (= \|(T|_Z)^{-1}\|^{-1}).$$

Then, $f(T,Z) > 0$ if and only if $T|_Z$ is an isomorphism, $f(T,Z) = \|T\| > 0$ if and only if $T|_Z$ is a multiple of an isometry and $\|T\| \geq f(T,Z_1) \geq f(T,Z_2)$ if $Z_1 \subset Z_2 \subset X$.

When $X = (\sum \ell_q)_{c_0}$, we use $\ell_q^{(n)}$ to denote the nth ℓ_q in the corresponding direct sum and for $x = (x_1, x_2, x_3, \ldots) \in X$, we define $\|x\| = \sup_i\{\|x_i\|_{\ell_q}\}$. Other notations and definitions can be found in [1, 15].

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Let X, Y and Z be Banach spaces. A bounded linear operator $T : X \to Y$ is *Z-strictly singular* if there is no subspace $Z_0 \subset X$ which is isomorphic to Z and such that $T|_{Z_0}$ is an isomorphism onto its range; T is *strictly singular* if there is no infinite-dimensional subspace $Z_0 \subset X$ such that $T|_{Z_0}$ is an isomorphism onto its range. So an operator is *strictly singular* if and only if it is *Z-strictly singular* for every infinite-dimensional space Z. (See [2–3, 16, 17] for more details on this topic.)

The proof of the next lemma is similar to the proof of Lemma 2.2 in [2]. An important ingredient is that if X is a subspace of $(\sum \ell_q)_{c_0}$ which is isomorphic to ℓ_q, then there exists a subspace Y of X so that Y is almost isometric to ℓ_q. That is, for any $\epsilon > 0$, there exists a subspace Y of X which is $(1+\epsilon)$-isomorphic to ℓ_q. This fact can be derived using the techniques in [6, 9].

LEMMA 2.1. *Let $1 < q < \infty$ and let $T : \ell_q \to (\sum \ell_q)_{c_0}$ be a bounded linear operator. Then for any $\epsilon > 0$, there exists a block subspace Z of ℓ_q so that $\|T|_Z\| < f(T, Z) + \epsilon$.*

PROOF. We divide the proof into two parts.

Case 1: T is a strictly singular operator. Then $f(T, Z) = 0$ for all infinite-dimensional subspaces $Z \subset \ell_q$. Let $\epsilon > 0$ and choose $\epsilon_i > 0$ such that $\sum \epsilon_i < \epsilon$. Let $(e_i)_{i=1}^{\infty}$ be the unit vector basis of ℓ_q. Since $f(T, \ell_q) = 0$, we can pick a norm one element x_1 from ℓ_q such that $\|Tx_1\| < \epsilon_1/2$. If $x_1 = \sum_{i=1}^{\infty} a_{1,i}e_i$, then we can choose $n_1 \in \mathbb{N}$ and define $y_1 = \sum_{i=1}^{n_1} a_{1,i}e_i$ so that $\|y_1\| > 1/2$ and $\|Ty_1\| < \epsilon_1/2$. Let $Z_1 = [(e_i)_{i=n_1+1}^{\infty}]$. Since $f(T, Z_1) = 0$, we can pick a norm one element x_2 from Z_1 such that $\|Tx_2\| < \epsilon_2/2$. If $x_2 = \sum_{i=n_1+1}^{\infty} a_{2,i}e_i$, then we can choose $n_2 \in \mathbb{N}$ and define $y_2 = \sum_{i=n_1+1}^{n_2} b_ie_i$ such that $\|y_2\| > 1/2$ and $\|Ty_2\| < \epsilon_2/2$. Define $Z_2 = [(e_i)_{i=n_2+1}^{\infty}]$. Continuing in this way, we obtain a block basic sequence (y_i) of (e_i) such that $\|y_i\| > 1/2$ and $\|Ty_i\| < \epsilon_i/2$ for all i. Let $Z = [(y_i)]$. Then Z is a block subspace of ℓ_q and hence isometric to ℓ_q. For any $z = \sum_{i=1}^{\infty} b_i(y_i/\|y_i\|)$ in S_Z, we have $|b_i| \le 1$ and

