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A DEDEKIND-STYLE AXIOMATIZATION AND THE CORRESPONDING UNIVERSAL PROPERTY OF AN ORDINAL NUMBER SYSTEM

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Abstract. In this paper, we give an axiomatization of the ordinal number system, in the style of Dedekind's axiomatization of the natural number system. The latter is based on a structure $(N, 0, s)$ consisting of a set N, a distinguished element $0 \in N$ and a function $s\colon N \to N$. The structure in our axiomatization is a triple (O, L, s), where O is a class, L is a class function defined on all s-closed 'subsets' of O, and s is a class function $s\colon O \to O$. In fact, we develop the theory relative to a Grothendieck-style universe (minus the power set axiom), as a way of bringing the natural and the ordinal cases under one framework. We also establish a universal property for the ordinal number system, analogous to the well-known universal property for the natural number system.

§1. Introduction. The introduction and study of ordinal numbers goes back to the pioneering works of Cantor in set theory [2, 3]. In modern language, Cantor's ordinal numbers are isomorphic classes of well-ordered sets, see e.g. [5]. There is also a 'concrete' definition of an ordinal number as a transitive set which is well-ordered under the element relation—see e.g. [7]. Such sets are usually called *von Neumann ordinals*. Natural numbers can be seen concretely as the finite ordinal numbers. In Dedekind's approach to the natural number system described in [4], the natural numbers are not defined as concrete objects, but rather as abstract entities organized in a certain structure; namely, a triple $(N, 0, s)$ consisting of a set N (a set of 'abstract' natural numbers), a distinguished element 0 of N (in [4], the distinguished element is 1), and a function $s\colon N \to N$, which names the 'successor' of each natural number. The axioms that such a system should satisfy were formulated by Dedekind, but are often referred to as Peano axioms today (see e.g. [16] for some historical background):

- 0 does not belong to the image of s.
- s is injective.
- $X = N$ for any subset X of N that is closed under s and contains 0.

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It is an observation due to Lawvere [10] that these axioms identify the natural number systems as initial objects in the category of all triples (X, x, t) where X is a set, $x \in X$ and t is a function $t \colon X \to X$. This 'universal property' of the natural number system, freed from its category-theoretic formulation, is actually the 'definition by induction' theorem already contained in [4]. A morphism $(N, 0, s) \to (X, x, t)$ of such triples is defined as a function $f : N \to X$ such that:

- $f(0) = x$,
- $f(s(n)) = t(f(n))$ for all $n \in N$.

The 'definition by induction' theorem states that there is exactly one morphism to any other triple (X, x, t) from the triple $(N, 0, s)$ satisfying the axioms stated above. This theorem is of course well known because of its practical use: it says that recursively defined functions exist and are uniquely determined by the recursion. Intuitively, the theorem can be understood as follows. A triple (X, x, t) can be viewed as an abstraction of the concept of counting—X is the set of figures used in counting, x is where counting begins and the function t names increments when counting. Without further restrictions on such 'counting systems', there are many non-isomorphic ones, some of which are quite different from the natural number system, but still useful; for instance, hours on a clock, where counting loops back to 1 once we pass 12. A morphism of these triples can be viewed as a 'translation' of one counting system to another. The universal property of the natural number system presents it as a 'universal' counting system, in the sense that it has a unique translation to any other counting system. Incidentally, such intuition is not particular to the natural number system: many structures in mathematics can be defined by natural universal properties—see [11].

Ordinal numbers exhibit a similar structure to natural numbers—there is a 'starting' ordinal (the natural number 0), and every ordinal number has a successor. The natural numbers $0, 1, 2, 3, \ldots$ are the first ordinal numbers. This set is closed under succession. The ordinal number system allows for another type of succession that can be applied to sets of ordinal numbers closed under succession, giving rise to the so-called 'limit' ordinal numbers. The infinite sequence of natural numbers is succeeded by a limit ordinal number, usually denoted by ω. Now, we can take the 'usual' successor of ω, call it $\omega + 1$, and keep taking its successors until we get another set that is closed under succession, after which we introduce another limit ordinal number—it will be $\omega + \omega = \omega \cdot 2$. The next limit ordinal number will be $\omega \cdot 3$. At some point, we reach $\omega \cdot \omega = \omega^2$, then ω^3, and so on until we reach ω^ω, then $\omega^\omega + 1$, and so on. The process is supposed to continue until all ordinal numbers that we have named no longer form a set. Let us also recall that von Neumann ordinals are defined as sets of preceding ordinals. Thus the first ordinal number, the number 0, is defined as $0 = \varnothing$ and the successor of an ordinal number n is defined as $\bigcup\{n, \{n\}\}$. A limit ordinal number is one that is the union of all preceding ordinal numbers. We can, in particular, think of 0 as a limit ordinal number given by the union of its predecessors, since the empty union equals the empty set. Equivalently, limit ordinal numbers are those whose sets of predecessors are closed under succession.

We may also think of the ordinal number system in terms of a triple (O, L, s)—this time, O is a class (since the collection of all ordinal numbers is no longer a set),

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L is a (class) function that specifies limit ordinals and is defined for those subclasses of O which form sets closed under the class function s, which specifies the successor of each ordinal. In this paper we show that the following three axioms on such a triple are suitable as analogues of the three Dedekind–Peano axioms for the ordinal number system:

- $L(I)$ does not belong to the image of s, and also $s(L(I)) \notin I$, for any I such that $L(I)$ is defined.
- s is injective and $L(I) = L(J)$ if and only if $\overline{I} = \overline{J}$ and $L(I), L(J)$ are defined, where $\overline{I}$ and $\overline{J}$ denote closures of I and J, respectively, under s and L predecessors.
- $X = O$ for any subclass X of O that is closed under s and that contains $L(I)$ for each $I \subseteq X$ such that $L(I)$ is defined.

In particular, we prove that:

- The system of von Neumann ordinals constitutes a triple (O, L, s) satisfying the three axioms above. There is nothing surprising here, as the result relies on the well-known properties of ordinal numbers.
- Any triple (O, L, s) satisfying these three axioms has an order which makes it order-isomorphic to the system of ordinal numbers. The order, in fact, is the specialization order of the topology given by the closure operator in the second axiom (without those axioms, this order is merely a preorder).
- The triple (O, L, s) satisfying the three axioms above is an initial object in the category of all triples (O', L', s') such that $\overline{I'} = \overline{J'}$ implies $L(I') = L(J')$, whenever those are defined (with the domain of L' being the class of all s'-closed subsets of O').

The idea for defining an ordinal number system abstractly goes back to Zermelo (see 'Seven notes on ordinal numbers and large cardinals' in [18]). His approach is to define it as a particular type of well-ordered class $(O, \leqslant)$. The following axioms, added to well-ordering, would suffice:

- For every $x \in O$, the class $\{x\}^{>} = \{y \in O \mid y < x\}$ is a set.
- For each subset X of O, the class $X^{<} = \{y \in O \mid \forall_{x \in X}[x < y]\}$ is nonempty.

There is, of course, an analogous presentation (although less known than the one given by Peano axioms) of the natural number system as a well-ordered set $(N, \leqslant)$ satisfying the following additional conditions:

- For each $x \in N$, the set $\{x\}^{>}$ is finite.
- For each $x \in N$, the set $\{x\}^{<}$ is nonempty.

This is an alternative approach to that of Dedekind, where there is greater emphasis on the order structure. The difference between our approach to ordinal numbers and the traditional approaches is similar, where in our approach we try to make minimal use of the order structure. Universal properties of the ordinal number system emerging from the more order-based approach have been established, in various forms, in [8, 12]. Our universal property is different from those.

The main new results of the paper are given in the last section. Before that, we redevelop the basic theory of ordinal numbers relative to the set-theoretic context in which these results are proved, ensuring that the paper is self-contained.

This paper grew out from the second author's rediscovery of Tarski's addition of sets [13] and of a universal property for ordinal numbers, known already from [12]—see [14] (although, at the time of writing [14], the author was not aware of [12]). Some of the chapters of [14] are based on the present paper (see also Remark 2).

§2. The context. There are a number of alternatives for a context in which the theory that we lay down in this paper could be developed. Elaboration of those contexts and comparison of results across the contexts as a future development of our work would certainly be worthwhile. In this paper, we have decided to stick to what we believe to be technically the most simple and intuitive context, given by a 'universe' inside the standard Zermelo–Fraenkel axiomatic set theory, including the axioms of foundation and choice (see e.g. [7]). This approach is not unusual. For instance, it is the approach followed in the exposition of category theory in [11]. The universes we work with, however, are slightly more general than the more commonly used Grothendieck universes [1, 6, 17]. The main difference is that our universes do not require closure under power sets and can be empty. Our context is in fact a particular instance of the quite general category-theoretic context used in [8]. Generalization of our results to that context is left for future work.

We remark that the definitions, theorems and their proofs contained in this paper could be adapted, after a straightforward modification to their formulation, to 'absolute' set theory, where our 'sets' could be replaced with 'classes' and elements of the fixed universe with 'sets'. We would then get the form of the definitions and theorems given in Section 1.

For a set X, by $\mathsf{P}X$ we denote the power set of X, and by $\bigcup X$ we denote the union of all elements of X. By $\mathbb{N}$ we denote the set of natural numbers. While we do not rely on any prior knowledge of facts about ordinal numbers (proofs of all needed facts are included in the paper), we do rely on knowledge of basic set-theoretic properties of the natural number system. In particular, we will make use of mathematical induction, definition by recursion, as well as the fact that any infinite set has a subset bijective to $\mathbb{N}$.

