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NILPOTENT, ALGEBRAIC AND QUASI-REGULAR ELEMENTS IN RINGS AND ALGEBRAS

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Abstract We prove that an integral Jacobson radical ring is always nil, which extends a well-known result from algebras over fields to rings. As a consequence we show that if every element x of a ring R is a zero of some polynomial p_x with integer coefficients, such that $p_x(1) = 1$, then R is a nil ring. With these results we are able to give new characterizations of the upper nilradical of a ring and a new class of rings that satisfy the Köthe conjecture: namely, the integral rings.

Keywords: π-algebraic element; nil ring; integral ring; quasi-regular element; Jacobson radical; upper nilradical

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1. Introduction

Let R be an associative ring or algebra. Every nilpotent element of R is quasi-regular and algebraic. In addition, the quasi-inverse of a nilpotent element is a value of some polynomial at this element. In the first part of this paper we will be interested in the connections between three notions: nilpotency, algebraicity and quasi-regularity. In particular, we will investigate how close algebraic elements are to being nilpotent and how close quasi-regular elements are to being nilpotent. We are motivated by the following two questions.

(Q1) Algebraic rings and algebras are usually thought of as nice and well behaved. For example, an algebraic algebra over a field, which has no zero divisors, is a division algebra. On the other hand, nil rings and algebras, which are of course algebraic, are hard to deal with since their structure is rather erratic. It is thus natural to ask: what makes the nil rings and algebras behave so poorly among all the algebraic ones?

The answer for algebras over fields is well known: namely, they are Jacobson radical. We generalize this to rings (and more generally to algebras over Jacobson rings) in two different ways: first, we show that nil rings are precisely those that are integral and Jacobson radical (see Theorem 3.11); and second, we show that the only condition needed for an algebraic ring to be nil is that its elements are zeros of polynomials p with $p(1) = 1$ (see Theorem 3.14).

(Q2) Can nilpotent elements be characterized by the property 'quasi-inverse of a is a value of some polynomial at a'?

It is somewhat obvious that element-by-element this will not be possible, however we are able to characterize the upper nilradical in this way (see Corollary 3.15).

One of the most important problems concerning nil rings is the Köthe conjecture. In 1930 Köthe conjectured that if a ring has no non-zero nil ideals, then it has no non-zero nil one-sided ideals. The question of whether this is true is still open. There are many statements that are equivalent to the Köthe conjecture and many classes of rings and algebras that are known to satisfy the Köthe conjecture (see [**3**,**5**,**6**] for an overview). We give yet another class of such rings: namely, the integral rings (see Corollary 3.17).

In the second part of the paper we investigate the structure of certain sets of elements of rings and algebras. In particular, we show that a subgroup of the group of quasi-regular elements (equipped with quasi-multiplication) is closed for ring addition if and only if it is closed for ring multiplication. This gives us some information on the structure of the set of all elements of a ring that are zeros of polynomials p with $p(1) = 1$.

2. Preliminaries

Throughout this paper we are dealing with associative rings and algebras, possibly non-unital and non-commutative. Given a ring or algebra $(R, +, \cdot)$, we define an operation $\circ$ on R, called *quasi-multiplication*, by

$$a \circ b = a + b - ab.$$

It is easy to see that $(R, \circ)$ is a monoid with identity element 0. An element $a \in R$ is called *quasi-regular* if it is invertible in $(R, \circ)$, i.e. if there exists $a' \in R$ such that $a \circ a' = a' \circ a = 0$. In this case we say that a' is the *quasi-inverse* of a. If R is unital, then this is equivalent to $1 - a$ being invertible in $(R, \cdot)$ with inverse $1 - a'$. In fact, the map $f\colon (R, \circ) \to (R, \cdot)$ given by $x \mapsto 1 - x$ is a monoid homomorphism, since $1 - a \circ b = (1 - a)(1 - b)$. The set of all quasi-regular elements of R will be denoted by $Q(R)$. Clearly, $(Q(R), \circ)$ is a group, since this is just the group of invertible elements of the monoid $(R, \circ)$. For every $a \in Q(R)$ and every $n \in \mathbb{Z}$, the nth power of a in $(Q(R), \circ)$ will be denoted by $a^{(n)}$ to distinguish it from a^n, the nth power of a in $(R, \cdot)$. In particular, $a^{(0)} = 0$ and $a^{(-1)}$ is the quasi-inverse of a. If R is unital, then $1 - a^{(-1)} = (1 - a)^{-1}$. A subset $S \subseteq R$ is called *quasi-regular* if $S \subseteq Q(R)$. The *Jacobson radical* of R is the largest quasi-regular ideal of R and will be denoted by $J(R)$.

The set of all nilpotent elements in R will be denoted by $N(R)$. Every nilpotent element is quasi-regular, so $N(R) \subseteq Q(R)$. In fact, if $x^n = 0$, then $-x - x^2 - \cdots - x^{n-1}$ is the quasi-inverse of x. A subset $S \subseteq R$ is called *nil* if $S \subseteq N(R)$. The *upper nilradical* of R is the largest nil ideal of R and will be denoted by $\mathrm{Nil}^*(R)$. If R is commutative, then $\mathrm{Nil}^*(R) = N(R)$.

The *lower nilradical* of R (also called the *prime radical*) is the intersection of all prime ideals of R and will be denoted by $\mathrm{Nil}_*(R)$. It can also be characterized as the lower radical determined by the class of all nilpotent rings (see [**2**] for details). For any ring R we have $\mathrm{Nil}_*(R) \subseteq \mathrm{Nil}^*(R) \subseteq J(R)$.

Let K be a commutative unital ring and let R be a K-algebra, possibly non-commutative and non-unital. An element $a \in R$ is *algebraic* over K if there exists a non-zero polynomial $p \in K[x]$ such that $p(0) = 0$ and $p(a) = 0$. If, in addition, p can be chosen monic (i.e. the leading coefficient of p is equal to 1), then a is called *integral* over K. The condition $p(0) = 0$ is necessary only because R may be non-unital, and then only polynomials with zero constant term can be evaluated at elements of R. The set of all algebraic elements of R will be denoted by $A_K(R)$; the set of all integral elements of R will be denoted by $I_K(R)$. A K-algebra R is *algebraic* (respectively *integral*) over K if every element of R is algebraic (respectively integral) over K. Note the special case of the above definitions when R is just a ring, in which case we consider it as an algebra over $K = \mathbb{Z}$. In this case we will also write $A(R) = A_{\mathbb{Z}}(R)$ and $I(R) = I_{\mathbb{Z}}(R)$. Clearly, every nilpotent element of R is integral, so $N(R) \subseteq I_K(R) \subseteq A_K(R)$. If F is a field, then $I_F(R) = A_F(R)$.

3. π-algebraic rings and algebras

Throughout this section K will always denote a commutative unital ring, F a field and R an algebra over K or F, unless specified otherwise. The two questions from the introduction motivate the following definition, which will play a crucial role in our considerations.

