Proceedings of the Royal Society of Edinburgh, **135A**, 1139–1167, *2005*

Existence, uniqueness and non-uniqueness of weak solutions of parabolic initial-value problems with discontinuous nonlinearities

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(MS received 21 September 2004; accepted 24 February 2005)

We deal with the initial-value problem for parabolic equations with discontinuous nonlinearities and establish the existence of its weak solution. Next, we show that for a suitable class of initial data, the weak solution is locally or globally unique in time. Lastly, we prove that there exist at least two different weak solutions in general if initial data do not belong to this class.

1. Introduction

In this paper we will study weak solutions of the initial-value problem

$$\left.\begin{aligned} u_t &= u_{xx} + f(u) - f(1)H(u-\lambda), && 0 < t < T,\ x \in \mathbb{R}, \\ u|_{t=0} &= u_0, && x \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where $0 < \lambda < 1$ is a constant, H is the function on $\mathbb{R}$ given by

$$H(u) = \begin{cases} 0 & \text{in } (-\infty, 0), \\ 1 & \text{in } (0, \infty) \end{cases} \quad \text{and} \quad 0 \leqslant H(0) \leqslant 1,$$

and f satisfies the following condition (see figures 1 and 2).

(A1) f is a Lipschitz continuous function on $\mathbb{R}$ and satisfies

$$f(0) = 0, \quad f(u) < 0 \text{ on } (0, \lambda] \quad \text{and} \quad f(u) - f(1) > 0 \text{ on } [\lambda, 1).$$

A problem like (1.1) arises as the model of best response dynamics in game theory [9]. A typical example of f in this model is $f(u) = -u$. Also, problem (1.1) with $f(u) = -u$ is a special case of the initial-value problem for the parabolic system:

$$\left.\begin{aligned} u_t &= u_{xx} - u + H(u-a) + v, \\ v_t &= bu - cv, \end{aligned}\right\} \tag{1.2}$$

with constants $0 < a < 1$, $b \leqslant 0$ and $c \geqslant 0$. Parabolic system (1.2) is a simplification by McKean [11] of the equations of FitzHugh [7] and Nagumo *et al.* [14],

$$\left.\begin{aligned} u_t &= u_{xx} + u(1-u)(u-a) + v, \\ v_t &= bu - cv, \end{aligned}\right\} \tag{1.3}$$

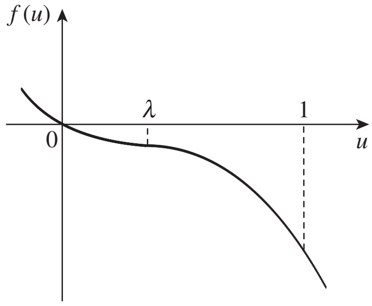

Figure 1. Condition (A1).

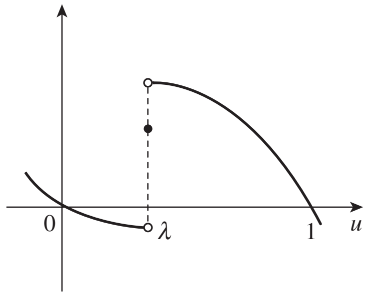

Figure 2. Graph of $u \mapsto f(u) - f(1)H(u - \lambda)$.

which were introduced as a model for the conduction of electrical impulses in the nerve axon.

As is well known, the method of upper and lower solutions combined with monotone iteration offers constructive existence results for a variety of differential equations with continuous nonlinearities. This method was first developed by Sattinger [15] for nonlinear parabolic boundary-value problems. Carl [2] showed that this method is available even for the construction of weak solutions belonging to the space $L^2((0,T); W^{1,2}(\Omega))$ of parabolic boundary-value problems with discontinuous nonlinearities, where Ω is a bounded domain of $\mathbb{R}^d$. A constructive existence result for parabolic boundary-value problems with more general forms was obtained by Carl and Heikkilä [3] in a similar space. Szilagyi [16] proved a constructive existence result for boundary-value problems for parabolic systems with discontinuous nonlinearities. For existence results for ordinary and elliptic equations with discontinuous nonlinearities, see Heikkilä and Lakshmikantham [8]. In this paper we are interested in bounded solutions taking values between 0 and 1 in $(0,T) \times \mathbb{R}$, for example, spatially constant solutions, from the viewpoint of best response dynamics. If there exists a solution $u \in C_{\mathrm{B}}([0,T) \times \mathbb{R})$ of problem (1.1), then, by a similar argument to the proof of lemma 2.5(a), the solution u satisfies the integral equation

$$u(t,x) = \int_{\mathbb{R}} K(t, x-y) u_0(y)\, \mathrm{d}y + \int_0^t \int_{\mathbb{R}} K(t-s, x-y)(f(u(s,y)) - f(1)H(u(s,y) - \lambda))\, \mathrm{d}y\, \mathrm{d}s \tag{1.4}$$

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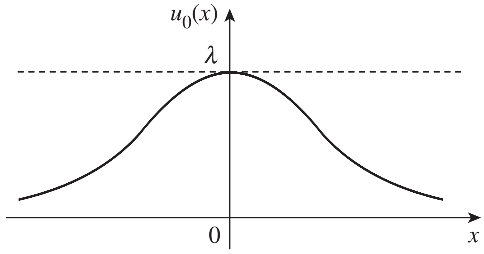

Figure 3. Graph of u_0.

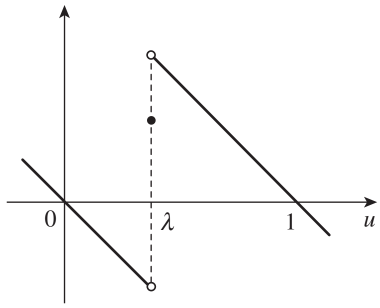

Figure 4. Graph of $u \mapsto -u + H(u - \lambda)$.

in $(0, T) \times \mathbb{R}$, where $C_B([0, T) \times \mathbb{R})$ is the space of bounded continuous functions on $[0, T) \times \mathbb{R}$, and K is the heat kernel. Hence, we find from this integral equation (1.4) that u belongs to the space $C^{0,1}((0, T) \times \mathbb{R})$ of continuous functions on $(0, T) \times \mathbb{R}$ that are continuously differentiable with respect to $x \in \mathbb{R}$. Therefore, we will study solutions of problem (1.1) in the space $C_B([0, T) \times \mathbb{R}) \cap C^{0,1}((0, T) \times \mathbb{R})$ under the following condition.

(B1) u_0 belongs to $C_B(\mathbb{R})$ and satisfies the inequality $0 \leqslant u_0(x) \leqslant 1$ on $\mathbb{R}$.

First, we will show that the monotone iterative method also is available for the construction of weak solutions belonging to the space $C_B([0, T) \times \mathbb{R}) \cap C^{0,1}((0, T) \times \mathbb{R})$ of problem (1.1). Key points of the proof are to show that weak solutions of an iteration scheme are expressed in explicit forms and, further, that an upper solution and a lower solution respectively satisfy integral inequalities.

Owing to the discontinuous nonlinearities, we cannot expect the uniqueness of weak solutions of problem (1.1) for general initial data. For example, for $f(u) = -u$, both

$$u_1(t) = (\lambda - 1)\mathrm{e}^{-t} + 1 \quad \text{and} \quad u_2(t) = \lambda \mathrm{e}^{-t}$$

are solutions if $u_0(x) = \lambda$ on $\mathbb{R}$. This corresponds to the fact that, in the best response dynamics, players have multiple locally best responses at the time $t = 0$ if the initial state is equal to λ. Secondly, we will find certain classes of initial data for which the weak solution of problem (1.1) is unique locally or globally in time. Several results on uniqueness of weak solutions of parabolic equations with discontinuous nonlinearities have been obtained for only one space-dimensional case. Feireisl and Norbury [6] and Feireisl [5] dealt with parabolic boundary-value problems. Feireisl

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and Norbury [6] showed that a certain class of initial data guarantees a uniqueness result for the nonlinearity $f(u) + cH(u-1)$, with a positive constant c and a non-decreasing Lipschitz continuous function f. Feireisl [5] considered more-general initial data and nonlinearities, but showed only a local uniqueness result. On the other hand, Terman [17] dealt with the initial-value problem (1.1) with $f(u) = -u$ (see figure 4), and showed the uniqueness of solutions for a certain class of initial data $u_0 \in C_{\mathrm{B}}(\mathbb{R}) \cap C^1(\mathbb{R})$ such that $u_0(0) > \lambda$, $u_0(x) = u_0(-x)$ in $\mathbb{R}$ and $u_0'(x) > 0$ in $(-\infty, 0)$. The case $u_0(0) = \lambda$ (see figure 3), in which the uniqueness of solutions was proved under the two conditions that $\liminf_{x\downarrow 0} 2x^{-2}[u_0(0) - u_0(x)] > 0$ and

$$\liminf_{x\downarrow 0} 2x^{-2}[u_0(0) - u_0(x)] + \lambda > \frac{1}{\sqrt{\pi}} \max_{p\geqslant 0} \int_0^1 \int_{-\sqrt{p}(1+\sqrt{\tau})/(2\sqrt{1-\tau})}^{-\sqrt{p}(1-\sqrt{\tau})/(2\sqrt{1-\tau})} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\tau, \tag{1.5}$$

was discussed by McKean and Moll [13]. McKean [12] showed the uniqueness of solutions of the initial-value problem for parabolic system (1.2) for a class of increasing initial data.

The main purpose of this paper is to study the non-uniqueness of weak solutions of problem (1.1). Let us consider problem (1.1) with $f(u) = -u$ (see figure 4) and initial data u_0 like figure 3. Then it is easy to see that problem (1.1) has a weak solution whose peak does not exceed λ. Furthermore, according to McKean and Moll [13], stated above, the weak solution is unique if $\liminf_{x\downarrow 0} 2x^{-2}[u_0(0) - u_0(x)] > 0$ and if inequality (1.5) holds. However, McKean and Moll [13] did not discuss in detail the case where inequality (1.5) does not hold, namely, the case that

$$\liminf_{x\downarrow 0} 2x^{-2}[u_0(0) - u_0(x)] + \lambda < \frac{1}{\sqrt{\pi}} \max_{p\geqslant 0} \int_0^1 \int_{-\sqrt{p}(1+\sqrt{\tau})/(2\sqrt{1-\tau})}^{-\sqrt{p}(1-\sqrt{\tau})/(2\sqrt{1-\tau})} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{1.6}$$

In this case we expect that there exists a weak solution whose peak exceeds λ. The existence of such a weak solution establishes the non-uniqueness of weak solutions, since problem (1.1) has a weak solution whose peak does not exceed λ, as stated above. Thirdly, we will consider problem (1.1) with more general f and initial data u_0, and will give a similar condition to (1.6) under which problem (1.1) has at least two different weak solutions. The non-uniqueness of weak solutions of parabolic boundary-value problems with discontinuous nonlinearities has been studied by Feireisl and Norbury [6] only for one space-dimensional case. They showed that a certain class of initial data guarantees a non-uniqueness result for the nonlinearity $f(u)+cH(u-1)$ with a positive constant c and a non-decreasing Lipschitz continuous function f satisfying $f(u) = 0$ for $u \leqslant 1$.

It should be mentioned that the treatment of discontinuous problems in unbounded domains is by no means a straightforward extension of corresponding problems in bounded domains, and in this sense the subject of the paper is challenging and worth pursuing.

The rest of this paper is organized as follows. In § 2, we first give the definition of a weak solution of problem (1.1) and, by using the monotone iterative method, we establish the existence of maximal and minimal weak solutions of problem (1.1) in an order interval (theorem 2.3). In § 3, we study the uniqueness of weak solutions of problem (1.1). In fact, we find a certain class of initial data so that the weak solution of problem (1.1) is unique locally in time (theorem 3.1). Furthermore, we prove two results on global uniqueness (theorems 3.3 and 3.4). In § 4, we prove that there exist at least two different weak solutions under some conditions (theorem 4.1). In § 5, we prove a result on the relationship between weak solutions of problem (1.1) and solutions of problem (1.1) formulated as a differential inclusion (proposition 5.4), from which we see that results similar to theorems 3.1, 3.3, 3.4 and 4.1 hold for solutions of problem (1.1) formulated as a differential inclusion (remark 5.6).