Recall that a set X is said to be *transitive* when $X \subseteq \mathsf{P}X$, or equivalently, when $\bigcup X \subseteq X$. For a function $f : X \to Y$ and a set $A \in \mathsf{P}X$, we write fA to denote the image of A under f:

$$fA = \{f(a) \mid a \in A\}.$$

Definition 1. A *universe* is a set $\mathfrak{U}$ satisfying the following:

(U1) $\mathfrak{U}$ is a transitive set.
(U2) If $X, Y \in \mathfrak{U}$ then $\{X, Y\} \in \mathfrak{U}$.
(U3) $\bigcup fI \in \mathfrak{U}$ for any $I \in \mathfrak{U}$ and any function $f : I \to \mathfrak{U}$.

These axioms imply that $\mathfrak{U}$ is closed under the following standard set-theoretic constructions:

- Singletons. Trivially, since $\{x\} = \{x, x\}$.
- Union. Because $\bigcup X = \bigcup 1_X X$.
- Subsets. Let $Y \in \mathfrak{U}$ and let $X \subseteq Y$. If $X = \varnothing$, then $X \in \mathfrak{U}$, because every nonempty transitive set contains the empty set thanks to the axiom of

foundation. If $X \neq \varnothing$, then let $x_0 \in X$. Consider the function $f\colon Y \to \mathfrak{U}$ defined by

$$f(y) = \begin{cases} \{y\}, & y \in X, \\ \{x_0\}, & y \notin X. \end{cases}$$

Notice that $\{x_0\} \in \mathfrak{U}$ since $x_0 \in X \subseteq Y \in \mathfrak{U}$ and $\mathfrak{U}$ is a transitive set closed under singletons. Then $X = \bigcup fY \in \mathfrak{U}$.

- Cartesian products (binary). Let $X \in \mathfrak{U}$ and $Y \in \mathfrak{U}$. Then

$$\{(x, y)\} = \{\{\{x, y\}, \{x\}\}\} \in \mathfrak{U}$$

for each $x \in X$ and $y \in Y$. For each $x \in X$ define a function $f_x\colon Y \to \mathfrak{U}$ by $f_x(y) = \{(x, y)\}$. Then $\{(x, y) \mid y \in Y\} = \bigcup f_x Y \in \mathfrak{U}$ for each $x \in X$. Now define a function $g\colon X \to \mathfrak{U}$ by $g(x) = \{\{(x, y) \mid y \in Y\}\}$. Then $X \times Y = \bigcup gX \in \mathfrak{U}$.

- Disjoint union. Given $X \in \mathfrak{U}$, the disjoint union $\sum X$ can be defined as

$$\sum X = \bigcup \{x \times \{x\} \mid x \in X\}.$$

Then $\sum X = \bigcup fX$ where $f\colon X \to \mathfrak{U}$ is defined by $f(x) = x \times \{x\}$.

- Quotient sets. Let $X \in \mathfrak{U}$ and let E be an equivalence relation on X. Then $X/E = \bigcup qX$, where q is the function $q\colon X \to \mathfrak{U}$ defined by $q(x) = \{[x]_E\}$.
- Replacement. Let $X \in \mathfrak{U}$ and let f be a function $X \to \mathfrak{U}$. By (a), we can define a function $g\colon \mathfrak{U} \to \mathfrak{U}$ such that $g(y) = \{y\}$ for each $y \in \mathfrak{U}$. Then $g \circ f$ is a function $X \to \mathfrak{U}$ such that for all $x \in X$,

$$(g \circ f)(x) = \{f(x)\}.$$

The image of X under $g \circ f$ is then

$$(g \circ f)X = \{\{f(x)\} \mid x \in X\}.$$

Then by (U3),

$$\begin{aligned} fX &= \{f(x) \mid x \in X\} \\ &= \bigcup \{\{f(x)\} \mid x \in X\} \\ &= \bigcup (g \circ f)X \in \mathfrak{U}. \end{aligned}$$ ⊣

From this it follows, of course, that when $\mathfrak{U}$ is not empty, it contains all natural numbers, assuming that they are defined by the recursion

$$\begin{aligned} 0 &= \varnothing, \\ n + 1 &= \bigcup \{n, \{n\}\}. \end{aligned}$$

Furthermore, when $\mathfrak{U}$ contains at least one infinite set, it also contains the set $\mathbb{N}$ of all natural numbers (as defined above).

The empty set $\varnothing$ is a universe. The sets whose transitive closure have cardinality less than a fixed infinite regular cardinal κ form a universe in the sense of the definition above (by Lemma 6.4 in [9]). In particular, hereditarily finite sets (where $\kappa = \aleph_0$) form a universe, as do hereditarily countable sets (where $\kappa = \aleph_1$). The so-called Grothendieck universes are exactly those universes in our sense, which are closed under power sets, i.e., if $X \in \mathfrak{U}$, then $\mathsf{P}X \in \mathfrak{U}$.

For any two sets A and B, we write

$$A \approx B$$

when there is a bijection from A to B. The *restricted power set* of a set X relative to a universe $\mathfrak{U}$ is the set

$$\mathsf{P}_{\mathfrak{U}}X = \{A \subseteq X \mid \exists_{B\in\mathfrak{U}}[A \approx B]\}.$$

The following lemmas will be useful.

LEMMA 2. *When $\mathfrak{U}$ is nonempty, for any set X and its finite subset $Y \subseteq X$, we have $Y \in \mathsf{P}_{\mathfrak{U}}X$.*

PROOF. This follows from the fact that when $\mathfrak{U}$ is nonempty, it contains a set of each finite size. ⊣

LEMMA 3. *If $A \subseteq B$ and $B \in \mathsf{P}_{\mathfrak{U}}X$, then $A \in \mathsf{P}_{\mathfrak{U}}X$.*

PROOF. If $A \subseteq B \approx C \in \mathfrak{U}$, then A is bijective to a subset of C. ⊣

LEMMA 4. *If $C \in \mathsf{P}_{\mathfrak{U}}\mathsf{P}_{\mathfrak{U}}X$ then $\bigcup C \in \mathsf{P}_{\mathfrak{U}}X$. In particular, this implies that if $I \in \mathsf{P}_{\mathfrak{U}}X$, then for any function $f : I \to \mathsf{P}_{\mathfrak{U}}X$, we have $\bigcup fI \in \mathsf{P}_{\mathfrak{U}}X$.*

PROOF. Suppose $C \in \mathsf{P}_{\mathfrak{U}}\mathsf{P}_{\mathfrak{U}}X$. Then there is a bijection $h\colon C' \to C$, where $C' \in \mathfrak{U}$. Since for each element $c \in C$ we have $c \in \mathsf{P}_{\mathfrak{U}}X$, by axiom of choice we have a function $g\colon C \to \mathfrak{U}$ such that $c \approx g(c)$ for each $c \in C$. Let g_c denote a bijection $g_c\colon c \to g(c)$ (we again use the axiom of choice to select such a bijection for each $c \in C$). Now, define a function $k\colon C' \to \mathfrak{U}$ as follows:

$$k(c') = \{(x, c') \mid x \in g(h(c'))\}.$$

Then $\bigcup kC' \in \mathfrak{U}$ by (U3). Consider the function $f : \bigcup kC' \to \bigcup C$ defined by

$$f(x, c') = g^{-1}_{h(c')}(x).$$

This function is a surjection. Indeed, for each $y \in \bigcup C$ there is a $c \in C$ such that $y \in c$ and so,

$$\begin{aligned} f(g_c(y), h^{-1}(c)) &= g^{-1}_{h(h^{-1}(c))}(g_c(y)) \\ &= g_c^{-1}(g_c(y)) \\ &= y. \end{aligned}$$

So $\bigcup C$ is bijective to a quotient set of $\bigcup kC'$, and thus $\bigcup C \in \mathsf{P}_{\mathfrak{U}}X$. This proves the first part of the lemma.

Now suppose $I \in \mathsf{P}_{\mathfrak{U}}X$ and let $f : I \to \mathsf{P}_{\mathfrak{U}}X$ be a function. Then $fI \in \mathsf{P}_{\mathfrak{U}}\mathsf{P}_{\mathfrak{U}}X$. By what we have just proved, $\bigcup fI \in \mathsf{P}_{\mathfrak{U}}X$. ⊣

LEMMA 5. *Given a function $f : X \to Y$,*

$$A \in \mathsf{P}_{\mathfrak{U}}X \quad \Rightarrow \quad fA \in \mathsf{P}_{\mathfrak{U}}Y.$$

PROOF. If $A \approx A' \in \mathfrak{U}$, then fA is bijective to a suitable quotient of A'. ⊣

LEMMA 6. *If $A \in \mathsf{P}_{\mathfrak{U}}X$ and $B \in \mathsf{P}_{\mathfrak{U}}Y$, then $A \times B \in \mathsf{P}_{\mathfrak{U}}(X \times Y)$.*

PROOF. This follows from the fact that $\mathfrak{U}$ is closed under cartesian products. ⊣

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LEMMA 7. $\mathsf{P}_{\mathfrak{U}}\mathfrak{U} = \mathfrak{U}$.

PROOF. Let $A \in \mathsf{P}_{\mathfrak{U}}\mathfrak{U}$, i.e., $A \subseteq \mathfrak{U}$ such that $A \approx B \in \mathfrak{U}$. Then a bijection $f : B \to A$ exists, and since $A \subseteq \mathfrak{U}$, this gives us $A = fB \in \mathfrak{U}$.

Now consider $A \in \mathfrak{U}$. Then $A \subseteq \mathfrak{U}$ by (U1), and since $A \approx A \in \mathfrak{U}$, this gives us $A \in \mathsf{P}_{\mathfrak{U}}\mathfrak{U}$. We can conclude that $\mathsf{P}_{\mathfrak{U}}\mathfrak{U} = \mathfrak{U}$. ⊣

REMARK. One could relax the context proposed in this section by dropping the axiom of foundation and adopting an axiom on $\mathfrak{U}$ stating that if it $\mathfrak{U}$ is non-empty, then it must contain the empty set—see [14]. We have refrained from adopting this weaker context in the present paper, as it would result in a slightly more technical presentation of the background material (especially the material on concrete ordinals), without a significant gain in generality.