Definition 3.1. An element a of a K-algebra R is *π-algebraic* (over K) if there exists a polynomial $p \in K[x]$ such that $p(0) = 0$, $p(1) = 1$ and $p(a) = 0$. In this case we will also say that a is *π-algebraic with polynomial* p. A subset $S \subseteq R$ is *π-algebraic* if every element in S is π-algebraic. The set of all π-algebraic elements of a K-algebra R will be denoted by $\pi_K(R)$.

When R is just a ring, we consider it as an algebra over $K = \mathbb{Z}$ and write $\pi(R) = \pi_{\mathbb{Z}}(R)$. The crucial condition in this definition is the condition $p(1) = 1$. The condition $p(0) = 0$ is there simply because R may be non-unital and then only polynomials with zero constant term can be evaluated at an element of R.

We first present some basic properties of π-algebraic elements along with some examples.

Lemma 3.2. *If R is a K-algebra, then $N(R) \subseteq \pi_K(R) \subseteq A_K(R) \cap Q(R)$. If R is an F-algebra, then $N(R) \subseteq \pi_F(R) = A_F(R) \cap Q(R)$. The quasi-inverse of a π-algebraic element a of R is a value of some polynomial at a.*

Proof. Clearly, every nilpotent element is π-algebraic and every π-algebraic element is algebraic. Suppose $a \in R$ is π-algebraic with a polynomial p. Then $P(x) = 1 - (1 - p(x))/(1 - x)$ is again a polynomial with $P(0) = 0$ (and suitable coefficients). Hence we may define $a' = P(a)$. Since $x \circ P(x) = x + P(x) - xP(x) = p(x)$, we have $a \circ a' = 0$. Similarly, we get $a' \circ a = 0$. Hence a' is the quasi-inverse of a. Now suppose R is an F-algebra and a is an element of $A_F(R) \cap Q(R)$. Let $r \in F[x]$ be the minimal polynomial of a. Suppose $r(1) = 0$. Then $r(x) = (1 - x)q(x) = q(x) - xq(x)$ for some polynomial

$q \in F[x]$ of degree less than that of r. It follows that $0 = r(a) - a'r(a) = q(a) - aq(a) - a'q(a) + a'aq(a) = q(a) - (a' \circ a)q(a) = q(a)$, which is a contradiction since r was the minimal polynomial for a. Thus $r(1)$ is an invertible element of F and hence the element a is π-algebraic with the polynomial $p(x) = r(1)^{-1}r(x)$. □

We shall see in the examples that the inclusion $\pi_K(R) \subseteq A_K(R) \cap Q(R)$ may be strict.

Lemma 3.3. *If R is a unital K-algebra, then $2 - \pi_K(R) \subseteq \pi_K(R)$. In particular, $0, 2 \in \pi_K(R)$ and $1 \notin \pi_K(R)$. If R is a unital F-algebra, then $2 - \pi_F(R) \subseteq \pi_F(R)$. In addition, $F \backslash \{1\} \subseteq \pi_F(R)$ and $1 \notin \pi_F(R)$.*

Proof. If a is π-algebraic with a polynomial p, then $2 - a$ is π-algebraic with the polynomial $q(x) = p(2-x)x$. We always have $0 \in \pi_K(R)$, hence $2 \in \pi_K(R)$. The identity element is never π-algebraic since it is not quasi-regular. If R is a unital F-algebra and $\lambda \neq 1$ is a scalar, then λ is π-algebraic with the polynomial $p(x) = (1-\lambda)^{-1}(x-\lambda)x$. □

Next we give a few examples.

Example 3.4. For a finite ring R, $\pi(R) = Q(R)$ and $J(R) = \mathrm{Nil}^*(R)$. To verify the first part observe that $(Q(R), \circ)$ is a finite group, say of order n. So for every $a \in Q(R)$ we have $a^{(n)} = 0$, hence every $a \in Q(R)$ is π-algebraic with the polynomial $p(x) = x^{(n)} = 1 - (1-x)^n$. The second part is well known and it also follows from the first part and Theorem 3.14.

Example 3.5. For any field F, $\pi_F(F) = F\backslash\{1\} = Q(F)$ by Lemmas 3.2 and 3.3. In particular, $\pi_{\mathbb{Q}}(\mathbb{Q}) = \mathbb{Q}\backslash\{1\} = Q(\mathbb{Q})$. On the other hand, we have $\pi(\mathbb{Q}) = \{1 + 1/n \colon n \in \mathbb{Z}\backslash\{0\}\}$. Indeed, if n is a non-zero integer, then $1 + 1/n$ is π-algebraic over $\mathbb{Z}$ with the polynomial $s(x) = (1 - n(x-1))x$. Conversely, suppose $a/b \in \mathbb{Q}$, with a and b coprime, is π-algebraic with a polynomial $p \in \mathbb{Z}[x]$ of degree d. Then $q(x) = b^d p(x/b)$ is a polynomial with integer coefficients. Hence $a - b$ divides $q(a) - q(b) = b^d p(a/b) - b^d p(1) = -b^d$. Since a and b are coprime, this is only possible if $a - b = \pm 1$ (any prime that would divide $a - b$ would divide b and hence a). Thus $a/b = 1 \pm 1/b$ as needed. Obviously, $A(\mathbb{Q}) = \mathbb{Q}$, so the inclusion $\pi(\mathbb{Q}) \subseteq A(\mathbb{Q}) \cap Q(\mathbb{Q})$ from Lemma 3.2 is strict here.

Example 3.6. Let $F \subseteq E$ be fields and let $M_n(E)$ be the ring of $n \times n$ matrices over E. Denote by $\sigma(X)$ the spectrum of $X \in M_n(E)$. Then

$$\begin{aligned} N(M_n(E)) &= \{X \in M_n(E) \colon \sigma(X) = \{0\}\}, \\ \pi_F(M_n(E)) &= \{X \in M_n(E) \colon \sigma(X) \subseteq \bar{F}\backslash\{1\}\}, \\ Q(M_n(E)) &= \{X \in M_n(E) \colon \sigma(X) \subseteq \bar{E}\backslash\{1\}\}, \end{aligned}$$

where $\bar{F} \subseteq \bar{E}$ are algebraic closures of F and E. By means of Jordan canonical form it is easy to see that a matrix X is nilpotent if and only if $\sigma(X) = \{0\}$. Moreover, X is quasi-regular if and only if $1 - X$ is invertible, i.e. X has no eigenvalue equal to 1. So in view of Lemma 3.2, to verify the above, we only need to prove that

$$A_F(M_n(E)) = \{X \in M_n(E) \colon \sigma(X) \subseteq \bar{F}\}.$$

If $A \in M_n(E)$ is algebraic over F, it clearly has eigenvalues in $\bar{F}$. So suppose $A \in M_n(E)$ has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n \in \bar{F}$. For every $i = 1, 2, \dots, n$, let p_i be the minimal polynomial of λ_i over F. Then the minimal polynomial m_A of A over E divides $P(x) = \prod_{i=1}^n p_i(x)$, hence $P(A) = 0$. Since P has coefficients in F, A is algebraic over F.

Next we exhibit a connection between π-algebraic and integral elements.