2. Existence theorem

Before we describe our results, let us first explain some notation and definitions that we will use. Let $C_{\mathrm{B}}(\mathbb{R})$ and $C_{\mathrm{B}}([0,T)\times\mathbb{R})$ denote the space of bounded continuous functions on $\mathbb{R}$ and $[0,T)\times\mathbb{R}$, respectively. Let $C^{0,1}((0,T)\times\mathbb{R})$ denote the space of continuous functions on $(0,T)\times\mathbb{R}$ that are continuously differentiable with respect to $x\in\mathbb{R}$. If $u,v\in C_{\mathrm{B}}([0,T)\times\mathbb{R})\cap C^{0,1}((0,T)\times\mathbb{R})$ and $u\leqslant v$ in $[0,T)\times\mathbb{R}$, then $[u,v]$ denotes the order interval $\{w\in C_{\mathrm{B}}([0,T)\times\mathbb{R})\cap C^{0,1}((0,T)\times\mathbb{R}) \mid u\leqslant w\leqslant v\}$. Let K denote the heat kernel, namely,

$$K(t,x)=\begin{cases}\dfrac{1}{2\sqrt{\pi t}}\exp\left(-\dfrac{x^2}{4t}\right), & \text{if } t>0 \text{ and } x\in\mathbb{R},\\ 0, & \text{if } t\leqslant 0 \text{ and } x\in\mathbb{R}.\end{cases}$$

We now define a weak solution of problem (1.1) as follows.

Definition 2.1. A function $u\in C_{\mathrm{B}}([0,T)\times\mathbb{R})\cap C^{0,1}((0,T)\times\mathbb{R})$ is said to be a *weak solution* of problem (1.1) if the following two conditions are satisfied:

(i) $u_t=u_{xx}+f(u)-f(1)H(u-\lambda)$ in $\mathcal{D}'((0,T)\times\mathbb{R})$, namely,

$$\int_0^T\int_{\mathbb{R}}(u\partial_t\varphi-\partial_x u\partial_x\varphi+(f(u)-f(1)H(u-\lambda))\varphi)\,\mathrm{d}x\,\mathrm{d}t=0,$$

for all $\varphi\in\mathcal{D}((0,T)\times\mathbb{R})$;

(ii) for all $x_0\in\mathbb{R}$,

$$\lim_{t\downarrow 0,x\to x_0}u(t,x)=u_0(x_0).$$

Definition 2.2. A function $u\in C_{\mathrm{B}}([0,T)\times\mathbb{R})\cap C^{0,1}((0,T)\times\mathbb{R})$ is called a *weak upper solution* of problem (1.1) if the following two conditions are satisfied:

(i′) for all non-negative functions $\varphi\in\mathcal{D}((0,T)\times\mathbb{R})$,

$$\int_0^T\int_{\mathbb{R}}(u\partial_t\varphi-\partial_x u\partial_x\varphi+(f(u)-f(1)H(u-\lambda))\varphi)\,\mathrm{d}x\,\mathrm{d}t\leqslant 0;$$

(ii′) for all $x_0\in\mathbb{R}$,

$$\lim_{t\downarrow 0,x\to x_0}u(t,x)\geqslant u_0(x_0).$$

A *weak lower solution* of problem (1.1) is defined by reversing the inequalities in the conditions (i′) and (ii′).

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Similar definitions to definitions 2.1 and 2.2 will be used for problem (1.1), with nonlinearities depending on t and x as well as u.

To show the existence of a weak solution of problem (1.1), we will impose the following condition on f.

(A2) Problem (1.1) has a weak upper solution $\bar{u}$ and a weak lower solution $\underline{u}$ such that $\underline{u} \leqslant \bar{u}$ in $(0,T) \times \mathbb{R}$. Furthermore, there exist a non-positive function $w_1 \in L^\infty((0,T) \times \mathbb{R})$ and a non-negative function $w_2 \in L^\infty((0,T) \times \mathbb{R})$ such that, for all non-negative functions $\varphi \in \mathcal{D}((0,T) \times \mathbb{R})$,

$$\int_0^T \int_{\mathbb{R}} (\bar{u}\partial_t \varphi - \partial_x \bar{u} \partial_x \varphi + (f(\bar{u}) - f(1)\tilde{H}(\bar{u} - \lambda) - w_1)\varphi) \,\mathrm{d}x \,\mathrm{d}t = 0,$$

$$\int_0^T \int_{\mathbb{R}} (\underline{u}\partial_t \varphi - \partial_x \underline{u} \partial_x \varphi + (f(\underline{u}) - f(1)\hat{H}(\underline{u} - \lambda) - w_2)\varphi) \,\mathrm{d}x \,\mathrm{d}t = 0,$$

where

$$\tilde{H}(u) = \begin{cases} 1, & \text{if } u \geqslant 0, \\ 0, & \text{if } u < 0, \end{cases} \quad \text{and} \quad \hat{H}(u) = \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{if } u \leqslant 0. \end{cases}$$

THEOREM 2.3. *Let $\lambda \in (0,1)$, and assume that conditions (A1), (A2) and (B1) are satisfied. There then exist the global maximal and minimal weak solutions U and u of problem (1.1) in the order interval $[\underline{u}, \bar{u}]$.*

For the proof, we need the following three lemmas.

LEMMA 2.4. *Let $u_0 \in C_{\mathrm{B}}(\mathbb{R})$ and let $g \in L^\infty((0,T) \times \mathbb{R})$. Then, for each $T > 0$, the problem*

$$\left.\begin{aligned} u_t &= u_{xx} + g(t,x), \qquad 0 < t < T, \quad x \in \mathbb{R}, \\ u|_{t=0} &= u_0, \qquad\qquad\qquad\qquad\quad x \in \mathbb{R}, \end{aligned}\right\} \tag{2.1}$$

has a unique weak solution $u \in C_{\mathrm{B}}([0,T) \times \mathbb{R}) \cap C^{0,1}((0,T) \times \mathbb{R})$, which is expressed in the form

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} K(t, x-y) u_0(y) \,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} K(t-s, x-y) g(s,y) \,\mathrm{d}y \,\mathrm{d}s \\ &=: A(t,x) + B(t,x). \end{aligned} \tag{2.2}$$

Moreover, we find that, for $0 < t < T$ and $x \in \mathbb{R}$,

$$|\partial_x A(t,x)| \leqslant \frac{1}{\sqrt{\pi t}} \|u_0\|_{L^\infty(\mathbb{R})}, \tag{2.3}$$

$$|\partial_t A(t,x)| \leqslant \frac{1}{t} \|u_0\|_{L^\infty(\mathbb{R})}, \tag{2.4}$$

$$|\partial_x B(t,x)| \leqslant 2\sqrt{\frac{t}{\pi}} \|g\|_{L^\infty((0,T)\times\mathbb{R})}, \tag{2.5}$$

and that, for $0 < r < t < T$ and $x \in \mathbb{R}$,

$$|B(t,x) - B(r,x)| \leqslant (2\sqrt{(t-r)r} + (t-r)) \|g\|_{L^\infty((0,T)\times\mathbb{R})}. \tag{2.6}$$

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Proof. We see that the function u given by (2.2) is a weak solution of problem (2.1), by using the fact that the heat kernel K satisfies the equation $K_t(t,x) = K_{xx}(t,x) + \delta(t,x)$ in $\mathcal{D}'((-T,T)\times\mathbb{R})$, where δ is the delta function. For the proof of uniqueness, see [1, theorem 2]. Thus, it remains to prove the four inequalities (2.3)–(2.6). Since the first three inequalities, (2.3)–(2.5), can be easily checked, the last one, (2.6), will be proved.

We can easily see that, for $0 < r < t < T$ and $x \in \mathbb{R}$,

$$\begin{aligned} |B(t,x) - B(r,x)| &\leqslant \int_0^r \int_{\mathbb{R}} |K(t-s,x-y) - K(r-s,x-y)||g(s,y)|\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_r^t \int_{\mathbb{R}} K(t-s,x-y)|g(s,y)|\,\mathrm{d}y\,\mathrm{d}s \\ &=: B_1(t,r,x) + B_2(t,r,x). \end{aligned} \tag{2.7}$$

(1) Estimate of $B_2(t,r,x)$: for $0 < r < t < T$ and $x \in \mathbb{R}$,

$$B_2(t,r,x) \leqslant (t-r)\|g\|_{L^\infty((0,T)\times\mathbb{R})}. \tag{2.8}$$

(2) Estimate of $B_1(t,r,x)$: we see that, for $0 < r < t < T$ and $x \in \mathbb{R}$,

$$\begin{aligned} &\int_0^r \int_{\mathbb{R}} |K(t-s,x-y) - K(r-s,x-y)|\,\mathrm{d}y\,\mathrm{d}s \\ &\quad = \int_0^r \int_{\mathbb{R}} \left| \int_r^t K_\tau(\tau-s,x-y)\,\mathrm{d}\tau \right| \mathrm{d}y\,\mathrm{d}s \\ &\quad \leqslant \int_0^r \int_{\mathbb{R}} \int_r^t \left(\frac{1}{4\pi(\tau-s)}\right)^{1/2} \frac{1}{2(\tau-s)} \left[1 + \frac{(x-y)^2}{2(\tau-s)}\right] \mathrm{e}^{-(x-y)^2/(4(\tau-s))}\,\mathrm{d}\tau\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

By the change of variable $x - y = 2\sqrt{\tau-s}\,\xi$,

$$\begin{aligned} &\int_0^r \int_{\mathbb{R}} \int_r^t \left(\frac{1}{4\pi(\tau-s)}\right)^{1/2} \frac{1}{2(\tau-s)} \left[1 + \frac{(x-y)^2}{2(\tau-s)}\right] \mathrm{e}^{-(x-y)^2/(4(\tau-s))}\,\mathrm{d}\tau\,\mathrm{d}y\,\mathrm{d}s \\ &\qquad = \int_0^r \int_r^t \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi}(\tau-s)} (1+2\xi^2)\mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\tau\,\mathrm{d}s \\ &\qquad = \int_0^r \log\frac{t-s}{r-s}\,\mathrm{d}s. \end{aligned}$$

Note that $\log x \leqslant \sqrt{x-1}$ if $x \geqslant 1$. Hence, we have

$$\int_0^r \log\frac{t-s}{r-s}\,\mathrm{d}s \leqslant \int_0^r \sqrt{\frac{t-r}{r-s}}\,\mathrm{d}s = 2\sqrt{(t-r)r}.$$

Therefore,

$$B_1(t,r,x) \leqslant 2\sqrt{(t-r)r}\,\|g\|_{L^\infty((0,T)\times\mathbb{R})}. \tag{2.9}$$

By combining inequalities (2.7)–(2.9) we get inequality (2.6). □

LEMMA 2.5. *Let $\lambda \in (0,1)$, and assume that conditions (A1) and (B1) are satisfied. Let $L \geqslant 0$ be any constant.*

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(a) *A function* $u \in C_{\mathrm{B}}([0,T)\times\mathbb{R})\cap C^{0,1}((0,T)\times\mathbb{R})$ *is a weak solution of problem (1.1) if and only if, for all* $(t,x)\in(0,T)\times\mathbb{R}$*, it satisfies the integral equation*

$$u(t,x) = \mathrm{e}^{-Lt}\int_{\mathbb{R}} K(t,x-y)u_0(y)\,\mathrm{d}y + \int_0^t\int_{\mathbb{R}} K(t-s,x-y)f_L(u(s,y))\mathrm{e}^{-L(t-s)}\,\mathrm{d}y\,\mathrm{d}s, \quad (2.10)$$

where $f_L(u) := f(u) - f(1)H(u-\lambda) + Lu$.

(b) *Let* $\bar{u} \in C_{\mathrm{B}}([0,T)\times\mathbb{R})\cap C^{0,1}((0,T)\times\mathbb{R})$ *be a weak upper solution of problem (1.1). If there exists a non-positive function* $w_1 \in L^\infty((0,T)\times\mathbb{R})$ *such that, for all non-negative functions* $\varphi\in\mathcal{D}((0,T)\times\mathbb{R})$,

$$\int_0^T\int_{\mathbb{R}}(\bar{u}\partial_t\varphi - \partial_x\bar{u}\partial_x\varphi + (f(\bar{u}) - f(1)\tilde{H}(\bar{u}-\lambda) - w_1)\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

then, for all $(t,x)\in(0,T)\times\mathbb{R}$*, the function* $\bar{u}$ *satisfies the integral inequality*

$$\bar{u}(t,x) \geqslant \mathrm{e}^{-Lt}\int_{\mathbb{R}} K(t,x-y)u_0(y)\,\mathrm{d}y + \int_0^t\int_{\mathbb{R}} K(t-s,x-y)\tilde{f}_L(\bar{u}(s,y))\mathrm{e}^{-L(t-s)}\,\mathrm{d}y\,\mathrm{d}s, \quad (2.11)$$

where $\tilde{H}$ *is as in condition (A2) and* $\tilde{f}_L(u) := f(u) - f(1)\tilde{H}(u-\lambda) + Lu$.