§3. Abstract ordinals. In this section we introduce an abstract notion of an ordinal number system relative to a universe and establish its basic properties. Consider a partially ordered set $(X, \leqslant)$. The relation $<$ for the partial order, as a relation from X to X, induces a Galois connection from $\mathsf{P}X$ to itself given by the mappings

$$S \mapsto S^{>} = \{x \in X \mid \forall_{y \in S}[x < y]\} \text{ and}$$
$$S \mapsto S^{<} = \{x \in X \mid \forall_{y \in S}[y < x]\}.$$

We call $S^{>}$ the *lower complement* of S, and $S^{<}$ the *upper complement* of S. Note that by '$a < b$' above we mean '$a \leqslant b \wedge a \neq b$', as usual. Since the two mappings above form a Galois connection, we have

$$S \subseteq (S^{>})^{<} \text{ and } S \subseteq (S^{<})^{>}$$

for any $S \subseteq X$. We define the *incremented join* $\overset{+}{\bigvee} S$ of a subset S of X (when it exists) as follows:

$$\overset{+}{\bigvee} S = \min S^{<}.$$

Note that since $S \cap S^{<} = \varnothing$, the incremented join of S is never an element of S.

It will be convenient to use the following abbreviations (where $x \in X$ and $S \subseteq X$):

$$x^{+} = \overset{+}{\bigvee}\{x\}, \quad S^{+} = \{x^{+} \mid x \in S\}.$$

The following basic laws are self-evident:

(L1) $x < x^{+}$,
(L2) there is no z such that $x < z < x^{+}$,
(L3) $x < y \;\Leftrightarrow\; x^{+} \leqslant y$,
(L4) $x = \overset{+}{\bigvee}\{x\}^{>}$ (for a total order),
(L5) $x < y^{+} \;\Leftrightarrow\; x \leqslant y$ (for a total order),
(L6) $x^{+} = y^{+} \;\Leftrightarrow\; x = y$ (for a total order),
(L7) $x < y \;\Leftrightarrow\; x^{+} < y^{+}$ (for a total order).

Using (L3), we can establish the following:

LEMMA 8. *For any $S \subseteq X$, if x^{+} exists for all $x \in S$, then $\bigvee S^{+}$ exists if and only if $\overset{+}{\bigvee} S$ exists, and when they exist,*

$$\bigvee S^{+} = \overset{+}{\bigvee} S.$$

The following (easy) lemma will also be useful:

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LEMMA 9. *Let $(X, \leqslant)$ be a poset and let $S \subseteq X$. Then*:

(i) *If S does not have a largest element, then $\bigvee S$ exists if and only if $\overset{+}{\bigvee} S$ exists, and when they exist, $\bigvee S = \overset{+}{\bigvee} S$. Conversely, if $\bigvee S = \overset{+}{\bigvee} S$, then S does not have a largest element.*

(ii) *If S has a largest element $x = \max S$, then $\overset{+}{\bigvee} S$ exists if and only if x^+ exists, and when they exist, $\overset{+}{\bigvee} S = x^+$. Conversely, when $\leqslant$ is a total order, if $\overset{+}{\bigvee} S = x^+$ then $x = \max S$.*

(iii) *If $\overset{+}{\bigvee} S^>$ exists then $\overset{+}{\bigvee} S^> = \min S$. Conversely, when $\leqslant$ is a total order, if $\min S$ exists, then $\min S = \overset{+}{\bigvee} S^>$.*

PROOF.

(i) If S has no largest element, then

$$\{x \in X \mid \forall_{y \in S}[y \leqslant x]\} = \{x \in X \mid \forall_{y \in S}[y < x]\}$$

and so

$$\bigvee S = \min\{x \in X \mid \forall_{y \in S}[y \leqslant x]\}$$

exists if and only if

$$\overset{+}{\bigvee} S = \min\{x \in X \mid \forall_{y \in S}[y < x]\}$$

exists, and when they exist, they are equal. Now suppose $\bigvee S = \overset{+}{\bigvee} S$. Since $\overset{+}{\bigvee} S$ is never an element of S, we conclude that S does not have a largest element.

(ii) Let $x = \max S$. Then $\{x\}^< = S^<$, and so $x^+ = \min\{x\}^<$ exists if and only if $\overset{+}{\bigvee} S = \min S^<$ exists, and when they exist, they are equal. Now suppose, in the case of total order, that $\overset{+}{\bigvee} S = x^+$ for some $x \in X$. Then S can only have elements that are strictly smaller than x^+, and thus each element of S is less than or equal to x (L5). Also, since $x^+ = \overset{+}{\bigvee} S = \min S^<$, it does not hold that $x \in S^<$, and thus x is not strictly larger than every element of S. We can conclude that $x = \max S$.

(iii) Suppose $\overset{+}{\bigvee} S^>$ exists. Since $S \subseteq (S^>)^<$, we must have $\overset{+}{\bigvee} S^> \leqslant x$ for each $x \in S$. This together with the fact that $\overset{+}{\bigvee} S^>$ cannot be an element of $S^>$ forces $\overset{+}{\bigvee} S^>$ to be an element of S. Hence $\overset{+}{\bigvee} S^> = \min S$. Suppose now that $\leqslant$ is a total order and $\min S$ exists. Consider an element $x \in (S^>)^<$. Then x is not an element of $S^>$ and so we cannot have $x < \min S$. Therefore, $\min S \leqslant x$. This proves $\min S = \overset{+}{\bigvee} S^>$. ⊣

DEFINITION 10. An *ordinal system* relative to a universe $\mathfrak{U}$ is a partially ordered set $\mathcal{O} = (\mathcal{O}, \leqslant)$ satisfying the following axioms:

(O1) For all $X \subseteq \mathcal{O}$, if $X \neq \varnothing$, then $X^> \in \mathsf{P}_{\mathfrak{U}}\mathcal{O}$.
(O2) $\overset{+}{\bigvee} X$ exists for each $X \in \mathsf{P}_{\mathfrak{U}}\mathcal{O}$.

We refer to elements of $\mathcal{O}$ as *ordinals*.

Note that if $\mathcal{O}$ is nonempty, then Axiom (O1) forces the universe $\mathfrak{U}$ to be nonempty as well. So for any ordinal $x \in \mathcal{O}$, we have $\{x\} \in \mathsf{P}_{\mathfrak{U}}\mathcal{O}$ (Lemma 2). Note also that Axiom (O1) is equivalent to its weaker form (the equivalence does not require (O2) and relies on Lemma 3):

(O1′) $\{x\}^> \in \mathsf{P}_{\mathfrak{U}}\mathcal{O}$ for all $x \in \mathcal{O}$.

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Axiom (O2) implies that the mapping $x \mapsto x^+$ is a function $\mathcal{O} \to \mathcal{O}$. We call this function the *successor function* of the ordinal system $\mathcal{O}$. For each $x \in \mathcal{O}$, an element of $\mathcal{O}$ that has the form x^+ is called a *successor ordinal* and the *successor of* x. We call an ordinal that is not a successor ordinal a *limit ordinal*.

Recall that a poset is a *well-ordered set* when each of its nonempty subsets has a smallest element.

THEOREM 11. *A poset* $(\mathcal{O}, \leqslant)$ *is an ordinal system relative to a universe* $\mathfrak{U}$ *if and only if it is a well-ordered set* (*and consecutively, a totally ordered set*) *satisfying* (O1′) *and such that* $X^< \neq \varnothing$ *for all* $X \in \mathsf{P}_{\mathfrak{U}}(\mathcal{O})$.

PROOF. Consider an ordinal system $\mathcal{O}$ relative to a universe $\mathfrak{U}$. Let X be a nonempty subset of $\mathcal{O}$. Then $X^> \in \mathsf{P}_{\mathfrak{U}}\mathcal{O}$ by (O1), and thus $\overset{+}{\bigvee} X^>$ exists by (O2). By Lemma 9(iii), $\overset{+}{\bigvee} X^> = \min X$. This proves the 'only if' part of the theorem. Note that any well-ordered set is totally ordered: having a smallest element of a two-element subset $\{x, y\}$ forces x and y to be comparable. The 'if' part is quite obvious. ⊣

We will use this theorem often without referring to it. One of its consequences is that each nonempty ordinal system has a smallest element. We denote this element by 0. Note that 0 is a limit ordinal. Since by the same theorem an ordinal system is a total order, the properties (L4–7) above apply to an ordinal system. In particular, we get that the successor function is injective. We also get the following:

LEMMA 12. *In an ordinal system, for an ordinal* x *the following conditions are equivalent*:

(i) x *is a limit ordinal.*
(ii) $\{x\}^>$ *is closed under successors.*
(iii) $x = \bigvee\{x\}^>$.

PROOF. We have $x = \overset{+}{\bigvee}\{x\}^>$ for any ordinal x (L4). If x is a limit ordinal then for each $y < x$ we have $y^+ < x$ (L3). So (i) ⇒ (ii). If (ii) holds, by (L1), we get that $\{x\}^>$ does not have a largest element. So $\overset{+}{\bigvee}\{x\}^> = \bigvee\{x\}^>$ (Lemma 9). This gives (ii) ⇒ (iii). Suppose now $x = \bigvee\{x\}^>$. If x were a successor ordinal $x = y^+$, then by (L5), y would be the join of $\{x\}^>$. However, $x \neq y$ by (L1), and therefore, x must be a limit ordinal. Thus, (iii) ⇒ (i). ⊣

The following theorem gives yet another way of thinking about an ordinal system.

THEOREM 13. *A poset* $(\mathcal{O}, \leqslant)$ *is an ordinal system relative to a universe* $\mathfrak{U}$ *if and only if* (O1) *holds along with the following axioms*:

(O2a) *For all* $X \in \mathsf{P}_{\mathfrak{U}}\mathcal{O}$, *the join* $\bigvee X$ *exists.*
(O2b) x^+ *exists for each* $x \in \mathcal{O}$.