Remark 3.7. As above, let R be a K-algebra. For a polynomial $p \in K[x]$ define $\hat{p}(x) = (x-1)^{\deg p} p(x/(x-1))$, which is again a polynomial in $K[x]$. Notice that $\hat{p}(1)$ equals the leading coefficient of p and the leading coefficient of $\hat{p}$ equals $p(1)$ (the sum of all coefficients of p) if $p(1) \neq 0$. In addition, $\hat{p}(0) = 0$ if and only if $p(0) = 0$.

We may assume that R is unital, otherwise we just adjoin a unit to R. Let a be a quasi-regular element. Then the inverse of $1-a$ is $1-a^{(-1)}$, so the term $x/(x-1)$ evaluated at a equals $-a(1-a^{(-1)}) = -a + aa^{(-1)} = a^{(-1)}$. Thus $\hat{p}(a) = (a-1)^{\deg p} p(a^{(-1)})$ and $\hat{p}(a^{(-1)}) = (a^{(-1)}-1)^{\deg p} p(a)$. This shows that $\hat{p}(a) = 0$ if and only if $p(a^{(-1)}) = 0$, since $1-a$ is invertible. Similarly, $\hat{p}(a^{(-1)}) = 0$ if and only if $p(a) = 0$.

Proposition 3.8. *Let R be a K-algebra. For any $a \in R$ the following conditions are equivalent:*

(i) *a is π-algebraic;*

(ii) *a is quasi-regular and $a^{(-1)}$ is integral; and*

(iii) *a is quasi-regular and $a^{(-1)}$ is a value of some polynomial at a.*

Proof. The equivalence of (i) and (ii) is a direct consequence of Remark 3.7. It follows from Lemma 3.2 that (i) implies (iii). Moreover, if $a^{(-1)} = P(a)$, where P is a polynomial in $K[x]$, then $a + P(a) - aP(a) = 0$, so a is π-algebraic with the polynomial $(x + P(x) - xP(x))x$. □

In particular, Proposition 3.8 implies the following (compare with Lemma 3.2).

Corollary 3.9. $\pi_K(R) = (Q(R) \cap I_K(R))^{(-1)}$.

By Lemma 3.2 an algebra over a field F is π-algebraic if and only if it is algebraic and Jacobson radical. So the following proposition is just a restatement of a well-known fact that any algebraic Jacobson radical F-algebra is nil (see, for example, [**7**, p. 144]). In fact, every algebraic element in the Jacobson radical of an F-algebra is nilpotent and its nilindex is equal to its degree.

Proposition 3.10. *Every π-algebraic F-algebra is nil.*

We now extend this result to algebras over Jacobson rings. Recall that a commutative unital ring K is a *Jacobson ring* (or a *Hilbert ring*) if every prime ideal of K is an intersection of maximal ideals of K. Examples of Jacobson rings are fields and polynomial rings over fields in finitely many commutative variables. In addition, any principal ideal domain with infinitely many irreducible elements is also a Jacobson ring. In particular, the ring of integers $\mathbb{Z}$ is a Jacobson ring.

Theorem 3.11. *If K is a Jacobson ring, then every integral Jacobson radical K-algebra is nil.*

Proof. Let R be an integral Jacobson radical K-algebra and let $a \in R$. Consider R as a subalgebra of some unital K-algebra R^1. Let $K[a]$ be a unital subalgebra of R^1 generated by a. Since K is a Jacobson ring and $K[a]$ is a finitely generated (commutative unital) K-algebra, $K[a]$ is a Jacobson ring by a version of Hilbert's Nullstellensatz [**1**, Theorem 4.19]. Hence $J(K[a]) = \mathrm{Nil}_*(K[a])$ is a nil ideal. It suffices to prove that $a \in J(K[a])$. Take any $r \in K[a]$. Since ar is an element of R, it is quasi-regular in R and its quasi-inverse $(ar)^{(-1)} \in R$ is integral. By Proposition 3.8, $(ar)^{(-1)}$ is a value of some polynomial at ar. But ar is a value of some polynomial at a, hence $(ar)^{(-1)} \in K[a]$, i.e. ar is quasi-regular in $K[a]$. Since r was arbitrary, we conclude that $a \in J(K[a])$. □

Remark 3.12. Without the assumption that the ring K is Jacobson, Theorem 3.11 fails. In fact, let P be a prime ideal of K that is not an intersection of maximal ideals. Then $J(K/P)$ is a non-zero K-algebra. In addition, it is Jacobson radical and integral because an element $k+P \in J(K/P)$ is integral over K with the polynomial $x^2 - kx$. Since K is commutative and P is a prime ideal, the algebra K/P has no non-zero nilpotent elements, hence $J(K/P)$ is not nil.

The assumption that the algebra is integral in Theorem 3.11 is also crucial. A merely algebraic Jacobson radical algebra over a Jacobson ring need not be nil.

Example 3.13. Consider $R = \{2m/(2n-1)\colon m, n \in \mathbb{Z}\}$ as a subring of rational numbers. The quasi-inverse of $2m/(2n-1)$ is $2m/(2m-2n+1)$, which is again an element of R. So R is a Jacobson radical ring algebraic over $\mathbb{Z}$, but it is not nil.

As a direct consequence of Theorem 3.11 and Proposition 3.8 we get the following.

Theorem 3.14. *If K is a Jacobson ring, then every π-algebraic K-algebra is nil.*

This answers question (Q1) in two ways: the property that distinguishes nil rings and algebras from all other algebraic ones is firstly that they are integral and Jacobson radical, and secondly that the polynomials ensuring algebraicity in the nil case have the sum of their coefficients equal to 1. It is perhaps interesting that this rather large family of polynomials with the sum of coefficients equal to 1 produces the same effect as the rather restrictive family $\{x, x^2, x^3, x^4, \dots\}$.

Observe that in an algebraic division F-algebra only the identity is not π-algebraic (since all other elements are quasi-regular). So if only one element in an algebra is not π-algebraic, then the algebra may be very nice instead of nil.

The next corollary addresses question (Q2), giving new characterizations of the upper nilradical of a ring in the process. We formulate it only for rings, though it is valid for all algebras over Jacobson rings.

Corollary 3.15. *For a ring R the following hold:*

(i) $\mathrm{Nil}^*(R)$ *is the largest π-algebraic ideal of R;*

(ii) $\mathrm{Nil}^*(R)$ *is the largest integral quasi-regular ideal of R; and*

(iii) $\mathrm{Nil}^*(R)$ *is the largest quasi-regular ideal of R such that the quasi-inverse of each element is a value of some polynomial at this element.*

Proof. If I is an ideal of R satisfying any of the above conditions, then I is π-algebraic by Proposition 3.8 and thus nil by Theorem 3.14. Hence $\mathrm{Nil}^*(R)$ is the largest such ideal. □

Corollary 3.16. *If R is an integral ring, then $J(R) = \mathrm{Nil}^*(R)$.*

A ring R is said to satisfy the Köthe conjecture if every nil one-sided ideal of R is contained in a nil two-sided ideal of R (see [**9**]). If $J(R) = \mathrm{Nil}^*(R)$ for a ring R, then R satisfies the Köthe conjecture since $J(R)$ contains every nil one-sided ideal. Corollary 3.16 thus implies the following.

