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ON A QUESTION OF REMESLENNIKOV

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Abstract. We give an example of an element r of a free group F, and an element s of minimal length in the normal closure of r in F, such that s is not conjugate to $r^{\pm 1}$ or to $[r^{\pm 1}, f]$, for any element f of F.

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1. Introduction. The question referred to in the title is Question **F16** of the list 'Open problems in combinatorial group theory' of Baumslag, Myasnikov and Shpilrain [**1**], and it reads as follows.

Let R be the normal closure of an element r in a free group F with the natural length function, and suppose that s is a (non-identity) element of minimal length in R. Is it true that s is conjugate to one of the following elements: $r, r^{-1}, [r, f]$, or $[r^{-1}, f]$, for some element f?

In [**1**] it is noted that this question was motivated by a well known result of Magnus (see e.g. [**2**]): if elements r and s of a free group F have the same normal closure, then s is conjugate to $r^{\pm 1}$. We would add that no general result classifying elements of (relatively) small length in the normal closure of a single element of a free group is known, and that such a result would be of great interest in the theory of one-relator groups.

The question is known to have a positive answer in a number of cases. Thus, for example, if r satisfies a suitable small cancellation condition (see Chapter V of [**2**]), then it is easily seen (e.g. by using a theorem of Greendlinger; see Theorem 4.5 of Chapter V of [**2**]) that any element of minimal length in R is conjugate to $r^{\pm 1}$, while in the free group F_2 with basis $\{a, b\}$, if $r = a^t b^2$ and $t \geq 5$, it can be easily shown, using arguments similar to those given below, that the elements of minimal length in R are conjugates of $[a^{\pm 1}, b^2]$, and these are the same as conjugates of $[r^{\pm 1}, a^{-1}r]$.

Let r be the element of F_2 given by $r = ba^t b^2 a^t$, where $t \geq 3$, and let $s = [b^3, a]$. We shall provide a negative answer to Remeslennikov's question by showing that s is of minimal length in R, and that s is not conjugate to $r^{\pm 1}$ or $[r^{\pm 1}, f]$, for any f in F_2.

2. The proof. Let G be the free product with amalgamation of two infinite cycles given by the presentation $\langle x, a \mid x^3 = a^{-t} \rangle$. We have

$$G = \langle x, a, b \mid x^3 = a^{-t}, b = x^2 \rangle = \langle x, a, b \mid b = x^2, x^{-1}b^2 = a^{-t} \rangle$$
$$= \langle a, b \mid b = b^2 a^t b^2 a^t \rangle = \langle a, b \mid ba^t b^2 a^t = 1 \rangle.$$

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We note that $s \in R$ (since $b^3 = x^6$ is central in G), s is cyclically reduced, and s is not conjugate to $r^{\pm 1}$. We now check that s is not conjugate to any element $[r^{\pm 1}, f]$ of F_2. If it were conjugate to such a commutator, say to $[r^{-1}, f] = r^{-1}frf^{-1}$, then we have $frf^{-1} = f_1 r_1 f_1^{-1}$ say, where r_1 is a cyclic permutation of r, and no cancellation occurs in $f_1 r_1 f_1^{-1}$. We have s is conjugate to $f_1^{-1} r^{-1} f_1 r_1$, and we see that f_1 must be cancelled completely by r^{-1}, since otherwise $f_1^{-1} r^{-1} f_1 r_1$, when reduced, would be cyclically reduced, but would have length greater than the length of r, and so could not be conjugate to s. Thus some cyclic permutation s_1 of s is the product $r_2^{-1} r_1$, for certain cyclic permutations r_1, r_2 of r. We distinguish seven types of cyclic permutations of r, namely:

$$ba^t b^2 a^t,\ a^t b^2 a^t b,\ a^{l_2} b^2 a^t b a^{l_1},\ b^2 a^t b a^t,\ ba^t b a^t b,\ a^t b a^t b^2,\ a^{m_2} b a^t b^2 a^{m_1},$$

where $l_1 + l_2 = m_1 + m_2 = t$ and l_1, l_2, m_1, m_2 are positive. It is now easy to check that no product $r_2^{-1} r_1$ (or $r_1 r_2^{-1}$) has length as small as 8, except in the case $t = 3$, and in case $t = 3$ the only such products of length as small as 8 are cyclic permutations of $[b, a^t]$. Thus s is not conjugate to any element of the form $[r^{\pm 1}, f]$.

It remains to show that no non-identity element w of R could have length less than 8. Suppose that such a w exists. We may assume that w is of the form $b^{l_1} a^{m_1} \dots b^{l_k} a^{m_k}$, where the l_i, m_i are non-zero, and $|l_1| + |m_1| + \dots + |l_k| + |m_k| < 8$. We have $w = 1$ in G. Replacing b by x^2, we see, since G is a free product with amalgamation, that either some l_i is a multiple of 3, or some m_i is a multiple of t. Suppose that the former occurs. There is no loss of generality in supposing that l_1 is a positive multiple of 3 (replacing w by a cyclic permutation of $w^{\pm 1}$ if necessary). If $l_1 > 3$, then w must be $b^6 a^{\pm 1}$; but $b^6 a^{\pm 1} \neq 1$ in G. Thus we have $l_1 = 3$, and $w = b^3 a^{m_1} b^{l_2} a^{m_2}$, where each l_i and m_i must have absolute value no more than 2; replacing b by x^2 we see that such a w is not the identity in G. Thus we have obtained a contradiction if some l_i is a multiple of 3. It is likewise easy to obtain a contradiction under the assumption that some m_i is a multiple of t. This proves our claim that s is of minimal length in R.

REFERENCES

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