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SHARP TWO-SIDED BOUNDS FOR DISTRIBUTIONS UNDER A HAZARD RATE CONSTRAINT

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Consider a continuous nonnegative random variable X with mean μ and hazard function h. Assume further that $a \leq h(t) \leq b$ for all $t \geq 0$. Under these constraints, we obtain sharp two-sided bounds for $\bar{F}(t) = \Pr(X > t)$. An application to birth and death processes is discussed.

1. INTRODUCTION

The problem of bounding a distribution subject to constraints has been widely studied. The book by Marshall and Olkin [4] lucidly discusses many important results. An earlier book by Barlow and Proschan [2, pp. 113–116] discusses bounds for various reliabilty classes of distributions. Our motivation for studying this particular variation of the problem arises from the study of first passage times for time-reversible Markov chains. There, certain first passage time distributions of interest have easily computable maximum hazard rates. In some examples, the minimum hazard rate as well as the mean are also computable. In contrast, the survival function is often difficult to compute. Our work provides the methodology for bounding the survival function in these cases.

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2. MAIN RESULT

Our approach is to first consider the inverse problem. Suppose for a fixed $t = t_0$ that $\bar{F}(t_0) = \exp(-\alpha t_0)$, where

$$\alpha = a + p(b - a) = qa + pb, (q = 1 - p). \tag{1}$$

Because, $\exp(-bt_0) \le \bar{F}(t_0) \le \exp(-at_0), \bar{F}(t)$ is of the above form for a unique $p \,\epsilon\, [0, 1]$. Given p, we seek to find the smallest and largest values of μ, among distributions in our class ($a \le h(t) \le b$).

LEMMA 1. *For $\bar{F}(t_0) = \exp(-\alpha t_0)$, with $\alpha = qa + pb$,*

$$H_*(t) \le H(t) = \int_0^t h(x)\, dx \le H^*(t) \quad \textit{for all } t, \tag{2}$$

where

$$H_*(t) = \begin{cases} at, & t \le qt_0 \\ qat_0 + b(t - qt_0), & qt_0 < t \le t_0 \\ \alpha t_0 + a(t - t_0), & t > t_0 \end{cases} \tag{3}$$

and,

$$H^*(t) = \begin{cases} bt, & t \le pt_0 \\ pbt_0 + a(t - pt_0), & pt_0 < t \le t_0 \\ \alpha t_0 + b(t - t_0), & t > t_0. \end{cases} \tag{4}$$

PROOF: We prove that $H \ge H_*$; the proof that $H \le H^*$ follows similarly. Since $a \le h \le b$, it follows that $H(t) \ge at = H_*(t)$ for $t \le qt_0$. As $H(t_0) = \alpha t_0$ (given), it follows that $H(t) \ge \alpha t_0 + a(t - t_0) = H_*(t)$ for $t > t_0$. For $qt_0 < t \le t_0$,

$$\alpha t_0 - H(t) = H(t_0) - H(t) = \int_t^{t_0} h(x)\, dx \le b(t_0 - t) = \alpha t_0 - H_*(t);$$

thus, $H(t) \ge H_*(t)$ in this range as well. ■

It follows from Lemma 1 that $\mu_*^{(p)} \stackrel{\text{def}}{=} \int_0^\infty \exp[-H^*(t)]\, dt \le \mu \le \int_0^\infty \exp[-H_*(t)]\, dt \stackrel{\text{def}}{=} \mu^*(p)$. This solves the inverse problem, in that given p, we found the range of μ.

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Represent μ_* and μ^* by

$$\mu_*(p) = \frac{1}{b} + \left(\frac{1}{a} - \frac{1}{b}\right) p_*(p) \tag{5}$$

$$\mu^*(p) = \frac{1}{b} + \left(\frac{1}{a} - \frac{1}{b}\right) p^*(p), \tag{6}$$

respectively, where

$$p_*(p) = \exp(-pbt_0) - \exp(-\alpha t_0) \tag{7}$$

$$p^*(p) = 1 - \exp(-qat_0) + \exp(-\alpha t_0). \tag{8}$$

The function $p_*(p)$ is strictly decreasing in p, with $p_*(0) = 1 - \exp(-at_0)$ and $p_*(1) = 0$. Thus, $p_*^{-1}(p)$, the inverse function of p_*, is uniquely defined for $0 \leq p \leq 1 - \exp(-at_0)$. Similarly, p^* is strictly decreasing in p, $p^*(0) = 1$, $p^*(1) = \exp(-bt_0)$, and $p^{*-1}(p)$ is uniquely defined for $\exp(-bt_0) \leq p \leq 1$.

Define

$$g_*(p) = \begin{cases} p_*^{-1}(p), & 0 \leq p \leq 1 - \exp(-at_0) \\ 0, & 1 - \exp(-at_0) < p \leq 1 \end{cases} \tag{9}$$

and,

$$g^*(p) = \begin{cases} p^{*-1}(p), & \exp(-bt_0) \leq p \leq 1 \\ 1, & 0 \leq p < \exp(-bt_0). \end{cases} \tag{10}$$

Now, we are prepared to address the original problem. Suppose that $X \geq 0$ with $a \leq h \leq b$ and that $EX = \mu$. Put μ in the form

$$\mu = b^{-1} + (a^{-1} - b^{-1})\tilde{p}, \tag{11}$$

with

$$\tilde{p} = (\mu - b^{-1})/(a^{-1} - b^{-1}). \tag{12}$$

As $b^{-1} \leq \mu \leq a^{-1}, \tilde{p}$ is uniquely defined by μ and lies in [0, 1].