(c) *Let* $\underline{u} \in C_{\mathrm{B}}([0,T)\times\mathbb{R})\cap C^{0,1}((0,T)\times\mathbb{R})$ *be a weak lower solution of problem (1.1). If there exists a non-negative function* $w_2 \in L^\infty((0,T)\times\mathbb{R})$ *such that, for all non-negative functions* $\varphi\in\mathcal{D}((0,T)\times\mathbb{R})$,

$$\int_0^T\int_{\mathbb{R}}(\underline{u}\partial_t\varphi - \partial_x\underline{u}\partial_x\varphi + (f(\underline{u}) - f(1)\hat{H}(\underline{u}-\lambda) - w_2)\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

then, for all $(t,x)\in(0,T)\times\mathbb{R}$*, the function* $\underline{u}$ *satisfies the integral inequality*

$$\underline{u}(t,x) \leqslant \mathrm{e}^{-Lt}\int_{\mathbb{R}} K(t,x-y)u_0(y)\,\mathrm{d}y + \int_0^t\int_{\mathbb{R}} K(t-s,x-y)\hat{f}_L(\underline{u}(s,y))\mathrm{e}^{-L(t-s)}\,\mathrm{d}y\,\mathrm{d}s,$$

where $\hat{H}$ *is as in condition (A2) and* $\hat{f}_L(u) := f(u) - f(1)\hat{H}(u-\lambda) + Lu$.

Proof.

Step 1. First, we show that assertion (a) of the lemma holds. Assume that $u \in C_{\mathrm{B}}([0,T)\times\mathbb{R})\cap C^{0,1}((0,T)\times\mathbb{R})$ is a weak solution of problem (1.1). Put $v(t,x) := u(t,x)\mathrm{e}^{Lt}$. Then v is a weak solution of the problem

$$\begin{aligned} v_t &= v_{xx} + f_L(v\mathrm{e}^{-Lt})\mathrm{e}^{Lt}, \qquad 0<t<T, \quad x\in\mathbb{R}, \\ v|_{t=0} &= u_0, \qquad\qquad\qquad\qquad\qquad\quad x\in\mathbb{R}. \end{aligned}$$

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We now consider the problem

$$\left.\begin{aligned} w_t &= w_{xx} + f_L(v\mathrm{e}^{-Lt})\mathrm{e}^{Lt}, \qquad 0<t<T, \quad x\in\mathbb{R},\\ w|_{t=0} &= u_0, \qquad\qquad\qquad\qquad\qquad\qquad x\in\mathbb{R}. \end{aligned}\right\} \tag{2.12}$$

By lemma 2.4, problem (2.12) has a unique weak solution, $w \in C_{\mathrm{B}}([0,T)\times\mathbb{R}) \cap C^{0,1}((0,T)\times\mathbb{R})$, which is expressed in the form

$$w(t,x) = \int_{\mathbb{R}} K(t,x-y)u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} K(t-s,x-y) f_L(v(s,y)\mathrm{e}^{-Ls})\mathrm{e}^{Ls}\,\mathrm{d}y\,\mathrm{d}s.$$

Hence, w must be equal to v in $(0,T)\times\mathbb{R}$. Since $v(t,x) := u(t,x)\mathrm{e}^{Lt}$, the function u satisfies the integral equation (2.10). The converse is obvious from lemma 2.4.

Step 2. Next, we show that assertion (b) holds. Put $\bar{v}(t,x) := \bar{u}(t,x)\mathrm{e}^{Lt}$. Then $\bar{v}$ satisfies the conditions that, for all non-negative functions $\varphi \in \mathcal{D}((0,T)\times\mathbb{R})$,

$$\int_0^T \int_{\mathbb{R}} (\bar{v}\partial_t\varphi - \partial_x\bar{v}\partial_x\varphi + (\tilde{f}_L(\bar{v}\mathrm{e}^{-Lt}) - w_1)\mathrm{e}^{Lt}\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0$$

and that, for all $x_0 \in \mathbb{R}$,

$$\lim_{t\downarrow 0, x\to x_0} \bar{v}(t,x) = \bar{u}(0,x_0).$$

Let us consider the problem in which, for all non-negative functions $\varphi \in \mathcal{D}((0,T)\times\mathbb{R})$, we have

$$\int_0^T \int_{\mathbb{R}} (\bar{w}\partial_t\varphi - \partial_x\bar{w}\partial_x\varphi + (\tilde{f}_L(\bar{v}\mathrm{e}^{-Lt}) - w_1)\mathrm{e}^{Lt}\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.13}$$

and, for all $x_0 \in \mathbb{R}$,

$$\lim_{t\downarrow 0, x\to x_0} \bar{w}(t,x) = \bar{u}(0,x_0). \tag{2.14}$$

We now prove that the function

$$\begin{aligned} \bar{w}(t,x) &= \int_{\mathbb{R}} K(t,x-y)\bar{u}(0,y)\,\mathrm{d}y \\ &\quad + \int_0^t \int_{\mathbb{R}} K(t-s,x-y)(\tilde{f}_L(\bar{v}(s,y)\mathrm{e}^{-Ls}) - w_1(s,y))\mathrm{e}^{Ls}\,\mathrm{d}y\,\mathrm{d}s \end{aligned} \tag{2.15}$$

is a unique solution of problem (2.13), (2.14) in $C_{\mathrm{B}}([0,T)\times\mathbb{R}) \cap C^{0,1}((0,T)\times\mathbb{R})$. Note that $(\tilde{f}_L(\bar{v}(t,x)\mathrm{e}^{-Lt}) - w_1(t,x))\mathrm{e}^{Lt}$ is bounded in $(0,T)\times\mathbb{R}$ and that $\bar{u}(0,x)$ is bounded and continuous on $\mathbb{R}$. Hence, by lemma 2.4, we see that $\bar{w}$, given by (2.15), is a solution of problem (2.13), (2.14) in $C_{\mathrm{B}}([0,T)\times\mathbb{R}) \cap C^{0,1}((0,T)\times\mathbb{R})$. To prove the uniqueness, it suffices to show that the zero solution is the only solution in $C_{\mathrm{B}}([0,T)\times\mathbb{R}) \cap C^{0,1}((0,T)\times\mathbb{R})$ of the problem in which, for all non-negative functions $\varphi \in \mathcal{D}((0,T)\times\mathbb{R})$,

$$\int_0^T \int_{\mathbb{R}} (w\partial_t\varphi - \partial_x w\partial_x\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.16}$$

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and, for all $x_0 \in \mathbb{R}$,

$$\lim_{t\downarrow 0, x\to x_0} w(t,x) = 0. \tag{2.17}$$

Assume that problem (2.16), (2.17) has a solution w that is different from the zero solution. Fix $\delta \in (0, \frac{1}{2}T)$ arbitrarily and assume that $\varepsilon \in (0,\delta)$. Let ψ be a fixed element of $\mathcal{D}(\mathbb{R}^2)$ such that $\psi \geqslant 0$, $\operatorname{supp}\psi \subset (-1,1)^2$ and

$$\iint \psi(t,x)\,\mathrm{d}x\,\mathrm{d}t = 1.$$

Put

$$\psi_\varepsilon(t,x) := \frac{1}{\varepsilon^2}\psi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$

and define a function $w^\varepsilon := w * \psi_\varepsilon$ on $[\delta, T-\delta]\times\mathbb{R}$. Then, by equation (2.16), we see that w^ε is a unique bounded classical solution of the problem

$$\left.\begin{aligned} w^\varepsilon_t &= w^\varepsilon_{xx}, & \delta < t < T-\delta, \quad & x\in\mathbb{R}, \\ w^\varepsilon|_{t=\delta} &= w^\varepsilon(\delta,x), & & x\in\mathbb{R}. \end{aligned}\right\}$$

Hence, w^ε is expressed in the form

$$w^\varepsilon(t,x) = \int_{\mathbb{R}} K(t-\delta, x-y) w^\varepsilon(\delta,y)\,\mathrm{d}y$$

in $(\delta, T-\delta)\times\mathbb{R}$. By the continuity of w, we see that w^ε converges uniformly to w on any compact subset of $(\delta, T-\delta)\times\mathbb{R}$ as $\varepsilon\downarrow 0$. Furthermore,

$$\int_{\mathbb{R}} K(t-\delta, x-y) w^\varepsilon(\delta,y)\,\mathrm{d}y \to \int_{\mathbb{R}} K(t-\delta,x-y) w(\delta,y)\,\mathrm{d}y \quad \text{as } \varepsilon\downarrow 0.$$

Hence, on taking the limit as $\delta\downarrow 0$, we find that $w=0$ in $(0,T)\times\mathbb{R}$. This contradicts the assumption that w is not the zero solution. Thus, $\bar{w}$ given by (2.15) is a unique solution of problem (2.13), (2.14) in $C_{\mathrm{B}}([0,T)\times\mathbb{R})\cap C^{0,1}((0,T)\times\mathbb{R})$, such that $\bar{w} = \bar{v}$ in $(0,T)\times\mathbb{R}$. Hence, by $\bar{v}(t,x) := \bar{u}(t,x)\mathrm{e}^{Lt}$ and equation (2.15), we have

$$\begin{aligned} \bar{u}(t,x) = \mathrm{e}^{-Lt}&\int_{\mathbb{R}} K(t,x-y)\bar{u}(0,y)\,\mathrm{d}y \\ &+ \int_0^t\int_{\mathbb{R}} K(t-s,x-y)(\tilde{f}_L(\bar{u}(s,y)) - w_1(s,y))\mathrm{e}^{-L(t-s)}\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

Since $\bar{u}(0,x) \geqslant u_0(x)$ on $\mathbb{R}$ and $w_1(t,x)$ is non-positive in $(0,T)\times\mathbb{R}$, the integral inequality (2.11) is obtained.

Assertion (c) can be proved similarly. □

The proof of lemma 2.6 is omitted, since it can be shown by a slight modification of the proof of [2, lemma 3].

Lemma 2.6. *Let $\lambda \in (0,1)$, and assume that conditions (A1) and (B1) are satisfied. Furthermore, let u be any weak solution of problem (1.1). Then the Lebesgue measure of $\{(t,x)\in(0,T)\times\mathbb{R} \mid u(t,x)=\lambda\}$ is zero if $f(\lambda) - f(1)H(0) \neq 0$.*

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Proof of theorem 2.3.

Step 1. First, we prove the existence of the maximal weak solution U in the order interval $[\underline{u}, \bar{u}]$. Let $\tilde{H}$ and $\hat{H}$ be as in condition (A2). By condition (A1), we can choose a constant $M \geqslant 0$ such that $u \mapsto f(u) + Mu$ is non-decreasing on $\mathbb{R}$. Put

$$\tilde{f}_M(u) := f(u) - f(1)\tilde{H}(u - \lambda) + Mu$$

and

$$\hat{f}_M(u) := f(u) - f(1)\hat{H}(u - \lambda) + Mu.$$

Then, by condition (A2) and lemma 2.5(b), the weak upper solution $\bar{u}$ satisfies the integral inequality

$$\bar{u}(t, x) \geqslant \mathrm{e}^{-Mt} \int_{\mathbb{R}} K(t, x - y) u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} K(t - s, x - y)\tilde{f}_M(\bar{u}(s, y))\mathrm{e}^{-M(t-s)}\,\mathrm{d}y\,\mathrm{d}s. \tag{2.18}$$

Similarly, we obtain the integral inequality

$$\underline{u}(t, x) \leqslant \mathrm{e}^{-Mt} \int_{\mathbb{R}} K(t, x - y) u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} K(t - s, x - y)\hat{f}_M(\underline{u}(s, y))\mathrm{e}^{-M(t-s)}\,\mathrm{d}y\,\mathrm{d}s. \tag{2.19}$$

We now fix $T > 0$ arbitrarily and consider the iteration scheme

$$\left.\begin{aligned} U_t^{n+1} - U_{xx}^{n+1} + MU^{n+1} &= \tilde{f}_M(U^n), \qquad 0 < t < T, \quad x \in \mathbb{R}, \\ U^{n+1}|_{t=0} &= u_0, \qquad\qquad\qquad\qquad\quad x \in \mathbb{R}, \end{aligned}\right\} \tag{2.20}$$

with the initial iteration $U^0 = \bar{u}$. We can then show that problem (2.20) may be uniquely solved for each $n \in \mathbb{N}_0$ and that the sequence $(U^n)_{n \in \mathbb{N}_0}$ converges to the maximal weak solution U of problem (1.1) in the order interval $[\underline{u}, \bar{u}]$. The proof is divided into three steps.