Corollary 3.17. *Every integral ring satisfies the Köthe conjecture.*

In what follows we will exhibit an even stronger connection between π-algebraic and nilpotent elements than that given by Theorem 3.14 if the ring K satisfies certain properties given by the following definition.

Definition 3.18. We shall say that a principal ideal domain (PID) K is *exceptional* if there is no non-constant polynomial $p \in K[x]$ such that $p(k)$ would be invertible in K for all $k \in K$.

Exceptional PIDs are quite common, here are some examples.

Proposition 3.19.

(i) *A field is an exceptional PID if and only if it is algebraically closed.*

(ii) *The ring of integers $\mathbb{Z}$ and the ring of Gaussian integers $\mathbb{Z}[i]$ are exceptional PIDs.*

(iii) *For any field F the polynomial ring $F[x]$ is an exceptional PID.*

(iv) *If K is an exceptional PID and $S \subseteq K$ is a multiplicatively closed subset multiplicatively generated by a finite number of elements, then the localization $S^{-1}K$ is an exceptional PID.*

Proof. Claim (i) is clear.

(ii) Let p be a polynomial in $\mathbb{Z}[x]$ such that $p(k)$ is invertible for all $k \in \mathbb{Z}$. Since there are only finitely many invertible elements in $\mathbb{Z}$, there exist an invertible element $u \in \mathbb{Z}$ such that $p(k) = u$ for infinitely many $k \in \mathbb{Z}$. But then the polynomial $p(x) - u$ has infinitely many zeros, so it must be zero. Hence p is a constant polynomial. The same proof works for $\mathbb{Z}[i]$.

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(iii) Let F be a field and let $P(y)$ be a non-constant polynomial in $(F[x])[y]$. Write $P(y) = p_0(x)+p_1(x)y+\cdots+p_n(x)y^n$, where $p_n(x) \neq 0$ and $n \geqslant 1$. Define $d_i = \deg p_i$ and $d = \max\{d_i \colon i = 0, 1, 2, \ldots, n\}$, where the degree of the zero polynomial is equal to $-\infty$. Let $p(x) = x^{d+1}$. The degree of $p_i(x)(p(x))^i$ is equal to $d_i + i(d+1)$. Since $d_n, d \neq -\infty$, we have $d_n + n(d+1) \geqslant n(d+1) > d + (n-1)(d+1) \geqslant d_i + i(d+1)$ for all $i < n$. This implies that the degree of $P(p(x))$ is equal to $d_n + n(d+1) \geqslant 1$, and hence $P(p(x))$ is not invertible in $F[x]$.

(iv) A localization of a PID is again a PID. Factor each generator of S into irreducible factors and let $S' \subseteq K$ be a multiplicatively closed subset multiplicatively generated by all the irreducible elements appearing in these factorizations. Since the localizations of K at S and S' are isomorphic, we may assume that $S = S'$, i.e. S is generated by a finite number of irreducible elements. Now suppose $p(x) \in S^{-1}K[x]$ is a polynomial such that $p(\hat{k})$ is invertible in $S^{-1}K$ for all $\hat{k} \in S^{-1}K$. Take any $s \in S$ such that the coefficients of $sp(x)$ are elements of K. Let t be the product of all irreducible elements in S. Observe that the coefficients of the polynomial $sp(sp(0)tx)$ are elements of K and are all divisible by $sp(0) \in K$. Hence,

$$P(x) = \frac{s}{sp(0)} p(sp(0)tx) = \frac{1}{p(0)} p(sp(0)tx)$$

is a polynomial with coefficients in K. Now take any $k \in K$. By the assumption, $p(0)$ and $p(sp(0)tk) = p(0)P(k)$ are invertible in $S^{-1}K$, and hence so is $P(k)$. But $P(k) \in K$, so the only irreducible elements that may divide $P(k)$ are those that lie in S. However, any such irreducible element divides t, and hence it divides all coefficients of P except $P(0) = 1$, so it cannot divide $P(k)$. This shows that $P(k)$ is invertible in K. Since K is an exceptional PID, $P(x)$, and consequently $p(x)$, must be constant polynomials. □

In a PID every non-zero prime ideal is maximal, so a PID is a Jacobson ring if and only if 0 is an intersection of maximal ideals, i.e. the Jacobson radical is 0.

Proposition 3.20. *If K is an exceptional PID, then $J(K) = 0$, i.e. K is a Jacobson ring. In particular, if K is not a field, then K has infinitely many non-associated irreducible elements.*

Proof. Let K be an exceptional PID. Suppose $J(K) \neq 0$ and take $0 \neq a \in J(K)$. Since K is commutative and unital, this implies that $1 - ak$ is invertible in K for every $k \in K$. But then the polynomial $p(x) = 1 - ax$ contradicts the definition of an exceptional PID. Hence $J(K) = 0$. Since K is commutative and unital, $J(K)$ is just the intersection of all maximal ideals of K. If K is not a field, then the maximal ideals of K are the principal ideals generated by the irreducible elements. If there are only finitely many such ideals, then their intersection is non-zero. □

The converse of Proposition 3.20 does not hold. There exist PIDs that are Jacobson rings but are not exceptional. The simplest example is given by any field that is not algebraically closed; however, fields are rather extremal among all PID, since they have no irreducible elements. Hence, we give an example which is not a field.

Example 3.21. Let $S \subseteq \mathbb{Z}$ be a multiplicatively closed subset multiplicatively generated by all primes p with $p = 2$ or $p \equiv 1 \pmod 4$, and let $K = S^{-1}\mathbb{Z}$ be the localization of $\mathbb{Z}$ at S. Then K has infinitely many non-associated irreducible elements, represented by the primes p with $p \equiv 3 \pmod 4$, hence $J(K) = 0$ and K is a Jacobson ring. Now let $p(x) = x^2 + 1$. To see that K is not exceptional, we will show that $p(k)$ is invertible in K for all $k \in K$. For $k = m/n \in K$ we have $p(k) = (m^2 + n^2)/n^2$. To see that this is invertible in K we need to show that any prime dividing $m^2 + n^2$ is contained in S. Suppose p is a prime with $p \equiv 3 \pmod 4$ that divides $m^2 + n^2$. Then $m^2 \equiv -n^2 \pmod p$. Since $n \in S$, this implies that both m and n are coprime to p. Hence we have $1 \equiv m^{p-1} \equiv (m^2)^{(p-1)/2} \equiv (-n^2)^{(p-1)/2} \equiv (-1)^{(p-1)/2} n^{p-1} \equiv (-1)^{(p-1)/2} \equiv -1 \pmod p$. This is a contradiction since $p \neq 2$, which finishes the proof.

Theorem 3.14 implies that if the subalgebra generated by an element a is π-algebraic, then a is a nilpotent element. The next proposition, which was our main motivation for the introduction of exceptional PIDs, considers the situation when only the submodule generated by a is assumed to be π-algebraic. It thus gives a stronger connection between π-algebraic and nilpotent elements for algebras over exceptional PIDs.