$$\|Tz\| = \left\| T \sum_{i=1}^{\infty} b_i\left(\frac{y_i}{\|y_i\|}\right)\right\| = \left\| \sum_{i=1}^{\infty} \frac{b_i}{\|y_i\|} T(y_i)\right\| \le 2\sum_{i=1}^{\infty} \|Ty_i\| < \sum_{i=1}^{\infty} \epsilon_i < \epsilon.$$

Since $f(T, Z) = 0$, we have $\|T|_Z\| < f(T, Z) + \epsilon$.

Case 2: T is not strictly singular. Then there is an infinite-dimensional subspace $Z_1 \subset \ell_q$ such that $T|_{Z_1}$ is an isomorphism onto its range. By [1, Theorem 2.2.1], Z_1 contains a closed subspace Z_2 which is isomorphic to ℓ_q. Using the fact that a subspace of $(\sum \ell_q)_{c_0}$ which is isomorphic to ℓ_q contains a smaller subspace almost isometric to ℓ_q, we deduce that $T(Z_2)$ contains a subspace Z_3 almost isometric to ℓ_q. Since $\epsilon > 0$, there is enough room for a small perturbation, so the problem reduces to the case where T maps ℓ_q into an isometric copy Y of ℓ_q.

Since T is bounded, $(Te_n)_{n=1}^{\infty}$ converges weakly to zero. By passing to a subsequence of $(e_n)_{n=1}^{\infty}$ and perturbing, we can assume that $(Te_n)_{n=1}^{\infty}$ is disjointly supported in Y. Let $\liminf_{n\to\infty} \|Te_n\| = \delta > 0$. Then by passing to a further subsequence of $(e_n)_{n=1}^{\infty}$ and perturbing again, we can assume that $\lim_{n\to\infty} \|Te_n\| = \delta$ and $\delta - \epsilon/2 < \|Te_n\| < \delta + \epsilon/2$ for all n and $Z = [(e_n)]$ is a block subspace of ℓ_q.

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Let $x = \sum_{n=1}^{\infty} a_n e_n \in Z$ with $\sum_{n=1}^{\infty} |a_n|^q = 1$. Then

$$\Big\|T\Big(\sum_{n=1}^{\infty} a_n e_n\Big)\Big\| = \Big(\sum_{n=1}^{\infty} |a_n|^q \|Te_n\|^q\Big)^{1/q} > \delta - \epsilon/2.$$

Hence, $f(T,Z) \geq \delta - \epsilon/2$ and $\delta \leq f(T,Z) + \epsilon/2$. However,

$$\Big\|T\Big(\sum_{n=1}^{\infty} a_n e_n\Big)\Big\| = \Big(\sum_{n=1}^{\infty} |a_n|^q \|Te_n\|^q\Big)^{1/q} < \delta + \epsilon/2.$$

So,

$$\|T|_Z\| \leq \delta + \epsilon/2 \leq f(T,Z) + \epsilon/2 + \epsilon/2 = f(T,Z) + \epsilon. \qquad \square$$

Next we will use Lemma 2.1 to prove Theorem 1.2.