PROOF. This can easily be proved using (i) and (ii) of Lemma 9. ⊣

Transfinite induction and recursion are well known for well-ordered sets. We formulate them here (in one particular form, out of many possibilities) in the case of ordinal systems, since we will use them later on in the paper. We have included our own direct proofs, for the sake of completeness, but we do not expect these proofs to have any new arguments that do not already exist in the literature.

THEOREM 14 (transfinite induction). *Let $\mathcal{O}$ be an ordinal system and let $X \subseteq \mathcal{O}$ satisfy the following conditions*:

(I1) $X^+ \subseteq X$;

(I2) *for every limit ordinal x, if $\{x\}^> \subseteq X$ then $x \in X$.*

Then $X = \mathcal{O}$.

PROOF. Since $\mathcal{O}$ is well-ordered, if $\mathcal{O} \setminus X$ is nonempty, then it has a smallest element y. By (I1), y cannot be a successor of any $z < y$ in X. By (I2), it also cannot be a limit ordinal. This is a contradiction, since a limit ordinal is defined as one that is not a successor ordinal. ⊣

Here is one of the immediate consequences of transfinite induction (the proof will require also the total order of an ordinal system, as well as the properties (L3) and (L4)):

COROLLARY 15. *Let $\mathcal{O}$ be an ordinal system relative to a universe $\mathfrak{U}$, and let $X \subseteq \mathcal{O}$ satisfy the following conditions*:

(S1) *X is down-closed in $\mathcal{O}$, i.e., if $x < y \in X$ then $x \in X$, for all $x, y \in \mathcal{O}$;*

(S2) *X is an ordinal system relative to $\mathfrak{U}$ under the restriction of the order of $\mathcal{O}$.*

Then $X = \mathcal{O}$.

THEOREM 16 (transfinite recursion). *Let $\mathcal{O}$ be an ordinal system and let $\mathbb{X} = (X, L, s)$, where X is a set, s is a function $X \to X$, and L is a function $\mathsf{P}_{\mathfrak{U}} X \to X$. Then there exists a unique function $f : \mathcal{O} \to X$ that satisfies the following conditions*:

(R1) *$f(x^+) = s(f(x))$ for any $x \in \mathcal{O}$;*

(R2) *$f(x) = L(\{f(y) \mid y < x\})$ for any limit ordinal x.*

PROOF. For each ordinal $x \in \mathcal{O}$, let F_x be the set consisting of all functions

$$f_x : \{y \in \mathcal{O} \mid y \leqslant x\} \to X$$

that satisfy the following conditions:

(i) $f(y^+) = s(f(y))$ for any successor ordinal $y^+ \leqslant x$, and

(ii) $f(y) = L(\{f(z) \mid z < y\})$ for any limit ordinal $y \leqslant x$.

Note that any function f satisfying (R1–2) must have a subfunction in each set F_x. Also, for any $x \in \mathcal{O}$, a function $f_x \in F_x$ must have a subfunction in each set F_y where $y < x$. We prove by transfinite induction that for each $x \in \mathcal{O}$, the set F_x contains exactly one function f_x.

Successor case: Suppose that for some ordinal x and each $y \leqslant x$ there exists a unique function $f_y \in F_y$. Then $g = \bigcup\{f_x, \{(x^+, s(f_x(x)))\}\}$ satisfies (i) and (ii), and thus $g \in F_{x^+}$.

Now consider any function $g' \in F_{x^+}$. Since g and g' must each have the unique function $f_x \in F_x$ as a subfunction, g and g' are identical on the domain $\{y \mid y \leqslant x\}$. However, since g and g' each satisfies condition (i), we get that $g'(x^+) = s(f_x(x)) = g(x^+)$, and thus $g' = g$ is the unique function in F_{x^+}.

Limit case: Suppose that for some limit ordinal x and each $y < x$ there exists a unique function $f_y \in F_y$. Then, for any two ordinals $y < y' < x$, the function f_y is a subfunction of $f_{y'}$, which is in turn a subfunction of every function in F_x. The relation

$$g = \bigcup\{\{(x, L(\{f_y(y) \mid y < x\}))\}, \bigcup\{f_y \mid y < x\}\}$$

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is then a function over the domain $\{y \mid y \leqslant x\}$ and, moreover, it is easy to see that $g \in F_x$.

Now consider any function $g' \in F_x$. Since for each ordinal $y < x$ the unique function $f_y \in F_y$ is a subfunction of both g and g', we see that they are identical on the domain $\{y \mid y < x\}$, and since they both satisfy condition (ii), the following holds:

$$g(x) = L(\{g(y) \mid y < x\}) = L(\{g'(y) \mid y < x\}) = g'(x).$$

We can conclude that $g' = g$ is the unique function in F_x.

We showed that for each ordinal x, exactly one function $f_x \in F_x$ exists. Construct a function $f : \mathcal{O} \to X$ as follows: for each ordinal x,

$$f(x) = f_x(x).$$

It satisfies (R1–2) because each f_x satisfies (i) and (ii). Since any function satisfying (R1–2) must have a subfunction in each set F_x, we can conclude that f is the unique function satisfying (R1–2). ⊣

§4. Concrete ordinals. For a universe $\mathfrak{U}$, define a $\mathfrak{U}$-*ordinal* to be a transitive set that belongs to the universe $\mathfrak{U}$ and is a well-ordered set under the relation

$$x \underline{\in} y \quad \Leftrightarrow \quad [x \in y] \vee [x = y].$$

Thus, a $\mathfrak{U}$-ordinal is nothing but a usual von Neumann ordinal number [15] that happens to be an element of $\mathfrak{U}$. In other words, it is a von Neumann ordinal number *internal* to the universe $\mathfrak{U}$. Thus, at least one direction in the following theorem is well known. We include a full proof for completeness.

THEOREM 17. *A set O is the set of all $\mathfrak{U}$-ordinals if and only if the following conditions hold*:

(a) *$O \subseteq \mathfrak{U}$, and if $\varnothing \in \mathfrak{U}$, then $\varnothing \in O$;*
(b) *O is a transitive set;*
(c) *O is an ordinal system relative to $\mathfrak{U}$ under the relation $\underline{\in}$, with the corresponding strict ordering given by $\in$.*

When these conditions hold, the incremented join of $X \in \mathsf{P}_{\mathfrak{U}}O$ is given by $\overset{+}{\bigvee} X = \bigcup\{X, \bigcup X\}$.

PROOF. We prove both directions. The proof of the last statement in the theorem is included in Step 1(c).

STEP 1. *We prove that if O is the set of all $\mathfrak{U}$-ordinals, then (a–c) hold. While proving (c), we show that the incremented join of a set of $\mathfrak{U}$-ordinals $X \in \mathsf{P}_{\mathfrak{U}}O$ is given by $\overset{+}{\bigvee} X = \bigcup\{X, \bigcup X\}$.*

Let O be the set of all $\mathfrak{U}$-ordinals.

(a) This follows easily from the definition of a $\mathfrak{U}$-ordinal as a transitive element of $\mathfrak{U}$.
(b) To prove transitivity, let $y \in x \in O$. We want to show that y is a $\mathfrak{U}$-ordinal (i.e., an element of $\mathfrak{U}$ that is transitive and well-ordered by $\underline{\in}$). Since $x \in \mathfrak{U}$, we have $y \in \mathfrak{U}$ by (U1). Since y is a subset of the well-ordered set x, y must also be well-ordered.

Next, we must prove that y is transitive. Let $z \in y$. Then $z \in x$, since x is transitive. We want to show that $z \subseteq y$, so suppose $t \in z$. Then $t \in x$, by transitivity of x. By the fact that $\underline{\in}$ is a total order on x, we must have $t \underline{\in} y$. We cannot have $t = y$, since $t \in z \in y$, and so $t \in y$.

(c) We use the characterization of an ordinal system given in Theorem 11(b) as a well-ordered set satisfying (O1′), in which $X^{<} \neq \varnothing$ holds for all $X \in \mathsf{P}_{\mathfrak{U}}O$.

Notice that the relation $\underline{\in}$ is a partial order on O: reflexivity is obvious, while transitivity follows from each element of O being a transitive set. By the axiom of foundation, $\in$ is the strict ordering on O corresponding to $\underline{\in}$.

We can easily prove, as follows, that O satisfies (O1′).

O1′. Consider $x \in O$. Then we have:

$$\{x\}^{>} = \{y \in O \mid y \in x\} \subseteq x.$$

Since x is a $\mathfrak{U}$-ordinal, $x \in \mathsf{P}_{\mathfrak{U}}O$ and so $\{x\}^{>} \in \mathsf{P}_{\mathfrak{U}}O$ (Lemma 3). Note that by (b) we actually have $x = \{x\}^{>}$, although we did not need this to establish (O1′).

Before we prove that O is well-ordered, we will first establish the following two facts.

FACT 1. $x \cap y = \mathsf{min}(y \setminus x)$ *for any two* $\mathfrak{U}$*-ordinals* x *and* y *such that* $y \setminus x \neq \varnothing$.

Let $a \in x \cap y$. Then $a \notin y \setminus x$, and hence $a \neq \mathsf{min}(y \setminus x)$. By the well-ordering of y, we then get that either $a \in \mathsf{min}(y \setminus x)$ or $\mathsf{min}(y \setminus x) \in a$. By transitivity of x and the fact that $\mathsf{min}(y \setminus x) \notin x$, the second option is excluded. So $a \in \mathsf{min}(y \setminus x)$. This shows that $x \cap y \subseteq \mathsf{min}(y \setminus x)$.