Lemma 3.22. *For any π-algebraic element r of an algebra R over a principal ideal domain K, there exist a non-zero polynomial $f \in K[x]$ and a non-zero element $c \in K$ such that $f(1) = 1$, $cf(r) = 0$ and f divides any polynomial of $K[x]$ that annihilates r.*

Proof. For a non-zero polynomial $f \in K[x]$, let $\delta(f)$ denote the greatest common divisor of all coefficients of f. Let r be π-algebraic with a polynomial $h \in K[x]$. Choose a non-zero polynomial $p \in K[x]$ of minimal degree such that $p(r) = 0$ and let $c = \delta(p)$ and $f(x) = p(x)/c \in K[x]$. Then $cf(r) = 0$ and $\delta(f) = 1$. Suppose $P \in K[x]$ is a polynomial with $P(r) = 0$. By the division algorithm there exists $0 \neq m \in K$ and polynomials $s, t \in K[x]$ with $\deg t < \deg f = \deg p$ such that $mP(x) = s(x)f(x) + t(x)$ (divide in $F[x]$, where F is the field of fractions of K, and multiply by a common denominator of all fractions). Multiplying by c we get $cmP(x) = cs(x)f(x) + ct(x) = s(x)p(x) + ct(x)$. The minimality of p now implies $ct(x) = 0$, hence $t(x) = 0$ and $mP(x) = s(x)f(x)$. By Gauss's lemma this implies $\delta(s) = m\delta(P)$ up to association, so m divides $\delta(s)$. Thus the polynomial $s(x)/m$ has coefficients in K and $P(x) = (s(x)/m)f(x)$, i.e. f divides P. In particular, f divides h, so there is a polynomial S such that $h(x) = S(x)f(x)$. Evaluating at 1 we get $1 = S(1)f(1)$, so $f(1)$ is invertible in K. We may assume that $f(1) = 1$, otherwise we just multiply f by $f(1)^{-1} = S(1)$. □

Proposition 3.23. *Let K be an exceptional principal ideal domain and let R be a K-algebra. If a is an element of R such that $Ka \subseteq \pi_K(R)$, then there exists $0 \neq k \in K$ such that ka is nilpotent. In particular, if R has no K-torsion, then a is nilpotent.*

Proof. By Lemma 3.22, for any $k \in K$ there exists $0 \neq c_k \in K$ and $0 \neq f_k \in K[x]$ such that $f_k(1) = 1$, $c_k f_k(ka) = 0$ and f_k divides any polynomial of $K[x]$ that annihilates ka. Let $k \neq 0$. Then f_1 divides $c_k f_k(kx)$, since $c_k f_k(kx)$ annihilates a. Similarly, $c_1 k^{\deg f_1} f_1(x/k)$ is a polynomial in $K[x]$ that annihilates ka, so f_k divides $c_1 k^{\deg f_1} f_1(x/k)$. This, in particular, implies that all these polynomials have the same

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degree, so there exists $d_k \in K$ such that $c_1 k^{\deg f_1} f_1(x/k) = d_k f_k(x)$. We have $f_k(1) = 1$, hence $\delta(f_k) = 1$. Consequently, $c_1\delta(k^{\deg f_1} f_1(x/k)) = d_k$ up to association. If k is coprime to the leading coefficient of f_1, then $\delta(k^{\deg f_1} f_1(x/k)) = 1$ since $\delta(f_1) = 1$. For such k we have $c_1 = d_k$ up to association, hence c_1 divides d_k and $u_k = d_k/c_1$ is invertible. In addition, $k^{\deg f_1} f_1(x/k) = u_k f_k(x)$. Evaluating at 1, we get $k^{\deg f_1} f_1(1/k) = u_k$. Now $p(x) = x^{\deg f_1} f_1(1/x)$ is a polynomial in $K[x]$ with $p(0)$ equal to the leading coefficient of f_1. Hence, we have proved above that $p(k)$ is invertible for every $k \neq 0$ coprime to $p(0)$. If we define $t(x) = p(p(0)x - 1) \in K[x]$, then $t(k)$ is invertible for all $k \in K$ ($p(0)k - 1 = 0$ means that $p(0)$ is invertible). Since K is exceptional, it follows that t is a constant polynomial and so is p. Hence, there exists $u \in K$ such that $f_1(1/x) = u/x^{\deg f_1}$, i.e. $f_1(x) = ux^{\deg f_1}$. Consequently, $c_1 u a^{\deg f_1} = 0$ and c_1ua is nilpotent. Clearly, $c_1 u \neq 0$. □

Remark 3.24. We shall later need a slightly modified version of Proposition 3.23 with $K = \mathbb{Z}$. Observe that the conclusion still holds if we assume only that $\mathbb{N}a \subseteq \pi(R)$ instead of $\mathbb{Z}a \subseteq \pi(R)$. Indeed, one just has to replace the polynomial $t(x) = p(p(0)x - 1)$ in the proof with the polynomial $\hat{t}(x) = p((p(0)x - 1)^2)$.

Remark 3.25. Without the assumption that K is exceptional, Proposition 3.23 fails. To see this, choose a non-constant polynomial $p \in K[x]$ such that $p(k)$ is invertible in K for all $k \in K$. Let F be the algebraic closure of the field of fractions of K. Clearly, F is a K-algebra. Since polynomial p is non-constant, the polynomial $P(x) = x^{\deg p} p(1/x) \in K[x]$ has a non-zero root $a \in F$. Clearly, ka is not nilpotent for any $0 \neq k \in K$. We want to show that $Ka \subseteq \pi_K(F)$. The zero element is always π-algebraic, so take any $0 \neq k \in K$. Observe that $\deg P = \deg p$, because $p(0)$ is invertible. Hence $Q(x) = k^{\deg p} P(x/k) \in K[x]$. Since $Q(1) = p(k)$ is invertible in K, the element ka is π-algebraic over K with the polynomial $Q(1)^{-1}Q(x)x \in K[x]$.

Recall that an algebra R is called *nil of bounded index less than or equal to* n if $a^n = 0$ for all $a \in R$. R is called *nil of bounded index* if there exists an integer n such that R is nil of bounded index less than or equal to n. Similarly, we will say that a K-algebra R is *π-algebraic of bounded degree less than or equal to* n (respectively *integral of bounded degree less than or equal to* n) if every element of R is π-algebraic (respectively integral) over K with some polynomial of degree less than or equal to n. R is *π-algebraic of bounded degree* (respectively *integral of bounded degree*) if there exists an integer n such that R is π-algebraic of bounded degree less than or equal to n (respectively integral of bounded degree less than or equal to n).

It follows from the proof of Proposition 3.8 that an algebra is π-algebraic of bounded degree less than or equal to n if and only if it is Jacobson radical and integral of bounded degree less than or equal to n. Theorem 3.14 raises the following natural question. If an algebra R over a Jacobson ring K is π-algebraic of bounded degree, is it nil of bounded index? The answer is positive for algebras with no K-torsion.