Proof of Theorem 1.2. First, we prove the theorem for the case when there is an infinite subset $M \subset \mathbb{N}$ such that $T|_{\ell_q^{(n)}}$ is strictly singular for all $n \in M$. Hence, $f(T,Z) = 0$ for any infinite-dimensional subspace Z of $\ell_q^{(n)}$. In particular, $f(T, \ell_q^{(n)}) = 0$. Now, let $\epsilon > 0$ and let $(\delta_n)_{n=1}^{\infty}$ be a sequence of positive reals decreasing to zero so that $\sum_{n\in M} \delta_n < \epsilon$. For each $n \in M$, choose $(\epsilon_{n,i})_{i=1}^{\infty}$ converging to zero so fast that $\sum_{i=1}^{\infty} \epsilon_{n,i} < \delta_n$. Fix $n \in M$ and pick a norm one element $x_1 = \sum_{i=1}^{\infty} a_{1,i} e_{n,i} \in \ell_q^{(n)}$ such that $\|Tx_1\| < \epsilon_{n,1}/2$. Choose $N_1 \in \mathbb{N}$ and define $y_1 = \sum_{i=1}^{N_1} a_{1,i} e_{n,i}$ so that $\|y_1\| > 1/2$ and $\|Ty_1\| < \epsilon_{n,1}/2$.

Let $Z_1 = [(e_{n,i})_{i=N_1+1}^{\infty}]$. Since $f(T,Z_1) = 0$, we can pick $x_2 = \sum_{i=N_1+1}^{\infty} a_{2,i} e_{n,i} \in Z_1$ with norm one such that $\|Tx_2\| < \epsilon_{n,2}/2$. Then we can find $N_2 \in \mathbb{N}$ such that $y_2 = \sum_{i=N_1+1}^{N_2} a_{2,i} e_{n,i}$, $\|y_2\| > 1/2$ and $\|Ty_2\| < \epsilon_{n,2}$. Let $Z_2 = [(e_{n,i})_{i=N_2+1}^{\infty}]$. Continuing in this way, we obtain a block basic sequence $(y_i)_{i=1}^{\infty}$ of the canonical basis of $\ell_q^{(n)}$. Let $X_n = [(y_i)]$. Then X_n is a block subspace of $\ell_q^{(n)}$ which is isometrically isomorphic to $\ell_q^{(n)}$ and it is easy to check that $\|T|_{X_n}\| < \delta_n$ and $X = \sum_{n\in M} X_n$ is isometrically isomorphic to $(\sum \ell_q)_{c_0}$. Moreover,

$$\|T|_X\| = \Big\|T|_{\sum_{n\in M} X_n}\Big\| = \Big\|\sum_{n\in M} T|_{X_n}\Big\| \leq \sum_{n\in M} \|T|_{X_n}\| < \sum_{n\in M} \delta_n < \epsilon.$$

This completes the proof for the particular case.

Now, suppose that $T|_{\ell_q^{(n)}}$ is not strictly singular for all but finitely many $n \in \mathbb{N}$. Discarding those finitely many $n \in \mathbb{N}$, we get a sequence of operators $\{T|_{\ell_q^{(n)}}\}_{n\in I}$ which are not strictly singular. Hence for each $n \in I$, there exists an infinite-dimensional subspace $Z_{n,1}$ of $\ell_q^{(n)}$ such that $T|_{Z_{n,1}}$ is an isomorphism. By [1, Theorem 2.2.1], $Z_{n,1}$ contains a subspace $Z_{n,2}$ which is isomorphic to ℓ_q. Let $(x_i)_{i=1}^{\infty}$ be a unit vector basis of $Z_{n,2}$ equivalent to the canonical basis of $\ell_q^{(n)}$. Then, $(x_i)_{i=1}^{\infty}$ converges weakly to zero. Passing to a subsequence and doing a small perturbation, without loss of generality, we may assume $(x_i)_{i=1}^{\infty}$ is a block basis of $\ell_q^{(n)}$. Hence, $Z_{n,3} = [(x_i)_{i=1}^{\infty}]$ is a block subspace of $\ell_q^{(n)}$ which is isometrically isomorphic to $\ell_q^{(n)}$. Since $T|_{Z_{n,3}}$ is an isomorphism, by

