Ergod. Th. & Dynam. Sys. (2021), **41**, 2241–2273
doi:10.1017/etds.2020.38

Relative bifurcation sets and the local dimension of univoque bases

PIETER ALLAART† and DERONG KONG‡

† *Mathematics Department, University of North Texas, 1155 Union Cir #311430, Denton, TX 76203-5017, USA*
(e-mail: allaart@unt.edu)

‡ *College of Mathematics and Statistics, Chongqing University, 401331, Chongqing, PR China*
(e-mail: derongkong@126.com)

(*Received* 21 *September* 2018 *and accepted in revised form* 10 *April* 2020)

Abstract. Fix an alphabet $A = \{0, 1, \ldots, M\}$ with $M \in \mathbb{N}$. The univoque set $\mathscr{U}$ of bases $q \in (1, M + 1)$ in which the number 1 has a unique expansion over the alphabet A has been well studied. It has Lebesgue measure zero but Hausdorff dimension one. This paper describes how the points in the set $\mathscr{U}$ are distributed over the interval $(1, M + 1)$ by determining the limit

$$f(q) := \lim_{\delta\to 0} \dim_{\mathrm{H}}(\mathscr{U} \cap (q - \delta, q + \delta))$$

for all $q \in (1, M + 1)$. We show in particular that $f(q) > 0$ if and only if $q \in \overline{\mathscr{U}}\setminus\mathscr{C}$, where $\mathscr{C}$ is an uncountable set of Hausdorff dimension zero, and f is continuous at those (and only those) points where it vanishes. Furthermore, we introduce a countable family of pairwise disjoint subsets of $\mathscr{U}$ called relative bifurcation sets, and use them to give an explicit expression for the Hausdorff dimension of the intersection of $\mathscr{U}$ with any interval, answering a question of Kalle *et al* [On the bifurcation set of unique expansions. Acta Arith. **188** (2019), 367–399]. Finally, the methods developed in this paper are used to give a complete answer to a question of the first author [On univoque and strongly univoque sets. *Adv. Math.* **308** (2017), 575–598] on strongly univoque sets.

Key words: univoque bases, Hausdorff dimension, bifurcation set, relative entropy plateau, strongly univoque set

2020 Mathematics Subject Classification: 11A63 (Primary); 37B10, 28A80 (Secondary)

1. *Introduction*

Fix an integer $M \geq 1$. For $q \in (1, M+1]$, any real number x in the interval $I_{M,q} := [0, M/(q-1)]$ can be represented as

$$x = \pi_q((d_i)) := \sum_{i=1}^{\infty} \frac{d_i}{q^i}, \tag{1.1}$$

where $d_i \in \{0, 1, \ldots, M\}$ for all $i \geq 1$. The infinite sequence $(d_i) = d_1 d_2 \cdots$ is called a *q-expansion* of x with *alphabet* $\{0, 1, \ldots, M\}$. Such non-integer base expansions have been studied since the pioneering work of Rényi [**36**] and Parry [**35**]. In the 1990s, the work by Erdős *et al* [**19–21**] inspired an explosion of research papers on the subject, covering unique expansions [**2**, **17**, **23**, **27**], finitely or countably many expansions [**11**, **12**, **26**, **39**], uncountably many expansions and random expansions [**14**, **37**]. Non-integer base expansions have furthermore been connected with Bernoulli convolutions [**24**], Diophantine approximation [**34**], singular self-affine functions [**3**], open dynamical systems [**38**], and intersections of Cantor sets [**32**].

Let

$$\mathscr{U} := \{q \in (1, M+1] : 1 \text{ has a unique } q\text{-expansion of the form (1.1)}\}.$$

Then for each $q \in \mathscr{U}$ there exists a unique sequence $(a_i) \in \Omega_M := \{0, 1, \ldots, M\}^{\mathbb{N}}$ such that $1 = \pi_q((a_i))$. The set $\mathscr{U}$ was extensively studied for over 25 years. Erdős *et al* [**19**] showed that $\mathscr{U}$ is uncountable and of zero Lebesgue measure. Daróczy and Kátai [**16**] proved that $\mathscr{U}$ has full Hausdorff dimension (see also [**27**]). Komornik and Loreti [**28**, **29**] found its smallest element $q_{\text{KL}} = q_{\text{KL}}(M)$, which is now called the *Komornik–Loreti constant* and is related to the Thue–Morse sequence (see equation (6.1) below). Later in [**30**] the same authors proved that its topological closure $\overline{\mathscr{U}}$ is a Cantor set, i.e. a non-empty compact set having neither interior nor isolated points. Recently, Dajani *et al* [**15**] proved that the algebraic difference $\mathscr{U} - \mathscr{U}$ contains an interval. Furthermore, the set $\mathscr{U}$ also has intimate connections with the kneading sequences of unimodal expanding maps (cf. [**8**, **9**]), and even with the real slice of the boundary of the Mandelbrot set [**13**].

The main purpose of this paper is to describe the distribution of $\mathscr{U}$. More precisely, we are interested in the *local dimensional function*

$$f(q) := \lim_{\delta \to 0} \dim_{\text{H}}(\mathscr{U} \cap (q-\delta, q+\delta)), \quad q \in (1, M+1],$$

as well as its one-sided analogs

$$f_-(q) := \lim_{\delta \to 0} \dim_{\text{H}}(\mathscr{U} \cap (q-\delta, q)), \quad f_+(q) := \lim_{\delta \to 0} \dim_{\text{H}}(\mathscr{U} \cap (q, q+\delta)),$$

which we call the *left and right local dimensional functions* of $\mathscr{U}$, respectively. Here $\dim_{\text{H}}$ stands for the Hausdorff dimension (cf. [**22**]). Note that $f = \max\{f_-, f_+\}$, and if $q \notin \overline{\mathscr{U}}$, then $f(q) = f_-(q) = f_+(q) = 0$. Extending a recent result by Baker and the authors [**6**], we compute $f(q)$, $f_-(q)$ and $f_+(q)$ for every $q \in (1, M+1]$ in terms of a kind of localized entropy. (See Figure 2 below for a rough graph of f.) As an application we compute the Hausdorff dimension of the intersection of $\mathscr{U}$ with any interval, answering a question of Kalle *et al* [**25**]. In addition, our methods allow us to give a complete answer to a question of the first author [**4**] on strongly univoque sets.

1.1. *Univoque set, entropy plateaus and the bifurcation set.* In order to state our main results, some notation is necessary. For $q \in (1, M+1]$ let $\mathcal{U}_q$ be the *univoque set* of $x \in I_{M,q}$ having a unique q-expansion as in equation (1.1). Let $\mathbf{U}_q$ be the set of corresponding sequences, i.e.

$$\mathbf{U}_q := \{(d_i) \in \Omega_M : \pi_q((d_i)) \in \mathcal{U}_q\}.$$

A useful tool in the study of unique expansions is the lexicographical characterization of $\mathbf{U}_q$ (cf. [**10**, **17**]): $(d_i) \in \mathbf{U}_q$ if and only if $(d_i) \in \Omega_M$ satisfies

$$\begin{aligned} d_{n+1}d_{n+2}\cdots \prec \alpha(q) \quad &\text{if } d_n < M, \\ d_{n+1}d_{n+2}\cdots \succ \overline{\alpha(q)} \quad &\text{if } d_n > 0, \end{aligned} \tag{1.2}$$

where $\alpha(q) = (\alpha_i(q)) \in \Omega_M$ is the lexicographically largest q-expansion of 1 not ending with 0^∞, called the *quasi-greedy* q-expansion of 1, and $\overline{\alpha(q)} := (M - \alpha_i(q))$. Here and throughout the paper we will use the lexicographical order between sequences and blocks in a natural way.

Note by equation (1.2) that any sequence $(d_i) \in \mathbf{U}_q \setminus \{0^\infty, M^\infty\}$ has a tail sequence in the set

$$\widetilde{\mathbf{U}}_q := \{(d_i) \in \Omega_M : \overline{\alpha(q)} \prec \sigma^n((d_i)) \prec \alpha(q) \ \forall\, n \geq 0\}, \tag{1.3}$$

where σ denotes the left shift map on Ω_M. Furthermore, $\mathbf{U}_q$ and $\widetilde{\mathbf{U}}_q$ have the same topological entropy, i.e. $h(\mathbf{U}_q) = h(\widetilde{\mathbf{U}}_q)$, where the *topological entropy* of a subset $X \subset \Omega_M$ is defined by

$$h(X) := \liminf_{n\to\infty} \frac{\log \# B_n(X)}{n}$$

(cf. [**33**]). Here $\# B_n(X)$ denotes the number of all length n blocks occurring in sequences from X, and 'log' denotes the natural logarithm. We may thus obtain all the relevant information about $\mathbf{U}_q$ by studying the simpler set $\widetilde{\mathbf{U}}_q$.

Since the map $q \mapsto \alpha(q)$ is strictly increasing on $(1, M+1]$ (see Lemma 2.1 below), equation (1.3) implies that the set-valued map $q \mapsto \widetilde{\mathbf{U}}_q$ is non-decreasing, and hence the entropy function $H : q \mapsto h(\widetilde{\mathbf{U}}_q)$ is non-decreasing. Recently, Komornik *et al* [**27**] and the present authors [**7**] proved the following theorem.

THEOREM 1.1. [**7**, **27**] *The graph of H (see Figure 1) is a Devil's staircase:*
(i) *H is non-decreasing and continuous on* $(1, M+1]$*;*
(ii) *H is locally constant almost everywhere on* $(1, M+1]$*;*
(iii) *$H(q) > 0$ if and only if $q > q_{\mathrm{KL}}$, where q_{KL} is the Komornik–Loreti constant.*

An interval $[p_{\mathrm{L}}, p_{\mathrm{R}}] \subset (1, M+1]$ is called an *entropy plateau* (or simply, a *plateau*) if it is a maximal interval (in the partial order of set inclusion) on which H is constant and positive. A complete characterization of all entropy plateaus was given by Alcaraz Barrera *et al* [**2**] (see also [**1**] for the case $M = 1$). Equivalently, they described the *bifurcation set*

$$\mathscr{B} := \{q \in (1, M+1] : H(p) \neq H(q) \ \forall\, p \neq q\}, \tag{1.4}$$

and showed that $\mathscr{B} \subset \mathscr{U}$, $\mathscr{B}$ is Lebesgue null, and $\dim_{\mathrm{H}} \mathscr{B} = 1$. From Theorem 1.1 and the definition of $\mathscr{B}$ it follows that

$$(1, M+1] \setminus \mathscr{B} = (1, q_{\mathrm{KL}}] \cup \bigcup [p_{\mathrm{L}}, p_{\mathrm{R}}], \tag{1.5}$$

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

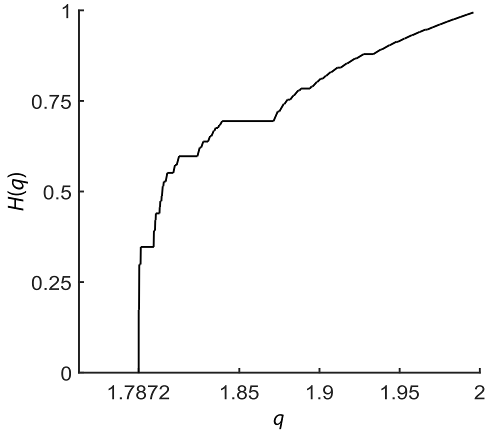

FIGURE 1. Graph of H for $M = 1$ (using the base 2 logarithm instead of the natural logarithm). It is positive above $q_{\mathrm{KL}} \approx 1.7872$.

where the union is taken over all plateaus $[p_{\mathrm{L}}, p_{\mathrm{R}}] \subset (q_{\mathrm{KL}}, M+1]$ of H. We emphasize that the plateaus are pairwise disjoint and therefore the union is countable.

Recall that our main objective is to find the local dimensional functions f, f_+ and f_-. The following result is due to Baker and the authors [**6**].

PROPOSITION 1.2. (Allaart, Baker and Kong [**6**]) *For any $q \in \mathscr{B} \setminus \{M+1\}$ we have*

$$f(q) = f_-(q) = f_+(q) = \dim_{\mathrm{H}} \mathcal{U}_q > 0,$$

and for any $q \in (1, M+1]$ we have $f(q) \le \dim_{\mathrm{H}} \mathcal{U}_q$. Furthermore, for $q = M+1$ we have $f(q) = f_-(q) = 1$ and $f_+(q) = 0$.

1.2. *Relative bifurcation sets and relative plateaus.* In order to describe the local dimensional function f of $\mathscr{U}$, we introduce the relative bifurcation sets, which provide finer information about the growth of $q \mapsto \widetilde{\mathbf{U}}_q$ inside entropy plateaus.

Definition 1.3. A word $a_1 \cdots a_m \in \{0, 1, \ldots, M\}^m$ with $m \ge 2$ is *admissible* if

$$\overline{a_1 \cdots a_{m-i}} \preccurlyeq a_{i+1} \cdots a_m \prec a_1 \cdots a_{m-i} \quad \text{for all } 1 \le i < m. \tag{1.6}$$

When $M \ge 2$, the 'word' $a_1 \in \{0, 1, \ldots, M\}$ is *admissible* if $\overline{a_1} \le a_1 < M$.

For any admissible word $\mathbf{a}$, there are bases q_{L} and q_{R} such that

$$\alpha(q_{\mathrm{L}}) = \mathbf{a}^\infty, \quad \alpha(q_{\mathrm{R}}) = \mathbf{a}^+(\overline{\mathbf{a}})^\infty.$$

Here, for a word $\mathbf{c} := c_1 \cdots c_n \in \{0, 1, \ldots, M\}^n$ with $c_n < M$ we set $\mathbf{c}^+ := c_1 \cdots c_{n-1}(c_n + 1)$. Similarly, for a word $\mathbf{c} := c_1 \cdots c_n \in \{0, 1, \ldots, M\}^n$ with $c_n > 0$ we shall write $\mathbf{c}^- := c_1 \cdots c_{n-1}(c_n - 1)$. We call $[q_{\mathrm{L}}, q_{\mathrm{R}}]$ a *basic interval* and say it is *generated by* the word $\mathbf{a}$. By [**2**, Lemma 4.8], any two basic intervals are either disjoint, or else one contains the other. For any basic interval I generated by an admissible word $\mathbf{a}$, we define the associated *de Vries–Komornik number* $q_c(I)$ by $\alpha(q_c(I)) = (\theta_i)$ (cf. [**31**]), where (θ_i) is given recursively by

(i) $\theta_1 \cdots \theta_m = \mathbf{a}^+$;
(ii) $\theta_{2^{k-1}m+1} \cdots \theta_{2^k m} = \overline{\theta_1 \cdots \theta_{2^{k-1}m}}^+$, for $k = 1, 2, \ldots$.
Thus,

$$\alpha(q_c(I)) = \mathbf{a}^+\overline{\mathbf{a}}\overline{\mathbf{a}^+}\mathbf{a}^+\overline{\mathbf{a}^+}\mathbf{a}\mathbf{a}^+\overline{\mathbf{a}} \cdots . \tag{1.7}$$

Note that $q_c(I)$ lies in the interior of I for each basic interval I; this is a direct consequence of Lemma 2.1 below. Observe also that different basic intervals can have the same associated de Vries–Komornik number.

We now construct a nested tree

$$\{J_\mathbf{i} : \mathbf{i} \in \{1, 2, \ldots\}^n;\, n \geq 1\}$$

of intervals, which we call *relative entropy plateaus*, or simply *relative plateaus*, as follows. At level 0, we set $J_\emptyset = [1, M+1]$. Next, at level 1, we put $J_0 = [1, q_{\mathrm{KL}}]$ and let $J_1, J_2, \ldots$ be an arbitrary enumeration of the entropy plateaus $[p_\mathrm{L}, p_\mathrm{R}]$ from equation (1.5). Note by [**2**] that these entropy plateaus are precisely the maximal basic intervals, which lie completely to the right of q_{KL}. We call J_0 a *null interval*, since $\mathscr{U} \cap (1, q_{\mathrm{KL}}) = \emptyset$.

From here, we proceed inductively as follows. Let $n \geq 1$, and for each $\mathbf{i} \in \{1, 2, \ldots\}^n$, assume $J_\mathbf{i}$ has already been defined and is a basic interval $[q_\mathrm{L}, q_\mathrm{R}]$. Then we set $J_{\mathbf{i}0} = [q_\mathrm{L}, q_c(J_\mathbf{i})]$, and let $J_{\mathbf{i}1}, J_{\mathbf{i}2}, \ldots$ be an arbitrary enumeration of the maximal basic intervals inside $[q_c(J_\mathbf{i}), q_\mathrm{R}]$. (It is not difficult to see that infinitely many such basic intervals exist.)

Note that for each fixed $n \geq 1$ the relative plateaus $J_\mathbf{i}, \mathbf{i} \in \{1, 2, \ldots\}^n$ are pairwise disjoint. Furthermore, for any word $\mathbf{i} \in \bigcup_{n=1}^\infty \{1, 2, \ldots\}^n$ we call $J_{\mathbf{i}0}$ a *null interval*, because it intersects $\mathscr{U}$ only in the single point $q_c(J_\mathbf{i})$. We emphasize that any basic interval generated by an admissible word $\mathbf{a}$ not of the form $\mathbf{b}\overline{\mathbf{b}}$ is a relative plateau.

We now define the sets

$$\mathscr{C}_\infty := \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i} \in \{1,2,\ldots\}^n} J_\mathbf{i}$$

and

$$\mathscr{C}_0 := \left\{ q_c(J_\mathbf{i}) : \mathbf{i} \in \bigcup_{n=0}^{\infty} \{1, 2, \ldots\}^n \right\}.$$

Thus $\mathscr{C}_\infty$ is the set of points that are contained in infinitely many relative plateaus, and $\mathscr{C}_0$ is the set of all de Vries–Komornik numbers (cf. [**31**]). The smallest element of $\mathscr{C}_0$ is the Komornik–Loreti constant $q_{\mathrm{KL}} = q_c(J_\emptyset)$. Finally, let

$$\mathscr{C} := \mathscr{C}_0 \cup \mathscr{C}_\infty.$$

For the proof of the following proposition, as well as examples of points in $\mathscr{C}$, we refer to §2.

PROPOSITION 1.4.
(i) *$\mathscr{C} \subset \mathscr{U}$;*
(ii) *$\mathscr{C}$ is uncountable and has no isolated points;*
(iii) $\dim_\mathrm{H} \mathscr{C} = 0$.