Step 1.1. First, we prove the following claim.

Claim 1. *Problem (2.20) has a unique weak solution U^{n+1}, for each $n \in \mathbb{N}_0$, which satisfies the following relationship:*

$$\underline{u} \leqslant U^{n+1} \leqslant U^n \leqslant \cdots \leqslant U^1 \leqslant U^0 = \bar{u}. \tag{2.21}$$

Proof. Using lemma 2.4 and induction on $n \in \mathbb{N}_0$, we see that problem (2.20) is uniquely solved and U^{n+1} can be expressed in the form

$$U^{n+1}(t, x) = \mathrm{e}^{-Mt} \int_{\mathbb{R}} K(t, x - y) u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} K(t - s, x - y)\tilde{f}_M(U^n(s, y))\mathrm{e}^{-M(t-s)}\,\mathrm{d}y\,\mathrm{d}s. \tag{2.22}$$

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By inequality (2.18) and equation (2.22) with $n = 0$, we obtain the relationship $U^1 \leqslant U^0 = \bar{u}$. Note that, by condition (A1), both $\tilde{f}_M$ and $\hat{f}_M$ are non-decreasing on $\mathbb{R}$ and $\tilde{f}_M \geqslant \hat{f}_M$ on $\mathbb{R}$. Furthermore, note that, by condition (A2), the relationship $\underline{u} \leqslant \bar{u}$ holds. Hence, we get the relationship $\underline{u} \leqslant U^1$ from inequality (2.19) and equation (2.22) with $n = 0$. A similar argument shows the relationship (2.21). □

Step 1.2. Secondly, we prove the following claim.

Claim 2. *The sequence* $(U^n)_{n\in\mathbb{N}_0}$ *obtained in claim 1 converges to a weak solution* U *of problem (1.1) in the order interval* $[\underline{u}, \bar{u}]$.

Proof. Let $A(t, x)$ and $B_n(t, x)$ be the first and second terms of the right-hand side of equation (2.22), respectively. Then, noting the monotonicity of $\tilde{f}_M$ and using claim 1 and lemma 2.4, we have the following four estimates: for $n \in \mathbb{N}_0$, $0 < t < T$ and $x \in \mathbb{R}$,

$$|\partial_x A(t,x)| \leqslant \frac{\mathrm{e}^{-Mt}}{\sqrt{\pi t}} \|u_0\|_{L^\infty(\mathbb{R})},$$

$$|\partial_t A(t,x)| \leqslant M\mathrm{e}^{-Mt}\|u_0\|_{L^\infty(\mathbb{R})} + \frac{\mathrm{e}^{-Mt}}{t}\|u_0\|_{L^\infty(\mathbb{R})},$$

$$|\partial_x B_n(t,x)| \leqslant 2\mathrm{e}^{-Mt}\sqrt{\frac{t}{\pi}}\kappa,$$

and for $n \in \mathbb{N}_0$, $0 < r < t < T$ and $x \in \mathbb{R}$,

$$|B_n(t,x) - B_n(r,x)| \leqslant \mathrm{e}^{-Mt}(2\sqrt{(t-r)r} + (t-r))\kappa + (\mathrm{e}^{-Mr} - \mathrm{e}^{-Mt})t\kappa,$$

where $\kappa = \max\{\|\tilde{f}_M(\underline{u}(t,x))\mathrm{e}^{Mt}\|_{L^\infty((0,T)\times\mathbb{R})}, \|\tilde{f}_M(\bar{u}(t,x))\mathrm{e}^{Mt}\|_{L^\infty((0,T)\times\mathbb{R})}\}$. Thus, U^n is equicontinuous on every compact subset of $(0, T) \times \mathbb{R}$. Hence, by the Arzelà–Ascoli theorem and a diagonal method, we can pass to the limit as $n \uparrow \infty$ along a subsequence, and so obtain a function U^∞ in $C_{\mathrm{B}}((0, T) \times \mathbb{R})$. Furthermore, by the monotonicity of $(U^n)_{n\in\mathbb{N}_0}$, the sequence itself must converge to U^∞ in the same space. Since $U^n(0, x) = u_0(x)$ on $\mathbb{R}$ for $n \in \mathbb{N}$, it follows that the limit U of $(U^n)_{n\in\mathbb{N}_0}$ on $[0, T) \times \mathbb{R}$ is given by

$$U(t,x) = \begin{cases} U^\infty(t,x), & \text{if } 0 < t < T \text{ and } x \in \mathbb{R}, \\ u_0(x), & \text{if } t = 0 \text{ and } x \in \mathbb{R}. \end{cases}$$

Thus, it remains to show that U is a weak solution of problem (1.1) in the order interval $[\underline{u}, \bar{u}]$. For this purpose, we consider the limit as $n \uparrow \infty$ in equation (2.22). By condition (A1) and the boundedness of $(U^n)_{n\in\mathbb{N}_0}$, we find that $(\tilde{f}_M(U^n(t,x)))_{n\in\mathbb{N}_0}$ is bounded in $(0, T) \times \mathbb{R}$. Note that the right continuity of $\tilde{H}$ means the right continuity of $\tilde{f}_M$. Hence, by using the fact that $(U^n)_{n\in\mathbb{N}_0}$ is non-increasing and converges to U in $(0, T) \times \mathbb{R}$, we find that

$$\tilde{f}_M(U^n(t,x)) \to \tilde{f}_M(U(t,x))$$

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in $(0,T)\times\mathbb{R}$ as $n\uparrow\infty$. Therefore, we can apply Lebesgue's dominated convergence theorem to get the integral equation

$$U(t,x)=\mathrm{e}^{-Mt}\int_{\mathbb{R}}K(t,x-y)u_0(y)\,\mathrm{d}y \\ +\int_0^t\int_{\mathbb{R}}K(t-s,x-y)\tilde{f}_M(U(s,y))\mathrm{e}^{-M(t-s)}\,\mathrm{d}y\,\mathrm{d}s \qquad (2.23)$$

in $(0,T)\times\mathbb{R}$. It is obvious from equation (2.23) that $\lim_{t\downarrow 0,x\to x_0}U(t,x)=u_0(x_0)$ for all $x_0\in\mathbb{R}$. Hence, U belongs to $C_{\mathrm{B}}([0,T)\times\mathbb{R})$ due to the definition of U. Again, from equation (2.23), we see that U belongs to $C^{0,1}((0,T)\times\mathbb{R})$. Hence, by using lemma 2.5(a), we find that U is a weak solution of the problem

$$\left.\begin{aligned} U_t=U_{xx}+f(U)-f(1)\tilde{H}(U-\lambda), &\qquad 0<t<T,\quad x\in\mathbb{R},\\ U|_{t=0}=u_0, &\qquad x\in\mathbb{R}. \end{aligned}\right\}$$

By condition (A1) and $\tilde{H}(0)=1$, we see that $f(\lambda)-f(1)\tilde{H}(0)\neq 0$. Hence, according to lemma 2.6, the Lebesgue measure of $\{(t,x)\in(0,T)\times\mathbb{R}\mid U(t,x)=\lambda\}$ is zero. Therefore, U must be a weak solution of problem (1.1). Also, it is obvious from claim 1 that U is contained in the order interval $[\underline{u},\bar{u}]$. □

Step 1.3. Finally, we prove the following claim.

Claim 3. *The weak solution U obtained in claim 2 is a maximal solution in the order interval $[\underline{u},\bar{u}]$.*

Proof. According to lemma 2.5(a), any weak solution $v\in[\underline{u},\bar{u}]$ of problem (1.1) satisfies the integral equation

$$v(t,x)=\mathrm{e}^{-Mt}\int_{\mathbb{R}}K(t,x-y)u_0(y)\,\mathrm{d}y \\ +\int_0^t\int_{\mathbb{R}}K(t-s,x-y)f_M(v(s,y))\mathrm{e}^{-M(t-s)}\,\mathrm{d}y\,\mathrm{d}s, \qquad (2.24)$$

where $f_M(u):=f(u)-f(1)H(u-\lambda)+Mu$. Hence, by equations (2.22) and (2.24), we get the relationship $v\leqslant U^1$. We can similarly derive the relationship $v\leqslant U^n$ for each $n\geqslant 2$. Therefore, the relationship $v\leqslant U$ holds. □

Step 2. Next, we prove the existence of the minimal weak solution u in the order interval $[\underline{u},\bar{u}]$. In this case, it suffices to consider the iteration scheme

$$\left.\begin{aligned} u_t^{n+1}-u_{xx}^{n+1}+Mu^{n+1}=\hat{f}_M(u^n), &\qquad 0<t<T,\quad x\in\mathbb{R},\\ u^{n+1}|_{t=0}=u_0, &\qquad x\in\mathbb{R}, \end{aligned}\right\} \qquad (2.25)$$

with the initial iteration $u^0=\underline{u}$. In fact, as in step 1, we can obtain the minimal weak solution u of problem (1.1) in the order interval $[\underline{u},\bar{u}]$ as the limit of a sequence $(u^n)_{n\in\mathbb{N}}$ of weak solutions of problem (2.25).

The proof of theorem 2.3 is now complete. □

Remark 2.7. There exist the maximal and minimal weak solutions U, $u \in [0,1]$, of problem (1.1) if $\lambda \in (0,1)$ and if conditions (A1) and (B1) are satisfied, since $\bar{u} \equiv 1$ and $\underline{u} \equiv 0$ are a weak upper solution and a weak lower solution, respectively, of problem (1.1) as in condition (A2). Remark that $[0,1]$ is an order interval.

The following proposition shows that any weak solution v of problem (1.1) is contained in the order interval $[0,1]$ under some conditions. Therefore, we see that v satisfies the relationship $u \leqslant v \leqslant U$ for the maximal and minimal weak solutions U, $u \in [0,1]$ of problem (1.1).

Proposition 2.8. *Let* $\lambda \in (0,1)$*, and assume that conditions (A1) and (B1) are satisfied. Any weak solution of problem (1.1) is then contained in the order interval* $[0,1]$.

Proof. Fix $T > 0$ arbitrarily. By condition (A1), we can find a constant $c > 0$ so that $|f(u) - f(1)H(u-\lambda)| \leqslant c(1+|u|)$ on $\mathbb{R}$. With this constant $c > 0$, we define $\bar{u}(t) := \mathrm{e}^{ct}(1+ct)$ and $\underline{u}(t) := -\mathrm{e}^{ct}(1+ct)$ on $[0,T)$. Then $\bar{u}$ and $\underline{u}$ are a weak upper solution and a weak lower solution, respectively, of problem (1.1) as in condition (A2). Furthermore, we can see that any weak solution u of problem (1.1) satisfies the inequality

$$\underline{u}(t) \leqslant u(t,x) \leqslant \bar{u}(t) \tag{2.26}$$

in $(0,T) \times \mathbb{R}$. In fact, by lemma 2.5(a), the weak solution u satisfies the integral equation

$$u(t,x) = \int_{\mathbb{R}} K(t,x-y)u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} K(t-s,x-y)(f(u(s,y)) - f(1)H(u(s,y)-\lambda))\,\mathrm{d}y\,\mathrm{d}s.$$

Hence, by the definition of c and condition (B1), we obtain, for $0 < t < T$,

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant 1 + \int_0^t c(1 + \|u(s,\cdot)\|_{L^\infty(\mathbb{R})})\,\mathrm{d}s.$$

On applying the Gronwall inequality, we obtain inequality (2.26). We now show that the maximal and minimal weak solutions U_1, $u_1 \in [\underline{u}, \bar{u}]$ of problem (1.1) are contained in the order interval $[0,1]$. As may be seen from the proof of theorem 2.3, the maximal weak solution $U_2 \in [\underline{u}, \bar{u}]$ of problem (1.1) with initial datum 1 satisfies $U_2 \geqslant U_1$ on $(0,T) \times \mathbb{R}$. Furthermore, by condition (A1), we see that $U_2 = 1$ on $(0,T) \times \mathbb{R}$. Similarly, the minimal weak solution $u_2 \in [\underline{u}, \bar{u}]$ of problem (1.1) with initial datum 0 satisfies $u_2 \leqslant u_1$ on $(0,T) \times \mathbb{R}$, and, furthermore, $u_2 = 0$ on $(0,T) \times \mathbb{R}$. Hence, U_1 and u_1 are contained in the order interval $[0,1]$, such that u is also contained in the order interval $[0,1]$. □

The following proposition shows certain monotonicity properties of the maximal and minimal weak solutions U and u of problem (1.1) in the order interval $[\underline{u}, \bar{u}]$.