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Lemma 2.1, we get a block subspace Z_n of $Z_{n,3}$ such that

$$\|T|_{Z_n}\| < f(T, Z_n) + 2^{-n}(\epsilon/2).$$

We claim that $\lim_{n\to\infty} f(T, Z_n) = 0$. Suppose this is not the case. Then, there exist a $\delta > 0$ and a sequence $(n_k)_{k=1}^{\infty} \subset \mathbb{N}$ such that $f(T, Z_{n_k}) > \delta$. For each $k \in \mathbb{N}$, choose $x_{n_k} \in Z_{n_k}$ with norm one such that $\|Tx_{n_k}\| \geq \delta$. Then, $(x_{n_k})_{k=1}^{\infty}$ is 1-equivalent to the canonical basis of c_0. Since T is bounded, $(T(x_{n_k}))_{k=1}^{\infty}$ is weakly null. Passing to a subsequence and doing a small perturbation again, we may assume that $(Tx_{n_k})_{k=1}^{\infty}$ is a block basic sequence which is equivalent to the canonical basis of ℓ_q. This contradicts the boundedness of T. Therefore, $\lim_{n\to\infty} f(T, Z_n) = 0$. Choose a subsequence (Z_{n_k}) of (Z_n) so that $f(T, Z_{n_k}) < 2^{-(k+1)}\epsilon$. Let $X = \sum_{k=1}^{\infty} Z_{n_k}$. Then X is isometric to $(\sum \ell_q)_{c_0}$ and

$$\|T|_X\| = \|T|_{\sum_{k=1}^{\infty} Z_{n_k}}\| \leq \sum_{k=1}^{\infty} \|T|_{Z_{n_k}}\| < \sum_{k=1}^{\infty} 2^{-(k+1)}\epsilon + 2^{-n_k}(\epsilon/2) < \epsilon. \qquad \square$$

For $m, n \in \mathbb{N} \cup \{\infty\}$ with $m \leq n$, let $P_{[m,n]}$ denote the natural projection on $(\sum \ell_q)_{c_0}$ so that $P_{[m,n]}(\sum_{i=1}^{\infty} x_i) = \sum_{i=m}^{n} x_i$ whenever $\sum_{i=1}^{\infty} x_i \in (\sum \ell_q)_{c_0}$ with $x_i \in \ell_q^{(i)}$ for all i.

LEMMA 2.2. *Let $1 < q < \infty$ and $T : (\sum \ell_q)_{c_0} \to (\sum \ell_q)_{c_0}$ be a bounded linear operator. Then for all $m \in \mathbb{N}$,*

$$\lim_{n\to\infty} \|P_{[1,m]} T P_{[n,\infty)}\| = 0.$$

PROOF. We will prove this by contradiction. Noting that the sequence of norms is monotone in n, we suppose there exists $\delta > 0$ and $m_0 \in \mathbb{N}$, such that $\|P_{[1,m_0]} T P_{[n,\infty)}\| > \delta$ for every $n \in \mathbb{N}$. Then there is a sequence $(x_n) \in (\sum l_q)_{c_0}$ with $\|x_n\| = 1$, such that

$$\|P_{[1,m_0]} T P_{[n,\infty)} x_k\| \geq \delta \quad \text{for every } n \in \mathbb{N}.$$

Then, by passing to a subsequence $(P_{[n_k,\infty)} x_k)_{k=1}^{\infty}$ of $(P_{[n,\infty)} x_n)_{n=1}^{\infty}$ and doing a truncation, without loss of generality, we can assume $(P_{[n_k,\infty)} x_k)_{k=1}^{\infty}$ is a block basis which converges to zero weakly, but not in norm. Therefore, $(P_{[n_k,\infty)} x_{n_k})_{k=1}^{\infty}$ is equivalent to the canonical basis of c_0. However, $(P_{[1,m_0]} T P_{[n_k,\infty)} x_{n_k})_{k=1}^{\infty}$ converges to zero weakly in ℓ_q, but not in norm. Hence by passing to a further subsequence, we may assume that $(P_{[1,m_0]} T P_{[n_k,\infty)} x_{n_k})_{k=1}^{\infty}$ is equivalent to the canonical basis of ℓ_q. However, this contradicts the boundedness of T. $\square$

PROOF OF THEOREM 1.3. We will prove the theorem by considering two cases.