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

1.3. *Main results.* Let $J=[q_L, q_R]$ be a relative plateau with $J \neq [1, M+1]$. Then there is an admissible word $\mathbf{a}=a_1 \cdots a_m$ such that $\alpha(q_L)=\mathbf{a}^\infty$ and $\alpha(q_R)=\mathbf{a}^+(\overline{\mathbf{a}})^\infty$. In particular, $\alpha(q)$ begins with the prefix $\mathbf{a}^+$ for each $q \in (q_L, q_R]$. Let

$$\widetilde{\mathbf{U}}_q(J) := \{(x_i) \in \widetilde{\mathbf{U}}_q : x_1 \cdots x_m = \alpha_1(q) \cdots \alpha_m(q) = a_1 \cdots a_m^+\}, \quad q \in (q_L, q_R]. \tag{1.8}$$

For the special case when $J = J_\emptyset = [1, M+1]$, we set $\widetilde{\mathbf{U}}_q(J) := \widetilde{\mathbf{U}}_q$. We are now ready to give a characterization of the local dimensional functions f, f_- and f_+.

THEOREM 1.

(i) *Let $q \in \overline{\mathscr{U}}$. Then*

$$f(q)=0 \iff f_-(q)=0 \iff q \in \mathscr{C}.$$

(ii) *Let $q \in \overline{\mathscr{U}} \setminus \mathscr{C}$. Then*

$$f_-(q) = \frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} > 0,$$

where $J=[q_L, q_R]$ is the smallest relative plateau such that $q \in (q_L, q_R]$. Furthermore,

$$f_+(q) = \begin{cases} 0 & \text{if } q \in \overline{\mathscr{U}} \setminus \mathscr{U}, \\ \dfrac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} > 0 & \text{if } q \in \mathscr{U} \setminus \mathscr{C}, \end{cases}$$

where $J=[q_L, q_R]$ is the smallest relative plateau such that $q \in (q_L, q_R)$. As a consequence,

$$f(q) = \frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} > 0,$$

where $J=[q_L, q_R]$ is the smallest relative plateau such that $q \in (q_L, q_R)$.

It is difficult to produce a meaningful graph of f, due to its highly discontinuous nature. We have made an attempt in Figure 2. The arc stretching from near the lower left corner to the upper right corner represents left and right endpoints of entropy plateaus; these points lie in $\overline{\mathscr{B}}$, so $f(q) = \dim_H \mathcal{U}_q = H(q)/\log q$ for these points. The near-vertical chains below this main arc represent endpoints of second- and third-level relative plateaus inside eight representative entropy plateaus. In reality, there are infinitely many such 'chains'. These values were computed using Theorem 1 and Proposition 3.8 below, and modeled after Example 4.7.

Remark 1.5. Note the asymmetry between f_- and f_+. This is caused by the very different roles played by the left and right endpoints q_L and q_R of a relative plateau. On the one hand, we have $f(q_L) = f_-(q_L) > 0$ while $f_+(q_L) = 0$. On the other hand, suppose $[q_L, q_R] = J$, and let I be the parent interval of J, that is, the relative plateau one level above J that contains J. Then

$$f_-(q_R) = \frac{h(\widetilde{\mathbf{U}}_{q_R}(J))}{\log q_R} > 0 \quad \text{and} \quad f_+(q_R) = \frac{h(\widetilde{\mathbf{U}}_{q_R}(I))}{\log q_R} > 0,$$

and since $\widetilde{\mathbf{U}}_{q_R}(J) \subset \widetilde{\mathbf{U}}_{q_R}(I)$, we have $f_-(q_R) \le f_+(q_R)$ so $f(q_R) = f_+(q_R)$. In fact, the inequality between $f_-(q_R)$ and $f_+(q_R)$ is almost always strict, with just one possible exception; see Example 4.7 below for more details.

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

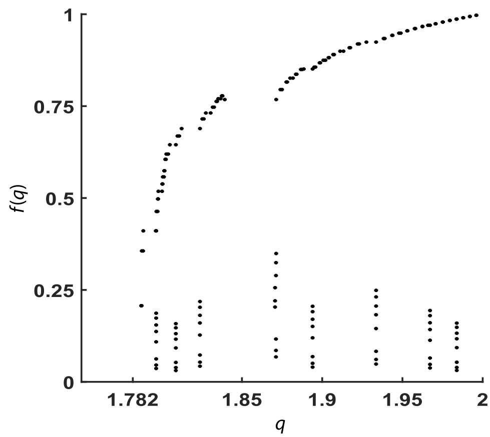

FIGURE 2. Approximate graph of the local dimensional function f for $M = 1$. Note that $f = 0$ almost everywhere; only non-zero values are shown. The graphs of f_- and f_+ look similar.

Theorem 1 suggests a closer investigation of the sets $\widetilde{\mathbf{U}}_q(J)$. Our next result gives a detailed description.

Recall the definition (1.7) of $q_c(J)$, and let $q_G(J)$ and $q_F(J)$ be the bases in (q_L, q_R) with

$$\alpha(q_G(J)) = (\mathbf{a}^+\overline{\mathbf{a}^+})^\infty, \quad \alpha(q_F(J)) = (\mathbf{a}^+\overline{\mathbf{a}}\,\overline{\mathbf{a}^+}\mathbf{a})^\infty.$$

Then $q_G(J) < q_F(J) < q_c(J)$.

THEOREM 2. *Let $J = [q_L, q_R]$ be a relative plateau generated by the admissible word* $\mathbf{a}$. *Then the entropy function*

$$H_J : q \mapsto h(\widetilde{\mathbf{U}}_q(J))$$

is a Devil's staircase on $(q_L, q_R]$, i.e. H_J is continuous, non-decreasing and locally constant almost everywhere on $(q_L, q_R]$. Furthermore, the set $\widetilde{\mathbf{U}}_q(J)$ has the following structure:

(i) *If $q_L < q \le q_G(J)$, then $\widetilde{\mathbf{U}}_q(J) = \emptyset$.*
(ii) *If $q_G(J) < q \le q_F(J)$, then $\widetilde{\mathbf{U}}_q(J) = \{(\mathbf{a}^+\overline{\mathbf{a}^+})^\infty\}$.*
(iii) *If $q_F(J) < q < q_c(J)$, then $\widetilde{\mathbf{U}}_q(J)$ is countably infinite.*
(iv) *If $q = q_c(J)$, then $\widetilde{\mathbf{U}}_q(J)$ is uncountable but $H_J(q) = 0$.*
(v) *If $q_c(J) < q \le q_R$, then $H_J(q) > 0$.*

Remark 1.6. Theorem 2 can be viewed as a generalization of Theorem 1.1 and the classical result of Glendinning and Sidorov [**23**] for the set $\mathcal{U}_q$ with $q \in (1, 2]$ and alphabet $\{0, 1\}$ (see Proposition 4.5 below).

Note that, while the function $H : q \mapsto h(\widetilde{\mathbf{U}}_q)$ is constant on each relative plateau J, the set-valued map $F : q \mapsto \widetilde{\mathbf{U}}_q$ is *not* constant on J. Since F is non-decreasing, it is natural to investigate the variation of the map $q \mapsto \dim_H(\widetilde{\mathbf{U}}_q \setminus \widetilde{\mathbf{U}}_{q_L})$ on $J = [q_L, q_R]$, where the Hausdorff dimension is well defined by equipping the symbolic space Ω_M with the metric ρ defined by

$$\rho((x_i), (y_i)) = 2^{-\inf\{i \ge 0 : x_{i+1} \ne y_{i+1}\}}. \tag{1.9}$$

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

As an application of Theorem 2 we have the following.

COROLLARY 1.7. *Let $J = [q_{\mathrm{L}}, q_{\mathrm{R}}]$ be a relative plateau. Then the function*

$$D_J : J \mapsto [0, \infty), \quad q \mapsto \dim_{\mathrm{H}}(\widetilde{\mathbf{U}}_q \setminus \widetilde{\mathbf{U}}_{q_{\mathrm{L}}})$$

is a Devil's staircase on J. Furthermore, $D_J(q) = 0$ if and only if $q \le q_c(J)$.

Remark 1.8. Unfortunately, the analogous statement for topological entropy in place of Hausdorff dimension fails: since sequences in $\widetilde{\mathbf{U}}_q \setminus \widetilde{\mathbf{U}}_{q_{\mathrm{L}}}$ can have arbitrarily long prefixes from any sequence in $\widetilde{\mathbf{U}}_q$, the difference set $\widetilde{\mathbf{U}}_q \setminus \widetilde{\mathbf{U}}_{q_{\mathrm{L}}}$ has the same entropy as $\widetilde{\mathbf{U}}_q$ for all $q \in J \backslash \{q_{\mathrm{L}}\}$.

Theorems 1 and 2 show that the local dimensional functions f, f_- and f_+ are highly discontinuous on $\overline{\mathscr{U}}$ (of course, they are everywhere continuous (and equal to zero) on $(1, M+1] \backslash \overline{\mathscr{U}}$):

COROLLARY 1.9. *The local dimensional function f is continuous at $q \in \overline{\mathscr{U}}$ if and only if $q \in \mathscr{C}$. The same statement holds for f_- and f_+.*

Next, for any relative entropy plateau J we define the *relative bifurcation set*

$$\mathscr{B}(J) := \{q \in J : h(\widetilde{\mathbf{U}}_p(J)) \ne h(\widetilde{\mathbf{U}}_q(J)) \ \forall p \in J, \ p \ne q\}.$$

As a special case, for $J = J_\emptyset = [1, M+1]$ we have $\mathscr{B}(J) = \mathscr{B}$.

THEOREM 3. *Let $J = J_{\mathbf{i}} = [q_{\mathrm{L}}, q_{\mathrm{R}}]$ be a relative plateau with generating word $\mathbf{a} = a_1 \cdots a_m$. Then the following statements hold.*

(i) $\mathscr{B}(J) = \mathscr{B}(J_{\mathbf{i}}) = J_{\mathbf{i}} \backslash \bigcup_{j=0}^{\infty} J_{\mathbf{i}j}$;

(ii) $\mathscr{B}(J) \subset \mathscr{U} \cap J$;

(iii) *$\mathscr{B}(J)$ is Lebesgue null;*

(iv) *$\mathscr{B}(J)$ has full Hausdorff dimension. Precisely,*

$$\dim_{\mathrm{H}} \mathscr{B}(J) = \dim_{\mathrm{H}}(\mathscr{U} \cap J) = \frac{\log 2}{m \log q_{\mathrm{R}}};$$

(v) *Let p_0 be the base with $\alpha(p_0) = \mathbf{a}^+ \overline{\mathbf{a}}^2 (\overline{\mathbf{a}^+} \mathbf{a} \mathbf{a}^+)^\infty$. Then*

$$\dim_{\mathrm{H}}((\mathscr{U} \cap J) \backslash \mathscr{B}(J)) = \frac{\log 2}{3m \log p_0}.$$

The representation of $\mathscr{B}(J)$ in (i) explains why we call the intervals $J_{\mathbf{i}j}$ relative entropy plateaus: they are the maximal intervals on which $h(\widetilde{\mathbf{U}}_q(J_{\mathbf{i}}))$ is positive and constant. Comparing the statements (i)–(iv) above with the properties of $\mathscr{B}$ given after equation (1.4), we can say that the set $\mathscr{B}(J)$ plays the same role on a local level (i.e. within J) as the bifurcation set $\mathscr{B}$ does on a global level. We may observe also that (v) is similar to [**6**, Theorem 4], which gives the Hausdorff dimension of $\mathscr{U} \backslash \mathscr{B}$.

From Proposition 1.4(i) and Theorem 3(i),(ii) we obtain the following decomposition of $\mathscr{U}$ into mutually disjoint subsets (recall that $\mathscr{U} \cap [q_{\mathrm{L}}, q_c(J)) = \emptyset$ while $q_c(J) \in \mathscr{C}$ for any relative plateau $J = [q_{\mathrm{L}}, q_{\mathrm{R}}]$):

$$\mathscr{U} = \mathscr{C} \cup \mathscr{B} \cup \bigcup_{n=1}^{\infty} \bigcup_{\mathbf{i} \in \{1,2,\ldots\}^n} \mathscr{B}(J_{\mathbf{i}}).$$

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

Using Theorems 1 and 2 we can answer an open question of Kalle *et al* [**25**], who asked on the Hausdorff dimension of $\mathscr{U} \cap [t_1, t_2]$ for any $t_1 < t_2$.

THEOREM 4. *For any* $1 < t_1 < t_2 \le M + 1$ *we have*

$$\dim_{\mathrm{H}}(\mathscr{U} \cap [t_1, t_2]) = \max\left\{\frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} : q \in \overline{\mathscr{B}(J) \cap [t_1, t_2]}\right\},$$

where $J = [q_{\mathrm{L}}, q_{\mathrm{R}}]$ *is the smallest relative plateau containing* $[t_1, t_2]$.

Remark 1.10. If (t_1, t_2) intersects the bifurcation set $\mathscr{B}$, then $J = [1, M + 1]$ and the expression in Theorem 4 simplifies to

$$\begin{aligned}\dim_{\mathrm{H}}(\mathscr{U} \cap [t_1, t_2]) &= \max\left\{\frac{h(\widetilde{\mathbf{U}}_q)}{\log q} : q \in \overline{\mathscr{B} \cap [t_1, t_2]}\right\}\\ &= \max\{\dim_{\mathrm{H}} \mathcal{U}_q : q \in \overline{\mathscr{B} \cap [t_1, t_2]}\}.\end{aligned}$$

Setting $t_1 = 1$ and noting that the map $q \mapsto \dim_{\mathrm{H}} \mathcal{U}_q$ is continuous on $(1, M + 1]$ and is decreasing inside each entropy plateau, we obtain Theorem 3 of [**25**], namely

$$\dim_{\mathrm{H}}(\mathscr{U} \cap [1, t]) = \max_{q \le t} \dim_{\mathrm{H}} \mathcal{U}_q \quad \text{for any } t \in [1, M + 1].$$

1.4. *Application to strongly univoque sets.* In 2011, Jordan *et al* [**24**] introduced the sets

$$\check{\mathbf{U}}_q := \bigcup_{k=1}^{\infty} \{(x_i) \in \Omega_M : \overline{\alpha_1(q) \cdots \alpha_k(q)} \prec x_{n+1} \cdots x_{n+k} \prec \alpha_1(q) \cdots \alpha_k(q)\ \forall n \ge 0\}. \tag{1.10}$$

(In fact, their definition was slightly different in that they require the above inequalities only for all sufficiently large n. They also defined $\check{\mathbf{U}}_q$ in a dynamical, rather than a symbolic way, but the definitions are easily seen to be equivalent.) Jordan *et al* used the sets $\check{\mathbf{U}}_q$ to study the multifractal spectrum of Bernoulli convolutions. Recently, the first author [**3**] used them to characterize the infinite derivatives of certain self-affine functions, and studied them in more detail in [**4**] where they were called *strongly univoque sets*.

In view of equation (1.3) it is clear that $\check{\mathbf{U}}_q \subseteq \widetilde{\mathbf{U}}_q$ for all $q \in (1, M + 1]$. On the other hand, $\check{\mathbf{U}}_q \supset \widetilde{\mathbf{U}}_p$ for all $p < q$ (see [**24**] or [**4**, Lemma 2.1]). It follows that

$$\check{\mathbf{U}}_q = \bigcup_{p<q} \widetilde{\mathbf{U}}_p, \tag{1.11}$$

and, since the function $q \mapsto \dim_{\mathrm{H}} \widetilde{\mathbf{U}}_q$ is continuous, that $\dim_{\mathrm{H}} \check{\mathbf{U}}_q = \dim_{\mathrm{H}} \widetilde{\mathbf{U}}_q$ for every q. A natural question now, is whether $\check{\mathbf{U}}_q$ could in fact equal $\widetilde{\mathbf{U}}_q$. Following [**4**], we define the difference set

$$\begin{aligned}\mathbf{W}_q &:= \widetilde{\mathbf{U}}_q \setminus \check{\mathbf{U}}_q\\ &= \bigcap_{k=1}^{\infty} \bigcup_{n=0}^{\infty} \{(x_i) \in \widetilde{\mathbf{U}}_q : x_{n+1} \cdots x_{n+k} = \alpha_1(q) \cdots \alpha_k(q) \text{ or } \overline{\alpha_1(q) \cdots \alpha_k(q)}\},\end{aligned} \tag{1.12}$$

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

and its projection, $\mathcal{W}_q := \pi_q(\mathbf{W}_q)$. One of the main results of [**4**] is that $\mathcal{W}_q \neq \emptyset$ if and only if $q \in \overline{\mathscr{U}}$, and then $\mathcal{W}_q$ is in fact uncountable. It is also shown in [**4**] that $\dim_H \mathcal{W}_q = 0$ whenever $q \in \mathscr{C}_0$ is a de Vries–Komornik number.

Using the techniques developed in this paper, we can improve on the results of [**4**] and completely characterize the Hausdorff dimension of $\mathcal{W}_q$.

THEOREM 5. *For any* $q \in (1, M+1]$ *we have*

$$\dim_H \mathcal{W}_q = f_-(q).$$

Remark 1.11.

(1) By Proposition 1.2 and Theorem 5 it follows that for each $q \in \mathscr{B}$ we have

$$\dim_H \mathcal{W}_q = \dim_H \mathcal{U}_q > 0.$$

This provides a negative answer to [**4**, Question 1.8], where it was conjectured that $\dim_H \mathcal{W}_q < \dim_H \mathcal{U}_q$ for all $q > q_{KL}$. Looking at equation (1.11), the above result is not too surprising, since the set-valued function $q \mapsto \widetilde{\mathbf{U}}_q$ is 'most discontinuous' at points of $\mathscr{B}$.

(2) Let $q \in \overline{\mathscr{U}}$. By Theorem 1(i) and Theorem 5 it follows that $\dim_H \mathcal{W}_q = 0$ if and only if $q \in \mathscr{C}$. This completely characterizes the set $\{q : \dim_H \mathcal{W}_q = 0\}$, extending [**4**, Theorem 1.5].

(3) In view of equation (1.11) and remark (2) above, we could say that, at points of $\overline{\mathscr{U}} \backslash \mathscr{C}$, the set-valued function $q \mapsto \widetilde{\mathbf{U}}_q$ 'jumps' by a set of positive Hausdorff dimension.

The remainder of this article is organized as follows. In §2 we prove Proposition 1.4 and give some examples of points in $\mathscr{C}_\infty$. In §3 we introduce for each relative plateau J a bijection Φ_J between symbolic spaces and its induced map $\hat{\Phi}_J$ between suitable sets of bases, and develop their properties. These maps allow us to answer questions about relative plateaus and relative bifurcation sets by relating them directly to the entropy plateaus $[p_L, p_R]$ and the bifurcation set $\mathscr{B}$ for the alphabet $\{0, 1\}$. This is done in §4, where we prove Theorems 1, 2 and 3. Section 5 contains a short proof of Theorem 4, and §6 is devoted to the proof of Theorem 5.

2. *Properties of the set* $\mathscr{C}$

In this section we prove Proposition 1.4. Recall that $\alpha(q)$ is the quasi-greedy expansion of 1 in base q. The following useful result is well known (cf. [**10**]).