Now suppose $a \in \mathsf{min}(y \setminus x)$. Then $a \in y$ by transitivity of y. If $a \notin x$, then $a \in y \setminus x$, which would give $\mathsf{min}(y \setminus x) \underline{\in} a$, which is clearly impossible. So $a \in x$. This proves $\mathsf{min}(y \setminus x) \subseteq x \cap y$, and hence $\mathsf{min}(y \setminus x) = x \cap y$.

From Fact 1 we will prove the following.

FACT 2. $x \underline{\in} y \Leftrightarrow x \subseteq y$ *for any two* $\mathfrak{U}$*-ordinals* x *and* y.

To see why this holds, consider $\mathfrak{U}$-ordinals x and y such that $x \subseteq y$. Then $x = x \cap y$, and either $y \setminus x$ is empty, in which case $x = y$, or

$$x = x \cap y = \mathsf{min}(y \setminus x) \in y,$$

by Fact 1. The other direction follows trivially from transitivity of y.

We now show that O is well-ordered under the relation $\underline{\in}$. First, we prove that O is totally ordered under $\underline{\in}$. Consider $x, y \in O$. If $x \neq y$, then either $x \setminus y$ or $y \setminus x$ is nonempty. Without loss of generality, suppose that $y \setminus x$ is nonempty. By Fact 1, $\mathsf{min}(y \setminus x) \subseteq x$. Then, by Fact 2, either $\mathsf{min}(y \setminus x) \in x$ or $\mathsf{min}(y \setminus x) = x$, and since $\mathsf{min}(y \setminus x) \in y \setminus x$, we can conclude that $x = \mathsf{min}(y \setminus x)$, and thus $x \in y$.

Now consider a nonempty $Y \subseteq O$, and any $y \in Y$. If $y \cap Y = \varnothing$, then for all $x \in Y$ we have $x \notin y$ and thus $y = \mathsf{min}\, Y$ by total ordering. If $y \cap Y \neq \varnothing$, then $\mathsf{min}(y \cap Y)$ exists, since $y \cap Y \subseteq y$. For all $x \in Y \setminus y$, it holds that $x \notin y$, and thus $y \subseteq x$ (by total ordering and Fact 2), which in turn implies $\mathsf{min}(y \cap Y) \in x$. We can conclude that $\mathsf{min}(y \cap Y) \in x$ for all $x \in Y$, and thus $\mathsf{min}\, Y = \mathsf{min}(y \cap Y)$.

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We will now complete the proof of (c) by showing that $X^<$ is nonempty for all $X \in \mathsf{P}_{\mathfrak{U}} O$, and simultaneously prove that $\mathop{\forall\!\!\!\!\!+} X = \bigcup\{X, \bigcup X\}$ for all $X \in \mathsf{P}_{\mathfrak{U}} O$.

Consider $X \in \mathsf{P}_{\mathfrak{U}} O$. By (a) and Lemma 7, $X \in \mathfrak{U}$. Then $\bigcup\{X, \bigcup X\} \in \mathfrak{U}$. To show that $\bigcup\{X, \bigcup X\}$ is a $\mathfrak{U}$-ordinal, we need to prove that it is a transitive set well-ordered under the relation $\underline{\in}$.

Since each element of X is transitive, $\bigcup\bigcup X \subseteq \bigcup X$ holds, which implies transitivity of $\bigcup\{X, \bigcup X\}$:

$$\begin{aligned}\bigcup\left(\bigcup\{X, \bigcup X\}\right) &= \bigcup\{\bigcup X, \bigcup\bigcup X\}\\ &= \bigcup X\\ &\subseteq \bigcup\{X, \bigcup X\}.\end{aligned}$$

Since O is transitive by (b), and $X \subseteq O$, each element of $\bigcup X$ is a $\mathfrak{U}$-ordinal, and thus each element of $\bigcup\{X, \bigcup X\}$ is a $\mathfrak{U}$-ordinal. Then the fact that $\bigcup\{X, \bigcup X\}$ is well-ordered under the relation $\underline{\in}$ follows from the fact that O is well-ordered under the same relation, as we have already proven.

Then, since $\bigcup\{X, \bigcup X\}$ is transitive and well-ordered under $\underline{\in}$, we have that $\bigcup\{X, \bigcup X\}$ is a $\mathfrak{U}$-ordinal.

Now, let us remark that for any $X \in \mathsf{P}_{\mathfrak{U}} O$,

$$\begin{aligned}\mathop{\forall\!\!\!\!\!+} X &= \min X^<\\ &= \min\{y \in O \mid \forall_{x \in X}\ x \in y\}\\ &= \min\{y \in O \mid X \subseteq y\}.\end{aligned}$$

Notice that $\bigcup\{X, \bigcup X\} \in \min X^<$, and consider any $y \in X^<$. Then $X \subseteq y$, and it is not hard to see that $\bigcup\{X, \bigcup X\} \subseteq y$, since $\bigcup X \subseteq y$ by transitivity of y. Then, by Fact 2, $\bigcup\{X, \bigcup X\} \underline{\in} y$, and so

$$\bigcup\{X, \bigcup X\} = \min X^< = \mathop{\forall\!\!\!\!\!+} X.$$

STEP 2. *We prove that if* (a)–(c) *hold, then O is the set of all $\mathfrak{U}$-ordinals.*

Let O be any set satisfying (a)–(c).

First we prove that every element of O is a $\mathfrak{U}$-ordinal, i.e., a transitive element of $\mathfrak{U}$ that is well-ordered by the relation $\underline{\in}$.

Let $x \in O$. By (c), the set O is ordered under the relation $\underline{\in}$. In this ordered set,

$$\{x\}^> = \{y \in O \mid y \in x\} = x \cap O.$$

By (b), $x \subseteq O$, and so $x = \{x\}^>$. By (a) and (c), $x \in \mathfrak{U}$. Furthermore, for any $y \in x$, the following holds:

$$y = \{y\}^> \subseteq \{x\}^> = x.$$

This shows that x is transitive.

Since O is well-ordered under $\underline{\in}$ (by (c) and Theorem 11) and $x \subseteq O$, we know x is also well-ordered under the same relation. Thus, every element of O is a $\mathfrak{U}$-ordinal.

Now we need to establish that every $\mathfrak{U}$-ordinal is in O. We already proved that the set $\mathcal{O}$ of all $\mathfrak{U}$-ordinals is an ordinal system, and since O is a set of $\mathfrak{U}$-ordinals, $O \subseteq \mathcal{O}$. The equality $O = \mathcal{O}$ then follows from Corollary 15. ⊣

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§5. A Dedekind-style axiomatization. A *limit-successor system* is a triple $X = (X, L, s)$, where X is a set, L is a partial function $L \colon \mathsf{P}X \to X$ called the *limit function* and s is a function $s \colon X \to X$ called the *successor function*. A *successor-closed* subset of a limit-successor system X is a subset I of X such that $sI \subseteq I$. For a subset $I \subseteq X$, write $s^{-1}I$ to denote

$$s^{-1}I = \{x \in X \mid s(x) \in I\}$$

and $L^{-1}I$ to denote

$$L^{-1}I = \{A \in \mathsf{dom}L \mid L(A) \in I\}.$$

A subset $I \subseteq X$ is said to be *closed* when

$$s^{-1}I \subseteq I \text{ and } \bigcup L^{-1}I \subseteq I.$$

It is not difficult to see that closed subsets form a topology on X; in fact, an Alexandrov topology (arbitrary unions of closed subset are closed). We denote the closure of a subset I in this topology by $\overline{I}$. Recall that the corresponding 'specialization preorder' given by

$$x \leqslant y \quad \Leftrightarrow \quad x \in \overline{\{y\}}$$

is a preorder (as it is for any Alexandrov topology) and that $x \in \overline{I}$ if and only if $x \leqslant y$ for some $y \in I$.

We abbreviate the operator s^{-1} composed with itself m times as s^{-m}, with the $m = 0$ case giving the identity operator. We write $s^{-\infty}$ for the operator defined by

$$s^{-\infty}I = \bigcup\{s^{-m}I \mid m \in \mathbb{N}\}$$

and $\bigcup L^{-1}$ for the operator $I \mapsto \bigcup L^{-1}I$. One can show that the closure of a subset $I \subseteq X$ can be computed as

$$\overline{I} = \bigcup\{s^{-\infty}\left[\bigcup L^{-1}s^{-\infty}\right]^k I \mid k \in \mathbb{N}\}.$$

This means that the specialization preorder 'breaks up' into two relations $\leqslant_s$ and $\prec_L$, each determined by s and L alone, as explained in what follows. These relations are defined by:

$$\begin{aligned} x \leqslant_s y \quad &\Leftrightarrow \quad \exists_{m \in \mathbb{N}}[s^m(x) = y] && (\Leftrightarrow x \in s^{-\infty}\{y\}),\\ x \prec_L y \quad &\Leftrightarrow \quad \exists_I[[x \in I] \wedge [L(I) = y]] && (\Leftrightarrow x \in \bigcup L^{-1}\{y\}). \end{aligned}$$

We then have $x \leqslant y$ if and only if

$$x = z_0 \leqslant_s z_1 \prec_L z_2 \leqslant_s z_3 \prec_L \dots z_{2k} \leqslant_s y$$

for some $z_0, \dots, z_{2k} \in X$, where k can be any natural number $k \geqslant 0$. Note that $\leqslant_s$ is both reflexive and transitive, although the same cannot be claimed for $\prec_L$.

Definition 18. Given a universe $\mathfrak{U}$, a $\mathfrak{U}$-*counting system* is a limit-successor system (X, L, s) satisfying the following conditions:

(C1) The domain of L is the set of all successor-closed subsets $I \in \mathsf{P}_{\mathfrak{U}}X$.
(C2) If I and J belong to the domain of L and $\overline{I} = \overline{J}$, then $L(I) = L(J)$.

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The structure above is the one that will be used for formulating the universal property of an ordinal system at the end of this section. Before that, we give a characterization of ordinal systems as counting systems having further internal properties.