Case 1: There is an infinite subset $M \subset \mathbb{N}$ so that $T|_{\ell_q^{(n)}}$ is strictly singular for all $n \in M$. Since the proof of the first case of Theorem 1.2 does not use any property of the range space of T, it also works here.

Case 2: For all but finitely many $n \in \mathbb{N}$, $T|_{\ell_q^{(n)}}$ is not strictly singular.

Discarding finitely many $n \in \mathbb{N}$ and following the same line of proof as in Theorem 1.2, for each $n \in \mathbb{N}$, we can prove the existence of block subspaces $Z_n \subset \ell_q^{(n)}$ such that

$$\|T|_{Z_n}\| < f(T, Z_n) + 2^{-n}(\epsilon/2).$$

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We claim that $\lim_{n\to\infty} f(T, Z_n) = 0$. If not, then there exist a $\delta > 0$ and a sequence of numbers (n_k) such that $f(T, Z_{n_k}) > \delta$ which implies $T|_{Z_{n_k}}$ is an isomorphism. Consider the operator $T(\sum Z_{n_k})_{c_0} \to (\sum \ell_q)_{c_0}$. By passing to further subspaces of each Z_{n_k} and perturbing, we can assume that Tx_1 and Tx_2 are disjointly supported in $(\sum \ell_q)_{c_0}$ whenever $x_1 \in Z_{n_{k_1}}$, $x_2 \in Z_{n_{k_2}}$ and $k_1 \neq k_2$. Let $x = \sum_k x_k \in (\sum Z_{n_k})_{c_0}$ with $x_k \in Z_{n_k}$ and let $k_0 \in \mathbb{N}$ be such that $\|x_{k_0}\| \geq \frac{1}{2}\|x\|$. Then

$$\|Tx\| = \Big\|T\Big(\sum_k x_k\Big)\Big\| = \Big\|\sum_k Tx_k\Big\| \geq \|Tx_{k_0}\| \geq \delta\|x_{k_0}\| \geq \frac{\delta}{2}\|x\|.$$

Thus $T|_{(\sum Z_{n_k})_{c_0}}$ is an isomorphism, which contradicts the fact that T is $(\sum \ell_q)_{c_0}$-strictly singular on $(\sum \ell_q)_{c_0}$.

Since $f(T, Z_n)$ converges to zero, by passing to a subsequence of $(Z_n)_{n=1}^{\infty}$ and relabelling, we can assume that $f(T, Z_n) < 2^{-n}(\epsilon/2)$ for all $n \in \mathbb{N}$. Thus,

$$\|T|_{Z_n}\| < 2^{-n}(\epsilon/2) + 2^{-n}(\epsilon/2).$$

So $X = (\sum Z_n)$ is isometrically isomorphic to $(\sum \ell_q)_{c_0}$ and

$$\|T|_X\| = \|T|_{\sum Z_n}\| = \Big\|\sum T|_{Z_n}\Big\| \leq \sum \|T|_{Z_n}\| < \sum 2^{-n}\epsilon = \epsilon. \qquad \square$$

3. Maximal ideal of $\mathcal{L}((\sum \ell_q)_{c_0})$

In this section, we will prove that $(\sum \ell_q)_{c_0}$ is complementably homogeneous. The following two lemmas will be used in the proof.