LEMMA 2.1. *The map* $q \mapsto \alpha(q)$ *is strictly increasing and bijective from* $(1, M+1]$ *to the set of sequences* $(a_i) \in \Omega_M$ *not ending with* 0^∞ *and satisfying*

$$\sigma^n((a_i)) \preccurlyeq (a_i) \quad \textit{for all } n \geq 0.$$

Proof of Proposition 1.4. (i) It is known that all de Vries–Komornik numbers belong to $\mathscr{U}$ (cf. [**31**]), i.e. $\mathscr{C}_0 \subset \mathscr{U}$. Now let $q \in \mathscr{C}_\infty$ and $\alpha(q) = \alpha_1\alpha_2\cdots$. Then q belongs to infinitely many relative plateaus. Hence, there are infinitely many integers $m_1 < m_2 < \cdots$ such that for each k, $\alpha_1 \cdots \alpha_{m_k}^-$ is admissible, since, if q lies in the relative plateau generated by

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

$b_1 \cdots b_n$, then $\alpha(q)$ must begin with $b_1 \cdots b_n^+$. It follows from equation (1.6) that for each k,

$$\overline{\alpha_1 \cdots \alpha_{m_k - i}} \prec \alpha_{i+1} \cdots \alpha_{m_k} \preccurlyeq \alpha_1 \cdots \alpha_{m_k - i} \quad \text{for all } 1 \le i < m_k.$$

This implies by induction that $\overline{\alpha(q)} \prec \sigma^i(\alpha(q)) \preccurlyeq \alpha(q)$ for all $i \in \mathbb{N}$, and hence $q \in \overline{\mathscr{U}}$ (cf. [**30**]). Recall from [**18**, Theorem 1.2] that for each $p \in \overline{\mathscr{U}} \setminus \mathscr{U}$ the quasi-greedy expansion $\alpha(p)$ of 1 can be written as

$$\alpha(p) = (a_1 \cdots a_m)^\infty$$

for some admissible word $a_1 \cdots a_m$ not of the form $\mathbf{b}\overline{\mathbf{b}}$. So $\overline{\mathscr{U}} \setminus \mathscr{U}$ contains only left endpoints of relative plateaus, and these points do not lie in $\mathscr{C}_\infty$. Therefore, $q \in \mathscr{U}$.

(ii) Clearly, by the construction of $\mathscr{C}_\infty$ it follows that $\mathscr{C}_\infty$ is uncountable, because each relative plateau of level n contains infinitely many pairwise disjoint relative plateaus of level $n+1$. That $\mathscr{C}$ has no isolated points follows since any right neighborhood of a de Vries–Komornik number contains infinitely many relative plateaus.

(iii) In [**6**], the following was proved: if $J = [q_L, q_R]$ is a relative plateau generated by $a_1 \cdots a_m$, then

$$\dim_H(\overline{\mathscr{U}} \cap [p, q_R]) = \frac{\log 2}{m \log q_R} \quad \text{for any } p \in [q_L, q_R). \tag{2.1}$$

(This was stated in [**6**] only for entropy plateaus, i.e. the first-level relative plateaus, but the proof carries over verbatim to any relative plateau.)

Observe that $\mathscr{C}_0$ is countable. Furthermore, for a relative plateau $J_{\mathbf{i}}$ with $\mathbf{i} \in \{1, 2, \ldots\}^n$, its generating block $a_1 \cdots a_m$ satisfies $m \ge n$. That $\dim_H \mathscr{C} = 0$ now follows from the definition of $\mathscr{C}_\infty$, the countable stability of Hausdorff dimension, and equation (2.1). □

Example 2.2. It is easy to find specific examples of points in $\mathscr{C}_\infty$. For instance, let $\mathbf{a} = a_1 \cdots a_m$ be an admissible word not of the form $\mathbf{b}\overline{\mathbf{b}}$ (e.g. $\mathbf{a} = 1110010$ when $M = 1$), and construct a sequence $\alpha_1\alpha_2 \cdots$ as follows: set $\alpha_1 \cdots \alpha_m = \mathbf{a}^+$, and recursively for $k = 0, 1, \ldots$, let

$$\alpha_{3^k m+1} \cdots \alpha_{2\cdot 3^k m} = \alpha_{2\cdot 3^k m+1} \cdots \alpha_{3^{k+1} m} = \overline{\alpha_1 \cdots \alpha_{3^k m}}^+.$$

Then $\alpha_1\alpha_2 \cdots = \alpha(q)$ for some q, and this q lies in $\mathscr{C}_\infty$.

More generally, one can find many more examples by the following procedure. Let again $\mathbf{a} = a_1 \cdots a_m$ be any admissible word not of the form $\mathbf{b}\overline{\mathbf{b}}$. Now let $\mathbf{w}$ be a word using the letters $\mathbf{a}^+$, $\overline{\mathbf{a}}$, $\overline{\mathbf{a}^+}$, $\mathbf{a}$ beginning with $\mathbf{a}^+$ such that $\mathbf{w}^-$ is admissible (e.g. $\mathbf{w} = \mathbf{a}^+\overline{\mathbf{a}}^2\overline{\mathbf{a}^+}\mathbf{a}^+\overline{\mathbf{a}^+}\mathbf{a}^+$). Put $\mathbf{v}_0 := \mathbf{w}$, and recursively, for $k = 0, 1, 2, \ldots$, let $\mathbf{v}_{i+1}$ be the word obtained from $\mathbf{v}_i$ by performing the substitutions

$$\mathbf{a} \mapsto \mathbf{v}_i^-, \quad \mathbf{a}^+ \mapsto \mathbf{v}_i, \quad \overline{\mathbf{a}} \mapsto \overline{\mathbf{v}_i}^+, \quad \overline{\mathbf{a}^+} \mapsto \overline{\mathbf{v}_i}.$$

Since $\mathbf{v}_{i+1}$ extends $\mathbf{v}_i$, the limit $\mathbf{v} := \lim_{i\to\infty} \mathbf{v}_i$ exists, and $\mathbf{v} = \alpha(q)$ for some q, as the interested reader may check using Lemma 2.1. Some reflection reveals that $q \in \mathscr{C}$. The de Vries–Komornik numbers are obtained from $\mathbf{w} = \mathbf{a}^+\overline{\mathbf{a}}$; all other examples obtained this way lie in $\mathscr{C}_\infty$, including the one given at the beginning of this example, which is obtained from $\mathbf{w} = \mathbf{a}^+\overline{\mathbf{a}}^2$. (For each i, q lies in the relative plateau $[q_L(i), q_R(i)]$ given by $\alpha(q_L(i)) = (\mathbf{v}_i^-)^\infty$ and $\alpha(q_R(i)) = \mathbf{v}_i(\overline{\mathbf{v}_i}^+)^\infty$; we leave the details to the interested reader.)

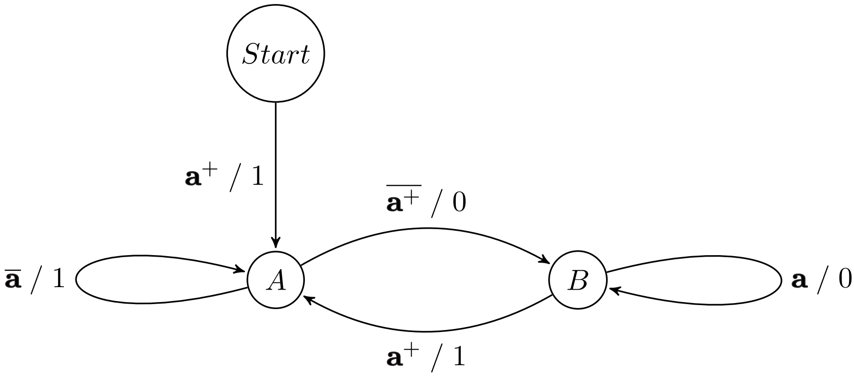

FIGURE 3. Labeled graph $\mathcal{G} = (G, \mathcal{L})$ with labels $\mathcal{L} = \{\mathbf{a}, \mathbf{a}^+, \overline{\mathbf{a}}, \overline{\mathbf{a}^+}\}$, and the reference labeled graph $\mathcal{G}^* = (G, \mathcal{L}^*)$ with labels $\mathcal{L}^* = \{0, 1\}$. The map $\phi : \mathcal{L} \to \mathcal{L}^*$ is defined by $\phi(\overline{\mathbf{a}^+}) = \phi(\mathbf{a}) = 0$ and $\phi(\mathbf{a}^+) = \phi(\overline{\mathbf{a}}) = 1$.

3. *Descriptions of the map Φ_J and the induced map $\hat{\Phi}_J$*

In this section we fix a relative plateau $J = [q_{\mathrm{L}}, q_{\mathrm{R}}]$ with

$$\alpha(q_{\mathrm{L}}) = \mathbf{a}^\infty \quad \text{and} \quad \alpha(q_{\mathrm{R}}) = \mathbf{a}^+(\overline{\mathbf{a}})^\infty$$

for some admissible word $\mathbf{a} = a_1 \cdots a_m$. Note by Definition 1.3 and Lemma 2.1 that q_{L} and q_{R} are well defined and $q_{\mathrm{L}} < q_{\mathrm{R}}$.

A special role in this paper is played by sets associated with the alphabet $\{0, 1\}$. When the alphabet $\{0, 1\}$ is intended, we will affix a superscript * to our notation. Thus, $\mathscr{B}^* = \mathscr{B}$ when $M = 1$, $\mathscr{U}^* = \mathscr{U}$ when $M = 1$, etc. We call $\mathscr{B}^*$ the *reference bifurcation set*. The key to the proofs of our main results, and the main methodological innovation of this paper, is the construction of a bijection $\hat{\Phi}_J$ from $\mathscr{B}(J)$ to $\mathscr{B}^*$. More generally, $\hat{\Phi}_J$ maps important points of J to important points of $(1, 2]$ for the case $M = 1$. Associated with $\hat{\Phi}_J$ is a symbolic map Φ_J which maps each set $\widetilde{\mathbf{U}}_q(J)$ to the symbolic univoque set $\widetilde{\mathbf{U}}^*_{\hat{q}}$ for $M = 1$, where $\hat{q} = \hat{\Phi}_J(q)$. By using properties of the maps Φ_J and $\hat{\Phi}_J$, many classical results on univoque sets with alphabet $\{0, 1\}$ can be transferred to the relative entropy plateaus and the sets $\widetilde{\mathbf{U}}_q(J)$.

Figure 3 shows a directed graph G with two sets of labels. The labeled graph $\mathcal{G} = (G, \mathcal{L})$ with labels in $\mathcal{L} := \{\mathbf{a}, \mathbf{a}^+, \overline{\mathbf{a}}, \overline{\mathbf{a}^+}\}$ is right-resolving, i.e. the out-going edges from the same vertex in $\mathcal{G}$ have different labels. Let $X(J)$ be the set of infinite sequences determined by the automata $\mathcal{G} = (G, \mathcal{L})$, beginning at the 'Start' vertex (cf. [**33**]). We emphasize that each digit $\mathbf{d}$ in $\mathcal{L}$ is a block of length m, and any sequence in $X(J)$ is an infinite concatenation of blocks from $\mathcal{L}$.

Likewise, the *reference labeled graph* $\mathcal{G}^* = (G, \mathcal{L}^*)$ with labels in $\mathcal{L}^* := \{0, 1\}$ is right-resolving. Hence for each $q \in (1, 2]$ the quasi-greedy expansion $\alpha^*(q)$ of 1 in base q is uniquely represented by an infinite path determined by the automata $\mathcal{G}^*$. Let $X^* \subset \{0, 1\}^{\mathbb{N}}$ be the set of all infinite sequences determined by the automata $\mathcal{G}^*$, and note that $X^* = \{(x_i) \in \{0, 1\}^{\mathbb{N}} : x_1 = 1\}$. Then $\{\alpha^*(q) : q \in (1, 2]\} \subset X^*$, the inclusion being proper in view of Lemma 2.1.

PROPOSITION 3.1. *$\widetilde{\mathbf{U}}_q(J) \subset X(J)$ for every $q \in (q_L, q_R]$.*

To prove the proposition we need the following.

LEMMA 3.2. *Any sequence $(x_i) \in \Omega_M$ satisfying $x_1 \cdots x_m = \mathbf{a}^+$ and*

$$\overline{\mathbf{a}^+}\mathbf{a}^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq \mathbf{a}^+(\overline{\mathbf{a}})^\infty \quad \text{for all } n \geq 0 \tag{3.1}$$

belongs to $X(J)$.

Proof. Take a sequence (x_i) satisfying $x_1 \cdots x_m = \mathbf{a}^+ = a_1 \cdots a_m^+$ and equation (3.1). Then by (3.1) with $n = 0$ and $n = m$ it follows that

$$\overline{\mathbf{a}^+} \preccurlyeq x_{m+1} \cdots x_{2m} \preccurlyeq \overline{\mathbf{a}}.$$

So, either $x_{m+1} \cdots x_{2m} = \overline{\mathbf{a}^+}$ or $x_{m+1} \cdots x_{2m} = \overline{\mathbf{a}}$.

(i) If $x_{m+1} \cdots x_{2m} = \overline{\mathbf{a}^+}$, then by equation (3.1) with $n = m$ and $n = 2m$ it follows that the next block $x_{2m+1} \cdots x_{3m} = \mathbf{a}^+$ or $\mathbf{a}$.

(ii) If $x_{m+1} \cdots x_{2m} = \overline{\mathbf{a}}$, then $x_1 \cdots x_{2m} = \mathbf{a}^+\overline{\mathbf{a}}$. By equation (3.1) with $n = 0$ and $n = 2m$ it follows that the next block $x_{2m+1} \cdots x_{3m} = \overline{\mathbf{a}^+}$ or $\overline{\mathbf{a}}$.

Iterating the above reasoning and referring to Figure 3 we conclude that $(x_i) \in X(J)$. □

Proof of Proposition 3.1. Take $q \in (q_L, q_R]$. Since $\alpha(q_L) = \mathbf{a}^\infty$ and $\alpha(q_R) = \mathbf{a}^+(\overline{\mathbf{a}})^\infty$, Lemma 2.1 implies that $\alpha_1(q) \cdots \alpha_m(q) = \mathbf{a}^+$ and $\alpha(q) \preccurlyeq \alpha(q_R) = \mathbf{a}^+(\overline{\mathbf{a}})^\infty$. Hence, by equations (1.8), (1.3) and Lemma 3.2, it follows that $\widetilde{\mathbf{U}}_q(J) \subset X(J)$. □

We next introduce the right bifurcation set $\mathscr{V}$ for the set-valued map $q \mapsto \widetilde{\mathbf{U}}_q$ (cf. [**17**]):

$$\mathscr{V} := \{q \in (1, M+1] : \widetilde{\mathbf{U}}_r \neq \widetilde{\mathbf{U}}_q \ \forall r > q\}.$$

Recall that $\mathscr{U}$ is the set of univoque bases. The following characterizations of $\mathscr{U}$ and $\mathscr{V}$ are proved in [**18**].

LEMMA 3.3.

(i) $q \in \mathscr{U} \setminus \{M+1\}$ *if and only if* $\overline{\alpha(q)} \prec \sigma^n(\alpha(q)) \prec \alpha(q)$ *for all* $n \geq 1$.

(ii) $q \in \mathscr{V}$ *if and only if* $\overline{\alpha(q)} \preccurlyeq \sigma^n(\alpha(q)) \preccurlyeq \alpha(q)$ *for all* $n \geq 1$.

Clearly, Lemma 3.3 implies that $\mathscr{U} \subset \mathscr{V}$. Furthermore, $\mathscr{V} \setminus \mathscr{U}$ is at most countable. Set

$$\mathbf{U}(J) := \{\alpha(q) : q \in \mathscr{U} \cap (q_L, q_R]\} \quad \text{and} \quad \mathbf{V}(J) := \{\alpha(q) : q \in \mathscr{V} \cap (q_L, q_R]\}.$$

As a consequence of Lemmas 3.2 and 3.3 we have the following.

PROPOSITION 3.4. $\mathbf{U}(J) \subset \mathbf{V}(J) \subset X(J)$. *Furthermore,*

$$\mathbf{U}(J) = \{(\mathbf{c}_i) \in X(J) : \overline{(\mathbf{c}_i)} \prec \sigma^n((\mathbf{c}_i)) \prec (\mathbf{c}_i) \ \forall n \geq 1\},$$
$$\mathbf{V}(J) = \{(\mathbf{c}_i) \in X(J) : \overline{(\mathbf{c}_i)} \preccurlyeq \sigma^n((\mathbf{c}_i)) \preccurlyeq (\mathbf{c}_i) \ \forall n \geq 1\}.$$

We shall also need the following sets. When $M = 1$ we denote $\mathscr{U}$ by $\mathscr{U}^*$ and $\mathscr{V}$ by $\mathscr{V}^*$, and we define

$$\mathbf{U}^* := \{\alpha^*(q) : q \in \mathscr{U}^*\}, \quad \mathbf{V}^* := \{\alpha^*(q) : q \in \mathscr{V}^*\}.$$

Then by Lemma 3.3 with $M = 1$ it follows that

$$\begin{aligned} \mathbf{U}^* \setminus \{1^\infty\} &= \{(a_i) \in \{0,1\}^{\mathbb{N}} : (1 - a_i) \prec \sigma^n((a_i)) \prec (a_i) \ \forall n \geq 1\}, \\ \mathbf{V}^* &= \{(a_i) \in \{0,1\}^{\mathbb{N}} : (1 - a_i) \preccurlyeq \sigma^n((a_i)) \preccurlyeq (a_i) \ \forall n \geq 1\}. \end{aligned} \tag{3.2}$$

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

3.1. *Description of Φ_J.* We now define a map $\phi : \mathcal{L} \to \mathcal{L}^*$ by

$$\phi(\overline{\mathbf{a}^+}) = \phi(\mathbf{a}) = 0, \quad \text{and} \quad \phi(\overline{\mathbf{a}}) = \phi(\mathbf{a}^+) = 1. \tag{3.3}$$

Then ϕ induces a block map $\Phi_J : X(J) \to X^*$ defined by

$$\Phi_J((\mathbf{d}_i)) := \phi(\mathbf{d}_1)\phi(\mathbf{d}_2)\cdots .$$

PROPOSITION 3.5. *The map $\Phi_J : X(J) \to X^*$ is strictly increasing and bijective. Furthermore,*

$$\Phi_J(\mathbf{U}(J)) = \mathbf{U}^* \quad \textit{and} \quad \Phi_J(\mathbf{V}(J)) = \mathbf{V}^*.$$

First we verify that Φ_J is a bijection.