LEMMA 19. *Let* $\mathfrak{U}$ *be a universe and let* (X, L, s) *be a* $\mathfrak{U}$*-counting system. Then* $\prec_L$ *is transitive and furthermore,*

$$x \leqslant_s y \prec_L z \quad \Rightarrow \quad x \prec_L z$$

for all $x, y, z \in X$.

PROOF. To prove transitivity of $\prec_L$, suppose $x \prec_L y$ and $y \prec_L z$. Then $x \in I$, $L(I) = y$, $y \in J$ and $L(J) = z$ for some successor-closed $I, J \in \mathsf{P}_{\mathfrak{U}}X$. Then the union $\bigcup\{I, J\} \in \mathsf{P}_{\mathfrak{U}}X$ (Lemma 4) is also successor-closed, and hence it belongs to the domain of L by (C1). Since $L(I) \in J$, we get that $I \subseteq \overline{J}$. This implies that $\overline{\bigcup\{I, J\}} = \overline{J}$. By (C2), $L(\bigcup\{I, J\}) = L(J)$. Having $x \in \bigcup\{I, J\}$ and $L(\bigcup\{I, J\}) = z$ means that $x \prec_L z$. This completes the proof of transitivity.

To prove the second property, suppose $x \leqslant_s y \prec_L z$. Then $s^m(x) = y$ and $y \in J$ where $L(J) = z$, for some $m \in \mathbb{N}$ and some successor-closed $J \in \mathsf{P}_{\mathfrak{U}}X$. Here we also expand J, this time adding to it all elements of the form $s^k(x)$, where $k \in \{0, \dots, m - 1\}$. The resulting set

$$K = \bigcup\{\{x, s(x), \dots, s^{m-1}(x)\}, J\}$$

is clearly successor-closed and belongs to $\mathsf{P}_{\mathfrak{U}}X$ (Lemmas 2 and 4). Then, since $\overline{K} = \overline{J}$, we get that $L(K) = L(J)$ by (C2). This implies $x \prec_L z$. ⊣

This lemma gives that in a $\mathfrak{U}$-counting system (X, L, s), for any $x, z \in X$ we have

$$x \leqslant z \quad \Leftrightarrow \quad [x \leqslant_s z] \vee \exists_y [x \prec_L y \leqslant_s z]$$

as a result of which the previous computation of the closure of a subset $I \subseteq X$ simplifies to

$$\overline{I} = \bigcup\{s^{-\infty}I, \bigcup L^{-1}s^{-\infty}I\}.$$

Note that $x \leqslant z$ now intuitively means that z is a finite successor of x, or z is a finite successor of the limit of a set containing x. Using this characterization of $\leqslant$, it is also easy to confirm the following two facts:

- s is increasing under $\leqslant$
- $x < y \Rightarrow s(x) \leqslant y$, for all $x, y \in X$.

THEOREM 20. *The specialization preorder of a* $\mathfrak{U}$*-counting system* (X, L, s) *makes* X *an ordinal system relative to* $\mathfrak{U}$*, provided the following conditions hold*:

(C3) $s^{-1}\{L(I)\} = \varnothing = I \cap \{s(L(I))\}$ *for all* I *such that* $L(I)$ *is defined.*
(C4) s *is injective and* L *has the property that if* $L(I) = L(J)$ *then* $\overline{I} = \overline{J}$.
(C5) $J = X$ *for any successor-closed set* J *having the property that* $I \subseteq J \Rightarrow L(I) \in J$ *every time* $L(I)$ *is defined.*

When these conditions hold, s *is the successor function of the ordinal system and* $L(I) = \bigvee I = \overset{+}{\bigvee} I$ *whenever* $L(I)$ *is defined*; *moreover, the limit ordinals are exactly the elements of* X *of the form* $L(I)$*. Furthermore, the closure of* $I \in \mathsf{P}_{\mathfrak{U}}X$ *is given by*

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$\overline{I} = \{\forall I\}^{>}$. *Finally, any ordinal system relative to* $\mathfrak{U}$ *arises this way from a* (*unique*) $\mathfrak{U}$-*counting system satisfying* (C3–5).

Before proving the theorem, let us illustrate axioms (C1–5) in the case when $\mathfrak{U}$ is the universe of hereditarily finite sets. Then L is only defined on finite successor-closed sets. Injectivity of s in (C4) forces every element x of such set to have the property $\exists_{m\in\mathbb{N}\setminus\{0\}}[s^m(x) = x]$. At the same time, by (C3), such x cannot lie in the image of L. So

$$J = \{x \in X \mid \forall_{m\in\mathbb{N}\setminus\{0\}}[s^m(x) \neq x]\}$$

has the second property in (C5). Moreover, by injectivity of s again, J is also successor-closed. Then, by (C5), $J = X$ and so L can only be defined on the empty set. With this provision, the triple (X, L, s) becomes a triple $(X, 0, s)$ where 0 is the unique element in the image of L, $0 = L(\varnothing)$. The axioms (C1–2) then trivially hold, while (C3–5) take the form of the axioms of Dedekind for a natural number system:

- The first equality in (C3) states that 0 does not belong to the image of s, while the second equality holds trivially.
- (C4) just states that s is injective.
- (C5) becomes the usual principle of mathematical induction.

Proof of Theorem 20. Suppose conditions (C1–5) hold.

Step 1. *As a first step, we prove that the specialization preorder is antisymmetric, i.e., that it is a partial order.*

For this, we first show that $\prec_L$ is 'antireflexive': it is impossible to have $x \prec_L x$. Indeed, suppose $x \in I$ and $L(I) = x$. Since I is successor-closed by (C1), $s(x) \in I$. But then $s(x) \in I \cap \{s(L(I))\}$, which is impossible by (C3).

Next, we show antisymmetry of $\leqslant_s$. Suppose $x \leqslant_s z \leqslant_s x$ and $x \neq z$. Then we get that $s^k(x) = x$ for $k > 1$. We will now show that this is not possible. In fact, we establish a slightly stronger property, which will be useful later on as well:

Property 1. $s^k(x) \neq x$ *for all* $x \in X$ *and all natural numbers* $k > 0$.

Actually, we have already established this property in the remark after the theorem. Here is a more detailed argument. Consider the set

$$J = \{x \in X \mid \forall_{k>0}[s^k(x) \neq x]\}.$$

We will use (C5) to show that $J = X$.

First, we show that J is successor-closed. Let $y \in J$, and suppose $s^k(s(y)) = s(y)$ for some $k > 0$. Then, by injectivity of s (which is required in (C4)), $s^{k-1}(s(y)) = y$, which is impossible. So $s(y) \in J$, showing that J is successor-closed.

Now let $I \in \mathsf{P}_{\mathfrak{U}}X$ be a successor-closed subset of J (by (C1), I is such if and only if $L(I)$ is defined). Then $L(I) \neq s^k(L(I))$ for all $k > 0$ by the first equality in (C3). Thus $L(I) \in J$, and we can apply (C5) to get $J = X$, as desired. Antisymmetry of $\leqslant_s$ has thus been established.

We are now ready to prove the antisymmetry of $\leqslant$. Suppose $x \leqslant z$ and $z \leqslant x$. There are four cases to consider:

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CASE 1. $(x \leqslant_s z \leqslant_s x)$. *Then $x = z$ by antisymmetry of $\leqslant_s$.*

CASE 2. $(x \prec_L y \leqslant_s z \leqslant_s x$ for some $y)$. *Then*

$$\begin{aligned} & y \leqslant_s z \leqslant_s x \prec_L y & \\ \Rightarrow\quad & y \leqslant_s x \prec_L y & \text{(by transitivity of } \leqslant_s) \\ \Rightarrow\quad & y \prec_L y, & \text{(by Lemma 19)} \end{aligned}$$

which we have shown to be impossible.

CASE 3. $(x \leqslant_s z \prec_L y \leqslant_s x$ for some $y)$. *Then, similarly, we get the impossible*

$$\begin{aligned} & y \leqslant_s x \leqslant_s z \prec_L y & \\ \Rightarrow\quad & y \leqslant_s z \prec_L y & \text{(by transitivity of } \leqslant_s) \\ \Rightarrow\quad & y \prec_L y. & \text{(by Lemma 19)} \end{aligned}$$

CASE 4. $(x \prec_L y \leqslant_s z \prec_L y' \leqslant_s x$ for some $y, y')$. *In this case, too, we get the impossible*

$$\begin{aligned} & y \leqslant_s z \prec_L y' \leqslant_s x \prec_L y & \\ \Rightarrow\quad & y \prec_L y' \prec_L y & \text{(by Lemma 19)} \\ \Rightarrow\quad & y \prec_L y. & \text{(by transitivity of } \prec_L \text{ from Lemma 19)} \end{aligned}$$

We have thus shown that the specialization preorder is antisymmetric. We will now establish the following property, which will be useful later on.

PROPERTY 2. *If $x < s(y)$ then $x \leqslant y$, for all $x, y \in X$.*

Suppose $x < s(y)$. Again, we have two cases:

CASE 1. $(x \leqslant_s s(y))$. *This, together with $x \neq s(y)$, gives $x \leqslant y$ by injectivity of s from (C4).*

CASE 2. $(x \prec_L x' \leqslant_s s(y)$ for some $x' \in X)$. *Since $x' \neq s(y)$ by the first equality in (C3), by injectivity of s (from (C4)), we must have $x' \leqslant_s y$. This gives us $x \leqslant y$.*

We get $x \leqslant y$ in both cases.

STEP 2. *We prove that the specialization preorder is a total order.*

We will prove this by simultaneously establishing the following:

PROPERTY 3. *Let $y \in X$, and let $I \subseteq X$. If $x < y$ for all $x \in I$ and $L(I)$ is defined, then $L(I) \leqslant y$.*

Let

$$J = \{x \in X \mid \forall_{y \in X}[y \leqslant x \vee x \leqslant y]\}.$$

We will use (C5) to show that $J = X$. For this, we first prove that J is successor-closed. Let $x \in J$, and consider any $y \in X$. Since $x \leqslant s(x)$, if $y \leqslant x$, then $y \leqslant s(x)$. If $x < y$, then $s(x) \leqslant y$, as remarked after Lemma 19. This proves that J is successor-closed.