Corollary 3.26. *Let K be a Jacobson ring. If R is a π-algebraic K-algebra of bound degree less than or equal to n with no K-torsion, then R is nil of bounded index less than or equal to n.*

Proof. Let R be a π-algebraic K-algebra of bounded degree less than or equal to n with no K-torsion. By the remark above, R is integral of bounded degree less than or equal to n, and by Theorem 3.14, R is nil. Take any $a \in R$. Let $p \in K[x]$ be a monic polynomial of degree less than or equal to n, such that $p(a) = 0$, and let m be the smallest integer such that $a^m = 0$. Suppose $m > n$. Write p in the form $p(x) = t(x)x^k$, where $t(0) \neq 0$ and $k \leqslant n < m$. Multiplying the equality $0 = t(a)a^k$ by a^{m-k-1}, we get $0 = t(a)a^{m-1} = t(0)a^{m-1}$, because $a^m = 0$. Since R has no K-torsion, this implies $a^{m-1} = 0$, which is in contradiction with the choice of m. Thus $m \leqslant n$ as needed. □

Perhaps surprisingly, the answer for general algebras over Jacobson rings is negative, as the following example shows.

Example 3.27. Let K be a Jacobson PID, which is not a field. Then K has infinitely many non-associated irreducible elements. Choose a countable set of non-associated irreducible elements $\{p_1, p_2, p_3, \dots\}$ and let $R = \bigoplus_{i=1}^{\infty} p_i K/p_i^i K$. Clearly, R is nil, but not of bounded index. Let $a = (a_i)_i$ be an element of R. By the Chinese remainder theorem there is an element $k \in K$ such that $k \equiv a_i \pmod{p_i^i}$ for all i with $a_i \neq 0$. Thus a is a zero of the monic polynomial $x^2 - kx$. This shows that R is integral of bounded degree less than or equal to 2, and hence it is also π-algebraic of bounded degree less than or equal to 2.

Nevertheless, the following holds for arbitrary algebras over Jacobson rings.

Proposition 3.28. *Let K be a Jacobson ring. If R is a π-algebraic K-algebra of bounded degree, then* $\mathrm{Nil}_*(R) = R$. *In particular, R is locally nilpotent.*

Proof. Suppose P is a prime ideal of R. We want to apply Corollary 3.26 to R/P. K-algebra R/P is again π-algebraic of bounded degree. Let $I = \{k \in K : k(R/P) = 0\}$. Clearly, I is an ideal of K and R/P becomes a K/I-algebra if we define $(k+I)(r+P) = k(r+P) = kr + P$. Observe that R/P is π-algebraic of bounded degree also over K/I. In addition, R/P has no K/I-torsion. Indeed, if $(k+I)(r+P) = 0$ for some $k \in K$ and $r \in R$ with $r + P \neq 0$, then $J = \{x + P \in R/P : k(x+P) = 0\}$ is a non-zero ideal of R/P. But $k(R/P) \cdot J = 0$ and R/P is a prime K-algebra, so $k(R/P) = 0$, i.e. $k + I = 0$ in K/I as needed. K/I is again a Jacobson ring, hence Corollary 3.26 implies that R/P is nil of bounded index. Thus, by a result of Levitzki [**4**, Theorem 4], we have $\mathrm{Nil}_*(R/P) = R/P$, but on the other hand, $\mathrm{Nil}_*(R/P) = 0$ since P is a prime ideal. So $P = R$, which shows that $\mathrm{Nil}_*(R) = R$. □

4. The structure of $\pi(R)$

In this section we investigate the structure of the set of all π-algebraic elements of an algebra. We restrict ourselves to algebras over fields and to rings. Throughout the section, F will always denote a field and R an F-algebra or a ring.

Recall that $(Q(R), \circ)$ is a group and, by Lemma 3.2, we have $N(R) \subseteq \pi(R) \subseteq Q(R)$. It is thus natural to ask under what conditions $N(R)$ and $\pi(R)$ are subgroups of $Q(R)$ and, more generally, what can be said about the structure of $\pi(R)$. In general, $\pi(R)$

will not be closed under $\circ$. We give a concrete example later (see Example 4.5), but the reason for this is that the integral elements of R do not have any structure in general (they do not form a subring). However, if R is commutative, then $\pi(R)$ will be closed under $\circ$. From here on $Q(R)$ will always be considered as a group with the operation $\circ$.

If R is unital, then $r \circ x \circ r^{(-1)} = (1-r)x(1-r^{(-1)})$ for all $x \in R$ and $r \in Q(R)$. Hence, the map $x \mapsto r \circ x \circ r^{(-1)}$ is an automorphism of R. Moreover, if R is non-unital, then we can adjoin a unit to R. Thus, we have the following.

Lemma 4.1. *For a quasi-regular element $r \in R$ the map $x \mapsto r \circ x \circ r^{(-1)}$ is an automorphism of R.*

Proposition 4.2.

(i) *If R is an F-algebra, then $N(R)$ and $\pi_F(R)$ are closed under conjugation and quasi-inversion. If R is commutative, then $N(R)$ and $\pi_F(R)$ are subgroups of $Q(R)$.*

(ii) *If R is a ring, then $N(R)$ and $\pi(R)$ are closed under conjugation. Moreover, $N(R)$ is closed under quasi-inversion. If R is commutative, then $N(R)$ and $\pi(R)$ are a subgroup and a submonoid of $Q(R)$, respectively.*

Proof. (i) Take any $a \in R$, $r \in Q(R)$ and $p \in F[x]$. Then $r \circ p(a) \circ r^{(-1)} = p(r \circ a \circ r^{(-1)})$, by Lemma 4.1, so $N(R)$ and $\pi_F(R)$ are closed under conjugation. Closure under quasi-inversion follows at once from the discussion in § 2 and from Lemma 3.2. If R is commutative, then $N(R)$ and $A_F(R)$ are subalgebras of R and, consequently, are closed under $\circ$. By Lemma 3.2, $\pi_F(R)$ is closed under $\circ$ as well.

(ii) We need to prove only the last claim because the rest of the proof runs as before. Suppose that R is commutative. Then $I(R)$ is a subring of R, whence $I(R)$ is closed under $\circ$. Moreover, it follows from Corollary 3.9 that $\pi(R)^{(-1)} = I(R) \cap Q(R)$, so $\pi(R)$ is closed under $\circ$. □

Remark 4.3. For a ring R the set $\pi(R)$ need not be closed under inversion. For example, the quasi-inverse of $1 + \frac{1}{2} \in \pi(\mathbb{Q})$ is $1 + 2$ and is not contained in $\pi(\mathbb{Q})$. In fact, we know that $\pi(R)^{(-1)} = I(R) \cap Q(R)$.

For a subset S of $Q(R)$ let $\langle S \rangle$ denote the subsemigroup of $Q(R)$ generated by S. Then an immediate consequence of Proposition 4.2 is the following.

Corollary 4.4. *$\langle N(R) \rangle$, $\langle \pi_F(R) \rangle$, $\langle \pi(R) \cup \pi(R)^{(-1)} \rangle$ and $\langle \pi(R) \cap \pi(R)^{(-1)} \rangle$ are normal subgroups of $Q(R)$.*

Example 4.5. Let F be an algebraically closed field and let $E = F(x)$ be the field of rational functions over F. By Example 3.6, $\pi_F(M_2(E))$ consists of matrices with eigenvalues in $F \backslash \{1\}$. Take matrices

$$A = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

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which both lie in $\pi_F(M_2(E))$, since they are nilpotent (see Lemma 3.2). Then

$$A \circ B = \begin{bmatrix} -x & x \\ 1 & 0 \end{bmatrix}$$

does not have eigenvalues in F, since its trace is $-x \notin F$. So $\pi_F(M_2(E))$ is not closed under $\circ$.