LEMMA 3.6. *The map $\Phi_J : X(J) \to X^*$ is strictly increasing and bijective.*

Proof. Note by Definition 1.3 that the blocks in $\mathcal{L}$ are ordered by $\overline{\mathbf{a}^+} \prec \overline{\mathbf{a}} \prec \mathbf{a} \prec \mathbf{a}^+$. Take two sequences $(\mathbf{c}_i), (\mathbf{d}_i) \in X(J)$ with $(\mathbf{c}_i) \prec (\mathbf{d}_i)$. Then $\mathbf{c}_1 = \mathbf{d}_1 = \mathbf{a}^+$, and there is an integer $k \geq 2$ such that $\mathbf{c}_1 \cdots \mathbf{c}_{k-1} = \mathbf{d}_1 \cdots \mathbf{d}_{k-1}$ and $\mathbf{c}_k \prec \mathbf{d}_k$. We will show that $\phi(\mathbf{c}_k) < \phi(\mathbf{d}_k)$. To this end we consider two cases (see Figure 3):

(I) If $\mathbf{c}_{k-1} = \mathbf{a}^+$ or $\overline{\mathbf{a}}$, then $\mathbf{c}_k = \overline{\mathbf{a}^+}$ and $\mathbf{d}_k = \overline{\mathbf{a}}$, and so $\phi(\mathbf{c}_k) = 0$ and $\phi(\mathbf{d}_k) = 1$.

(II) If $\mathbf{c}_{k-1} = \mathbf{a}$ or $\overline{\mathbf{a}^+}$, then $\mathbf{c}_k = \mathbf{a}$ and $\mathbf{d}_k = \mathbf{a}^+$, so again $\phi(\mathbf{c}_k) = 0$ and $\phi(\mathbf{d}_k) = 1$.

Thus, Φ_J is strictly increasing on $X(J)$. Finally, since the labeled graphs $\mathcal{G}$ and $\mathcal{G}^*$ are both right-resolving, the definitions of $X(J)$ and X^* imply that Φ_J is bijective. □

LEMMA 3.7. *The following statements are equivalent for sequences $(\mathbf{c}_i), (\mathbf{d}_i) \in X(J)$.*

(i) $\overline{(\mathbf{d}_i)} \prec \sigma^n((\mathbf{c}_i)) \prec (\mathbf{d}_i)$ *for all $n \geq 0$;*

(ii) $\overline{(\mathbf{d}_i)} \prec \sigma^{mn}((\mathbf{c}_i)) \prec (\mathbf{d}_i)$ *for all $n \geq 0$;*

(iii) *The image sequences $(x_i) := \Phi_J((\mathbf{c}_i))$ and $(y_i) := \Phi_J((\mathbf{d}_i))$ in X^* satisfy*

$$(1 - y_i) \prec \sigma^n((x_i)) \prec (y_i) \textit{ for all } n \geq 0.$$

Proof. Since $\mathbf{a} = a_1 \cdots a_m$ is admissible, Definition 1.3 implies

$$\overline{a_1 \cdots a_m^+} \prec a_{i+1} \cdots a_m^+ \overline{a_1 \cdots a_i} \prec a_1 \cdots a_m^+$$

and

$$\overline{a_1 \cdots a_m^+} \prec a_{i+1} \cdots a_m a_1 \cdots a_i \prec a_1 \cdots a_m^+$$

for all $1 \leq i < m$. Using $\mathbf{d}_1 = \mathbf{a}^+ = a_1 \cdots a_m^+$ and $(\mathbf{c}_i) \in X(J)$ we prove the equivalence (i) $\Leftrightarrow$ (ii).

Next, we prove (ii) $\Rightarrow$ (iii). We only verify the second inequality in (iii); the first one can be proved in the same way. Take $(\mathbf{c}_i), (\mathbf{d}_i) \in X(J)$ satisfying the inequalities in (ii), and let $(x_i) := \Phi_J((\mathbf{c}_i))$ and $(y_i) := \Phi_J((\mathbf{d}_i))$. Fix $n \geq 0$. If $\mathbf{c}_{n+1} \in \{\overline{\mathbf{a}^+}, \mathbf{a}\}$, then by $\mathbf{d}_1 = \mathbf{a}^+$ we have $x_{n+1} = \phi(\mathbf{c}_{n+1}) = 0 < 1 = \phi(\mathbf{d}_1) = y_1$. Furthermore, if $\mathbf{c}_{n+1} = \mathbf{a}^+$, then $\sigma^{mn}((\mathbf{c}_i)) \in X(J)$ and the second inequality in (iii) follows from Lemma 3.6. Therefore, the critical case is when $\mathbf{c}_{n+1} = \overline{\mathbf{a}}$, which we assume for the remainder of the proof.

Since $(\mathbf{c}_i) \in X(J)$, there is $0 \leq j < n$ such that $\mathbf{c}_{j+1} \cdots \mathbf{c}_{n+1} = \mathbf{a}^+(\overline{\mathbf{a}})^{n-j}$. (See Figure 3.) Furthermore, since $\mathbf{c}_{j+1}\mathbf{c}_{j+2} \cdots \prec \mathbf{d}_1\mathbf{d}_2 \cdots \preccurlyeq \mathbf{a}^+(\overline{\mathbf{a}})^\infty$, there is a number $k \geq n - j$ such that

$$\mathbf{c}_{j+1} \cdots \mathbf{c}_{j+k+2} = \mathbf{a}^+(\overline{\mathbf{a}})^k \overline{\mathbf{a}^+}, \quad \text{and} \quad \mathbf{d}_1 \cdots \mathbf{d}_{k+1} = \mathbf{a}^+(\overline{\mathbf{a}})^k. \tag{3.4}$$

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

The second equality in equation (3.4) yields $y_1 \cdots y_{k+1} = \phi(\mathbf{d}_1) \cdots \phi(\mathbf{d}_{k+1}) = 1^{k+1}$, and the first equality implies

$$\mathbf{c}_{n+1} \cdots \mathbf{c}_{j+k+2} = (\overline{\mathbf{a}})^{k-(n-j)+1}\overline{\mathbf{a}^+}.$$

Hence,

$$\begin{aligned} x_{n+1} \cdots x_{j+k+2} &= \phi(\mathbf{c}_{n+1}) \cdots \phi(\mathbf{c}_{j+k+2}) = 1^{k-(n-j)+1}0 \prec 1^{k-(n-j)+2} \\ &= y_1 \cdots y_{k-(n-j)+2}, \end{aligned}$$

since $j < n$ implies $k - (n - j) + 2 \le k + 1$. Therefore, $\sigma^n((x_i)) \prec (y_i)$, which gives the second inequality in (iii).

Finally, we prove (iii) $\Rightarrow$ (ii). First we verify the second inequality in (ii). Let $(x_i), (y_i) \in X^*$ and let $(\mathbf{c}_i), (\mathbf{d}_i) \in X(J)$ such that $\Phi_J((\mathbf{c}_i)) = (x_i)$ and $\Phi_J((\mathbf{d}_i)) = (y_i)$. Fix $n \ge 0$. We may assume $\mathbf{c}_{n+1} = \mathbf{a}^+$, as otherwise the inequality is trivial. But then $\mathbf{c}_{n+1}\mathbf{c}_{n+2} \cdots \in X(J)$, and since $x_{n+1}x_{n+2} \cdots \prec y_1 y_2 \cdots$ it follows from Lemma 3.6 that $\mathbf{c}_{n+1}\mathbf{c}_{n+2} \cdots \prec \mathbf{d}_1\mathbf{d}_2 \cdots$. This proves the second inequality in (ii). The first inequality is verified analogously. □

Proof of Proposition 3.5. In view of Lemma 3.6 it remains to prove

$$\Phi_J(\mathbf{U}(J)) = \mathbf{U}^* \quad \text{and} \quad \Phi_J(\mathbf{V}(J)) = \mathbf{V}^*.$$

Since the proof of the second equality is similar, we only prove the first one.

Let $(\mathbf{c}_i) \in \mathbf{U}(J)$, and $(x_i) := \Phi_J((\mathbf{c}_i))$. Then by Proposition 3.4 it follows that

$$\mathbf{c}_1 = \mathbf{a}^+, \quad \text{and} \quad \overline{(\mathbf{c}_i)} \prec \sigma^n((\mathbf{c}_i)) \prec (\mathbf{c}_i) \text{ for all } n \ge 1.$$

By Lemma 3.7 with $(\mathbf{c}_i) = (\mathbf{d}_i)$ this is equivalent to

$$x_1 = 1, \quad \text{and} \quad (1 - x_i) \prec \sigma^n((x_i)) \prec (x_i) \text{ for all } n \ge 1.$$

So, by equation (3.2) we have $(x_i) \in \mathbf{U}^*$, and thus $\Phi_J(\mathbf{U}(J)) \subseteq \mathbf{U}^*$.

Conversely, take $(x_i) \in \mathbf{U}^* \subset X^*$. By Lemma 3.6 there exists a (unique) sequence $(\mathbf{c}_i) \in X(J)$ such that $\Phi_J((\mathbf{c}_i)) = (x_i)$. If $(x_i) = 1^\infty$, then $(\mathbf{c}_i) = \mathbf{a}^+(\overline{\mathbf{a}})^\infty = \alpha(q_R) \in \mathbf{U}(J)$. If $(x_i) \in \mathbf{U}^* \setminus \{1^\infty\}$, then by equation (3.2), Lemma 3.7, and the same argument as above it follows that $(\mathbf{c}_i) \in \mathbf{U}(J)$. Hence, $\Phi_J(\mathbf{U}(J)) = \mathbf{U}^*$. □

3.2. *Description of the induced map* $\hat{\Phi}_J$. Recall from Proposition 3.4 that $\mathbf{V}(J) \subset X(J)$. Hence Proposition 3.5 implies that the bijective map $\Phi_J : \mathbf{V}(J) \to \mathbf{V}^*$ induces an increasing bijective map (see Figure 4)

$$\hat{\Phi}_J : \mathscr{V} \cap (q_L, q_R] \to \mathscr{V}^*; \quad q \mapsto (\alpha^*)^{-1} \circ \Phi_J \circ \alpha(q).$$

The relevance of the map $\hat{\Phi}_J$ is made clear by the following proposition. Here, for $M = 1$ and $q \in (1, 2]$ we write $\widetilde{\mathbf{U}}_q^* := \widetilde{\mathbf{U}}_q$.

PROPOSITION 3.8.
(i) $\hat{\Phi}_J : \mathscr{V} \cap (q_L, q_R] \to \mathscr{V}^*$ *is a strictly increasing homeomorphism;*
(ii) $\hat{\Phi}_J(\mathscr{U} \cap (q_L, q_R]) = \hat{\Phi}_J(\mathscr{U} \cap J) = \mathscr{U}^*$*;*

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

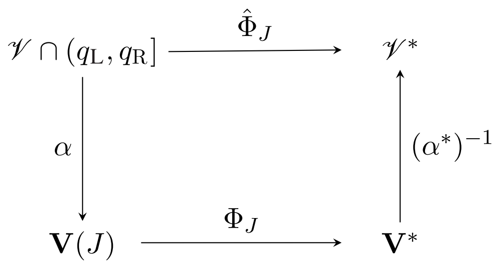

FIGURE 4. Exchange map between $\hat{\Phi}_J$ and Φ_J.

(iii) *For any* $q \in \mathscr{V} \cap (q_L, q_R]$, *and* $\hat{q} := \hat{\Phi}_J(q)$ *we have*

$$\Phi_J(\widetilde{\mathbf{U}}_q(J)) = \{(x_i) \in \widetilde{\mathbf{U}}^*_{\hat{q}} : x_1 = 1\} \quad \textit{and} \quad h(\widetilde{\mathbf{U}}_q(J)) = \frac{h(\widetilde{\mathbf{U}}^*_{\hat{q}})}{m}.$$

Remark 3.9. In the special case when $M = 1$, Proposition 3.8(ii) implies that $\mathscr{U}$ can be viewed as an attractor of an inhomogeneous infinite iterated function system: since $\mathscr{U}^* = \mathscr{U}$ in this case, we can write

$$\mathscr{U} = \bigcup_{i=1}^{\infty} \hat{\Phi}_{J_i}^{-1}(\mathscr{U}) \cup (\mathscr{B} \cup \{q_{KL}\}),$$

using equation (1.5) and the definition of J_i.

Part (i) of Proposition 3.8 follows from the following lemma, which proves something stronger: it implies Hölder properties of the maps $\hat{\Phi}_J$ and $\hat{\Phi}_J^{-1}$. These will be important later for Hausdorff dimension calculations.

LEMMA 3.10. *There exist constants* $c_1, c_2 > 0$ *such that for any* $q_1, q_2 \in \mathscr{V} \cap (q_L, q_R)$ *with* $q_1 < q_2$ *we have*

$$c_1(q_2 - q_1)^{(\log \hat{q}_2)/(m \log q_2)} \le \hat{\Phi}_J(q_2) - \hat{\Phi}_J(q_1) \le c_2(q_2 - q_1)^{(\log \hat{q}_2)/(m \log q_2)}, \tag{3.5}$$

where $\hat{q}_i := \hat{\Phi}_J(q_i)$ *for* $i = 1, 2$.

Proof. We first demonstrate the second inequality in equation (3.5). Recall that $J = [q_L, q_R]$ is a relative plateau generated by $\mathbf{a} = a_1 \cdots a_m$. Let $q_1, q_2 \in \mathscr{V} \cap (q_L, q_R)$ with $q_1 < q_2$, and let $\hat{q}_i := \hat{\Phi}_J(q_i)$, $i = 1, 2$. Then $\hat{q}_1 < \hat{q}_2$ by the monotonicity of $\hat{\Phi}_J$. Furthermore, Lemma 2.1 gives $\alpha(q_1) \prec \alpha(q_2)$. Note by Proposition 3.4 that $\alpha(q_1), \alpha(q_2) \in \mathbf{V}(J) \subset X(J)$. Therefore, we can write $\alpha(q_1) = (\mathbf{c}_i)$ and $\alpha(q_2) = (\mathbf{d}_i)$ with $\mathbf{c}_1 = \mathbf{d}_1 = \mathbf{a}^+$, and $\mathbf{c}_i, \mathbf{d}_i \in \{\mathbf{a}, \mathbf{a}^+, \overline{\mathbf{a}}, \overline{\mathbf{a}^+}\}$ for all $i \ge 2$. Since $\alpha(q_1) \prec \alpha(q_2)$, there exists $n \ge 2$ such that

$$\mathbf{c}_1 \cdots \mathbf{c}_{n-1} = \mathbf{d}_1 \cdots \mathbf{d}_{n-1} \quad \text{and} \quad \mathbf{c}_n \prec \mathbf{d}_n. \tag{3.6}$$

Observe that $\alpha(q_2) \in \mathbf{V}(J)$. By Proposition 3.4 it follows that

$$\sigma^{mn}(\alpha(q_2)) \succcurlyeq \overline{\alpha(q_2)} \succcurlyeq \overline{\mathbf{a}^+}\mathbf{a}^\infty \succcurlyeq 0^m 10^\infty,$$

which implies

$$1 = \sum_{i=1}^{\infty} \frac{\alpha_i(q_2)}{q_2^i} \ge \sum_{i=1}^{mn} \frac{\alpha_i(q_2)}{q_2^i} + \frac{1}{q_2^{mn+m+1}}.$$

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

Therefore, by equation (3.6) with $\alpha(q_1) = (\mathbf{c}_i)$ and $\alpha(q_2) = (\mathbf{d}_i)$ it follows that

$$\begin{aligned}\frac{1}{q_2^{mn+m+1}} &\le 1 - \sum_{i=1}^{mn} \frac{\alpha_i(q_2)}{q_2^i} = \sum_{i=1}^{\infty} \frac{\alpha_i(q_1)}{q_1^i} - \sum_{i=1}^{mn} \frac{\alpha_i(q_2)}{q_2^i} \\ &\le \sum_{i=1}^{mn} \left(\frac{\alpha_i(q_2)}{q_1^i} - \frac{\alpha_i(q_2)}{q_2^i} \right) \\ &\le \sum_{i=1}^{\infty} \left(\frac{M}{q_1^i} - \frac{M}{q_2^i} \right) = \frac{M(q_2 - q_1)}{(q_1 - 1)(q_2 - 1)}.\end{aligned}$$

Since $q_L < q_1 < q_2 < q_R$, we obtain

$$\frac{1}{q_2^{mn}} \le \frac{M q_R^{m+1}}{(q_L - 1)^2}(q_2 - q_1). \tag{3.7}$$

Write $(x_i) := \Phi_J((\mathbf{c}_i))$ and $(y_i) := \Phi_J((\mathbf{d}_i))$. Then equation (3.6) and Lemma 3.6 imply

$$x_1 \cdots x_{n-1} = y_1 \cdots y_{n-1} \quad \text{and} \quad x_n < y_n. \tag{3.8}$$

Note that $(x_i), (y_i) \in \mathbf{V}^*$. By the definition of $\hat{\Phi}_J$ we have $(x_i) = \Phi_J(\alpha(q_1)) = \alpha^*(\hat{\Phi}_J(q_1)) = \alpha^*(\hat{q}_1)$, and similarly $(y_i) = \alpha^*(\hat{q}_2)$. So, by equation (3.8) it follows that

$$\begin{aligned}\hat{\Phi}_J(q_2) - \hat{\Phi}_J(q_1) = \hat{q}_2 - \hat{q}_1 &= \sum_{i=1}^{\infty} \frac{y_i}{\hat{q}_2^{i-1}} - \sum_{i=1}^{\infty} \frac{x_i}{\hat{q}_1^{i-1}} \\ &\le \sum_{i=1}^{n-1} \left(\frac{y_i}{\hat{q}_2^{i-1}} - \frac{x_i}{\hat{q}_1^{i-1}} \right) + \sum_{i=n}^{\infty} \frac{y_i}{\hat{q}_2^{i-1}} \\ &\le \frac{1}{\hat{q}_2^{n-2}} \le \frac{4}{\hat{q}_2^{n}}.\end{aligned}$$

Here the second inequality follows from the definition of the quasi-greedy expansion $\alpha^*(\hat{q}_2) = (y_i)$. This, together with equation (3.7), yields

$$\hat{\Phi}_J(q_2) - \hat{\Phi}_J(q_1) \le 4 \left(\frac{1}{q_2^{mn}} \right)^{(\log \hat{q}_2)/(m \log q_2)} \le c_2(q_2 - q_1)^{(\log \hat{q}_2)/(m \log q_2)}$$

for some constant c_2 independent of q_1 and q_2. This proves the second inequality in equation (3.5).