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Now consider $L(I)$, where $I \subseteq J$. To prove that $L(I) \in J$, we proceed as follows. Let $x \in X$. If $x \leqslant y$ for at least one $y \in I$, then $x \leqslant L(I)$, since $y \prec_L L(I)$. Thus, it suffices to prove that the set

$$K_I = \{x \in X \mid [\forall_{y \in I}\ y < x] \Rightarrow [L(I) \leqslant x]\},$$

is the entire $K_I = X$. This we prove using (C5).

First, we show that K_I is successor-closed. Suppose $x \in K_I$. If $y < s(x)$ for all $y \in I$, then by Property 2, $y \leqslant x$ for all $y \in I$. If $y = x$ for some $y \in I$, then $x \in I$, and, since I is successor-closed by (C1), we will have $s(x) \in I$, which will violate the assumption that $y < s(x)$ for all $y \in I$. So we get that $y < x$ for all $y \in I$. Then $L(I) \leqslant x$, since $x \in K_I$. This implies $L(I) \leqslant s(x)$, thus proving that K_I is successor-closed.

Now let $H \subseteq K_I$ be such that $L(H)$ is defined. Suppose $y < L(H)$ for all $y \in I$. From the first equality in (C3), we get that $y \prec_L L(H)$ for each $y \in I$. So for each $y \in I$, there exists G_y such that $y \in G_y$ and $L(G_y) = L(H)$. This implies that $\overline{G_y} = \overline{H}$ for each $y \in I$ (by (C4)), and so $I \subseteq \overline{H}$. Since $I \subseteq J$, each element of H is comparable with each element of I. We consider two cases.

Case 1. (for every $h \in H$, there exists $y \in I$ such that $h \leqslant y$). *Then $H \subseteq \overline{I}$. This would then give $\overline{I} = \overline{H}$, and so by (C2), $L(I) = L(H)$, showing that $L(I) \leqslant L(H)$, as desired.*

Case 2. (there exists $h \in H$ such that $y < h$ for every $y \in I$). *Since $H \subseteq K_I$, this gives us $L(I) \leqslant h$. This, together with $h \prec_L L(H)$, will give $L(I) \leqslant L(H)$.*

In both cases, $L(I) \leqslant L(H)$. We have thus shown that K_I has the required properties for us to apply (C5) to conclude that $K_I = X$. This, then, shows that $L(I) \in J$, and so J has the required properties to conclude that $J = X$. The proof that the specialization preorder is a total order is now complete. At the same time, since $J = X$, and for each $I \subseteq J$ such that $L(I)$ is defined, $K_I = X$, we have also established Property 3.

Step 3. *We now show that $L(I) = \bigvee I = \overset{+}{\bigvee} I$ whenever $L(I)$ is defined, and that $s(x) = x^+$ for all $x \in X$.*

Properties 1 and 3 show that $L(I)$ is the join of I for any I such that $L(I)$ is defined. Indeed, if $x \leqslant y$ for all $x \in I$, then for each $x \in I$, we also have $s(x) \leqslant y$. Since $x < s(x)$, as clearly $x \leqslant s(x)$ and by Property 1, $x \neq s(x)$, we get that $x < y$ for all $x \in I$. Then, by Property 3, $L(I) \leqslant y$, thus showing that $L(I)$ is a join of I.

Furthermore, when $L(I)$ is defined, I is successor-closed and so it cannot have a largest element, by Property 1. Then the join $L(I)$ of I must also be the incremented join of I (Lemma 9).

Finally, the property $x < s(x)$, together with the fact that $s(x) \leqslant y$ whenever $x < y$, implies that $s(x) = x^+$ for each $x \in X$. Thus, once we prove that X is an ordinal system under the specialization preorder, we have that s is its successor function and L is given by the join.

Step 4. *We prove that X is an ordinal system under the specialization preorder, where limit ordinals are exactly the elements of the form $L(I)$.*

Consider the set

$$J = \{x \in X \mid \{x\}^{>} \in \mathsf{P}_{\mathfrak{U}}X\}.$$

If $x \in J$, then by Property 2, $\{s(x)\}^{>} = \bigcup\{\{x\}^{>}, \{x\}\} \in \mathsf{P}_{\mathfrak{U}}X$ (Lemmas 2 and 4), and so $s(x) \in J$. For any $x \in J$, we therefore have

$$\begin{aligned}\overline{\{x\}} &= \{y \in X \mid y \leqslant x\}\\ &= \{y \in X \mid y < s(x)\} && \text{(by Property 2)}\\ &= \{s(x)\}^{>} \in \mathsf{P}_{\mathfrak{U}}X.\end{aligned}$$

Suppose $I \subseteq J$ is such that $L(I)$ is defined, and define a function $f : I \to \mathsf{P}_{\mathfrak{U}}X$ by $f(x) = \overline{\{x\}}$. Then, by Lemma 4,

$$\bigcup fI = \bigcup\{\overline{\{x\}} \mid x \in I\} \in \mathsf{P}_{\mathfrak{U}}X.$$

Notice that $\overline{I} \subseteq \{\bigvee I\}^{>}$, since $\overline{I}$ is the down-closure of I under the specialization preorder.

Then, note that $\overline{\bigcup I} = \bigcup\{\overline{x} \mid x \in I\}$ holds for all $I \subseteq X$, as is true in any Alexandrov topology, and thus

$$\begin{aligned}\{L(I)\}^{>} &= \left\{\bigvee I\right\}^{>} && \text{(by Step 3)}\\ &= \overline{I} && \text{(since } X \text{ is totally ordered)}\\ &= \overline{\bigcup\{\{x\} \mid x \in I\}}\\ &= \bigcup\{\overline{\{x\}} \mid x \in I\} \in \mathsf{P}_{\mathfrak{U}}X,\end{aligned}$$

and so $L(I) \in J$. By (C5), $J = X$, and thus (O1′) holds.

To prove that X is an ordinal system under the specialization preorder, it remains to show that it satisfies (O2), i.e., that for any $Y \in \mathsf{P}_{\mathfrak{U}}X$, the incremented join of Y exists in X. If Y has a largest element, then the successor of that element is the incremented join of Y (Lemma 9). If $\mathfrak{U}$ does not contain an infinite set, then Y is finite, and so it has a largest element.

Now consider $Y \in \mathsf{P}_{\mathfrak{U}}X$ that has no largest element, with $\mathfrak{U}$ containing an infinite set. We define:

$$s^{\infty}Y = \{s^{n}(x) \mid [x \in Y] \wedge [n \in \mathbb{N}]\}.$$

This is, of course, the closure of Y under s. Consider a function $f: Y \times \mathbb{N} \to X$ defined by $f(x,n) = s^{n}(x)$. Then, by Lemmas 5 and 6,

$$\begin{aligned}s^{\infty}Y &= \{s^{n}(x) \mid [x \in Y] \wedge [n \in \mathbb{N}]\}\\ &= \{f(x,n) \mid (x,n) \in Y \times \mathbb{N}\}\\ &= f(Y \times \mathbb{N}) \in \mathsf{P}_{\mathfrak{U}}X.\end{aligned}$$

Thus $L(s^{\infty}Y)$ is defined, by (C1). Since $Y \subseteq s^{\infty}Y$, it holds that $y < L(s^{\infty}Y)$ for all $y \in Y$. Let $y \in Y$. We prove by induction on n that for each $n \in \mathbb{N}$, we have $s^{n}(y) < z$ for some $z \in Y$.

Base case. *If $n = 0$, this follows from the fact that Y does not have a largest element.*

INDUCTION STEP. *Suppose $s^n(y) < z$ for some $z \in Y$. Then $s^{n+1}(y) = s(s^n(y)) \leqslant z$. Since z cannot be the largest element of Y, we must have $z < z'$ for some $z' \in Y$. Then $s^{n+1}(y) < z'$.*

What we have shown implies that the incremented join $L(s^\infty Y)$ of $s^\infty Y$ is also the incremented join of Y.

We have thus proven that the specialization preorder of a $\mathfrak{U}$-counting system satisfying (C3–5) makes it an ordinal system relative to $\mathfrak{U}$, with s as its successor function, and L given equivalently by join and by incremented join. This also establishes that, if an ordinal system relative to $\mathfrak{U}$ arises this way from a $\mathfrak{U}$-counting system satisfying (C3–5), then this $\mathfrak{U}$-counting system is unique.

We now prove the existence of such a $\mathfrak{U}$-counting system. Actually, before doing that, note that by (C3), no element of X of the form $L(I)$ can be a successor ordinal, and so it must be a limit ordinal. Conversely, for a limit ordinal x we have $x = L(\{x\}^>)$ (Lemma 12). This shows that limit ordinals are precisely the ordinals of the form $L(I)$.

For an ordinal system $\mathcal{O}$ relative to $\mathfrak{U}$, consider the limit-successor system $(\mathcal{O}, \bigvee, \_^+)$, where $\bigvee$ is the usual join restricted to the domain required by (C1) (i.e., successor-closed elements of $\mathsf{P}_\mathfrak{U} X$).