LEMMA 3.1 (Johnson and Schechtman [2, Lemma 2.5]). *Suppose that X has an unconditionally monotone basis with p-convexity constant one and that $(x_k)_{k=1}^{n}$, for $n \in \mathbb{N} \cup \{\infty\}$, is a disjoint sequence in X so that for some θ with $0 < \theta < 1$ and all scalars (α_k),*

$$\theta\Big(\sum_k |\alpha_k|^p\Big)^{1/p} \leq \Big\|\sum_k \alpha_k x_k\Big\| \leq \Big(\sum_k |\alpha_k|^p\Big)^{1/p}.$$

Then there is an unconditionally monotone norm $!\cdot!$ on X with p-convexity constant one so that for all scalars (α_k),

(1) $\theta ! x ! \leq \|x\| \leq ! x !$ *for all* $x \in X$;
(2) $(\sum_k |\alpha_k|^p)^{1/p} = ! \sum_k \alpha_k x_k !$.

LEMMA 3.2 (Johnson and Schechtman [2, Lemma 2.6]). *Suppose that X has an unconditionally monotone basis with p-convexity constant one $(1 \leq p < \infty)$ and $(x_k)_{k=1}^{n}$, for $n \in \mathbb{N} \cup \{\infty\}$, is a disjoint sequence of unit vectors in X which is isometrically equivalent to the unit vector basis for ℓ_p. Then $\overline{\text{span}}\, x_k$ is norm one complemented in X.*

PROOF OF THEOREM 1.1. Let $\epsilon > 0$ be given and (ϵ_j) be a sequence of positive real numbers decreasing to 0 so fast that $\epsilon_j < \epsilon$ for each j. Write $X = \sum X_j$, where

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X is C-isomorphic to $(\sum \ell_q)_{c_0}$ and each X_j maps onto ℓ_q under this isomorphism. By the stability of ℓ_q, by passing to a subspace for each X_j, we can assume that X_j is $(1 + \epsilon_j)$-isomorphic to ℓ_q. Let $(x_{i,j})_i$ be a normalised basis of X_j which is $(1 + \epsilon_j)$-equivalent to the canonical basis of ℓ_q. Again, by passing to a subspace for each X_j and perturbing, we can assume that X_j is a block subspace of $(\sum \ell_q)_{c_0}$. By passing to a further subspace and perturbing, we can assume that the X_j subspaces are disjointly supported with respect to the canonical basis of $(\sum \ell_q)_{c_0}$. Let $(e_{i,j})_{i,j}$ be the canonical basis of $(\sum \ell_q)_{c_0}$, where $(e_{i,j})_i$ is the standard basis for $\ell_q^{(j)}$ and define

$$J_j = \bigcup_{i=1}^{\infty} \text{Support}(x_{i,j}).$$

Define norm one projections

$$P_{J_j} : \Big(\sum \ell_q\Big)_{c_0} \to \Big(\sum \ell_q\Big)_{c_0} \quad \text{by } P_{J_j}(x) = \sum_{(i,j)\in J_j} a_{i,j} e_{i,j},$$

for all $x = \sum_{i,j} a_{i,j} e_{i,j} \in (\sum \ell_q)_{c_0}$. Define $A_j = [(e_{i,j})_{(i,j)\in J_j}]$ which has an unconditionally monotone basis with q-convexity constant one. Since Support$(X_j) \subset J_j$, X_j is a subspace of A_j. By Lemma 3.1, we can define a new norm $!\cdot!$ on A_j such that $!\cdot!$ is $(1 + \epsilon_j)$-equivalent to $\|\cdot\|$ and the sequence $(x_{i,j})_{i=1}^{\infty}$ under the new norm is 1-equivalent to the canonical basis of ℓ_q. By Lemma 3.2, there exists a projection $Q_j : A_j \to X_j$ with $!Q_j! = 1$. Since the formal identity $I : (A_j, !\cdot!) \to (A_j, \|\cdot\|)$ is an onto isomorphism with isomorphism constant $1 + \epsilon_j$, we see that X_j is also complemented in A_j under the original norm $\|\cdot\|$ and $\|Q_j\| \le 1 + \epsilon_j$. Now, consider the projection $\sum_{j=1}^{\infty} Q_j P_{J_j} : (\Sigma \ell_q)_{c_0} \to \Sigma X_i$. We have

$$\begin{aligned}\Big\|\sum_{j=1}^{\infty} Q_j P_{J_j}\Big\| &= \sup\Big\{\Big\|\sum_{j=1}^{\infty} Q_j P_{J_j} x\Big\| : \|x\| = 1\Big\}\\ &= \sup\Big\{C \sup_j \|Q_j P_{J_j} x\| : \|x\| = 1\Big\} < C(1 + \epsilon).\end{aligned}$$ □