For the first inequality we proceed similarly. Fix bases $q_1, q_2 \in \mathscr{V} \cap (q_L, q_R)$ with $q_1 < q_2$. Then the quasi-greedy expansions $\alpha(q_1) = (\mathbf{c}_i)$ and $\alpha(q_2) = (\mathbf{d}_i)$ satisfy equation (3.6), and therefore,

$$\begin{aligned}q_2 - q_1 &= \sum_{i=1}^{\infty} \frac{\alpha_i(q_2)}{q_2^{i-1}} - \sum_{i=1}^{\infty} \frac{\alpha_i(q_1)}{q_1^{i-1}} \\ &\le \sum_{i=1}^{m(n-1)} \left(\frac{\alpha_i(q_2)}{q_2^{i-1}} - \frac{\alpha_i(q_1)}{q_1^{i-1}} \right) + \sum_{i=m(n-1)+1}^{\infty} \frac{\alpha_i(q_2)}{q_2^{i-1}} \\ &\le \frac{1}{q_2^{mn-m-1}} \le \frac{M^{m+1}}{q_2^{mn}}.\end{aligned} \tag{3.9}$$

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

Let $\hat{q}_i = \hat{\Phi}_J(q_i)$ for $i = 1, 2$. Then $\hat{q}_1 < \hat{q}_2$, and the quasi-greedy expansions $\alpha^*(\hat{q}_1) = (x_i)$ and $\alpha^*(\hat{q}_2) = (y_i)$ satisfy equation (3.8). Since $q_2 < q_{\mathrm{R}}$, we have $\hat{q}_2 < 2$. So there exists a positive integer N such that $\alpha^*(\hat{q}_2) \prec 1^N 0^\infty$. Note that $\hat{q}_2 \in \mathscr{V}^*$. Then

$$\sigma^n((y_i)) = \sigma^n(\alpha^*(\hat{q}_2)) \succcurlyeq \overline{\alpha^*(\hat{q}_2)} \succ 0^N 1^\infty,$$

which implies

$$1 = \sum_{i=1}^{\infty} \frac{y_i}{\hat{q}_2^i} \geq \sum_{i=1}^{n} \frac{y_i}{\hat{q}_2^i} + \frac{1}{\hat{q}_2^{n+N+1}}.$$

Thus, by equation (3.8) and using $\alpha^*(\hat{q}_1) = (x_i)$, it follows that

$$\begin{aligned} \frac{1}{\hat{q}_2^{n+N+1}} &\leq 1 - \sum_{i=1}^{n} \frac{y_i}{\hat{q}_2^i} = \sum_{i=1}^{\infty} \frac{x_i}{\hat{q}_1^i} - \sum_{i=1}^{n} \frac{y_i}{\hat{q}_2^i} \\ &\leq \sum_{i=1}^{n} \left(\frac{y_i}{\hat{q}_1^i} - \frac{y_i}{\hat{q}_2^i} \right) \leq \sum_{i=1}^{\infty} \left(\frac{1}{\hat{q}_1^i} - \frac{1}{\hat{q}_2^i} \right) \\ &= \frac{\hat{q}_2 - \hat{q}_1}{(\hat{q}_1 - 1)(\hat{q}_2 - 1)} \leq \frac{\hat{q}_2 - \hat{q}_1}{(\hat{q}_{\mathrm{G}} - 1)^2}, \end{aligned} \tag{3.10}$$

where $\hat{q}_{\mathrm{G}} = (1 + \sqrt{5})/2$ is the golden ratio. Here we use that $\hat{q}_1, \hat{q}_2 \in \mathscr{V}^*$, and $\mathscr{V}^* \subset [\hat{q}_{\mathrm{G}}, 2]$. Combining equations (3.9) and (3.10) we conclude that

$$\begin{aligned} \hat{\Phi}_J(q_2) - \hat{\Phi}_J(q_1) = \hat{q}_2 - \hat{q}_1 &\geq \frac{(\hat{q}_{\mathrm{G}} - 1)^2}{2^{N+1}} \left(\frac{1}{q_2^{mn}} \right)^{(\log \hat{q}_2)/(m \log q_2)} \\ &\geq c_1 (q_2 - q_1)^{(\log \hat{q}_2)/(m \log q_2)} \end{aligned}$$

for some constant c_1 independent of q_1 and q_2. This establishes the first inequality in equation (3.5). □

Proof of Proposition 3.8. That $\hat{\Phi}_J$ is increasing and bijective follows since it is the composition of increasing and bijective maps. By Lemma 3.10, $\hat{\Phi}_J$ and $\hat{\Phi}_J^{-1}$ are continuous. Thus, we have proved (i). Since $q_{\mathrm{L}} \notin \mathscr{U}$, we have $\mathbf{U}(J) = \{\alpha(q) : q \in \mathscr{U} \cap J\}$. Thus, the statement (ii) is a direct consequence of Proposition 3.5. It remains only to establish (iii).

Take $q \in \mathscr{V} \cap (q_{\mathrm{L}}, q_{\mathrm{R}}]$. Then by Proposition 3.4 we have $\alpha(q) \in \mathbf{V}(J) \subset X(J)$. Note by Proposition 3.1 that $\widetilde{\mathbf{U}}_q(J) \subset X(J)$. Now take a sequence $(\mathbf{c}_i) \in X(J)$ and let $(x_i) := \Phi_J((\mathbf{c}_i)) \in X^*$. Then we have the equivalences

$$\begin{aligned} (\mathbf{c}_i) \in \widetilde{\mathbf{U}}_q(J) &\iff \mathbf{c}_1 = \mathbf{a}^+ \quad \text{and} \quad \overline{\alpha(q)} \prec \sigma^n((\mathbf{c}_i)) \prec \alpha(q) \quad \text{for all } n \geq 0 \\ &\iff x_1 = 1 \quad \text{and} \quad \Phi_J(\overline{\alpha(q)}) \prec \sigma^n((x_i)) \prec \Phi_J(\alpha(q)) \quad \text{for all } n \geq 0 \\ &\iff x_1 = 1 \quad \text{and} \quad (1 - \alpha_i^*(\hat{q})) \prec \sigma^n((x_i)) \prec \alpha^*(\hat{q}) \quad \text{for all } n \geq 0 \\ &\iff x_1 = 1 \quad \text{and} \quad (x_i) \in \widetilde{\mathbf{U}}_{\hat{q}}^*. \end{aligned}$$

Here the second equivalence follows by Lemma 3.7 with $(\mathbf{d}_i) = \alpha(q)$, and the third equivalence follows since $\alpha^*(\hat{q}) = \Phi_J(\alpha(q))$. As a result, $\Phi_J(\widetilde{\mathbf{U}}_q(J)) = \{(x_i) \in \widetilde{\mathbf{U}}_{\hat{q}}^* : x_1 = 1\}$.

For the entropy statement we observe that the map

$$\Phi_J : \widetilde{\mathbf{U}}_q(J) \to \widetilde{\mathbf{U}}^*_{\hat{q}}(1) := \{(x_i) \in \widetilde{\mathbf{U}}^*_{\hat{q}} : x_1 = 1\}; \quad (\mathbf{c}_i) \mapsto (\phi(\mathbf{c}_i))$$

is a bijective m-block map. (Recall that m is the length of the admissible word, which generates the relative plateau J.) Furthermore, $\widetilde{\mathbf{U}}^*_{\hat{q}}$ is the disjoint union of $\widetilde{\mathbf{U}}^*_{\hat{q}}(1)$ with its reflection $\{(1 - x_i) : (x_i) \in \widetilde{\mathbf{U}}^*_{\hat{q}}(1)\}$. This implies $h(\widetilde{\mathbf{U}}_q(J)) = h(\widetilde{\mathbf{U}}^*_{\hat{q}})/m$. □

Remark 3.11. In the recent paper [**5**], the first author takes the approach using the maps Φ_J and $\hat{\Phi}_J$ a bit further in order to obtain new information about the nature of the entropy plateaus $[p_{\mathrm{L}}, p_{\mathrm{R}}]$ and to present a more streamlined proof of the characterization of the entropy plateaus given in [**2**].

4. *Proofs of Theorems 1, 2, and 3*

Our first goal is to prove Theorem 1. We begin with a useful lemma.

LEMMA 4.1. *Let $J = J_{\mathbf{i}} = [q_{\mathrm{L}}, q_{\mathrm{R}}]$ be a relative entropy plateau. Then the union $\bigcup_{j=1}^{\infty} J_{\mathbf{i}j}$ is dense in $(q_c(J), q_{\mathrm{R}}]$.*

Proof. Recall from [**2**] that the entropy plateaus J^*_j, $j \in \mathbb{N}$ are dense in $(q^*_{\mathrm{KL}}, 2]$. Note that we may order the intervals $J_{\mathbf{i}j}$, $j \in \mathbb{N}$ so that $\hat{\Phi}_J(\mathscr{V} \cap J_{\mathbf{i}j}) = \mathscr{V}^* \cap J^*_j$ for each j. Hence, the result follows from the continuity of $\hat{\Phi}_J^{-1}$ (cf. Lemma 3.10). □

For $M = 1$ and $q \in (1, 2]$ we denote the left and right local dimensional functions by

$$f^*_-(q) := \lim_{\delta \to 0} \dim_{\mathrm{H}}(\mathscr{U}^* \cap (q - \delta, q)) \quad \text{and} \quad f^*_+(q) := \lim_{\delta \to 0} \dim_{\mathrm{H}}(\mathscr{U}^* \cap (q, q + \delta)),$$

respectively.

LEMMA 4.2. *Let $J = [q_{\mathrm{L}}, q_{\mathrm{R}}]$ be a relative plateau generated by a word $a_1 \cdots a_m$, and $q \in \overline{\mathscr{U}} \cap (q_{\mathrm{L}}, q_{\mathrm{R}}]$. Then*

$$f_-(q) = \frac{\log \hat{q}}{m \log q} f^*_-(\hat{q}),$$

where $\hat{q} := \hat{\Phi}_J(q)$.

Proof. Since $q \in \overline{\mathscr{U}} \cap (q_{\mathrm{L}}, q_{\mathrm{R}}]$, there is a sequence (p_i) in $\mathscr{V} \cap J$ such that $p_i < q$ for each i, and $p_i \nearrow q$ (cf. [**18**, Theorem 1.3 (ii)]). Let $\hat{p}_i := \hat{\Phi}_J(p_i)$. Then by Proposition 3.8(i), $\hat{p}_i < \hat{q}$ for each i and $\hat{p}_i \nearrow \hat{q}$.

Observe from Lemma 3.10 that for each i, $\hat{\Phi}_J$ is Hölder continuous on $[p_i, q]$ with exponent $\log \hat{p}_i/(m \log q)$, and $\hat{\Phi}_J^{-1}$ is Hölder continuous on $[\hat{p}_i, \hat{q}]$ with exponent $m \log p_i / \log \hat{q}$. It follows on the one hand that

$$\dim_{\mathrm{H}}(\mathscr{U}^* \cap (\hat{p}_i, \hat{q})) = \dim_{\mathrm{H}} \hat{\Phi}_J(\mathscr{U} \cap (p_i, q)) \le \frac{m \log q}{\log \hat{p}_i} \dim_{\mathrm{H}}(\mathscr{U} \cap (p_i, q)),$$

so letting $i \to \infty$ we obtain

$$f^*_-(\hat{q}) \le \frac{m \log q}{\log \hat{q}} f_-(q).$$

On the other hand,

$$\dim_H(\mathscr{U}\cap(p_i,q))=\dim_H\hat{\Phi}_J^{-1}(\mathscr{U}^*\cap(\hat{p}_i,\hat{q}))\le\frac{\log\hat{q}}{m\log p_i}\dim_H(\mathscr{U}^*\cap(\hat{p}_i,\hat{q})),$$

so letting $i\to\infty$ gives

$$f_-(q)\le\frac{\log\hat{q}}{m\log q}f_-^*(\hat{q}).$$

Hence, the lemma follows. □

For the right local dimensional function f_+ we have a similar relationship, but with a subtle difference for the domain of q.

LEMMA 4.3. *Let $J=[q_L,q_R]$ be a relative plateau generated by a word $a_1\cdots a_m$, and $q\in\overline{\mathscr{U}}\cap(q_L,q_R)$. Then*

$$f_+(q)=\frac{\log\hat{q}}{m\log q}f_+^*(\hat{q}),$$

where $\hat{q}:=\hat{\Phi}_J(q)$.

Proof. The proof is analogous to that of Lemma 4.2. If $q\in\mathscr{U}\cap(q_L,q_R)$, then we can approximate q from the right by a sequence of points (r_i) from $\mathscr{V}\cap J$, and use the Hölder properties of $\hat{\Phi}_J$ and $\hat{\Phi}_J^{-1}$ in much the same way as before. On the other hand, if $q\in(\overline{\mathscr{U}}\backslash\mathscr{U})\cap(q_L,q_R)$, then q is a left endpoint of some relative plateau inside J. In this case, $\hat{q}$ is the left endpoint of an entropy plateau in $(1,2]$, and we have $f_+(q)=f_+^*(\hat{q})=0$, so the identity in the lemma holds trivially. □

Motivated by [**6**] we introduce the *left and right bifurcation sets* $\mathscr{B}_L$ and $\mathscr{B}_R$, defined by

$$\begin{aligned}\mathscr{B}_L&:=\{q\in(1,M+1]:h(\mathbf{U}_p)\ne h(\mathbf{U}_q)\text{ for all }p<q\},\\ \mathscr{B}_L&:=\{q\in(1,M+1]:h(\mathbf{U}_r)\ne h(\mathbf{U}_q)\text{ for all }r>q\}.\end{aligned}\tag{4.1}$$

Then $\mathscr{B}=\mathscr{B}_L\cap\mathscr{B}_L$. Furthermore, any $q\in\mathscr{B}_L\setminus\mathscr{B}$ is a left endpoint of an entropy plateau, and any $q\in\mathscr{B}_L\setminus\mathscr{B}$ is a right endpoint of an entropy plateau. As usual, when $M=1$ we write $\mathscr{B}_L^*=\mathscr{B}_L$ and $\mathscr{B}_R^*=\mathscr{B}_L$. Below, we will need the following extension of Proposition 1.2, which follows from the main results of [**6**].

PROPOSITION 4.4. [**6**]
(i) If $q\in\mathscr{B}_L$, then $f_-(q)=\dim_H\mathcal{U}_q$;
(ii) If $q\in\mathscr{B}_L$, then $f_+(q)=\dim_H\mathcal{U}_q$.

Proof of Theorem 1. Note by equation (2.1) that for any $q\in\mathscr{C}_\infty$ we have $f(q)=f_-(q)=f_+(q)=0$. Suppose $q\in\mathscr{C}_0$, i.e. q is a de Vries–Komornik number. Then $f_-(q)=0$ since $\mathscr{U}\cap(q-\varepsilon,q)=\emptyset$ for sufficiently small $\varepsilon>0$. Furthermore, $q=q_c(J)$ for some relative plateau J, so $\hat{\Phi}_J(q)=q_{KL}^*$. Since $f_+^*(q_{KL}^*)=0$ (see [**6**, Theorem 2]) and $q\in\overline{\mathscr{U}}$, it follows by Lemma 4.3 that $f_+(q)=0$. Thus, the proof will be complete once we establish (ii).

Consider first f_-. Take $q\in\overline{\mathscr{U}}\backslash\mathscr{C}$, and let $J=[q_L,q_R]$ be the smallest relative plateau such that $q\in(q_L,q_R]$. If $J=[1,M+1]$, then $q\in\mathscr{B}_L$ and by Proposition 4.4(i),

$$f_-(q)=\dim_H\mathcal{U}_q=\frac{h(\mathbf{U}_q)}{\log q}=\frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q}>0.$$

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

Otherwise, put $\hat{q} := \hat{\Phi}_J(q)$. Then $\hat{q} \in \mathscr{B}_L^*$. By Proposition 4.4(i) it follows that

$$f_-^*(\hat{q}) = \dim_H \mathcal{U}_{\hat{q}}^* = \frac{h(\mathbf{U}_{\hat{q}}^*)}{\log \hat{q}} = \frac{h(\widetilde{\mathbf{U}}_{\hat{q}}^*)}{\log \hat{q}} > 0.$$

Hence, Lemma 4.2 along with Proposition 3.8(iii) gives

$$f_-(q) = \frac{h(\widetilde{\mathbf{U}}_{\hat{q}}^*)}{m \log q} = \frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} > 0.$$

Consider next f_+. Take again $q \in \overline{\mathscr{U}} \backslash \mathscr{C}$. If $q \in \overline{\mathscr{U}} \backslash \mathscr{U}$, then q is a left endpoint of a relative plateau and $f_+(q) = 0$. So assume $q \in \mathscr{U} \backslash \mathscr{C}$, and let $J = [q_L, q_R]$ now be the smallest relative plateau such that $q \in (q_L, q_R)$. If $J = [1, M+1]$, then $q \in \mathscr{B}_L$ and by Proposition 4.4(ii),

$$f_+(q) = \dim_H \mathcal{U}_q = \frac{h(\mathbf{U}_q)}{\log q} = \frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} > 0.$$

Otherwise, put $\hat{q} := \hat{\Phi}_J(q)$. Then $\hat{q} \in \mathscr{B}_L^*$, and using Proposition 4.4 (ii) and Lemma 4.3 it follows in the same way as above that

$$f_+(q) = \frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} > 0.$$

The statement about $f(q)$ is a direct consequence of the statements about f_- and f_+. □

We next prepare to prove Theorem 2. Fix a relative plateau $J = [q_L, q_R]$ generated by $\mathbf{a} = a_1 \cdots a_m$. Recall from §1 that the bases $q_G(J), q_F(J) \in J$ satisfy

$$\alpha(q_G(J)) = (\mathbf{a}^+\overline{\mathbf{a}^+})^\infty \quad \text{and} \quad \alpha(q_F(J)) = (\mathbf{a}^+\overline{\mathbf{a}}\overline{\mathbf{a}^+}\mathbf{a})^\infty.$$

Furthermore, the de Vries–Komornik number $q_c(J) = \min(\mathscr{U} \cap J)$ satisfies

$$\alpha(q_c(J)) = \mathbf{a}^+\overline{\mathbf{a}}\overline{\mathbf{a}^+}\mathbf{a}^+\overline{\mathbf{a}^+}\mathbf{a}\mathbf{a}^+\overline{\mathbf{a}} \cdots .$$

By Lemma 3.3, the bases $q_G(J)$, $q_F(J)$, and $q_c(J)$ all belong to $\mathscr{V} \cap (q_L, q_R]$, so we may define their image bases in $\mathscr{V}^*$ by

$$\hat{q}_G := \hat{\Phi}_J(q_G(J)), \quad \hat{q}_F := \hat{\Phi}_J(q_F(J)), \quad \text{and} \quad \hat{q}_c := \hat{\Phi}_J(q_c(J)).$$

The quasi-greedy expansions of these bases are given by

$$\alpha^*(\hat{q}_G) = (10)^\infty, \quad \alpha^*(\hat{q}_F) = (1100)^\infty, \quad \text{and} \quad \alpha^*(\hat{q}_c) = 11010011\,00101101 \cdots .$$

We have $\hat{q}_G = (1+\sqrt{5})/2 \approx 1.61803$, $\hat{q}_F \approx 1.75488$, and $\hat{q}_c \approx 1.78723$. Note that $\hat{q}_c$ is simply the Komornik–Loreti constant q_{KL}^*. The following result is due to Glendinning and Sidorov [**23**] and Komornik *et al* [**27**]; see also [**7**].