Example 4.6. From Example 3.5 it is easy to calculate that $\langle \pi(\mathbb{Q}) \cup \pi(\mathbb{Q})^{(-1)} \rangle = Q(\mathbb{Q}) = \mathbb{Q}\backslash\{1\}$ and $\langle \pi(\mathbb{Q}) \cap \pi(\mathbb{Q})^{(-1)} \rangle = \{0, 2\}$.

Example 4.7. Recall that a complex matrix A is called *unipotent* if $I - A$ is nilpotent, where I denotes the identity matrix. In [**8**] it was shown that a complex matrix is a finite product of unipotent matrices if and only if it has determinant 1. This shows that

$$\langle N(M_n(\mathbb{C})) \rangle = \{A \in M_n(\mathbb{C})\colon \det(I - A) = 1\},$$

which is a proper subgroup of

$$\pi_{\mathbb{C}}(M_n(\mathbb{C})) = Q(M_n(\mathbb{C})) = \{A \in M_n(\mathbb{C})\colon \det(I - A) \neq 0\}.$$

Next we investigate what can be said about addition. Direct verification shows that for all quasi-regular element x and y of any ring we have $xy = x \circ (x^{(-1)} + y^{(-1)}) \circ y$, $x + y = x \circ (x^{(-1)} y^{(-1)}) \circ y$ and $-x = (2x^{(-1)}) \circ x$. Thus, we obtain the following.

Theorem 4.8. *Let R be a ring. For any subgroup S of $Q(R)$ the following are equivalent:*

(i) *S is closed under addition;*

(ii) *S is closed under multiplication; and*

(iii) *S is a subring of R.*

As a corollary to Theorem 4.8 we have the following.

Corollary 4.9. *Let F be a field of characteristic* 0 *and let R be a commutative F-algebra. If $\pi_F(R)$ is closed under addition, then $\pi_F(R) = N(R)$.*

Proof. Since R is commutative, $\pi_F(R)$ is a subgroup of $Q(R)$ by Proposition 4.2. If $\pi_F(R)$ is closed under addition, then it is a subring of R by Theorem 4.8. Let $a \in R$ be π-algebraic with a polynomial p and let λ be a non-zero scalar. Since F is of characteristic 0, there exists a positive integer n such that $n\lambda^{-1}$ is not a zero of p. Hence $n^{-1}\lambda a$ is π-algebraic with the polynomial $p(n\lambda^{-1})^{-1} p(n\lambda^{-1} x)$. Since $\pi_F(R)$ is closed under addition and λa is a multiple of $n^{-1}\lambda a$, λa is π-algebraic as well. So $\pi_F(R)$ is in fact a subalgebra of R. Thus $\pi_F(R)$ is nil by Proposition 3.10 and $\pi_F(R) = N(R)$ follows. □

The next proposition shows that the conclusion of Corollary 4.9 remains true for rings.

Proposition 4.10. *Let R be a commutative ring. If $\pi(R)$ is closed under addition, then $\pi(R) = N(R)$.*

Proof. First we show that $\pi(R)$ is also closed under negation. If a is π-algebraic, then $\mathbb{N}a \subseteq \pi(R)$ since $\pi(R)$ is closed under addition. By Proposition 3.23 and Remark 3.24 there exists a non-zero integer n such that na is nilpotent. Thus $-|n|a$ is nilpotent and hence π-algebraic. So $-a = -|n|a + (|n| - 1)a$ is π-algebraic as well, since $(|n| - 1)a$ is a non-negative multiple of a. The commutativity of R implies that $\pi(R)$ is closed under $\circ$ by Proposition 4.2. Since $xy = x + y - x \circ y$, $\pi(R)$ is closed under multiplication as well. So $\pi(R)$ is a π-algebraic subring of R, and hence it is nil by Theorem 3.14. □

We are now left with the case of algebras over fields of prime characteristic. We were not able to obtain an analogue of Corollary 4.9 for arbitrary fields of prime characteristic, but only for algebraic extensions of prime fields.

Corollary 4.11. *Let p be a prime number, let F be an algebraic field extension of the prime field $\mathbb{Z}/p\mathbb{Z}$, and let R be a commutative F-algebra. If $\pi_F(R)$ is closed under addition, then $\pi_F(R) = N(R)$.*

Proof. Since F is algebraic over $\mathbb{Z}/p\mathbb{Z}$, we have $A_F(R) = A_{\mathbb{Z}/p\mathbb{Z}}(R)$, so $\pi_F(R) = \pi_{\mathbb{Z}/p\mathbb{Z}}(R)$ by Lemma 3.2. Now let $a \in R$ be π-algebraic over $\mathbb{Z}/p\mathbb{Z}$ with a polynomial $\hat{f}$ and let f be a polynomial with integer coefficients that represents $\hat{f}$. Since $\hat{f}(1) = 1$, there exists an integer k such that $f(1) = kp + 1$. If we set $F(x) = f(x) - kpx$, then $F(0) = 0$, $F(1) = 1$ and $F(a) = 0$, since $pa = 0$. So a is π algebraic over $\mathbb{Z}$. Hence $\pi_{\mathbb{Z}/p\mathbb{Z}}(R) \subseteq \pi(R)$ and clearly $\pi(R) \subseteq \pi_{\mathbb{Z}/p\mathbb{Z}}(R)$. This implies that $\pi_F(R) = \pi(R)$ and so $\pi_F(R) = N(R)$ by Proposition 4.10. □

This was one extremal situation, when every π-algebraic element is in fact nilpotent. The other extremal situation would be when there are no nilpotent elements, but many π-algebraic ones. As we have mentioned before, in an algebraic division algebra there are no non-zero nilpotent elements although all elements except the unit are π-algebraic. Next we investigate when something similar happens in general algebras. The question is whether $\pi_F(R) \cup \{1\}$ will form a division subring of a unital F-algebra R. When R is just a ring, we can ask a similar question; however, it seems more natural to consider the set $\langle \pi(R) \cup \pi(R)^{(-1)} \rangle \cup (\mathbb{Z} \cdot 1)$ in this case, since the elements in $(\mathbb{Z} \cdot 1) \backslash \{1\}$ need not be automatically contained in $\langle \pi(R) \cup \pi(R)^{(-1)} \rangle$. In certain situations though, they are.

In the proof of Theorem 4.14 we will need the following auxiliary proposition, which may be of independent interest. We formulate it slightly more generally than needed for the proof of the theorem. Recall that an integral domain K is called a *factorization domain* (also an *atomic domain*) if every non-zero non-unit of K can be written as a finite product of irreducible elements.

