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Generalized Majority Colourings of Digraphs

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We almost completely solve a number of problems related to a concept called majority colouring recently studied by Kreutzer, Oum, Seymour, van der Zypen and Wood. They raised the problem of determining, for a natural number k, the smallest number $m = m(k)$ such that every digraph can be coloured with m colours where each vertex has the same colour as at most a $1/k$ proportion of its out-neighbours. We show that $m(k) \in \{2k-1, 2k\}$. We also prove a result supporting the conjecture that $m(2) = 3$. Moreover, we prove similar results for a more general concept called majority choosability.

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For a natural number $k \geqslant 2$, a $1/k$*-majority colouring* of a digraph is a colouring of the vertices such that each vertex receives the same colour as at most a $1/k$ proportion of its out-neighbours. We say that a digraph D is $1/k$*-majority* m*-colourable* if there exists a $1/k$-majority colouring of D using m colours. The following natural question was recently raised by Kreutzer, Oum, Seymour, van der Zypen and Wood [6].

Question 1. *Given $k \geqslant 2$, determine the smallest number $m = m(k)$ such that every digraph is $1/k$-majority m-colourable.*

In particular, they asked whether $m(k) = O(k)$. Let us first observe that $m(k) \geqslant 2k-1$. Consider a tournament on $2k-1$ vertices where every vertex has out-degree $k-1$. Any $1/k$-majority colouring of this tournament must be a proper vertex-colouring, and hence it needs at least $2k-1$ colours. Conversely, we prove that $m(k) \leqslant 2k$.

Theorem 2. *Every digraph is $1/k$-majority $2k$-colourable for all $k \geqslant 2$.*

This is an immediate consequence of a result of Keith Ball (see [3]) about partitions of matrices. We shall use a slightly more general version proved by Alon [1].

Lemma 3. *Let $A = (a_{ij})$ be an $n \times n$ real matrix where $a_{ii} = 0$ for all i, $a_{ij} \geqslant 0$ for all $i \neq j$, and $\sum_j a_{ij} \leqslant 1$ for all i. Then, for every t and all positive reals $c_1, \ldots, c_t$ whose sum is 1, there is a partition of $\{1,2,\ldots,n\}$ into pairwise disjoint sets $S_1, S_2, \ldots, S_t$, such that for every r and every $i \in S_r$, we have $\sum_{j \in S_r} a_{ij} \leqslant 2c_r$.*

Proof of Theorem 2. Let D be a digraph on n vertices with vertex set $\{v_1, v_2, \ldots, v_n\}$, and write $d^+(v_i)$ for the out-degree of v_i. Let $A = (a_{ij})$ be an $n \times n$ matrix where $a_{ij} = 1/d^+(v_i)$ if there is a directed edge from v_i to v_j and $a_{ij} = 0$ otherwise. We apply Lemma 3 with $t = 2k$ and $c_i = 1/2k$ for $1 \leqslant i \leqslant 2k$, obtaining a partition of $\{1,2,\ldots,n\}$ into sets $S_1, S_2, \ldots, S_{2k}$, such that for every r and every $i \in S_r$, $\sum_{j \in S_r} a_{ij} \leqslant 1/k$. Equivalently, the number of out-neighbours of v_i that have the same colour as v_i is at most $d^+(v_i)/k$, where the colouring of D is defined by the partition $S_1 \cup S_2 \cup \cdots \cup S_{2k}$. □

Question 1 has now been reduced to whether $m(k)$ is $2k-1$ or $2k$.

Question 4. *Is every digraph $1/k$-majority $(2k-1)$-colourable?*

Surprisingly, this is open even for $k = 2$. Kreutzer, Oum, Seymour, van der Zypen and Wood [6] gave an elegant argument showing that every digraph is $1/2$-majority 4-colourable and they conjectured that $m(2) = 3$.

Conjecture 5. *Every digraph is $1/2$-majority 3-colourable.*

We provide evidence for this conjecture by proving that tournaments are *almost* $1/2$-majority 3-colourable.

Theorem 6. *Every tournament can be 3-coloured in such a way that all but at most 205 vertices receive the same colour as at most half of their out-neighbours.*

Proof. The proof relies on an observation that in a tournament T, the set

$$S_i = \{x \in V(T) : 2^{i-1} \leqslant d^+(x) < 2^i\}$$

has size at most 2^{i+1}. Indeed, the sum of the out-degrees of the vertices of S_i is at least $\binom{|S_i|}{2}$, the number of edges inside S_i. On the other hand, this sum is at most $(2^i - 1)|S_i|$ by the definition of

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S_i. Therefore

$$\binom{|S_i|}{2} \leqslant (2^i - 1)|S_i|,$$

and hence $|S_i| \leqslant 2^{i+1} - 1$.