PROPOSITION 4.5. *Let* $q \in (1, 2]$. *Then the entropy function*

$$H : q \mapsto h(\widetilde{\mathbf{U}}_q^*)$$

is a Devil's staircase, i.e. H is continuous, non-deceasing, and locally constant almost everywhere on $(1, 2]$.

(i) *If* $1 < q \le \hat{q}_{\mathrm{G}}$, *then* $\widetilde{\mathbf{U}}_q^* = \emptyset$.
(ii) *If* $\hat{q}_{\mathrm{G}} < q \le \hat{q}_{\mathrm{F}}$, *then* $\widetilde{\mathbf{U}}_q^* = \{(01)^\infty, (10)^\infty\}$.
(iii) *If* $\hat{q}_{\mathrm{F}} < q < \hat{q}_c$, *then* $\widetilde{\mathbf{U}}_q^*$ *is countably infinite.*
(iv) *If* $q = \hat{q}_c$, *then* $\widetilde{\mathbf{U}}_q^*$ *is uncountable but* $h(\widetilde{\mathbf{U}}_q^*) = 0$.
(v) *If* $\hat{q}_c < q \le 2$, *then* $h(\widetilde{\mathbf{U}}_q^*) > 0$.

Proof of Theorem 2. Recall from Proposition 3.8(iii) that for each $q \in \mathscr{V} \cap (q_{\mathrm{L}}, q_{\mathrm{R}}]$ we have

$$\widetilde{\mathbf{U}}_q(J) = \Phi_J^{-1}(\{(x_i) \in \widetilde{\mathbf{U}}_{\hat{q}}^* : x_1 = 1\}) \quad \text{and} \quad h(\widetilde{\mathbf{U}}_q(J)) = \frac{h(\widetilde{\mathbf{U}}_{\hat{q}}^*)}{m}, \tag{4.2}$$

where $\hat{q} := \hat{\Phi}_J(q)$. Since $\mathscr{B}^* \subset \mathscr{U}^*$, the function $q \mapsto h(\widetilde{\mathbf{U}}_q^*)$ is constant on each connected component of $(1, 2]\backslash\mathscr{U}^*$. Recalling from Proposition 3.8(ii) that $\hat{\Phi}_J(\mathscr{U} \cap J) = \mathscr{U}^*$, it follows by equation (4.2) that the function $H_J : q \mapsto h(\widetilde{\mathbf{U}}_q(J))$ is constant on each connected component of $(q_{\mathrm{L}}, q_{\mathrm{R}}]\backslash(\mathscr{U} \cap J)$. Since $\mathscr{U}$ is Lebesgue null, this implies that H_J is almost everywhere locally constant on J. That H_J is also continuous follows since $\mathscr{U} \cap J$ has no isolated points, and the restriction of H_J to $\mathscr{U} \cap J$ is the composition of the map $q \mapsto h(\widetilde{\mathbf{U}}_q^*)$ with $\hat{\Phi}_J$; the former is continuous by Proposition 4.5, the latter is continuous by Lemma 3.10. Therefore, the entropy function H_J is a Devil's staircase.

Statements (i)–(v) of Theorem 2 now follow from the corresponding statements of Proposition 4.5. For example, if $q_{\mathrm{L}} < q \le q_{\mathrm{G}}(J)$, then by equation (4.2) it follows that

$$\widetilde{\mathbf{U}}_q(J) \subset \widetilde{\mathbf{U}}_{q_{\mathrm{G}}(J)}(J) = \Phi_J^{-1}(\{(x_i) \in \widetilde{\mathbf{U}}_{\hat{q}_{\mathrm{G}}}^* : x_1 = 1\}) = \emptyset,$$

where the last equality follows from Proposition 4.5 (i).

Similarly, for (ii) we take $q \in (q_{\mathrm{G}}(J), q_{\mathrm{F}}(J)]$. Then by equation (4.2) and Proposition 4.5(ii) it follows that

$$\widetilde{\mathbf{U}}_q(J) \subset \widetilde{\mathbf{U}}_{q_{\mathrm{F}}(J)}(J) = \Phi_J^{-1}(\{(x_i) \in \widetilde{\mathbf{U}}_{\hat{q}_{\mathrm{F}}}^* : x_1 = 1\}) = \Phi_J^{-1}(\{(10)^\infty\}) = \{(\mathbf{a}^+\overline{\mathbf{a}^+})^\infty\}.$$

Vice versa, one checks easily using equation (1.3) that $\{(\mathbf{a}^+\overline{\mathbf{a}^+})^\infty\} \subset \widetilde{\mathbf{U}}_q(J)$.

For (iii), we take $q \in (q_{\mathrm{F}}(J), q_c(J))$. Then

$$\{(\mathbf{a}^+\overline{\mathbf{a}^+})^k(\mathbf{a}^+\overline{\mathbf{a}}\overline{\mathbf{a}^+}\mathbf{a})^\infty : k \in \mathbb{N}\} \subset \widetilde{\mathbf{U}}_q(J).$$

On the other hand, we can find a sequence (q_n) in $\mathscr{V}$ that converges from the left to $q_c(J)$: if $\alpha(q_c(J)) = \theta_1\theta_2\cdots$, we can take q_n with $\alpha(q_n) = (\theta_1 \cdots \theta_{2^n m}^-)^\infty$. Then for large enough n, $q < q_n$ and $\hat{q}_n := \hat{\Phi}_J(q_n) < q_{\mathrm{KL}}^*$, so $\widetilde{\mathbf{U}}_q(J)$ is countable by Proposition 4.5(iii) and (4.2).

Statement (iv) is immediate from equation (4.2) and Proposition 4.5(iv), since $q_c(J) \in \mathscr{V}$.

For (v), we first note that $q_c(J) \in \mathscr{V} \cap (q_{\mathrm{L}}, q_{\mathrm{R}}]$ and $\hat{q}_c \in \mathscr{V}^*$. Furthermore, there exists a sequence (r_i) in $\mathscr{V} \cap (q_{\mathrm{L}}, q_{\mathrm{R}}]$ such that $r_i \searrow q_c(J)$. This follows from Lemma 4.1, since the endpoints of relative plateaus lie in $\mathscr{V}$. Accordingly, the image sequence $(\hat{r}_i)$ in $\mathscr{V}^*$ satisfies $\hat{r}_i \searrow \hat{q}_c$, where $\hat{r}_i = \hat{\Phi}_J(r_i)$. So, for any $q \in (q_c(J), q_{\mathrm{R}}]$ there exists $r_i \in \mathscr{V} \cap (q_c(J), q)$ such that

$$\widetilde{\mathbf{U}}_q(J) \supset \widetilde{\mathbf{U}}_{r_i}(J) \quad \text{and} \quad h(\widetilde{\mathbf{U}}_{r_i}(J)) = \frac{h(\widetilde{\mathbf{U}}_{\hat{r}_i}^*)}{m} > 0.$$

This proves (v). □

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

Proof of Corollary 1.7. Take $q \in (q_L, q_R]$. Then $\alpha_1(q)\cdots\alpha_m(q) = a_1\cdots a_m^+$. Note that $\alpha(q_L) = (a_1\cdots a_m)^\infty$. Then by the definitions of $\widetilde{\mathbf{U}}_q$ and $\widetilde{\mathbf{U}}_{q_L}$ it follows that $\widetilde{\mathbf{U}}_q(J) \subset \widetilde{\mathbf{U}}_q \setminus \widetilde{\mathbf{U}}_{q_L}$. Furthermore, any sequence $(x_i) \in \widetilde{\mathbf{U}}_q \setminus \widetilde{\mathbf{U}}_{q_L}$ or its reflection $\overline{(x_i)}$ has a tail sequence in $\widetilde{\mathbf{U}}_q(J) \cup \{\mathbf{a}^\infty\}$. Therefore,

$$\dim_H(\widetilde{\mathbf{U}}_q \setminus \widetilde{\mathbf{U}}_{q_L}) = \dim_H \widetilde{\mathbf{U}}_q(J).$$

Hence, the result follows from Theorem 2. □

Proof of Corollary 1.9. Fix $q_0 \in \overline{\mathscr{U}}$. If $q_0 \in \mathscr{C}_\infty$, the same argument based equation on (2.1) that we used to prove $f(q_0) = 0$ shows also that $f(q) \to 0$ as $q \to q_0$. Hence f is continuous at q_0. If $q_0 \in \mathscr{C}_0$, then $q_0 = q_c(J)$ for some relative plateau J. Since $\widetilde{\mathbf{U}}_q(I) \subset \widetilde{\mathbf{U}}_q(J)$ whenever $I \subset J$, Theorem 1 implies that

$$f(q) \le \frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} \quad \text{for all } q \in J,\ q \ne q_0.$$

But by Theorem 2, $h(\widetilde{\mathbf{U}}_q(J)) \to 0$ as $q \to q_0 = q_c(J)$. Hence, $f(q) \to 0 = f(q_0)$. This shows that f is continuous on $\mathscr{C}$.

Now suppose $q_0 \in \overline{\mathscr{U}} \backslash \mathscr{C}$. Then, by Lemma 4.1, there is a sequence of relative plateaus $[p_L(i), p_R(i)]$ such that $p_L(i) \nearrow q_0$ as $i \to \infty$. Each of these plateaus contains a point $q_i \in \mathscr{C}$ (in fact, infinitely many), so that $q_i \nearrow q_0$. By Theorem 1, we obtain $f(q_0) > 0 = \lim_{i\to\infty} f(q_i)$. Therefore, f is discontinuous at q_0. The corresponding statements for f_- and f_+ follow in the same way. □

Finally, we prove Theorem 3.

Proof of Theorem 3. (i) Let $J = J_{\mathbf{i}} = [q_L, q_R]$ be a relative plateau generated by $\mathbf{a} = a_1\cdots a_m$. We show that the next level relative plateaus $J_{\mathbf{i}j}$, $j = 1, 2, \ldots$ are exactly the maximal intervals on which $h(\widetilde{\mathbf{U}}_q(J))$ is positive and constant; this, along with Theorem 2, will imply (i). Fix $j \in \mathbb{N}$, and write $I := J_{\mathbf{i}j} = [p_L, p_R]$. Then $p_L, p_R \in \mathscr{V}$, so we may put $\hat{p}_L := \hat{\Phi}_J(p_L)$ and $\hat{p}_R := \hat{\Phi}_J(p_R)$. Then $\hat{I} := [\hat{p}_L, \hat{p}_R]$ is an entropy plateau in $(1, 2]$, and so $h(\widetilde{\mathbf{U}}^*_{\hat{q}})$ is positive and constant on $\hat{I}$. By Proposition 3.8(iii), it follows that $h(\widetilde{\mathbf{U}}_q(J))$ is positive and constant on I.

By Lemma 4.1 the union $\bigcup_{j\in\mathbb{N}} J_{\mathbf{i}j}$ is dense in $(q_c(J), q_R]$. As a result, I is a *maximal* interval on which $h(\widetilde{\mathbf{U}}_q(J))$ is constant.

(ii) Since $\bigcup_{j\in\mathbb{N}} J_{\mathbf{i}j}$ is dense in $(q_c(J), q_R]$, each $q \in \mathscr{B}(J)$ is an accumulation point of the set of endpoints of the intervals $J_{\mathbf{i}j}$. Since these endpoints lie in $\mathscr{V}$ and $\mathscr{V}$ is closed, it follows that $\mathscr{B}(J) \subset \mathscr{V}$. Hence, $\hat{\Phi}_J(q)$ is well defined for all $q \in \mathscr{B}(J)$. It now follows immediately from part (i) that

$$\hat{\Phi}_J(\mathscr{B}(J)) = \mathscr{B}^*. \tag{4.3}$$

Since $\mathscr{B}^* \subset \mathscr{U}^*$, it follows from Proposition 3.8(ii) that $\mathscr{B}(J) \subset \mathscr{U} \cap J$.

(iii) That $\mathscr{B}(J)$ is Lebesgue null is now obvious from (ii), since $\mathscr{U}$ is Lebesgue null.

(v) By (i), (ii) and the countable stability of Hausdorff dimension,

$$\dim_H((\mathscr{U} \cap J)\backslash\mathscr{B}(J)) = \sup_{j\in\mathbb{N}} \dim_H(\mathscr{U} \cap J_{\mathbf{i}j}).$$

If $J_{\mathbf{i}j} = [p_L, p_R]$ is generated by the block $b_1 \cdots b_l$, then by equation (2.1)

$$\dim_H(\mathscr{U} \cap J_{\mathbf{i}j}) = \frac{\log 2}{l \log p_R}. \tag{4.4}$$

Furthermore, $b_1 \cdots b_l$ must be a concatenation of words from $\mathcal{L} = \{\mathbf{a}, \mathbf{a}^+, \overline{\mathbf{a}}, \overline{\mathbf{a}^+}\}$, so l is a multiple of m. Since $b_1 \cdots b_l$ is admissible and $\alpha(p_L) > q_c(J)$, it follows from equation (1.7) that $l \geq 3m$. (See Figure 3.) Moreover, the only relative plateau among the $J_{\mathbf{i}j}$ with $l = 3m$ is the one with generating word $b_1 \cdots b_l = \mathbf{a}^+\overline{\mathbf{a}}\overline{\mathbf{a}^+}$, whose right endpoint is p_0.

It remains to check that this plateau is maximal for equation (4.4) to hold. To this end, take any other relative plateau $[p_L, p_R] \subset J$ generated by a block of length $l = km$. If $p_R \geq p_0$, then $l \log p_R \geq 3m \log p_0$. On the other hand, suppose $p_R < p_0$. Then $\alpha(q_c(J)) \prec \alpha(p_L) \prec (\mathbf{a}^+\overline{\mathbf{a}}\overline{\mathbf{a}^+})^\infty$, and since $\alpha(p_L)$ must correspond to an infinite path in the labeled graph $\mathcal{G} = (G, \mathcal{L})$ from Figure 3, this is only possible when $k \geq 5$. In **[6]** it was observed that $q_{KL} \geq (M+2)/2$. Estimating p_R below by q_{KL} and p_0 above by $M+1$, we thus obtain for all $M \geq 2$,

$$l \log p_R \geq 5m \log q_{KL} \geq 5m \log\left(\frac{M+2}{2}\right) \geq 3m \log(M+1) > 3m \log p_0,$$

where we used the algebraic inequality $(M+2)^5 \geq 32(M+1)^3$, valid for $M \geq 2$. For the case $M = 1$ we can use the better estimate $q_{KL} > 1.78$, giving $5 \log q_{KL} > 2.8 > 3 \log 2 > 3 \log p_0$, where we have used the natural logarithm. Thus, in all cases, $l \log p_R \geq 3m \log p_0$, as was to be shown.

Finally, we prove (iv). Since

$$p_0^3 > q_{KL}^3 \geq \left(\frac{M+2}{2}\right)^3 \geq M+1 > q_R,$$

it follows from part (v) and (2.1) that

$$\dim_H((\mathscr{U} \cap J) \setminus \mathscr{B}(J)) = \frac{\log 2}{3m \log p_0} < \frac{\log 2}{m \log q_R} = \dim_H(\mathscr{U} \cap J).$$

This implies (iv). □

Remark 4.6. Note by equation (4.3) and Proposition 3.8(i) that the relative bifurcation sets $\mathscr{B}(J_{\mathbf{i}}) : \mathbf{i} \in \{1, 2, \ldots\}^n, n \in \mathbb{N}$ are mutually homeomorphic.

To end this section, we illustrate how Theorem 1 can be combined with the entropy 'bridge' of Proposition 3.8(iii) to compute $f(q)$ explicitly at some special points.

Example 4.7. Let $J = [p_L, p_R]$ be a relative plateau generated by the word $\mathbf{a} = a_1 \cdots a_m$. For any integer $k \geq 3$, let $[q_L, q_R]$ be the relative plateau generated by the admissible word $\mathbf{b} := \mathbf{a}^+\overline{\mathbf{a}}^{k-2}\overline{\mathbf{a}^+}$. Then $[q_L, q_R] \subset J$, and J is the parent interval of $[q_L, q_R]$. Note that $q_L \in \mathscr{V}$. Hence, by Theorem 1 and Proposition 3.8(iii),

$$f(q_L) = f_-(q_L) = \frac{h(\widetilde{\mathbf{U}}_{q_L}(J))}{\log q_L} = \frac{h(\widetilde{\mathbf{U}}^*_{\hat{q}_L})}{m \log q_L},$$

where $\hat{q}_L := \hat{\Phi}_J(q_L)$. Note that

$$\alpha^*(\hat{q}_L) = \Phi_J(\alpha(q_L)) = \Phi_J((\mathbf{a}^+\overline{\mathbf{a}}^{k-2}\overline{\mathbf{a}^+})^\infty) = (1^{k-1}0)^\infty.$$

Define the sets

$$\widetilde{\mathbf{V}}^*_{\hat{q}} := \{(x_i) \in \{0, 1\}^{\mathbb{N}} : \overline{\alpha^*(\hat{q})} \preceq \sigma^n((x_i)) \preceq \alpha^*(\hat{q})\ \forall n \geq 0\}, \quad \hat{q} \in (1, 2].$$

It is well known (see [**27**] or [**7**]) that $h(\widetilde{\mathbf{U}}^*_{\hat{q}}) = h(\widetilde{\mathbf{V}}^*_{\hat{q}})$. Moreover, $\widetilde{\mathbf{V}}^*_{\hat{q}_L}$ is a subshift of finite type and it consists of precisely those sequences in $\{0, 1\}^{\mathbb{N}}$ which do not contain the word 1^k or 0^k. A standard argument (see [**33**] or [**6**, Lemma 4.2]) now shows that $h(\widetilde{\mathbf{V}}^*_{\hat{q}_L}) = \log \varphi_{k-1}$, where for each $j \in \mathbb{N}$, φ_j is the unique root in $(1, 2)$ of $1 + x + \cdots + x^{j-1} = x^j$. Therefore,

$$f(q_L) = f_-(q_L) = \frac{\log \varphi_{k-1}}{m \log q_L}.$$

Of course, $f_+(q_L) = 0$. Similarly, since $h(\widetilde{\mathbf{U}}_q(J))$ is constant on $[q_L, q_R]$, Theorem 1 gives

$$f(q_R) = f_+(q_R) = \frac{\log \varphi_{k-1}}{m \log q_R}.$$

On the other hand, by equation (2.1),

$$f_-(q_R) = \frac{\log 2}{mk \log q_R},$$

since the generating word $\mathbf{b}$ of $[q_L, q_R]$ has length mk. Observe that $f_-(q_R) < f_+(q_R)$. This last inequality holds generally, for any relative plateau $[q_L, q_R]$ in J: if $[q_L, q_R]$ has generating block $\mathbf{b}$ of length l, then $l = mk$ for some $k \in \mathbb{N}$. Again putting $\hat{q}_L := \hat{\Phi}_J(q_L)$, Lemma 3.1(ii) in [**6**] gives

$$h(\widetilde{\mathbf{V}}^*_{\hat{q}_L}) > \frac{\log 2}{k},$$

and so

$$f_-(q_R) = \frac{\log 2}{mk \log q_R} < \frac{h(\widetilde{\mathbf{V}}^*_{\hat{q}_L})}{m \log q_R} = \frac{h(\widetilde{\mathbf{V}}^*_{\hat{q}_R})}{m \log q_R} = f_+(q_R).$$

(There is one exception: if $[q_L, q_R]$ is a first-level relative plateau (i.e. an entropy plateau) generated by $\mathbf{a} = a_1 \cdots a_m$, then the parent interval J is $J_\emptyset = [1, M+1]$. In this case, there is no map Φ_J relating $h(\widetilde{\mathbf{U}}_{q_L}(J))$ to the alphabet $\{0, 1\}$. Instead,

$$f_+(q_R) = \frac{h(\widetilde{\mathbf{U}}_{q_R})}{\log q_R} = \frac{h(\widetilde{\mathbf{U}}_{q_L})}{\log q_R}, \quad \text{and} \quad f_-(q_R) = \frac{\log 2}{m \log q_R}.$$

As shown in [**6**, Lemma 3.1 (ii)], these two quantities are equal if (and only if) $M = 2j + 1 \geq 3$, and $\mathbf{a} = a_1 := j + 1$.)