PROOF OF COROLLARY 1.4. First, we show that the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is a linear subspace of $\mathcal{L}((\sum \ell_q)_{c_0})$. Let T and Q be two $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$. If $T + Q$ is not a $(\sum \ell_q)_{c_0}$-strictly singular operator on $(\sum \ell_q)_{c_0}$, then there exists a subspace X of $(\sum \ell_q)_{c_0}$, isomorphic to $(\sum \ell_q)_{c_0}$ such that $(T + Q)|_X$ is an isomorphism. Thus, there exists a $\delta > 0$ such that

$$\|(T + Q)(x)\| \ge \delta\|x\|, \quad \text{for all } x \in X.$$

Since T is $(\sum \ell_q)_{c_0}$-strictly singular on $(\sum \ell_q)_{c_0}$, by Theorem 1.3, there exists a subspace Y of X which is isomorphic to $(\sum \ell_q)_{c_0}$ such that $\|T|_Y\| < \delta/2$. Similarly, there exists a subspace Z of Y which is isomorphic to $(\sum \ell_q)_{c_0}$ such that $\|Q|_Z\| < \delta/2$. Now, for $z \in Z$, observe that

$$\|(T + Q)(z)\| \le \|T(z)\| + \|Q(z)\| < \delta\|z\|.$$

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This is a contradiction. Therefore, $T + Q$ is a $(\sum \ell_q)_{c_0}$-strictly singular operator on $(\sum \ell_q)_{c_0}$. It is easy to see that αT is a $(\sum \ell_q)_{c_0}$-strictly singular operator on $(\sum \ell_q)_{c_0}$ for all scalars α and the ideal property of the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators is also trivial.

Next we prove that the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is maximal. Let T be an operator in $\mathcal{L}((\sum \ell_q)_{c_0})$ which is not $(\sum \ell_q)_{c_0}$-strictly singular. Then, there exists a subspace X of $(\sum \ell_q)_{c_0}$, which is isomorphic to $(\sum \ell_q)_{c_0}$ such that $T|_X$ is an isomorphism. Hence by Theorem 1.1, the subspace TX contains a subspace Z which is isomorphic to $(\sum \ell_q)_{c_0}$ and complemented in $(\sum \ell_q)_{c_0}$. Let $Q_1 : Z \to (\sum \ell_q)_{c_0}$ be an onto isomorphism and let $P : (\sum \ell_q)_{c_0} \to Z$ be a continuous projection onto Z. Since Z is isomorphic to $(\sum \ell_q)_{c_0}$, $W = X \cap T^{-1}(Z)$ is isomorphic to $(\Sigma\ell_q)_{c_0}$. Let $Q_2 : (\Sigma\ell_q)_{c_0} \to W$ be defined by $Q_2 = (T|_W)^{-1} \circ Q_1^{-1}$. Then Q_2 is an onto isomorphism. By the definition of Q_2, the identity map on $(\sum \ell_q)_{c_0}$ is equal to $(Q_1 \circ P) \circ T \circ Q_2$. Since Q_2 and $Q_1 \circ P$ are in $\mathcal{L}((\sum \ell_q)_{c_0})$, the identity map belongs to in any ideal containing T. Hence, any ideal containing T must coincide with $\mathcal{L}((\Sigma\ell_q)_{c_0})$. Therefore, the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is the unique maximal ideal. □

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