We proceed by randomly assigning one of three colours to each vertex independently with probability $1/3$. Given a vertex x, let B_x be the number of out-neighbours of x which receive the same colour as x. We say that x is *bad* if $B_x > d^+(x)/2$. Trivially $\mathbb{E}(B_x) = d^+(x)/3$, and hence by a Chernoff-type bound, it follows that, for $x \in S_i$,

$$\begin{aligned}\mathbb{P}(x \text{ is bad}) &= \mathbb{P}(B_x > d^+(x)/2) = \mathbb{P}(B_x > (1+1/2)\mathbb{E}(B(x))) \\ &\leqslant \exp\left(-\frac{(1/2)^2}{3}\mathbb{E}(B_x)\right) = \exp(-d^+(x)/36) \leqslant \exp(-2^{i-1}/36).\end{aligned}$$

Notice that if $i \geqslant 11$ then $\mathbb{P}(x \text{ is bad}) \leqslant 2^{-(2i-7)}$. Let X denote the total number of bad vertices. Since the vertices of out-degree 0 cannot be bad,

$$\begin{aligned}\mathbb{E}(X) &= \sum_{i\geqslant 1}\sum_{x\in S_i}\mathbb{P}(x \text{ is bad}) \leqslant \sum_{i=1}^{10} 2^{i+1}\exp(-2^{i-1}/36) + \sum_{i\geqslant 11} 2^{i+1}2^{-(2i-7)} \\ &\leqslant 205 + \sum_{i\geqslant 11} 2^{-i+8} = 205 + \frac{1}{4} < 206.\end{aligned}$$

Hence, there is a 3-colouring such that all but at most 205 vertices receive the same colour as at most half of their out-neighbours. □

Observe also that the same argument proves a special case of Conjecture 5.

Theorem 7. *Every tournament with minimum out-degree at least* 2^{10} *is* $1/2$*-majority* 3*-colourable.*

We remark that Theorem 6 can be strengthened (205 can be replaced by 7) by solving a linear programming problem. Recall that the expected number of bad vertices of out-degree at least 1024 is at most $1/4$. We shall use linear programming to show that the expected number of bad vertices of out-degree less than 1024 is less than 7.75. Let V_i be the set of vertices of out-degree i for $i \in \{1,2,\dots,1023\}$ and note that the expected number of bad vertices of out-degree at most 1023 is $f(v_1,v_2,\dots,v_{1023}) = \sum_{i=1}^{1023} v_i p_i$ where $v_i = |V_i|$ and

$$p_i = \sum_{j=\lceil (i+1)/2\rceil}^{i} \binom{i}{j}(1/3)^j(2/3)^{i-j}.$$

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As before, observe that the number of vertices of degree at most i is at most $2i+1$, and therefore, $\sum_{j=1}^{i} v_i \leqslant 2i+1$, leading to the following linear program:

$$\text{Maximize: } f(v_1, v_2, \ldots, v_{1023})$$
$$\text{Subject to: } \sum_{j=1}^{i} v_j \leqslant 2i+1, \text{ for } i \in \{1,2,\ldots,1023\}$$
$$\text{Subject to: } v_i \geqslant 0, \text{ for } i \in \{1,2,\ldots,1023\}.$$

See the Appendix for the source code. Similarly, we can replace 2^{10} in Theorem 7 by 55, by using the same linear program to show that the expected number of bad vertices of out-degree in $[55, 1023]$ is less than $3/4$.

Let us now change direction to a more general concept of majority choosability. A digraph is *$1/k$-majority m-choosable* if, for any assignment of lists of m colours to the vertices, there exists a $1/k$-majority colouring where each vertex gets a colour from its list. In particular, a $1/k$-majority m-choosable digraph is $1/k$-majority m-colourable. Kreutzer, Oum, Seymour, van der Zypen and Wood [6] asked whether there exists a finite number m such that every digraph is $1/2$-majority m-choosable. Anholcer, Bosek and Grytczuk [2] showed that the statement holds with $m = 4$. We generalize their result as follows.