The above procedure generalizes to other relative plateaus: $\widetilde{\mathbf{V}}^*_{\hat{q}_L}$ is always a subshift of finite type of $\{0, 1\}^{\mathbb{N}}$, so its topological entropy can be calculated, numerically at least, by writing down the corresponding adjacency matrix and computing its spectral radius; see [**33**, Ch. 5]. This method was used to produce the graph in Figure 2.

5. *Proof of Theorem 4*

Proof of Theorem 4. Let $1 < t_1 < t_2 \leq M + 1$, and let $J = J_{\mathbf{i}} = [q_L, q_R]$ be the smallest relative plateau containing $[t_1, t_2]$. Define

$$g_J(t_1, t_2) := \max\left\{\frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} : q \in \overline{\mathscr{B}(J) \cap [t_1, t_2]}\right\},$$

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

so we need to show that

$$\dim_H(\mathscr{U} \cap [t_1, t_2]) = g_J(t_1, t_2). \tag{5.1}$$

Note first that, if $t_1 = q_L$, then there exists $\delta > 0$ such that $\mathscr{U} \cap [t_1, t_1 + \delta] = \emptyset$, and hence $\mathscr{B}(J) \cap [t_1, t_2] = \mathscr{B}(J) \cap [t_1 + \delta, t_2]$. Therefore, both sides of equation (5.1) remain unchanged upon replacing t_1 with $t_1 + \delta$. Consequently, we may assume that $t_1 > q_L$.

We first demonstrate the lower bound. Since $\mathscr{B}(J) \subset \mathscr{U}$, we may assume without loss of generality that $\mathscr{U} \cap (t_1, t_2) \neq \emptyset$. Then by the definition of J we also have $\mathscr{B}(J) \cap [t_1, t_2] \neq \emptyset$. Since $t_1 > q_L$, Theorem 1 gives for any $q \in \mathscr{B}(J) \cap [t_1, t_2]$ that

$$f_-(q) = \frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} > 0.$$

Since $\mathscr{B}(J) \cap [t_1, t_2] \subset \mathscr{U} \cap [t_1, t_2]$, this implies

$$\dim_H(\mathscr{U} \cap [t_1, t_2]) \geq \sup\{f_-(q) : q \in \mathscr{B}(J) \cap [t_1, t_2]\} = g_J(t_1, t_2),$$

where in the last step we used the continuity of the map $q \mapsto h(\widetilde{\mathbf{U}}_q(J))$ (cf. Theorem 2). This proves the lower bound.

For the upper bound, we use a compactness argument similar to that used in [**25**]. Recall from Theorem 3(i) that

$$(q_L, q_R] = \mathscr{B}(J) \cup (q_L, q_c(J)] \cup \bigcup_{j=1}^{\infty} J_{\mathbf{i}_j}.$$

Let $J_{\mathbf{i}_j} = [p_L, p_R]$ be a relative plateau that intersects $[t_1, t_2]$ in more than one point. Then either p_L or p_R lies in (t_1, t_2), so at least one of these two points lies in $\overline{\mathscr{B}(J) \cap [t_1, t_2]}$. Then by the proof of [**6**, Theorem 4.1] it follows that

$$\dim_H(\mathscr{U} \cap [p_L, p_R]) = \frac{\log 2}{m \log p_R} \leq \min\left\{\frac{h(\widetilde{\mathbf{U}}_{p_L})}{\log p_L}, \frac{h(\widetilde{\mathbf{U}}_{p_R})}{\log p_R}\right\} \leq g_J(t_1, t_2).$$

By the countable stability of Hausdorff dimension, we obtain

$$\dim_H\bigg(\mathscr{U} \cap \bigcup_{j=1}^{\infty} J_{\mathbf{i}_j} \cap [t_1, t_2]\bigg) \leq g_J(t_1, t_2). \tag{5.2}$$

Now let $\varepsilon > 0$. Then for each $q \in \overline{\mathscr{B}(J) \cap [t_1, t_2]}$ there is a number $\delta(q) > 0$ such that

$$\dim_H(\mathscr{U} \cap (q - \delta(q), q + \delta(q))) \leq f(q) + \varepsilon \leq \frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} + \varepsilon \leq g_J(t_1, t_2) + \varepsilon.$$

The intervals $(q - \delta(q), q + \delta(q))$ form an open cover of $\overline{\mathscr{B}(J) \cap [t_1, t_2]}$, and since $\overline{\mathscr{B}(J) \cap [t_1, t_2]}$ is compact, this open cover contains a finite subcover. Therefore,

$$\dim_H(\mathscr{U} \cap \overline{\mathscr{B}(J) \cap [t_1, t_2]}) \leq g_J(t_1, t_2) + \varepsilon. \tag{5.3}$$

Letting $\varepsilon \to 0$, equations (5.2) and (5.3) together give the upper bound in equation (5.1), since $\mathscr{U} \cap (q_L, q_c(J)) = \emptyset$. □

6. *Proof of Theorem 5*

Recall the definitions (1.10) and (1.12) of $\check{\mathbf{U}}_q$ and $\mathbf{W}_q$, and that $\mathcal{W}_q = \pi_q(\mathbf{W}_q)$. When $M = 1$ we write $\mathbf{W}_q^* := \mathbf{W}_q$. We will prove Theorem 5 indirectly, by showing that $\dim_{\mathrm{H}} \mathcal{W}_q = 0$ for $q \in \mathscr{C}$, and that if $q \in \overline{\mathscr{U}}\backslash\mathscr{C}$, then

$$\dim_{\mathrm{H}} \mathcal{W}_q = \frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q},$$

where $J = [q_{\mathrm{L}}, q_{\mathrm{R}}]$ is the smallest relative plateau such that $q \in (q_{\mathrm{L}}, q_{\mathrm{R}}]$. The result then follows from Theorem 1.

Recall that on the sequence space Ω_M we are using the metric ρ from equation (1.9). The following lemma allows us to work with subsets of Ω_M rather than sets in Euclidean space.

LEMMA 6.1. *Let $q \in (1, M+1]$. For any subset $F \subset \widetilde{\mathbf{U}}_q$, we have*

$$\dim_{\mathrm{H}} \pi_q(F) = \frac{\log 2}{\log q} \dim_{\mathrm{H}} F.$$

Proof. It is well known (see [**24**, Lemma 2.7] or [**4**, Lemma 2.2]) that π_q is bi-Lipschitz on $\widetilde{\mathbf{U}}_q$ with respect to the metric

$$\rho_q((x_i), (y_i)) := q^{-\inf\{i \geq 0 : x_{i+1} \neq y_{i+1}\}}.$$

Hence, with respect to the metric ρ_q on Ω_M, F, and $\pi_q(F)$ have the same Hausdorff dimension for any $F \subset \widetilde{\mathbf{U}}_q$. The lemma now follows since $\rho_q = \rho^{\log q/\log 2}$. □

In view of Lemma 6.1, it suffices to compute $\dim_{\mathrm{H}} \mathbf{W}_q$. The next lemma facilitates this.

LEMMA 6.2. *Let $J = [q_{\mathrm{L}}, q_{\mathrm{R}}]$ be a relative plateau generated by $\mathbf{a} = a_1 \cdots a_m$, and $q \in \mathscr{V} \cap (q_{\mathrm{L}}, q_{\mathrm{R}}]$. Then*

$$\dim_{\mathrm{H}} \mathbf{W}_q = \frac{1}{m} \dim_{\mathrm{H}} \mathbf{W}_{\hat{q}}^*,$$

where $\hat{q} := \hat{\Phi}_J(q)$.

Proof. Since $\mathbf{W}_q \subset \widetilde{\mathbf{U}}_q$ and every sequence in $\mathbf{W}_q$ must eventually contain the word $\alpha_1(q) \cdots \alpha_m(q)$, we have

$$\dim_{\mathrm{H}} \mathbf{W}_q = \dim_{\mathrm{H}}(\mathbf{W}_q \cap \widetilde{\mathbf{U}}_q(J)).$$

By a trivial extension of Proposition 3.8(iii),

$$\Phi_J(\mathbf{W}_q \cap \widetilde{\mathbf{U}}_q(J)) = \mathbf{W}_{\hat{q}}^* \cap \widetilde{\mathbf{U}}_{\hat{q}}^*(1),$$

where $\widetilde{\mathbf{U}}_{\hat{q}}^*(1) := \{(x_i) \in \widetilde{\mathbf{U}}_{\hat{q}}^* : x_1 = 1\}$. Since Φ_J is bi-Hölder continuous with exponent $1/m$, it follows that

$$\dim_{\mathrm{H}} \mathbf{W}_q = \frac{1}{m} \dim_{\mathrm{H}}(\mathbf{W}_{\hat{q}}^* \cap \widetilde{\mathbf{U}}_{\hat{q}}^*(1)) = \frac{1}{m} \dim_{\mathrm{H}} \mathbf{W}_{\hat{q}}^*,$$

as desired. □

We first consider the case when $q \in \mathscr{C}$.

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

PROPOSITION 6.3. *If $q \in \mathscr{C}$, then* $\dim_H \mathbf{W}_q = 0$.

Proof. If $q = q_{KL}$, then $\dim_H \mathbf{W}_q \le \dim_H \widetilde{\mathbf{U}}_q = 0$ by Proposition 4.5(iv), which holds also for larger alphabets (cf. [**32**]). And if $q \in \mathscr{C}_0 \backslash \{q_{KL}\}$, then $q = q_c(J)$ for some relative plateau J, so that $\hat{q} := \hat{\Phi}_J(q) = q^*_{KL}$ and the result follows from Lemma 6.2 and Proposition 4.5(iv).

Suppose $q \in \mathscr{C}_\infty$. Then $q \in \mathscr{U} \subset \mathscr{V}$ by Proposition 1.4, and there are infinitely many relative plateaus $J = [q_L, q_R]$ such that $q \in (q_L, q_R]$. If J is one such relative plateau generated by a word of length m, then Lemma 6.2 gives

$$\dim_H \mathbf{W}_q = \frac{1}{m} \dim_H \mathbf{W}^*_{\hat{q}} \le \frac{1}{m} \dim_H \{0, 1\}^{\mathbb{N}} = \frac{1}{m}.$$

Letting $m \to \infty$, we obtain $\dim_H \mathbf{W}_q = 0$. □

Recall the definition of $\mathscr{B}_L$ (and $\mathscr{B}^*_L$) from equation (4.1).

PROPOSITION 6.4. *Let $q \in \mathscr{B}_L$. Then*

$$\dim_H \mathbf{W}_q = \frac{h(\widetilde{\mathbf{U}}_q)}{\log 2} = \dim_H \widetilde{\mathbf{U}}_q.$$

The proof uses the following lemma.

LEMMA 6.5. *Let $G = (V, E)$ be a finite strongly connected directed graph with adjacency matrix A, and let γ be the spectral radius of A. Let $\mathcal{P}_k^{u,v}$ be the set of all directed paths of length k in G starting from vertex u and ending at vertex v. Then there are constants $0 < C_1 < C_2$ such that the following hold:*

(i) *For each vertex $v \in V$ and for each $K \in \mathbb{N}$, there is an integer $k \ge K$ such that*

$$\#\mathcal{P}_k^{v,v} \ge C_1 \gamma^k.$$

(ii) *For all $k \in \mathbb{N}$,*

$$\sum_{u,v \in V} \#\mathcal{P}_k^{u,v} \le C_2 \gamma^k.$$

Proof. By the Perron–Frobenius theorem, γ is an eigenvalue of A and there is a strictly positive left eigenvector $\boldsymbol{\xi} = [\xi_1 \cdots \xi_N]$ of A corresponding to γ, where $N := \#V < \infty$. We may normalize $\boldsymbol{\xi}$ so that $\max \xi_i = 1$. Clearly, for any two vertices u and v in V, there is a path from u to v of length at most N. Fix $v \in V$ and $K \in \mathbb{N}$. Without loss of generality order V so that v is the first vertex. Let $\mathbf{e}_1 = [1\ 0\ \cdots\ 0]^T$ be the first standard unit vector in $\mathbb{R}^N$, and let $\mathbf{1} = [1\ 1\ \cdots\ 1]$ be the row vector of all 1 in $\mathbb{R}^N$. The number of paths in G of length K starting anywhere in G but ending at v is

$$\mathbf{1} A^K \mathbf{e}_1 \ge \boldsymbol{\xi} A^K \mathbf{e}_1 = \gamma^K \boldsymbol{\xi} \mathbf{e}_1 \ge \gamma^K \min \xi_i.$$

Hence there is a vertex u in V such that

$$\#\mathcal{P}_K^{u,v} \ge N^{-1} \gamma^K \min \xi_i.$$

Let L be the length of the shortest path in G from v to u; then $L \le N$. Set $k := K + L$. It follows that

$$\#\mathcal{P}_k^{v,v} \ge N^{-1} \gamma^K \min \xi_i = N^{-1} \gamma^{k-L} \min \xi_i \ge \frac{\min \xi_i}{N \gamma^N} \gamma^k =: C_1 \gamma^k.$$

This proves (i). The proof of (ii) is standard (cf. [**33**, Ch. 4]). □

Recall from [**29**] that the Komornik–Loreti constant $q_{\mathrm{KL}} = q_{\mathrm{KL}}(M)$ satisfies

$$\alpha(q_{\mathrm{KL}}) = \lambda_1\lambda_2\cdots, \tag{6.1}$$

where for each $i \geq 1$,

$$\lambda_i = \lambda_i(M) := \begin{cases} k + \tau_i - \tau_{i-1} & \text{if } M = 2k, \\ k + \tau_i & \text{if } M = 2k+1. \end{cases}$$

Here $(\tau_i)_{i=0}^{\infty} = 0110100110010110\cdots$ is the classical Thue–Morse sequence.

In the proof below we use the sets

$$\widetilde{\mathbf{V}}_q := \{(x_i) \in \Omega_M : \overline{\alpha(q)} \preccurlyeq \sigma^n((x_i)) \preccurlyeq \alpha(q)\ \forall n \geq 0\}, \quad q \in (1, M+1].$$

It is well known (see [**27**] or [**6**]) that $\dim_{\mathrm{H}} \widetilde{\mathbf{U}}_q = \dim_{\mathrm{H}} \widetilde{\mathbf{V}}_q$ for every q.

Proof of Proposition 6.4. Fix $q \in \mathscr{B}_{\mathrm{L}}$. Then $q > q_{\mathrm{KL}}$, so $\alpha(q) \succ \lambda_1\lambda_2\cdots$, and hence there is a number $l_0 \geq 1$ such that $\alpha_1\cdots\alpha_{l_0-1} = \lambda_1\cdots\lambda_{l_0-1}$ and $\alpha_{l_0} > \lambda_{l_0}$, where for brevity we put $\alpha_i := \alpha_i(q)$.

By [**2**, Lemma 3.16] (see also [**6**]), there is an increasing sequence (l_n) of integers with $l_n > l_0$ such that for each n, there is an entropy plateau $[p_{\mathrm{L}}(n), p_{\mathrm{R}}(n)]$ with $\alpha(p_{\mathrm{L}}(n)) = (\alpha_1\cdots\alpha_{l_n}^-)^\infty$, and moreover $p_{\mathrm{L}}(n) \nearrow q$. By the continuity of the function $p \mapsto \dim_{\mathrm{H}} \widetilde{\mathbf{U}}_p$ it is enough to prove that $\dim_{\mathrm{H}} \mathbf{W}_q \geq \dim_{\mathrm{H}} \widetilde{\mathbf{U}}_{p_{\mathrm{L}}(n)}$ for each n.

Fix $n \in \mathbb{N}$, and put $p := p_{\mathrm{L}}(n)$ and $l := l_n$. Then $\widetilde{\mathbf{V}}_p$ is a subshift of finite type, characterized by

$$(x_i) \in \widetilde{\mathbf{V}}_p \quad \Leftrightarrow \quad \overline{\alpha_1\cdots\alpha_l} \prec x_{k+1}\cdots x_{k+l} \prec \alpha_1\cdots\alpha_l \quad \text{for all } k \geq 0.$$

We represent $\widetilde{\mathbf{V}}_p$ by a labeled directed graph $G = (V, E, L)$ in the usual way: the set V of vertices consists of allowed words in $\widetilde{\mathbf{V}}_p$ of length $l-1$, and there is an edge uv from $u = x_1\cdots x_{l-1}$ to $v = y_1\cdots y_{l-1}$ if and only if $x_2\cdots x_{l-1} = y_1\cdots y_{l-2}$ and $x_1\cdots x_{l-1}y_{l-1}$ is an allowed word in $\widetilde{\mathbf{V}}_p$, in which case we label the edge uv with y_{l-1}.

Case 1. Assume first that $\widetilde{\mathbf{V}}_p$ is transitive, so the graph G is strongly connected. Let γ be the spectral radius of the adjacency matrix of G, and let C_1, C_2 be the constants from Lemma 6.5. Put $C := \max\{C_2, C_1^{-1}\}$. Let $\mathbf{u} = \alpha_1\cdots\alpha_{l-1}$ be the lexicographically largest vertex in V.

Next, fix $0 < s < \dim_{\mathrm{H}} \widetilde{\mathbf{U}}_p$. We will construct a subset $\mathbf{Y}$ of $\mathbf{W}_q$ such that $\dim_{\mathrm{H}} \mathbf{Y} \geq s$. Since the Hausdorff dimension of a subshift of finite type is given by its entropy, we have

$$s < \dim_{\mathrm{H}} \widetilde{\mathbf{U}}_p = \dim_{\mathrm{H}} \widetilde{\mathbf{V}}_p = \log_2 \gamma. \tag{6.2}$$

Let $(m_j)_{j\in\mathbb{N}}$ be any strictly increasing sequence of positive integers with $m_1 > l$ such that $\alpha_1\cdots\alpha_{m_j}^-$ is admissible for each j. We claim that for each j there exists a connecting block $b_1\cdots b_{n_j}$ such that $\alpha_1\cdots\alpha_{m_j}^- b_1\cdots b_{n_j}\mathbf{u}$ is an allowed word in $\widetilde{\mathbf{U}}_q$. This follows essentially from the proof of [**2**, Proposition 3.17], but for the reader's convenience we sketch the main idea.