STEP 5. *We show that* (C1–5) *hold for the limit-successor system* $(\mathcal{O}, \bigvee, \_^+)$ *and that the corresponding specialization preorder matches with the order of $\mathcal{O}$. In this step we also show that $\overline{I} = \{\overset{+}{\bigvee} I\}^>$ holds for each $I \in \mathsf{P}_\mathfrak{U}\mathcal{O}$.*

By Theorem 13, L is indeed defined over the entire domain required in (C1). To prove (C2), first we establish that

$$\overline{I} = \{\overset{+}{\bigvee} I\}^>$$

for each $I \in \mathsf{P}_\mathfrak{U}\mathcal{O}$. It is easy to see that $\{\overset{+}{\bigvee} I\}^>$ is closed, so $\overline{I} \subseteq \{\overset{+}{\bigvee} I\}^>$. To show $\{\overset{+}{\bigvee} I\}^> \subseteq \overline{I}$, let $x \in \{\overset{+}{\bigvee} I\}^>$. We have well-ordering and hence total order by Theorem 11. Then $x < \overset{+}{\bigvee} I$ and so $x \leqslant y \in I$ for some y. Consider

$$y' = \min\{y \in \overline{I} \mid x \leqslant y\}.$$

We consider two cases:

CASE 1. (y' is a successor ordinal). *Then $y' = y''^+$ for some ordinal y''. Since $y' \in \overline{I}$, we must have $y'' \in \overline{I}$. Then $y'' < x$ and so $y' \leqslant x$ by* (L3). *This gives $x = y'$ and so $x \in \overline{I}$.*

CASE 2. (y' is a limit ordinal). *Then $\{y'\}^>$ is successor-closed. Furthermore, we have*

$$y' = \overset{+}{\bigvee}\{y'\}^> = \bigvee(\{y'\}^>)^+ = \bigvee\{y'\}^>.$$

By closure of $\overline{I}$, we get $\{y'\}^> \subseteq \overline{I}$. Since $y < x$ for all $y < y'$, we get $y' \leqslant x$. This gives $x = y'$ and so $x \in \overline{I}$.

We have thus established that the equality $\overline{I} = \{\overset{+}{\bigvee} I\}^>$ holds for each $I \in \mathsf{P}_\mathfrak{U}\mathcal{O}$. From this it follows that the specialization preorder matches with the order of $\mathcal{O}$. We then get that (C2) holds by the fact that if down-closures of two subsets of a poset are equal, then so are their joins. Thus $(\mathcal{O}, \bigvee, \_^+)$ is a $\mathfrak{U}$-counting system.

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It remains to show that (C3–5) hold. Consider a successor-closed $I \in \mathsf{P}_{\mathfrak{U}}\mathcal{O}$. I has no maximum element thanks to (L1) and thus, $\bigvee I = \overset{+}{\bigvee} I$ by Lemma 9. By the same lemma, $\overset{+}{\bigvee} I$ cannot be a successor if I has no maximum. Thus, $(\_^{+})^{-1}\{\bigvee I\} = \varnothing$, which is the first part of (C3). Since $(\bigvee I)^{+} > \bigvee I \geqslant x$ for each $x \in I$, we have that $(\bigvee I)^{+} \notin I$, which means the second part of (C3) also holds, i.e., $I \cap \{(\bigvee I)^{+}\} = \varnothing$.

We already know that $\_^{+}$ is injective, so to see that (C4) holds, consider another successor-closed $J \in \mathsf{P}_{\mathfrak{U}}\mathcal{O}$. If $\bigvee I = \bigvee J$, then

$$\overline{I} = \{\overset{+}{\bigvee} I\}^{>} = \{\bigvee I\}^{>} = \{\bigvee J\}^{>} = \{\overset{+}{\bigvee} J\}^{>} = \overline{J}.$$

Thus (C4) holds. Finally, consider a successor-closed subset J of $\mathcal{O}$ where $\bigvee I \in J$ for all successor-closed subsets I of J such that $I \in \mathsf{P}_{\mathfrak{U}}\mathcal{O}$. Then $J = \mathcal{O}$ if it satisfies (I1) and (I2) in our formulation of transfinite induction. We check both:

I1. $J^{+} \subseteq J$ follows from the fact that J is successor-closed.

I2. Let x be a limit ordinal such that $\{x\}^{>} \subseteq J$. Then $x = \bigvee\{x\}^{>} \in J$ by Lemma 12.

Since both of these conditions hold, we can conclude that $J = \mathcal{O}$, and thus (C5) holds. This completes the proof. ⊣

Given two $\mathfrak{U}$-counting systems (X_1, L_1, s_1) and (X_2, L_2, s_2), a function $f\colon X_1 \to X_2$ that preserves the successor function $(f s_1 = s_2 f)$ automatically preserves successor-closed subsets, so for any successor-closed $I \in \mathsf{P}_{\mathfrak{U}}X_1$, both sides of the equality

$$L_2(fI) = f(L_1(I))$$

are defined (Lemma 5). When this equality holds for any such I, along with f preserving successors, we say that f is a *morphism* of $\mathfrak{U}$-counting systems and represent f as an arrow

$$f\colon (X_1, L_1, s_1) \to (X_2, L_2, s_2).$$

It is not difficult to see that $\mathfrak{U}$-counting systems and morphisms between them form a category, under the usual composition of functions. Isomorphisms in this category are bijections between $\mathfrak{U}$-counting systems which preserve both succession and limiting. Call a $\mathfrak{U}$-counting system an *ordinal* $\mathfrak{U}$-*counting* system when conditions (C3–5) hold. Clearly, the property of being an ordinal $\mathfrak{U}$-counting system is stable under isomorphism of $\mathfrak{U}$-counting systems. By Theorems 17 and 20, an ordinal $\mathfrak{U}$-counting system exists and is given by the $\mathfrak{U}$-ordinals. We will now see that ordinal $\mathfrak{U}$-counting systems are precisely the initial objects in the category of $\mathfrak{U}$-counting systems.

THEOREM 21. *For any universe* $\mathfrak{U}$, *a* $\mathfrak{U}$-*counting system is an initial object in the category of* $\mathfrak{U}$-*counting systems if and only if it is an ordinal* $\mathfrak{U}$-*counting system.*

PROOF. Since we know that an ordinal $\mathfrak{U}$-counting system exists (Theorem 17) and that the property of being an ordinal $\mathfrak{U}$-counting system is stable under isomorphism, it suffices to show that any ordinal $\mathfrak{U}$-counting system is an initial object in the category of $\mathfrak{U}$-counting systems. By Theorem 20, an ordinal $\mathfrak{U}$-counting system has the form $(\mathcal{O}, \bigvee, \_^{+})$, where $\mathcal{O}$ is an ordinal system relative to $\mathfrak{U}$ and $\bigvee$ is the join defined for exactly the successor-closed subsets $I \in \mathsf{P}_{\mathfrak{U}}\mathcal{O}$ in the $\mathfrak{U}$-counting system.

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For any $\mathfrak{U}$-counting system (X, L, s), if a morphism $(\mathcal{O}, \bigvee, \_^+) \to (X, L, s)$ exists, it must be the unique function f defined by the transfinite recursion

(i) $f(x^+) = s(f(x))$ for any $x \in \mathcal{O}$;
(ii) $f(x) = L(\{f(y) \mid y < x\})$ for any limit ordinal x (Lemma 12).

We now prove that the function f defined by the recursion above is a morphism. It preserves succession by (i). Consider $I \in \mathsf{P}_{\mathfrak{U}}\mathcal{O}$ closed under successors. Then $\bigvee I$ is a limit ordinal and $\overline{I} = \{\bigvee I\}^>$, by Theorem 20. By definition of f, we then have

$$f(\bigvee I) = L(\{f(y) \mid y < \bigvee I\}) = L(f\overline{I}).$$

We will now prove $\overline{f\overline{I}} = \overline{fI}$. We clearly have $fI \subseteq \overline{f\overline{I}}$, so it suffices to show that $f\overline{I} \subseteq \overline{fI}$. This is equivalent to showing $\overline{I} \subseteq f^{-1}\overline{fI}$, which would follow if we prove $f^{-1}\overline{fI}$ is closed. If $x^+ \in f^{-1}\overline{fI}$, then $s(f(x)) = f(x^+) \in \overline{fI}$. Therefore, $f(x) \in \overline{fI}$ and so $x \in f^{-1}\overline{fI}$. If $\bigvee J \in f^{-1}\overline{fI}$, then (as $\bigvee J$ is a limit ordinal by Theorem 20)

$$L(\{f(y) \mid y < \bigvee J\}) = f(\bigvee J) \in \overline{fI},$$

which implies $\{f(y) \mid y < \bigvee J\} \subseteq \overline{fI}$. This gives $J \subseteq \{\bigvee J\}^> \subseteq f^{-1}\overline{fI}$. Note that we have the first of these two subset inclusions due to the fact that $\bigvee J = \overset{+}{\bigvee} J$ thanks to Theorem 20. This proves that $f^{-1}\overline{fI}$ is closed. So $\overline{f\overline{I}} = \overline{fI}$. We therefore get $f(\bigvee I) = L(f\overline{I}) = L(fI)$, showing that f is indeed a morphism $(\mathcal{O}, \bigvee, \_^+) \to (X, L, s)$. ⊣

Consider the case when every element in $\mathfrak{U} \neq \varnothing$ is a finite set (e.g. $\mathfrak{U}$ could be the universe of hereditarily finite sets). Then each triple $(X, 0, s)$, where X is a set, s is a function $s\colon X \to X$, and $0 \in X$, can be seen as a $\mathfrak{U}$-counting system for the same s, with $L(I) = 0$ for each finite I. A morphism $f\colon (X_1, 0_1, s_1) \to (X_2, 0_2, s_2)$ between such $\mathfrak{U}$-counting systems is a function $f\colon X_1 \to X_2$ such that $s_2 f = f s_1$ and $f(0_1) = 0_2$. The natural number system $(\mathbb{N}, 0, s)$, with its usual successor function $s(n) = n + 1$, is an initial object in the category of such $\mathfrak{U}$-counting systems. Theorem 21 presents the natural number system as an initial object in the category of all $\mathfrak{U}$-counting systems. It is not surprising that the natural number system is initial in this larger category too, since the empty set is the only finite successor-closed subset of $\mathbb{N}$.

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