Proposition 2.9. *In addition to the assumptions in theorem 2.3, let* u_0 *not be a constant.*

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(a) *If $U \in [\underline{u}, \bar{u}]$ is the maximal weak solution of problem (1.1), then $U(t,x)$ is increasing in x in cases when both $u_0(x)$ and $\bar{u}(t,x)$ are non-decreasing in x, and $U(t,x)$ is decreasing in x in cases when both $u_0(x)$ and $\bar{u}(t,x)$ are non-increasing in x.*

(b) *If $u \in [\underline{u}, \bar{u}]$ is the minimal weak solution of problem (1.1), then $u(t,x)$ is increasing in x in cases when both $u_0(x)$ and $\underline{u}(t,x)$ are non-decreasing in x, and $u(t,x)$ is decreasing in x in cases when both $u_0(x)$ and $\underline{u}(t,x)$ are non-increasing in x.*

Proof. We will prove only assertion (a) of the proposition in the case when both $u_0(x)$ and $\bar{u}(t,x)$ are non-decreasing in x. The other cases can be proved similarly.

As may be seen from the proof of theorem 2.3, the maximal weak solution $U \in [\underline{u}, \bar{u}]$ is obtained as the limit of a sequence $(U^n)_{n\in\mathbb{N}}$ of weak solutions of problem (2.20) with $U^0 = \bar{u}$. Let us recall that U^{n+1} is expressed in the form (2.22):

$$U^{n+1}(t,x) = \mathrm{e}^{-Mt} \int_{\mathbb{R}} K(t, x-y) u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} K(t-s, x-y) \tilde{f}_M(U^n(s,y)) \mathrm{e}^{-M(t-s)}\,\mathrm{d}y\,\mathrm{d}s$$

in $(0,T)\times\mathbb{R}$. Since u_0 is non-decreasing and is not constant on $\mathbb{R}$, the first term on the right-hand side of equation (2.22) is increasing in x. Note that, by condition (A1), the function $\tilde{f}_M$ is non-decreasing on $\mathbb{R}$ and that $U^0(t,x)$ is non-decreasing in x. Hence, $\tilde{f}_M(U^0(t,x))$ is non-decreasing in x and so the second term on the right-hand side of equation (2.22) with $n=0$ is non-decreasing in x. Therefore, by equation (2.22) with $n=0$, we see that $U^1(t,x)$ is increasing in x. We can repeat this process to derive the result that $U^n(t,x)$ is increasing in x for all $n \geqslant 2$. Therefore, $U(t,x)$ is non-decreasing in x. However, from equation (2.23), we see that $U(t,x)$ is increasing in x. In fact, the first term on the right-hand side of equation (2.23) is increasing in x, since u_0 is non-decreasing and is not constant on $\mathbb{R}$. Furthermore, the second term on the right-hand side of equation (2.23) is non-decreasing in x, since $\tilde{f}_M(U(t,x))$ is non-decreasing in x. Thus, we find that $U(t,x)$ is increasing in x. □

3. Uniqueness theorem

In this section, we will investigate the uniqueness of weak solutions of problem (1.1). We will first prove that there exists a certain class of initial data for which the weak solution of problem (1.1) is locally unique in time. For this purpose, we impose the following three conditions on u_0.

(B2) $u_0 \in C^1(\mathbb{R})$.

(B3) u_0 and u_0' are Lipschitz continuous on $\mathbb{R}$, respectively.

(B4) There exist $\delta > 0$ and $\eta > 0$ such that

$$|u_0'(x)| \geqslant \delta > 0 \quad \text{whenever } u_0(x) \in [\lambda - \eta, \lambda + \eta],$$

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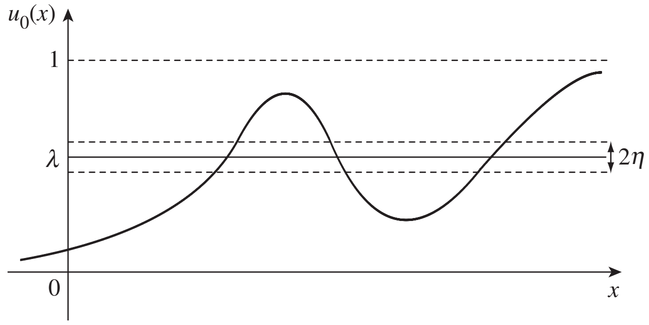

Figure 5. Condition (B4).

and the Lebesgue measure of $\{x \in \mathbb{R} \mid u_0(x) \in [\lambda - \eta, \lambda + \eta]\}$ is finite (see figure 5).

THEOREM 3.1. *Let $\lambda \in (0,1)$, and assume that conditions (A1), (B1), (B2), (B3) and (B4) are satisfied. Then the weak solution of problem (1.1) is locally unique in time.*

For the proof, we need the following lemma.

LEMMA 3.2. *Let $\lambda \in (0,1)$, and assume that conditions (A1), (B1), (B2) and (B3) are satisfied. Assume that $u \in C_{\mathrm{B}}([0,T)\times\mathbb{R}) \cap C^{0,1}((0,T)\times\mathbb{R})$ is any weak solution of problem (1.1). Then $u(t,x)$ and $u_x(t,x)$ converge uniformly to $u_0(x)$ and $u_0'(x)$ on $\mathbb{R}$ as $t \downarrow 0$, respectively. More precisely, we have the two assertions:*

$$\|u(t,\cdot) - u_0(\cdot)\|_{L^\infty(\mathbb{R})} = O(\sqrt{t}), \tag{3.1}$$

$$\|u_x(t,\cdot) - u_0'(\cdot)\|_{L^\infty(\mathbb{R})} = O(\sqrt{t}). \tag{3.2}$$

Proof. By condition (A1), we can choose a constant $c > 0$ such that

$$|f(u) - f(1)H(u-\lambda)| \leqslant c(1+|u|) \quad \text{on } \mathbb{R}.$$

Hence, by lemma 2.5(a) and proposition 2.8, the difference $u(t,x) - u_0(x)$ satisfies the inequality

$$|u(t,x) - u_0(x)| \leqslant \int_{\mathbb{R}} K(t,x-y)|u_0(y) - u_0(x)|\,\mathrm{d}y + 2ct$$

in $(0,T)\times\mathbb{R}$. Since, by condition (B3), the function u_0 is Lipschitz continuous on $\mathbb{R}$, there exists a constant $L_{u_0} > 0$ such that, for $0 < t < T$ and $x \in \mathbb{R}$,

$$\begin{aligned}|u(t,x) - u_0(x)| &\leqslant \int_{\mathbb{R}} K(t,x-y)L_{u_0}|y-x|\,\mathrm{d}y + 2ct \\ &= \frac{2L_{u_0}}{\sqrt{\pi}}\sqrt{t} + 2ct.\end{aligned}$$

Therefore, we obtain assertion (3.1). In a similar way, we can derive the inequality

$$|u_x(t,x) - u_0'(x)| \leqslant \frac{2\sqrt{t}}{\sqrt{\pi}}(L_{u_0'} + 2c)$$

with some constant $L_{u_0'} > 0$ in $(0,T)\times\mathbb{R}$. Hence, assertion (3.2) follows. □

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Proof of theorem 3.1. We use an argument similar to that in the proof of [5, theorem 1]. Assume that we have two weak solutions u and v of problem (1.1). Define $E(t) := \|u - v\|_{L^\infty((0,t)\times\mathbb{R})}$ and $I_{s,\lambda,t} := \{y \in \mathbb{R} \mid |u(s, y) - \lambda| \leqslant E(t)\}$. Then, by lemma 2.5(a), the difference $u(t, x) - v(t, x)$ satisfies

$$\begin{aligned} u(t,x) - v(t,x) &\\ &= \int_0^t \int_{\mathbb{R}} K(t-s, x-y)(f(u(s,y)) - f(v(s,y)))\,\mathrm{d}y\,\mathrm{d}s \\ &\quad - \int_0^t \int_{I_{s,\lambda,t}} K(t-s, x-y) f(1)(H(u(s,y)-\lambda) - H(v(s,y)-\lambda))\,\mathrm{d}y\,\mathrm{d}s \\ &=: A(t,x) + B(t,x). \end{aligned} \tag{3.3}$$

(1) Estimate of $A(t, x)$: since, by condition (A1), the function f is Lipschitz continuous on $\mathbb{R}$, there exists a constant $L_f > 0$ such that, for $0 < t < T$ and $x \in \mathbb{R}$,

$$\begin{aligned} |A(t,x)| &\leqslant \int_0^t \int_{\mathbb{R}} K(t-s, x-y) L_f |u(s,y) - v(s,y)|\,\mathrm{d}y\,\mathrm{d}s \\ &\leqslant L_f t \|u - v\|_{L^\infty((0,t)\times\mathbb{R})}. \end{aligned} \tag{3.4}$$

(2) Estimate of $B(t, x)$: by lemma 3.2 and the assumptions on u_0, there exist $T_1 > 0$ and $\nu > 0$ such that $|u_x(t, x)| \geqslant \nu > 0$ on $\{(t, x) \in (0, T_1)\times\mathbb{R} \mid |u(t, x)-\lambda| \leqslant E(T_1)\}$ and

$$\sup_{0<t<T_1} \mu(I_{t,\lambda,T_1}) = \sup_{0<t<T_1} \mu(\{x \in \mathbb{R} \mid |u(t,x) - \lambda| \leqslant E(T_1)\}) < \infty. \tag{3.5}$$

Furthermore, by estimate (3.5) and the fact that u_x is bounded in $(0, T_1) \times \mathbb{R}$, there exists $m \in \mathbb{N}_0$ such that, for any $0 < t < T_1$, the set $\{x \in \mathbb{R} \mid u(t, x) = \lambda\}$ consists of at most m different points. Hence, for any $0 < t < T_1$ and any $0 < s < t$, we get the inequality

$$\mu(I_{s,\lambda,t}) = \mu(\{y \in \mathbb{R} \mid |u(s,y) - \lambda| \leqslant E(t)\}) \leqslant \frac{2mE(t)}{\nu}. \tag{3.6}$$

Therefore, by inequality (3.6), the absolute value of $B(t, x)$ satisfies the inequality

$$\begin{aligned} |B(t,x)| &\leqslant \frac{2m|f(1)|E(t)}{\nu} \int_0^t \frac{1}{\sqrt{\pi(t-s)}}\,\mathrm{d}s \\ &= \frac{4m|f(1)|\sqrt{t}}{\sqrt{\pi}\nu} \|u - v\|_{L^\infty((0,t)\times\mathbb{R})} \end{aligned} \tag{3.7}$$

in $(0, T_1) \times \mathbb{R}$.

By combining (3.3), (3.4) and (3.7), we obtain, for $0 < t < T_1$,

$$\|u - v\|_{L^\infty((0,t)\times\mathbb{R})} \leqslant \left(\frac{4m|f(1)|\sqrt{t}}{\sqrt{\pi}\nu} + L_f t \right) \|u - v\|_{L^\infty((0,t)\times\mathbb{R})}.$$

Therefore, we find that $u - v = 0$ in $(0, T_0) \times \mathbb{R}$ by choosing $0 < T_0 < T_1$ such that $4m|f(1)|\sqrt{T_0}/(\sqrt{\pi}\nu) + L_f T_0$ is less than 1. □

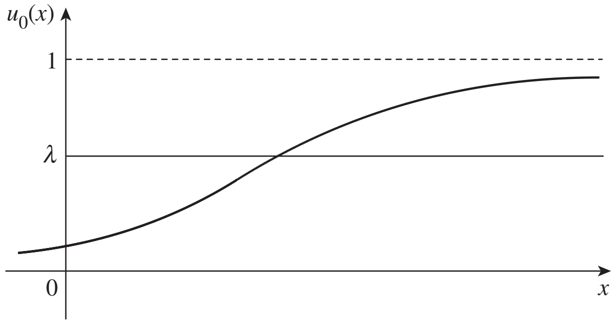

Figure 6. Condition (B5-1).