Theorem 8. *Every digraph is $1/k$-majority $2k$-choosable for all $k \geqslant 2$.*

Theorem 8 was independently proved by Fiachra Knox and Robert Šámal [5]. We prove Theorem 8 using a slight modification of Lemma 3.

Lemma 9. *Let $A = (a_{ij})$ be an $n \times n$ real matrix where $a_{ii} = 0$ for all i, $a_{ij} \geqslant 0$ for all $i \neq j$, and $\sum_j a_{ij} \leqslant 1$ for all i. Then, for every m and subsets $L_1, L_2, \ldots, L_n \subset \mathbb{N}$ of size m, there is a function $f : \{1,2,\ldots,n\} \to \mathbb{N}$ such that, for every i, $f(i) \in L_i$ and $\sum_{j \in f^{-1}(r)} a_{ij} \leqslant 2/m$ where $r = f(i)$.*

Proof. By increasing some of the numbers a_{ij}, if needed, we may assume that $\sum_j a_{ij} = 1$ for all i. We may also assume, by an obvious continuity argument, that $a_{ij} > 0$ for all $i \neq j$. Thus, by the Perron–Frobenius theorem, 1 is the largest eigenvalue of A with right eigenvector $(1,1,\ldots,1)$ and left eigenvector $(u_1, u_2, \ldots, u_n)$ in which all entries are positive. It follows that $\sum_i u_i a_{ij} = u_j$. Define $b_{ij} = u_i a_{ij}$; then

$$\sum_i b_{ij} = u_j, \quad \sum_j b_{ij} = u_i \left(\sum_j a_{ij} \right) = u_i.$$

Let $f : \{1,2,\ldots,n\} \to \mathbb{N}$ be a function such that $f(i) \in L_i$ and f minimizes the sum

$$\sum_{r \in \mathbb{N}} \sum_{i,j \in f^{-1}(r)} b_{ij}.$$

By minimality, the value of the sum will not decrease if we change $f(i)$ from r to l where $l \in L_i$. Therefore, for any $i \in f^{-1}(r)$ and $l \in L_i$, we have

$$\sum_{j \in f^{-1}(r)} (b_{ij} + b_{ji}) \leqslant \sum_{j \in f^{-1}(l)} (b_{ij} + b_{ji}).$$

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Summing over all $l \in L_i$, we conclude that

$$m \sum_{j \in f^{-1}(r)} (b_{ij} + b_{ji}) \leqslant \sum_{j \in f^{-1}(L_i)} (b_{ij} + b_{ji}) \leqslant \sum_{j=1}^{n} (b_{ij} + b_{ji}) = 2u_i.$$

Hence,

$$\sum_{j \in f^{-1}(r)} u_i a_{ij} = \sum_{j \in f^{-1}(r)} b_{ij} \leqslant \sum_{j \in f^{-1}(r)} (b_{ij} + b_{ji}) \leqslant \frac{2u_i}{m}.$$

Dividing by u_i, the desired result follows. □

Proof of Theorem 8. The proof is the same as that of Theorem 2, using Lemma 9 instead of Lemma 3. □

In fact, the same statement also holds when the size of the lists is odd.

Corollary 10. *Every digraph is $2/m$-majority m-choosable for all $m \geqslant 2$.*

This statement generalizes a result of Anholcer, Bosek and Grytczuk [2] where they prove the case $m = 3$, which says that, given a digraph with colour lists of size three assigned to the vertices, there is a colouring from these lists such that each vertex has the same colour as at most two-thirds of its out-neighbours.

We have established that the $1/k$-majority choosability number is either $2k - 1$ or $2k$. Let us end this note with an analogue of Question 4.

Question 11. *Is every digraph $1/k$-majority $(2k-1)$-choosable?*

Appendix: Linear program

We use the toolkit [4] to solve the linear program with the following source code:

```
param N := 1024;

param comb 'n choose k' {n in 0..N, k in 0..n} :=
    if k = 0 or k = n then 1 else comb[n-1,k-1] + comb[n-1,k];
param prob 'probability' {n in 0..N} :=
    sum{k in (floor(n/2)+1)..n} comb[n,k]*((1/3)^k)*((2/3)^(n-k));

var x{1..N}, integer, >= 0;

subject to constraint{i in 1..N}: sum{j in 1..i} x[j] <= 2*i+1;

maximize expectation: sum{i in 1..N} x[i]*prob[i];

solve;

end;
```

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