Proposition 4.12. *Let R be a unital ring and let K be a commutative subring of R with $1 \in K$ such that $R \backslash K \subseteq R^{-1}$. If K is a factorization domain, then one of the following holds:*

(i) $R = K$;

(ii) *R is a local ring with maximal ideal $M \subseteq K$ and K is a local ring with maximal ideal M;*

(iii) *R is a division ring.*

Proof. Suppose that $R \neq K$ and that R is not a division ring. Then there exist $r \in R \backslash K \subseteq R^{-1}$ and $0 \neq a \in K \backslash R^{-1}$. Since K is a factorization domain, we may assume that the element a is irreducible. We will prove that $K^{-1} = K \cap R^{-1}$. Let x be an arbitrary element of K that is invertible in R and set $y = x^{-1}a$. Then y is not invertible in R, since a is not. But $R \backslash K \subseteq R^{-1}$, so $y \in K$. Thus $a = xy$ is a factorization of a in K. Since a was irreducible and y is not invertible, x must be invertible in K, as needed. Now let M be the set of all elements of K that are not invertible in K. Since $R \backslash K \subseteq R^{-1}$, M is also the set of all non-invertible elements of R. If $x \in M$ and $k \in K$, then xk is not invertible in K, otherwise x would be invertible due to the commutativity of K. So $MK \subseteq M$. If $x, y \in M$, then by the above x and y are not invertible in R. By the choice of r this implies that xr and yr are not invertible in R, so $xr, yr \in K$. Thus $(x-y)r \in K$. But $x - y \in K$ and $r \notin K$, hence $x - y$ cannot be invertible in K, so $x - y \in M$. This proves that M in an ideal in K, so K is local with maximal ideal M. Now let $x \in M$ and $s \in R$, so by the above x is not invertible in R. If $s \in K$, then $sx, xs \in M$ by what we have just proved. If $s \notin K$, then s is invertible in R. So sx and xs are not invertible in R, hence $sx, xs \in M$. This shows that M is also an ideal of R and R is local with maximal ideal M. □

Remark 4.13. There exist examples where case (ii) of Proposition 4.12 occurs in a non-trivial way and K is not the maximal ideal of R. Take, for example, $R = E[\![x]\!]$ and $K = F + E[\![x]\!]x \subseteq R$, where $F \subsetneq E$ are fields. Every non-zero non-unit in K is contained in $E[\![x]\!]x$ and factors as $x^n g(x)$ for some non-negative integer n and some $g(x)$ of the form $\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots$ with $\alpha_1 \neq 0$.

Theorem 4.14. *Let R be a unital ring of characteristic 0. For any subgroup S of $Q(R)$ with $\{0, 2\} \subsetneq S$ the following are equivalent:*

(i) *$S \cup \mathbb{Z}$ is closed under addition;*

(ii) *$S \cup \{1\}$ is a division subring of R.*

Proof. Clearly, (ii) implies (i), since in this case $S \cup \{1\} = S \cup \mathbb{Z}$. So assume (i) holds. First we show that $S \cup \mathbb{Z}$ is a subring. If $x \in S \cup \mathbb{Z}$, then $2 \circ x = 2 - x \in S \cup \mathbb{Z}$, since $2 \in S \cap \mathbb{Z}$. So if $x \in S \cup \mathbb{Z}$, then $-x = 2 - (2 + x) \in S \cup \mathbb{Z}$ by (i). Thus $S \cup \mathbb{Z}$ is closed under negation. S and $\mathbb{Z}$ are both closed under $\circ$. If $x \in S$ and $n \in \mathbb{Z}$, then $x \circ n = n \circ x = n + x - nx \in S \cup \mathbb{Z}$, since nx is a multiple of x or $-x$ and $S \cup \mathbb{Z}$

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is closed under addition. So $S \cup \mathbb{Z}$ is closed under $\circ$ and also under multiplication since $xy = x + y - x \circ y$. This shows that $S \cup \mathbb{Z}$ is a subring of R. Now, every element in S is quasi-regular with quasi-inverse in S, thus every element in $1 - S$ is invertible in $S \cup \mathbb{Z}$. Since $S \cup \mathbb{Z}$ is a subring, we have $1 - (S\backslash\mathbb{Z}) = S\backslash\mathbb{Z}$. So every element in $S\backslash\mathbb{Z}$ is invertible in $S \cup \mathbb{Z}$. By Proposition 4.12, either $S \subseteq \mathbb{Z}$ or $S \cup \mathbb{Z}$ is a division ring. Suppose $S \subseteq \mathbb{Z}$. Then the quasi-inverse of every element in $S \subseteq \mathbb{Z}$ lies again in $S \subseteq \mathbb{Z}$, so $S \subseteq Q(\mathbb{Z}) = \{0, 2\}$, which contradicts our assumption. Therefore, $S \cup \mathbb{Z}$ is a division ring. It remains to prove that $\mathbb{Z}\backslash\{1\} \subseteq S$. Let $n \in \mathbb{Z}\backslash\{1\}$. If $n = 0$ or $n = 2$, then $n \in S$ by assumption. So suppose $n \neq 0, 2$. Since $S \cup \mathbb{Z}$ is a division ring, $1 - n$ is invertible in $S \cup \mathbb{Z}$. Since $1 - n \neq \pm 1$, the fact that the characteristic of R is 0 implies $(1 - n)^{-1} \notin \mathbb{Z}$, i.e. $1 - (1 - n)^{-1} \in S$. Consequently, $n = (1 - (1 - n)^{-1})^{(-1)} \in S$, since S is a subgroup of $Q(R)$. □

Theorem 4.15. *Let R be a unital ring of prime characteristic p. For any subgroup S of $Q(R)$ the following are equivalent:*

(i) *$S \cup \mathbb{Z}/p\mathbb{Z}$ is closed under addition;*

(ii) *$S \cup \mathbb{Z}/p\mathbb{Z}$ is a division subring of R.*

Proof. In this case $S \cup \mathbb{Z}/p\mathbb{Z}$ is automatically closed under negation, since $-x = (p-1)x$ is a multiple of x. The proof is now the same as that of Theorem 4.14 except for the case $S \subseteq \mathbb{Z}/p\mathbb{Z}$, but in this case $S \cup \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/p\mathbb{Z}$ is automatically a division ring. □

Corollary 4.16. *Let F be a field and let R be a unital commutative F-algebra. If $\pi_F(R) \cup \{1\}$ is closed under addition, then it is a field that is a subring of R.*

Proof. This follows directly from Proposition 4.2 and Theorems 4.14 and 4.15, since $(\mathbb{Z} \cdot 1)\backslash\{1\} \subseteq \pi_F(R)$ by Lemma 3.3. □

Corollary 4.17. *Let R be a unital commutative ring of prime or 0 characteristic with $\pi(R) \neq \{0, 2\}$. If $\pi(R) \circ \pi(R)^{(-1)} \cup (\mathbb{Z} \cdot 1)$ is closed under addition, then it is a field that is a subring of R.*

Proof. The commutativity of R implies $\langle \pi(R) \cup \pi(R)^{(-1)} \rangle = \pi(R) \circ \pi(R)^{(-1)}$ by Proposition 4.2. Since $2 \in \pi(R)$, the result follows from Theorems 4.14 and 4.15. □

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