To state another uniqueness theorem, we impose the following condition on u_0:

(B5) u_0 satisfies either condition (B5-1) (see figure 6) or condition (B5-2):

(B5-1) u_0 is increasing on $\mathbb{R}$ and satisfies $u_0(-\infty) < \lambda < u_0(\infty)$;

(B5-2) u_0 is decreasing on $\mathbb{R}$ and satisfies $u_0(-\infty) > \lambda > u_0(\infty)$.

THEOREM 3.3. *Let $\lambda \in (0, 1)$, and assume that conditions (A1), (B1), (B2) and (B5) are satisfied. The weak solution of problem (1.1) is then globally unique in time.*

Proof. The proof will be given only for the case that u_0 satisfies condition (B5-1). The case that u_0 satisfies condition (B5-2) can be proved similarly.

Fix $T > 0$ arbitrarily. From theorem 2.3, remark 2.7 and proposition 2.9, it follows that there exists a weak solution $u \in [0, 1]$ of problem (1.1) such that $u_x(t, x) > 0$ in $(0, T) \times \mathbb{R}$. Assume that there exists a weak solution v different from u and define $T_1 := \sup\{t \in [0, T) \mid u(s, x) = v(s, x) \text{ on } [0, t] \times \mathbb{R}\}$. Then $u(T_1, x) = v(T_1, x)$ on $\mathbb{R}$. We will show that $u(t, x) = v(t, x)$ in $(T_1, T_2) \times \mathbb{R}$ for some $T_1 < T_2 < T$.

By lemma 2.5(a), the function u satisfies the integral equation

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} K(t, x-y) u_0(y)\,\mathrm{d}y \\ &\quad + \int_0^t \int_{\mathbb{R}} K(t-s, x-y)(f(u(s,y)) - f(1)H(u(s,y)-\lambda))\,\mathrm{d}y\,\mathrm{d}s \\ &=: A(t,x) + B(t,x). \end{aligned} \tag{3.8}$$

By condition (A1), we can find a constant $c > 0$ so that

$$|f(u(t,x)) - f(1)H(u(t,x)-\lambda)| \leqslant c(1 + |u(t,x)|) \quad \text{in } (0,T) \times \mathbb{R}.$$

Hence, using proposition 2.8 and inequalities (2.4) and (2.6) in lemma 2.4, we find that, for $0 < t < T$ and $x \in \mathbb{R}$,

$$|\partial_t A(t,x)| \leqslant \frac{1}{t},$$

and that, for $0 < r < t < T$ and $x \in \mathbb{R}$,

$$|B(t,x) - B(r,x)| \leqslant 2c(2\sqrt{(t-r)r} + (t-r)).$$

Thus, the family $(u(\cdot,x))_{x\in\mathbb{R}}$ is equicontinuous on every compact subset of $(0,T)$. Therefore, by the Arzelà–Ascoli theorem and a diagonal method, we can pass to the limit as $x\uparrow\infty$ along a subsequence and so obtain a function $u(\cdot,\infty)$ in $C_{\mathrm{B}}((0,T))$. Furthermore, by the monotonicity of $u(t,x)$ in x, the family itself must converge to $u(\cdot,\infty)$ in the same space. We now consider the limit as $x\uparrow\infty$ in equation (3.8). By condition (A1) and the boundedness of u, we find that $f(u(t,x))-f(1)H(u(t,x)-\lambda)$ is bounded in $(0,T)\times\mathbb{R}$. Furthermore, using the fact that $u(t,x)$ is increasing in x and converges to $u(t,\infty)$ in $(0,T)$, we have

$$f(u(t,x))-f(1)H(u(t,x)-\lambda)\to f(u(t,\infty))-f(1)\hat{H}(u(t,\infty)-\lambda)$$

in $(0,T)$ as $x\uparrow\infty$, where $\hat{H}$ is as in condition (A2). Therefore, we can apply Lebesgue's dominated convergence theorem to get the integral equation

$$u(t,\infty)=u_0(\infty)+\int_0^t(f(u(s,\infty))-f(1)\hat{H}(u(s,\infty)-\lambda))\,\mathrm{d}s \tag{3.9}$$

in $(0,T)$. Hence, $u(t,\infty)>\lambda$ on $(0,T)$, since $u_0(\infty)>\lambda$ by condition (B5-1). A similar argument shows that $u(t,-\infty)<\lambda$ on $(0,T)$. Therefore, u satisfies $|u_x(t,x)|\geqslant\nu>0$ on $\{(t,x)\in(T_1,T_3)\times\mathbb{R}\mid |u(t,x)-\lambda|\leqslant E(T_3)\}$ for some $\nu>0$ and for some $T_1<T_3<T$, where E is as in the proof of theorem 3.1. Furthermore, by the monotonicity of $u(t,x)$ in x, for any $t\in(T_1,T_3)$ the cardinal number of $\{x\in\mathbb{R}\mid u(t,x)=\lambda\}$ is equal to 1. Hence, we can apply the same argument as in the proof of theorem 3.1 to get $u(t,x)=v(t,x)$ in $(T_1,T_2)\times\mathbb{R}$ for some $T_1<T_2<T_3$. This contradicts the definition of T_1 and so the assertion follows. □

We consider the following condition on u_0.

(B6) u_0 satisfies one of the following three conditions:

- (B6-1) u_0 is non-decreasing on $\mathbb{R}$, and $u_0(-\infty)<\lambda<u_0(\infty)$;
- (B6-2) u_0 is non-increasing on $\mathbb{R}$, and $u_0(-\infty)>\lambda>u_0(\infty)$;
- (B6-3) u_0 is a constant on $\mathbb{R}$.

We then obtain the following uniqueness theorem by combining the arguments in the proofs of theorems 3.1 and 3.3.

THEOREM 3.4. *Let* $\lambda\in(0,1)$*, and assume that conditions (A1), (B1), (B2), (B3), (B4) and (B6) are satisfied. The weak solution of problem (1.1) is then globally unique in time.*

Proof. First, we discuss the case that condition (B6-1) is satisfied. Fix $T>0$ arbitrarily. From theorem 2.3, remark 2.7 and proposition 2.9, it follows that there exists a weak solution $u_1\in[0,1]$ of problem (1.1) such that $(u_1)_x(t,x)>0$ in $(0,T)\times\mathbb{R}$. Assume that there exists a weak solution v_1 different from u_1 and define $T_1:=\sup\{t\in[0,T)\mid u_1(s,x)=v_1(s,x)$ on $[0,t]\times\mathbb{R}\}$. Since the assumptions in theorem 3.4 are stronger than the ones in theorem 3.1, it follows that $T_1>0$. Obviously, $u_1(T_1,x)=v_1(T_1,x)$ on $\mathbb{R}$, and $u_1(T_1,x)=v_1(T_1,x)$ satisfies conditions (B1) and (B2). Furthermore, we can apply a similar argument to the proof

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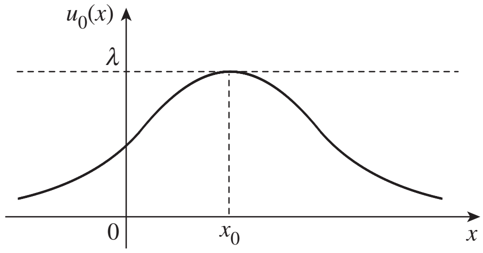

Figure 7. Graph of u_0.

of theorem 3.3 to find that $u_1(T_1, x) = v_1(T_1, x)$ satisfies condition (B5-1). It follows from theorem 3.3 that $u_1(t, x) = v_1(t, x)$ in $(T_1, T) \times \mathbb{R}$. This contradicts the definition of T_1. The case that condition (B6-2) is satisfied can be argued similarly.

Next, we discuss the case that condition (B6-3) is satisfied. Fix $T > 0$ arbitrarily and consider the integral equation

$$u_2(t) = u_0 + \int_0^t (f(u_2(s)) - f(1)H(u_2(s) - \lambda))\,\mathrm{d}s. \tag{3.10}$$

Since, by condition (B4), the constant u_0 is not equal to λ, equation (3.10) has a unique classical solution $u_2 \in [0, 1]$ on $[0, T)$. Furthermore, by lemma 2.5(a) it is easy to see that u_2 is a weak solution of problem (1.1). Now, assume that there exists a weak solution v_2 different from u_2 and define $T_2 := \sup\{t \in [0, T) \mid u_2(s) = v_2(s, x) \text{ on } [0, t] \times \mathbb{R}\}$. We then see that $u_2(T_2) = v_2(T_2, x)$ on $\mathbb{R}$ and that $u_2(T_2) = v_2(T_2, x)$ satisfies conditions (B1), (B2) and (B3). Since $u_2(t)$ does not belong to $[\lambda - \eta, \lambda + \eta]$ on $[0, T)$ for sufficiently small $\eta > 0$, we see that $u_2(T_2) = v_2(T_2, x)$ satisfies condition (B4). Hence, we can apply theorem 3.1 to find that $u_2(t) = v_2(t, x)$ in $(T_2, T_3) \times \mathbb{R}$ for some $T_2 < T_3 < T$. This contradicts the definition of T_2. Thus, the assertion follows. □

Remark 3.5. Results on global uniqueness for other classes of initial data are obtained by Terman [17], McKean and Moll [13] and Deguchi [4].

4. Non-uniqueness theorem

In this section, we will study the non-uniqueness of weak solutions of problem (1.1) with initial data like figure 7 and nonlinearities like figure 8. More precisely, we impose the following conditions on u_0 and f:

(B7) u_0 satisfies the inequality $v_0(x) \leqslant u_0(x) \leqslant \lambda$ on $\mathbb{R}$, where v_0 is a function on $\mathbb{R}$ such that, for some $x_0 \in \mathbb{R}$ (see figure 7),

$$v_0 \in C_{\mathrm{B}}(\mathbb{R}) \cap C^2(\mathbb{R}), \qquad v_0', v_0'' \in C_{\mathrm{B}}(\mathbb{R}), \qquad v_0(x + x_0) \text{ is even on } \mathbb{R},$$
$$v_0'(x) \geqslant 0 \quad \text{and} \quad 0 \leqslant v_0(x) < \lambda \text{ in } (-\infty, x_0), \qquad v_0(x_0) = \lambda.$$

(C1) For v_0 and x_0 given in condition (B7), there exists a constant $p > 0$ such that

$$\frac{-(p+2)v_0''(x_0) - \inf_{0<u<\lambda} f(u)}{f(\lambda) - f(1) - \inf_{0<u<\lambda} f(u)} < \frac{1}{\sqrt{\pi}} \int_0^1 \int_{-\sqrt{p}(1+\sqrt{\tau})/(2\sqrt{1-\tau})}^{-\sqrt{p}(1-\sqrt{\tau})/(2\sqrt{1-\tau})} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\tau.$$

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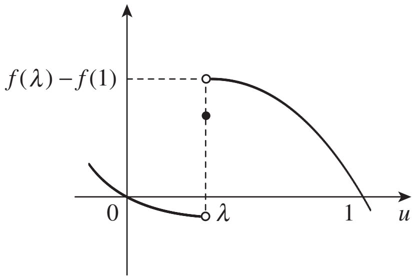

Figure 8. Graph of $u \mapsto f(u) - f(1)H(u-\lambda)$.

THEOREM 4.1. *Let $\lambda \in (0,1)$, and assume that conditions (A1), (B1), (B7) and (C1) are satisfied. Then the maximal weak solution $U \in [0,1]$ of problem (1.1) is different from the minimal weak solution $u \in [0,1]$, where $[0,1]$ is an order interval.*

REMARK 4.2. Let $f(u) = -u$. Assume that condition (B7) with $u_0 = v_0$ and $x_0 = 0$ holds. Then condition (C1) for non-uniqueness becomes the condition that there exists a constant $p > 0$ such that

$$-(p+2)u_0''(0) + \lambda < \frac{1}{\sqrt{\pi}} \int_0^1 \int_{-\sqrt{p}(1+\sqrt{\tau})/(2\sqrt{1-\tau})}^{-\sqrt{p}(1-\sqrt{\tau})/(2\sqrt{1-\tau})} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\tau.$$

On the other hand, according to McKean and Moll [13], the weak solution of problem (1.1) is unique if $u_0''(0) < 0$ and if

$$-u_0''(0) + \lambda > \frac{1}{\sqrt{\pi}} \max_{p \geqslant 0} \int_0^1 \int_{-\sqrt{p}(1+\sqrt{\tau})/(2\sqrt{1-\tau})}^{-\sqrt{p}(1-\sqrt{\tau})/(2\sqrt{1-\tau})} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\tau.$$

Proof of theorem 4.1. In fact, we will show that the minimal weak solution $u \in [0,1]$ of problem (1.1) satisfies

$$u(t,x) \leqslant \lambda \tag{4.1}$$

for $(t,x) \in (0,T) \times \mathbb{R}$, and further that the maximal weak solution $U \in [0,1]$ satisfies

$$U(t,x) > \lambda \tag{4.2}$$

for $(t,x) \in (0,T_0) \times [-\sqrt{pt} + x_0, \sqrt{pt} + x_0]$ with some $T_0 > 0$.