Set $i_0 := m_j$. Recursively, for $\nu = 0, 1, 2, \ldots,$ proceed as follows. If $i_\nu < l_0$, then stop; otherwise, let $i_{\nu+1}$ be the largest integer i such that

$$\alpha_{i_\nu - i+1}\cdots\alpha_{i_\nu} = \overline{\alpha_1\cdots\alpha_i}^+.$$

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

(If no such i exists, set $i_{\nu+1}=0$.) We now argue that

$$i_{\nu+1} < i_\nu \quad \text{for every } \nu. \tag{6.3}$$

This will follow once we show that $\alpha_1\cdots\alpha_k \succ \overline{\alpha_1\cdots\alpha_k}^+$ for every $k\ge l_0$. This inequality is clear for $k\ge 2$, since $q>q_{KL}$ implies $\alpha_1>\overline{\alpha_1}$. On the other hand, if $l_0=1$, then $\alpha_1>\lambda_1\ge\overline{\lambda_1}^+>\overline{\alpha_1}^+$, yielding the inequality for $k=1$ as well.

In view of equation (6.3), this process eventually stops, say after $N=N(j)$ steps, with $i_N<l_0$. It is easy to check that $\alpha_1\cdots\alpha_{i_\nu}^-$ is admissible for each $\nu<N$. Since $q\in\mathscr{B}_L$ and $\alpha(q)\succ(\alpha_1\cdots\alpha_{i_\nu}^-)^\infty$, it follows that

$$\alpha(q)\succ\alpha_1\cdots\alpha_{i_\nu}(\overline{\alpha_1\cdots\alpha_{i_\nu}}^+)^\infty,\qquad \nu=1,2,\ldots,N-1.$$

Hence there is a positive integer k_ν such that

$$\alpha(q)\succ\alpha_1\cdots\alpha_{i_\nu}(\overline{\alpha_1\cdots\alpha_{i_\nu}}^+)^{k_\nu},\qquad \nu=1,2,\ldots,N-1, \tag{6.4}$$

where by $\alpha(q)\succ\beta_1\cdots\beta_i$ we mean that $\alpha_1\cdots\alpha_i\succ\beta_1\cdots\beta_i$. Put

$$B_\nu:=(\alpha_1\cdots\alpha_{i_\nu}^-)^{k_\nu},\qquad \nu=1,\ldots,N-1,$$

and $b_1\cdots b_{n_j}:=B_1B_2\cdots B_{N-1}$, where if $N=1$ we take $B_1B_2\cdots B_{N-1}$ to be the empty word.

Since $|\mathbf{u}|=l-1\ge l_0$, it can be verified using the admissibility of $\alpha_1\cdots\alpha_{i_\nu}^-$ for each ν that $\alpha_1\cdots\alpha_{m_j}^- b_1\cdots b_{n_j}\mathbf{u}$ is an allowed word in $\widetilde{\mathbf{U}}_q$. Here we emphasize that the length n_j of the connecting block depends only on m_j, since the word $\mathbf{u}$ is fixed throughout.

We now construct sequences (r_j) and (R_j) as follows: set $R_0=m_1+n_1$, and inductively, for $j=1,2,\ldots$, we can choose by equation (6.2) and Lemma 6.5 an integer r large enough so that

$$(\log_2\gamma-s)r\ge(R_{j-1}+m_{j+1}+n_{j+1}+l-1)s+(j+2)\log_2 C \tag{6.5}$$

and

$$\#\mathcal{P}_r^{\mathbf{u},\mathbf{u}}\ge C^{-1}\gamma^r. \tag{6.6}$$

Put

$$r_j:=r,\quad\text{and}\quad R_j:=R_{j-1}+r_j+m_{j+1}+n_{j+1},$$

to complete the induction step. We also set

$$M_j:=\sum_{i=1}^{j}(m_i+n_i+r_i),\qquad N_j:=M_j+m_{j+1}\quad\text{for } j\ge 0.$$

Observe that $M_{j+1}-R_j=r_{j+1}$ for $j\ge 0$.

Now let $\mathbf{Y}$ be the set of sequences (y_i) in Ω_M satisfying the following requirements for all $j\ge 0$:

(1) $y_{M_j+1}\cdots y_{M_j+m_{j+1}}=\alpha_1\cdots\alpha_{m_{j+1}}^-$;

(2) $y_{N_j+1}\cdots y_{N_j+n_{j+1}}=b_1\cdots b_{n_{j+1}}$;

(3) $y_{R_j+1}\cdots y_{R_j+l-1}=\mathbf{u}$;

(4) $y_{R_j+l}\cdots y_{M_{j+1}+l-1}=$ the word formed by reading the labels of any path of length r_{j+1} in G that starts and ends at $\mathbf{u}$.

Note that (4) is consistent with (1) despite the overlapping definitions, since for each j, $\mathbf{u}$ is a prefix of $\alpha_1 \cdots \alpha_{m_j}^-$. By the construction of the connecting block $b_1 \cdots b_{n_{j+1}}$, the word $y_{M_j+1} \cdots y_{M_{j+1}}$ is allowed in $\widetilde{\mathbf{U}}_q$, for each j. It now follows easily that $\mathbf{Y} \subset \mathbf{W}_q$.

Next, we construct a mass distribution on $\mathbf{Y}$. Let t_j denote the number of words of length r_{j+1} satisfying the requirement of (4) above, and note that by equation (6.6),

$$t_j \geq C^{-1}\gamma^{r_{j+1}}, \quad j \geq 0. \tag{6.7}$$

Define a measure μ on $\mathbf{Y}$ by

$$\mu([y_1 \cdots y_k]) = \frac{\tilde{t}_j(y_1 \cdots y_k)}{\prod_{i=0}^{j} t_i} \quad \text{for } j \geq 0 \text{ and } R_j + l - 1 \leq k \leq M_{j+1}, \tag{6.8}$$

where $[y_1 \cdots y_k] := \{(x_i) \in \mathbf{Y} : x_1 \cdots x_k = y_1 \cdots y_k\}$ is the cylinder generated by $y_1 \cdots y_k$, and $\tilde{t}_j(y_1 \cdots y_k)$ is the number of paths in G of length $M_{j+1} + l - 1 - k$ starting at vertex $y_{k-l+2} \cdots y_k$ and ending at $\mathbf{u}$. Note that $\tilde{t}_j(y_1 \cdots y_{R_j+l-1}) = t_j$ and $\tilde{t}_{j-1}(y_1 \cdots y_{M_j}) = 1$, so that

$$\mu([y_1 \cdots y_{R_j+l-1}]) = \mu([y_1 \cdots y_{M_j}]). \tag{6.9}$$

Observe furthermore that

$$\tilde{t}_j(y_1 \cdots y_k) \leq C\gamma^{M_{j+1}+l-1-k}. \tag{6.10}$$

We complete the definition of μ by setting $\mu([y_1 \cdots y_k]) = 1$ for $k < R_0 + l - 1$, and

$$\mu([y_1 \cdots y_k]) = \mu([y_1 \cdots y_{M_j}]) \quad \text{for } j \geq 1 \text{ and } M_j < k < R_j + l - 1. \tag{6.11}$$

It is easy to see that Kolmogorov's consistency conditions are satisfied, so that μ defines a unique mass distribution on $\mathbf{Y}$. We claim that

$$\mu([y_1 \cdots y_k]) \leq \tilde{C}(\operatorname{diam}([y_1 \cdots y_k]))^s \tag{6.12}$$

for any $k \in \mathbb{N}$ and any cylinder $[y_1 \cdots y_k]$, where $\tilde{C} := C^2 2^{(R_0+l-1)s}$. Note that $\operatorname{diam}([y_1 \cdots y_k]) = 2^{-k}$. In view of equation (6.9), it is sufficient to check equation (6.12) for $R_j + l - 1 \leq k \leq M_{j+1}$, where $j \geq 0$. Assuming k is in this range, equation (6.8) and the estimates (6.7), (6.10) give

$$\begin{aligned}
&\log_2 \mu([y_1 \cdots y_k]) + ks \\
&\quad \leq (j+2)\log_2 C + \left(M_{j+1} + l - 1 - k - \sum_{i=1}^{j+1} r_i\right)\log_2 \gamma + ks \\
&\quad = (j+2)\log_2 C + (R_j + l - 1 - k)\log_2 \gamma - \sum_{i=1}^{j} r_i \log_2 \gamma + ks \\
&\quad \leq (R_j + l - 1)s + (j+2)\log_2 C - \sum_{i=1}^{j} r_i \log_2 \gamma,
\end{aligned}$$

using that $\log_2 \gamma > s$ and $M_{j+1} = R_j + r_{j+1}$. For $j = 0$ this last expression reduces to $(R_0 + l - 1)s + 2\log_2 C = \log_2 \tilde{C}$. For $j \geq 1$, it can be written as

$$(R_{j-1} + m_{j+1} + n_{j+1} + l - 1)s + (j+2)\log_2 C - \sum_{i=1}^{j-1} r_i \log_2 \gamma - r_j(\log_2 \gamma - s),$$

which is ≤ 0 by equation (6.5). Thus, in both cases, we obtain equation (6.12).

By the mass distribution principle, equation (6.12) implies $\dim_H \mathbf{W}_q \geq \dim_H \mathbf{Y} \geq s$, as required. Finally, since $s < \dim_H \widetilde{\mathbf{U}}_p$ was arbitrary, we conclude that $\dim_H \mathbf{W}_q \geq \dim_H \widetilde{\mathbf{U}}_p$.

Case 2. When $\widetilde{\mathbf{V}}_p$ is not transitive, $\widetilde{\mathbf{V}}_p$ contains by [**2**, Lemma 5.9] a transitive subshift $\mathbf{Z}_p$ of finite type with the same entropy $\log \gamma$, and $\alpha(p) \in \mathbf{Z}_p$. Hence the directed graph associated with $\mathbf{Z}_p$ still contains the vertex $\alpha_1 \cdots \alpha_{l-1}$, and the above argument goes through with $\mathbf{Z}_p$ replacing $\widetilde{\mathbf{V}}_p$. □

Proof of Theorem 5. Note that $\mathbf{W}_q = \emptyset$ for any $q \in (1, M+1] \setminus \overline{\mathscr{U}}$. In view of Propositions 6.3 and 6.4 it remains to prove the theorem for $q \in \overline{\mathscr{U}} \setminus (\mathscr{C} \cup \mathscr{B}_L)$. Then $q \in \overline{\mathscr{U}} \cap (q_L, q_R]$ for some relative plateau $[q_L, q_R]$. Assume $J = J_{\mathbf{i}} = [q_L, q_R]$ is the *smallest* such plateau, and let its generating word be $a_1 \cdots a_m$. Then either $q \in \mathscr{B}(J)$ or q is the left endpoint of $J_{\mathbf{i}j}$ for some $j \in \mathbb{N}$. Let $\hat{q} := \hat{\Phi}_J(q)$. Then $\hat{q} \in \mathscr{B}_L^*$. So using Lemma 6.2, Proposition 6.4, and Proposition 3.8(iii) we obtain

$$\dim_H \mathbf{W}_q = \frac{1}{m} \dim_H \mathbf{W}^*_{\hat{q}} = \frac{1}{m} \dim_H \widetilde{\mathbf{U}}^*_{\hat{q}} = \frac{h(\widetilde{\mathbf{U}}^*_{\hat{q}})}{m \log 2} = \frac{h(\widetilde{\mathbf{U}}_q(J))}{\log 2}.$$

By Lemma 6.1 and Theorem 1 this implies

$$\dim_H \mathcal{W}_q = \frac{\log 2}{\log q} \dim_H \mathbf{W}_q = \frac{h(\widetilde{\mathbf{U}}_q(J))}{\log q} = f_-(q),$$

completing the proof. □

Acknowledgements. The authors wish to thank the anonymous referee for many useful remarks and suggestions. P. Allaart was partially sponsored by NWO visitor's travel grant 040.11.647/4701. P. Allaart furthermore wishes to thank the mathematics department of Utrecht University, and in particular Karma Dajani, for their warm hospitality during a sabbatical visit in the spring of 2018 when much of this work was undertaken. Kong was supported by NSFC No. 11971079, and the Fundamental Research Project of Chongqing No. cstc2019jcyj-msxmX0338 and cx2019067. D. Kong would like to thank the Mathematical Institute of Leiden University. All correspondence should be addressed to D. Kong.

References

[**1**] R. A. Barrera. Topological and ergodic properties of symmetric sub-shifts. *Discrete Contin. Dyn. Syst.* **34**(11) (2014), 4459–4486.
[**2**] R. A. Barrera, S. Baker and D. Kong. Entropy, topological transitivity, and dimensional properties of unique q-expansions. *Trans. Amer. Math. Soc.* **371** (2019), 3209–3258.
[**3**] P. C. Allaart. The infinite derivatives of Okamoto's self-affine functions: an application of β-expansions. *J. Fractal Geom.* **3**(1) (2016), 1–31.
[**4**] P. C. Allaart. On univoque and strongly univoque sets. *Adv. Math.* **308** (2017), 575–598.
[**5**] P. C. Allaart. An algebraic approach to entropy plateaus in non-integer base expansions. *Discrete Contin. Dyn. Syst. A* **39**(11) (2019), 6507–6522.
[**6**] P. C. Allaart, S. Baker and D. Kong. Bifurcation sets arising from non-integer base expansions. *J. Fractal Geom.* **6**(4) (2019), 301–341.

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press

[**7**] P. C. Allaart and D. Kong. On the continuity of the Hausdorff dimension of the univoque set. *Adv. Math.* **354** (2019), 106729.
[**8**] J.-P. Allouche and M. Cosnard. Itérations de fonctions unimodales et suites engendrées par automates. *C. R. Acad. Sci. Paris Sér. I Math.* **296**(3) (1983), 159–162.
[**9**] J.-P. Allouche and M. Cosnard. Non-integer bases, iteration of continuous real maps, and an arithmetic self-similar set. *Acta Math. Hungar.* **91**(4) (2001), 325–332.
[**10**] C. Baiocchi and V. Komornik. Greedy and quasi-greedy expansions in non-integer bases. *Preprint*, 2007, arXiv:0710.3001v1.
[**11**] S. Baker. On small bases which admit countably many expansions. *J. Number Theory* **147** (2015), 515–532.
[**12**] S. Baker and N. Sidorov. Expansions in non-integer bases: lower order revisited. *Integers* **14** (2014), Paper No. A57, 15.
[**13**] C. Bonanno, C. Carminati, S. Isola and G. Tiozzo. Dynamics of continued fractions and kneading sequences of unimodal maps. *Discrete Contin. Dyn. Syst.* **33**(4) (2013), 1313–1332.
[**14**] K. Dajani and M. de Vries. Invariant densities for random β-expansions. *J. Eur. Math. Soc. (JEMS)* **9**(1) (2007), 157–176.
[**15**] K. Dajani, V. Komornik, D. Kong and W. Li. Algebraic sums and products of univoque bases. *Indag. Math.* **29** (2018), 1087–1104.
[**16**] Z. Daróczy and I. Kátai. On the structure of univoque numbers. *Publ. Math. Debrecen* **46**(3-4) (1995), 385–408.
[**17**] M. de Vries and V. Komornik. Unique expansions of real numbers. *Adv. Math.* **221**(2) (2009), 390–427.
[**18**] M. de Vries, V. Komornik and P. Loreti. Topology of the set of univoque bases. *Topology Appl.* **205** (2016), 117–137.
[**19**] P. Erdős, I. Joó and V. Komornik. Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems. *Bull. Soc. Math. France* **118** (1990), 377–390.
[**20**] P. Erdős, M. Horváth and I. Joó. On the uniqueness of the expansions $1 = \sum q^{-n_i}$. *Acta Math. Hungar.* **58**(3–4) (1991), 333–342.
[**21**] P. Erdős and I. Joó. On the number of expansions $1 = \sum q^{-n_i}$. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **35** (1992), 129–132.
[**22**] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons Ltd., Chichester, 1990.
[**23**] P. Glendinning and N. Sidorov. Unique representations of real numbers in non-integer bases. *Math. Res. Lett.* **8** (2001), 535–543.
[**24**] T. Jordan, P. Shmerkin and B. Solomyak. Multifractal structure of Bernoulli convolutions. *Math. Proc. Cambridge Phil. Soc.* **151**(3) (2011), 521–539.
[**25**] C. Kalle, D. Kong, W. Li and F. Lü. On the bifurcation set of unique expansions. *Acta Arith.* **188** (2019), 367–399.
[**26**] V. Komornik and D. Kong. Bases in which some numbers have exactly two expansions. *J. Number Theory* **195** (2019), 226–268.
[**27**] V. Komornik, D. Kong and W. Li. Hausdorff dimension of univoque sets and devil's staircase. *Adv. Math.* **305** (2017), 165–196.
[**28**] V. Komornik and P. Loreti. Unique developments in non-integer bases. *Amer. Math. Monthly* **105**(7) (1998), 636–639.
[**29**] V. Komornik and P. Loreti. Subexpansions, superexpansions and uniqueness properties in non-integer bases. *Period. Math. Hungar.* **44**(2) (2002), 197–218.
[**30**] V. Komornik and P. Loreti. On the topological structure of univoque sets. *J. Number Theory* **122**(1) (2007), 157–183.
[**31**] D. Kong and W. Li. Hausdorff dimension of unique beta expansions. *Nonlinearity* **28**(1) (2015), 187–209.
[**32**] D. Kong, W. Li and F. M. Dekking. Intersections of homogeneous Cantor sets and beta-expansions. *Nonlinearity* **23**(11) (2010), 2815–2834.
[**33**] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.
[**34**] F. Lü and J. Wu. Diophantine analysis in beta-dynamical systems and Hausdorff dimensions. *Adv. Math.* **290** (2016), 919–937.
[**35**] W. Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416.
[**36**] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar.* **8** (1957), 477–493.
[**37**] N. Sidorov. Almost every number has a continuum of β-expansions. *Amer. Math. Monthly* **110**(9) (2003), 838–842.
[**38**] N. Sidorov. Combinatorics of linear iterated function systems with overlaps. *Nonlinearity* **20**(5) (2007), 1299–1312.
[**39**] N. Sidorov. Expansions in non-integer bases: lower, middle and top orders. *J. Number Theory* **129**(4) (2009), 741–754.

https://doi.org/10.1017/etds.2020.38 Published online by Cambridge University Press