STEP 1. First, we show inequality (4.1). Let $\hat{H}$ be as in condition (A2). Then by condition (A1) and $\hat{H}(0) = 0$, we see that $f(\lambda) - f(1)\hat{H}(0) < 0$. Put $w(t,x) := \lambda$ in $(0,T) \times \mathbb{R}$. Then w satisfies the integral inequality

$$\begin{aligned} w(t,x) \geqslant \mathrm{e}^{-Mt} &\int_{\mathbb{R}} K(t,x-y)u_0(y)\,\mathrm{d}y \\ &+ \int_0^t \int_{\mathbb{R}} K(t-s,x-y)\hat{f}_M(w(s,y))\mathrm{e}^{-M(t-s)}\,\mathrm{d}y\,\mathrm{d}s, \end{aligned} \tag{4.3}$$

where $M \geqslant 0$ is a constant such that $u \mapsto f(u) + Mu$ is non-decreasing on $\mathbb{R}$, and $\hat{f}_M(u) := f(u) - f(1)\hat{H}(u-\lambda) + Mu$. Note that $u(t,x)$ is obtained as the limit of

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the sequence $(u^n)_{n\in\mathbb{N}}$ given by

$$u^{n+1}(t,x) = \mathrm{e}^{-Mt}\int_{\mathbb{R}} K(t,x-y)u_0(y)\,\mathrm{d}y + \int_0^t\int_{\mathbb{R}} K(t-s,x-y)\hat{f}_M(u^n(s,y))\mathrm{e}^{-M(t-s)}\,\mathrm{d}y\,\mathrm{d}s \tag{4.4}$$

with $u^0 = 0$. Since $u^0 \leqslant w$ in $(0,T)\times\mathbb{R}$, it follows from inequality (4.3) and equation (4.4) that $u^1 \leqslant w$ in $(0,T)\times\mathbb{R}$. We can repeat this process to get $u^n \leqslant w = \lambda$ in $(0,T)\times\mathbb{R}$ for $n \geqslant 2$. Thus, we obtain inequality (4.1).

Step 2. Next, we show inequality (4.2). As seen from the proof of theorem 2.3, the relationship $U \geqslant v$ holds for any weak lower solution $v \leqslant 1$ of problem (1.1) as in condition (A2). Hence, it suffices to construct a weak lower solution $v \leqslant 1$ of problem (1.1) that satisfies condition (A2) and inequality (4.2).

By condition (A1), we have $\inf_{0<u<\lambda} f(u) < 0$. Furthermore, by conditions (A1) and (C1), we can choose small $\delta_1 > 0$ so that $\inf_{\lambda<u<\lambda+\delta_1} f(u) - f(1) > 0$ and

$$\frac{-(p+2)v_0''(x_0) - \inf_{0<u<\lambda} f(u)}{\inf_{\lambda<u<\lambda+\delta_1} f(u) - f(1) - \inf_{0<u<\lambda} f(u)} < \frac{1}{\sqrt{\pi}}\int_0^1\int_{-\sqrt{p}(1+\sqrt{\tau})/(2\sqrt{1-\tau})}^{-\sqrt{p}(1-\sqrt{\tau})/(2\sqrt{1-\tau})} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{4.5}$$

Put $d_1 := \inf_{\lambda<u<\lambda+\delta_1} f(u) - f(1) > 0$ and $d_2 := -\inf_{0<u<\lambda} f(u) > 0$. By using the constants d_1 and d_2, we define

$$g(t,x) = \begin{cases} d_1, & \text{if } -\sqrt{pt}+x_0 \leqslant x \leqslant \sqrt{pt}+x_0, \\ -d_2, & \text{otherwise,} \end{cases}$$

in $(0,T)\times\mathbb{R}$. We consider the problem

$$\left.\begin{aligned} v_t &= v_{xx} + g(t,x), \qquad 0<t<T, \quad x\in\mathbb{R}, \\ v|_{t=0} &= v_0, \qquad\qquad\qquad\qquad\quad x\in\mathbb{R}, \end{aligned}\right\} \tag{4.6}$$

with the function v_0 given in condition (B7). According to lemma 2.4, problem (4.6) has a unique weak solution v, which is expressed in the form

$$v(t,x) = \int_{\mathbb{R}} K(t,x-y)v_0(y)\,\mathrm{d}y + \int_0^t\int_{\mathbb{R}} K(t-s,x-y)g(s,y)\,\mathrm{d}y\,\mathrm{d}s. \tag{4.7}$$

The function v then satisfies the properties stated above. The proof is divided into four steps.

Step 2.1. First, we prove the following claim.

Claim 1. *$v(t,x+x_0)$ is even on $\mathbb{R}$ for each $0<t<T$.*

Proof. By condition (B7), we see that $v_0(x+x_0)$ is even on $\mathbb{R}$. Furthermore, by the definition of g, we see that $g(t,x+x_0)$ is even on $\mathbb{R}$ for each $0<t<T$, so that $v(t,x+x_0)$ is also even on $\mathbb{R}$ for each $0<t<T$. □

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STEP 2.2. Second, we prove the following claim.

CLAIM 2. *$v_x(t, x + x_0) > 0$ for $0 < t < T$ and $x < 0$.*

Proof. Differentiating $v(t, x + x_0)$ in x, we have

$$\begin{aligned} v_x(t, x + x_0) &= \int_{\mathbb{R}} K_x(t, x + x_0 - y)v_0(y)\,\mathrm{d}y \\ &\quad + \int_0^t \int_{\mathbb{R}} K_x(t - s, x + x_0 - y)g(s, y)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{\mathbb{R}} K_x(t, x - \xi)v_0(\xi + x_0)\,\mathrm{d}\xi \\ &\quad + \int_0^t \int_{-\sqrt{ps}+x_0}^{\sqrt{ps}+x_0} K_x(t - s, x + x_0 - y)(d_1 + d_2)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{\mathbb{R}} K(t, x - \xi)v_0'(\xi + x_0)\,\mathrm{d}\xi \\ &\quad + (d_1 + d_2)\int_0^t (K(t - s, x + \sqrt{ps}) - K(t - s, x - \sqrt{ps}))\,\mathrm{d}s \\ &=: A(t, x) + B(t, x). \end{aligned}$$

We can easily see that $B(t, x) > 0$ for $0 < t < T$ and $x < 0$. Furthermore, we have $A(t, x) \geqslant 0$ for $0 < t < T$ and $x < 0$. Indeed, since $v_0'(\xi + x_0)$ is odd in ξ, the term $A(t, x)$ can be rewritten as

$$A(t, x) = \int_0^\infty (K(t, x - \xi) - K(t, x + \xi))v_0'(\xi + x_0)\,\mathrm{d}\xi.$$

Note that $v_0'(\xi + x_0) \leqslant 0$ for $\xi \geqslant 0$ by condition (B7) and that $K(t, x - \xi) - K(t, x + \xi) \leqslant 0$ for $0 < t < T$, $x < 0$ and $\xi \geqslant 0$. Hence, we have $A(t, x) \geqslant 0$ for $0 < t < T$ and $x < 0$. Thus, claim 2 follows. □

STEP 2.3. Third, we prove the following claim.

CLAIM 3. *$v(t, x) > \lambda$ for $(t, x) \in (0, T_1) \times [-\sqrt{pt} + x_0, \sqrt{pt} + x_0]$ with some $T_1 > 0$.*

Proof. By claims 1 and 2, it suffices to show that $v(t, -\sqrt{pt} + x_0) > \lambda$ in $(0, T_1)$ for some $T_1 > 0$.

By equation (4.7), we have

$$\begin{aligned} v(t, -\sqrt{pt} + x_0) &= \int_{\mathbb{R}} K(t, -\sqrt{pt} + x_0 - y)v_0(y)\,\mathrm{d}y \\ &\quad + \int_0^t \int_{\mathbb{R}} K(t - s, -\sqrt{pt} + x_0 - y)g(s, y)\,\mathrm{d}y\,\mathrm{d}s \\ &=: C(t) + D(t). \end{aligned}$$

(1) Estimate of $C(t)$: we rewrite $C(t)$ as

$$C(t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\xi^2} v_0(-\sqrt{pt} + x_0 - 2\sqrt{t}\xi)\,\mathrm{d}\xi.$$

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Note that $v_0(x_0) = \lambda$ and $v_0'(x_0) = 0$ by condition (B7). Hence, by the mean value theorem, we have

$$\begin{aligned}
C(t) - \lambda &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\xi^2} (v_0(-\sqrt{pt} + x_0 - 2\sqrt{t}\xi) - v_0(x_0))\,\mathrm{d}\xi \\
&= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\xi^2} (v_0'(x_0 - (\sqrt{pt} + 2\sqrt{t}\xi)\kappa_1) - v_0'(x_0))(-\sqrt{pt} - 2\sqrt{t}\xi)\,\mathrm{d}\xi \\
&= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\xi^2} v_0''(x_0 - (\sqrt{pt} + 2\sqrt{t}\xi)\kappa_1\kappa_2)(\sqrt{pt} + 2\sqrt{t}\xi)^2\kappa_1\,\mathrm{d}\xi \\
&\geqslant \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\xi^2} v_0''(x_0 - (\sqrt{pt} + 2\sqrt{t}\xi)\kappa_1\kappa_2)(\sqrt{p} + 2\xi)^2 t\,\mathrm{d}\xi
\end{aligned}$$

for some κ_1, $\kappa_2 \in (0,1)$ depending only on t and ξ. Therefore, we see that

$$\liminf_{t\downarrow 0} \frac{C(t) - \lambda}{t} \geqslant v_0''(x_0)(p+2). \tag{4.8}$$

(2) Estimate of $D(t)$:

$$\begin{aligned}
D(t) &= \int_0^t \int_{-\sqrt{ps}+x_0}^{\sqrt{ps}+x_0} K(t-s, -\sqrt{pt} + x_0 - y)(d_1 + d_2)\,\mathrm{d}y\,\mathrm{d}s - d_2 t \\
&= \frac{d_1 + d_2}{\sqrt{\pi}} \int_0^t \int_{-\sqrt{p}(\sqrt{t}+\sqrt{s})/(2\sqrt{t-s})}^{-\sqrt{p}(\sqrt{t}-\sqrt{s})/(2\sqrt{t-s})} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\,\mathrm{d}s - d_2 t \\
&= \biggl[\frac{d_1 + d_2}{\sqrt{\pi}} \int_0^1 \int_{-\sqrt{p}(1+\sqrt{\tau})/(2\sqrt{1-\tau})}^{-\sqrt{p}(1-\sqrt{\tau})/(2\sqrt{1-\tau})} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\tau - d_2\biggr] t.
\end{aligned} \tag{4.9}$$

From inequalities (4.5) and (4.8) and equation (4.9), it follows that

$$\begin{aligned}
&\liminf_{t\downarrow 0} \frac{C(t) - \lambda + D(t)}{t} \\
&\qquad \geqslant \frac{d_1 + d_2}{\sqrt{\pi}} \int_0^1 \int_{-\sqrt{p}(1+\sqrt{\tau})/(2\sqrt{1-\tau})}^{-\sqrt{p}(1-\sqrt{\tau})/(2\sqrt{1-\tau})} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\tau - d_2 + v_0''(x_0)(p+2) > 0.
\end{aligned}$$

This means that

$$\frac{C(t) - \lambda + D(t)}{t} > 0$$

in $(0, T_1)$ with sufficiently small $T_1 > 0$. Hence, we get

$$v(t, -\sqrt{pt} + x_0) = C(t) + D(t) > \lambda$$

for $0 < t < T_1$. □

Step 2.4. Finally, we prove the following claim.

Claim 4. *v satisfies $v \leqslant 1$ in $(0, T_0) \times \mathbb{R}$ and is a weak lower solution of problem (1.1) in $(0, T_0) \times \mathbb{R}$ as in condition (A2) for some $0 < T_0 < T_1$.*

Proof. By equation (4.7) and condition (B7), we obtain, for $(t,x) \in (0,T) \times \mathbb{R}$,

$$-d_2 t < v(t,x) < \lambda + d_1 t.$$

Hence, $v \leqslant 1$ in $(0,(1-\lambda)/d_1) \times \mathbb{R}$. Thus, it remains to prove that v is a weak lower solution of problem (1.1) in $(0,T_0) \times \mathbb{R}$ as in condition (A2) for some $0 < T_0 < \min\{T_1,(1-\lambda)/d_1\}$. This can be proved as follows. By claim 3 we have

$$f(v(t,x)) - f(1)\hat{H}(v(t,x)-\lambda) - g(t,x) \geqslant \inf_{\lambda<u<\lambda+\delta_1} f(u) - f(1) - d_1 = 0$$

for $(t,x) \in (0,\min\{T_1,(1-\lambda)/d_1,\delta_1/d_1\}) \times [-\sqrt{pt}+x_0, \sqrt{pt}+x_0]$. By condition (A1), there exists a constant $\delta_2 > 0$ such that

$$\inf_{-\delta_2<u<\lambda} f(u) = \inf_{0<u<\lambda} f(u).$$

Hence, we have

$$f(v(t,x)) - f(1)\hat{H}(v(t,x)-\lambda) - g(t,x) \geqslant \inf_{0<u<\lambda} f(u) + d_2 = 0$$

for $(t,x) \in (0,\min\{T_1,(1-\lambda)/d_1,\delta_1/d_1,\delta_2/d_2\}) \times (\mathbb{R} \setminus [-\sqrt{pt}+x_0, \sqrt{pt}+x_0])$. Furthermore, $f(v(t,x)) - f(1)\hat{H}(v(t,x)-\lambda) - g(t,x)$ is bounded in $(0,T) \times \mathbb{R}$ and, by condition (B7), the inequality $v_0(x) \leqslant u_0(x)$ holds on $\mathbb{R}$. Hence, v is a weak lower solution of problem (1.1) in $(0,T_0) \times \mathbb{R}$ as in condition (A2) for $T_0 := \min\{T_1,(1-\lambda)/d_1,\delta_1/d_1,\delta_2/d_2\}$. □

The proof of theorem 4.1 is now complete. □

REMARK 4.3. In the case that the maximal and minimal weak solutions U, $u \in [0,1]$ of problem (1.1) are different, results on their asymptotic behaviour may be obtained from [4]. On the other hand, results on asymptotic behaviour of unique weak solutions of problem (1.1) with $f(u) = -u$ have been obtained by Terman [17] and McKean and Moll [13].

5. Concluding remarks

Hofbauer and Simon [10] dealt with problem (1.1) with a bounded and Borel measurable function as the nonlinearity and a bounded, uniformly continuous function as initial datum. They formulated the first equation in problem (1.1) as a differential inclusion, and showed the existence of its solution. To distinguish weak solutions of problem (1.1) from solutions of problem (1.1) formulated as a differential inclusion, we call the latter 'HS-solutions'. In this section, we will study the relationship between weak solutions and HS-solutions. For this purpose, in addition to conditions (A1) and (B1), we will impose the following conditions on u_0 and f.

(A3) f satisfies the condition

$$f(u) = \begin{cases} 0 & \text{on } (-\infty,0), \\ f(1) & \text{on } (1,\infty). \end{cases}$$

(B8) u_0 is uniformly continuous on $\mathbb{R}$.

https://doi.org/10.1017/S0308210500004315 Published online by Cambridge University Press

As stated in § 1, we are interested in solutions taking values between 0 and 1 from the viewpoint of best response dynamics. Hence, $f(u) - f(1)H(u - \lambda)$ may be regarded as zero outside the interval $[0, 1]$, and so in this sense condition (A3) is not strong. The definition of an HS-solution of problem (1.1) is given as follows.

Definition 5.1 (Hofbauer and Simon [10, definition 1]). A function

$$u \in C([0, T) \times \mathbb{R}) \cap C^{0,1}((0, T) \times \mathbb{R})$$

is said to be an *HS-solution* of problem (1.1) if the following two conditions are satisfied:

(i) there exists a bounded Borel measurable function k on $(0, T) \times \mathbb{R}$ such that, for all $\varphi \in \mathcal{D}((0, T) \times \mathbb{R})$,

$$\int_0^T \int_{\mathbb{R}} (u\partial_t\varphi - \partial_x u \partial_x \varphi + (f(u) + k(t, x))\varphi) \,\mathrm{d}x \,\mathrm{d}t = 0$$

and that

$$-f(1)\hat{H}(u(t, x) - \lambda) \leqslant k(t, x) \leqslant -f(1)\tilde{H}(u(t, x) - \lambda) \quad \text{a.e. in } (0, T) \times \mathbb{R},$$

where $\tilde{H}$ and $\hat{H}$ are as in condition (A2).

(ii) for all $x_0 \in \mathbb{R}$,

$$\lim_{t \downarrow 0, x \to x_0} u(t, x) = u_0(x_0).$$

The following theorem is a special case of [10, theorem 1].

Theorem 5.2. *Let $\lambda \in (0, 1)$, and assume that conditions (A1), (A3), (B1) and (B8) are satisfied. Then there exists an HS-solution of problem (1.1).*

We now briefly explain their proof of theorem 5.2. Let ψ be a fixed element of $\mathcal{D}(\mathbb{R})$ such that $\psi \geqslant 0$, $\operatorname{supp} \psi \subset (-1, 1)$ and $\int \psi(u) \,\mathrm{d}u = 1$. For each $n \in \mathbb{N}$, put $\psi_n(u) := n\psi(nu)$ and define two functions,

$$f_n(u) := \int_{-\infty}^{\infty} f(v)\psi_n(u - v) \,\mathrm{d}v,$$
$$H_n(u - \lambda) := \int_{-\infty}^{\infty} H(v - \lambda)\psi_n(u - v) \,\mathrm{d}v,$$

on $\mathbb{R}$. Since $u \mapsto f_n(u) - f(1)H_n(u - \lambda)$ is Lipschitz continuous on $\mathbb{R}$ for each $n \in \mathbb{N}$, there exists a unique bounded classical solution u^n of the problem

$$\left.\begin{aligned} u^n_t &= u^n_{xx} + f_n(u^n) - f(1)H_n(u^n - \lambda), && 0 < t < T, \quad x \in \mathbb{R}, \\ u^n|_{t=0} &= u_0, && x \in \mathbb{R}. \end{aligned}\right\} \tag{5.1}$$

They proved that a subsequence of $(u^n)_{n \in \mathbb{N}}$ converges to an HS-solution of problem (1.1). From this, we can obtain the following proposition.

Proposition 5.3. *Let $\lambda \in (0, 1)$, and assume that conditions (A1), (A3), (B1) and (B8) are satisfied. Then the HS-solution u obtained by Hofbauer and Simon [10] of problem (1.1) is contained in the order interval* $[0, 1]$.

Proof. By condition (A3), it is easy to see that $f_n(1+1/n)-f(1)H_n(1+1/n-\lambda)=0$ and $f_n(-1/n)-f(1)H_n(-1/n-\lambda)=0$ for each $n\in\mathbb{N}$. Hence, $u_1^n(t,x):=1+1/n$ and $u_2^n(t,x):=-1/n$ are classical solutions of problem (5.1) corresponding to initial data $1+1/n$ and $-1/n$, respectively. Furthermore, for each $n\in\mathbb{N}$, note that $-1/n<u_0(x)<1+1/n$ on $\mathbb{R}$ and that $u\mapsto f_n(u)-f(1)H_n(u-\lambda)$ is Lipschitz continuous on $\mathbb{R}$. Therefore, we can apply a comparison theorem to obtain the relationship

$$u_2^n(t,x)=-\frac{1}{n}\leqslant u^n(t,x)\leqslant u_1^n(t,x)=1+\frac{1}{n}$$

for each $n\in\mathbb{N}$. Hence, on taking the limit as $n\uparrow\infty$, we obtain the assertion. □

It is easy to see that the maximal and minimal weak solutions U, $u\in[0,1]$ obtained in theorem 2.3 of problem (1.1) are HS-solutions of problem (1.1). The following proposition shows the relationship between them and other HS-solutions.

PROPOSITION 5.4. *Let $\lambda\in(0,1)$, and assume that conditions (A1), (A3), (B1) and (B8) are satisfied. Then the maximal and minimal weak solutions U, $u\in[0,1]$ obtained in theorem 2.3 of problem (1.1) are the maximal and minimal HS-solutions, respectively, of problem (1.1) in the order interval* $[0,1]$.

Proof. We will prove only that the maximal weak solution $U\in[0,1]$ of problem (1.1) is the maximal HS-solution of problem (1.1) in the order interval $[0,1]$. The assertion for the minimal weak solution $u\in[0,1]$ can be proved similarly.

Let $v\in[0,1]$ be any HS-solution of problem (1.1). Then, by definition 5.1, there exists a bounded Borel measurable function k on $(0,T)\times\mathbb{R}$ such that v is a weak solution of the problem

$$\left.\begin{aligned} v_t&=v_{xx}+f(v)+k(t,x), && 0<t<T, && x\in\mathbb{R},\\ v|_{t=0}&=u_0, && && x\in\mathbb{R}.\end{aligned}\right\}$$

Since, by condition (A1), the function f is Lipschitz continuous on $\mathbb{R}$, there exists a constant $M\geqslant 0$ such that $u\mapsto f(u)+Mu$ is non-decreasing on $\mathbb{R}$. With this constant $M\geqslant 0$, we define $w(t,x):=v(t,x)\mathrm{e}^{Mt}$. We then find that w is a weak solution of the problem

$$\left.\begin{aligned} w_t&=w_{xx}+(f(w\mathrm{e}^{-Mt})+Mw\mathrm{e}^{-Mt}+k(t,x))\mathrm{e}^{Mt}, && 0<t<T, && x\in\mathbb{R},\\ w|_{t=0}&=u_0, && && x\in\mathbb{R}.\end{aligned}\right\}$$

We can apply a similar argument to the proof of lemma 2.5(a) in order to find that

$$\begin{aligned} w(t,x)={}&\int_{\mathbb{R}}K(t,x-y)u_0(y)\,\mathrm{d}y\\ &+\int_0^t\int_{\mathbb{R}}K(t-s,x-y)(f(w\mathrm{e}^{-Ms})+Mw\mathrm{e}^{-Ms}+k(s,y))\mathrm{e}^{Ms}\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

Note that $w(t,x):=v(t,x)\mathrm{e}^{Mt}$ and that

$$k(t,x)\leqslant -f(1)\tilde{H}(v(t,x)-\lambda)\quad\text{a.e. in }(0,T)\times\mathbb{R}.$$

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Hence,

$$v(t,x) \leqslant \mathrm{e}^{-Mt}\int_{\mathbb{R}} K(t,x-y)u_0(y)\,\mathrm{d}y + \int_0^t\int_{\mathbb{R}} K(t-s,x-y)\tilde{f}_M(v(s,y))\mathrm{e}^{-M(t-s)}\,\mathrm{d}y\,\mathrm{d}s,$$

where $\tilde{f}_M(v) := f(v) - f(1)\tilde{H}(v-\lambda) + Mv$. Therefore, as in step 1.3 of the proof of theorem 2.3, we can show that $v \leqslant U$ in $(0,T)\times\mathbb{R}$. □

Remark 5.5. By propositions 5.3 and 5.4, the HS-solution v obtained by Hofbauer and Simon [10] of problem (1.1) satisfies the relationship $u \leqslant v \leqslant U$ for the maximal and minimal weak solutions U, $u \in [0,1]$ obtained in theorem 2.3 of problem (1.1).

Remark 5.6. By proposition 5.4, theorems 3.1, 3.3 and 3.4, which are local or global uniqueness theorems for weak solutions of problem (1.1), hold for HS-solutions of problem (1.1) in the order interval $[0,1]$. Furthermore, theorem 4.1, which is a non-uniqueness theorem for weak solutions of problem (1.1), explains the non-uniqueness of HS-solutions of problem (1.1) in the order interval $[0,1]$.

Acknowledgments

The author expresses hearty thanks to Professor Izumi Kubo of the Hiroshima Institute of Technology for guidance on this paper. This research was partly supported by Grant-in-Aid for JSPS Fellows (no. 0300097).

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(*Issued 16 December 2005*)