Given μ, equivalently $\tilde{p}$, the range of $\bar{F}(t_0)$ consists of

$$\exp(-\alpha t_0) \quad \text{with } \alpha = qa + pb \tag{13}$$

and p satisfying $\mu_*(p) \leq \mu \leq \mu^*(p)$. This in turn is equivalent to

$$\exp[-t_0(a + (b-a)g^*(\tilde{p}))] \leq \bar{F}(t_0) \leq \exp[-t_0(a + (b-a)g_*(\tilde{p}))], \tag{14}$$

where $\tilde{p}$ is given by (12) and g_* and g^* are given by (9) and (10), respectively.

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To study (14) in greater detail, consider the case $\exp(-bt_0) \leq 1 - \exp(-at_0)$. Here, we have the following:

(i) If $\tilde{p} < \exp(-bt_0)$, then

$$\exp(-bt_0) \leq \bar{F}(t_0) \leq \exp(-t_0(a + (b-a)p_*^{-1}(\tilde{p}))).$$

(ii) If $\exp(-bt_0) \leq \bar{F}(t_0) \leq 1 - \exp(-at_0)$, then

$$\exp(-t_0(a + (b-a)p^{*-1}(\tilde{p}))) \leq \bar{F}(t_0) \leq \exp(-t_0(a + (b-a)p_*^{-1}(\tilde{p}))).$$

(iii) If $\tilde{p} > 1 - \exp(-at_0)$, then

$$\exp(-t_0)(a + (b-a)p^{*-1}(\tilde{p})) \leq \bar{F}(t_0) \leq \exp(-at_0).$$

The inversion of p_* and p^* can be simply performed. For example, the Solver program on the TI83 Plus calculator gives the solution quickly and accurately for any choice of $(a, b, t_0, \tilde{p})$.

3. BIRTH AND DEATH PROCESS APPLICATION

Consider a birth and death process with states $\{0, 1, 2, 3, 4\}$. The birth rates are given by $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (5, 2, 2, 1)$ and the death rates are given by $(\mu_1, \mu_2, \mu_3, \mu_4) = (4, 1, 1, 10)$.

Define T to be the first passage time to state 2, starting from steady state restricted to states $\{0, 1, 3, 4\}$. (This is known as the ergodic exit distribution.) This conditional steady-state distribution works as

$$(\pi^*(0), \pi^*(1), \pi^*(3), \pi^*(4)) \leq \frac{1}{31}(4, 5, 20, 2).$$

The hazard rate of T at time 0 is given by $b = \lambda_1\pi^*(1) + \lambda_3\pi^*(3) = \frac{30}{31}$. T is known to be DFR (decreasing failure rate) [3]. The quantity $b = h(0) = \frac{30}{31}$ is thus the largest value of the hazard rate function of T.

It is also known [1, p. 8] that $\lim_{t\to\infty} h(t) = a$, where a is the smallest eigenvalue of the matrix

$$-Q^* = \begin{pmatrix} 5 & -5 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -10 & 10 \end{pmatrix}.$$

The value in this example is $a = 6 - \sqrt{26}$. Thus,

$$6 - \sqrt{26} = a \leq h(t) \leq b = 30|31 \tag{15}$$

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for all $t \geq 0$. Next,

$$ET = \sum_{i \in \{0,1,3,4\}} \pi^*(i) E_i T, \tag{16}$$

where $E_i T$ is the mean of T starting in state i. By standard birth and death process methodology [5, p. 351], these four quantities are computed and plugged into (16), giving

$$\mu = ET = 333/310.$$

Next, by (12),

$$\tilde{p} = \frac{\mu - b^{-1}}{a^{-1} - b} = 0.5336417988.$$

Applying our methodology to bound $\bar{F}(2)$, we find

$$g_*(\tilde{p}) = p_*^{-1}(\tilde{p}) = 0.22938544945$$
$$g^*(\tilde{p}) = p^{*-1}(\tilde{p}) = 0.7311121555$$

and, consequently, that

$$0.1496312727 \leq \bar{F}(2) \leq 0.159998765. \tag{17}$$

In this example, the sharp DFR upper bound based on μ [2, p. 116], but not utilizing a and b, equals 0.1976 to four decimal places. This significantly exceeds the upper bound in (17). The simple bound

$$\exp(-2b) \leq \bar{F}(2) \leq \exp(-2a),$$

which does not utilize μ, in this case works out to [0.1444, 0.1650] (to four decimal places), which is about twice the width of (17).

With some computational effort, we can employ the spectral representation for the distribution of T [1, p. 10] and find that $\bar{F}(2) = 0.1558$, to four decimal places. This is 0.0010 greater than the midpoint of (17).

We further remark that $\exp(-t|\mu)$ lies in the interval given in (14) for all t. Applying this to the above example,

$$\begin{aligned} |\bar{F}(2) - \exp(-2|\mu)| &\leq \max(0.1600 - 0.1554, 0.1554 - 0.1496) \\ &= \max(0.0046, 0.0058) = 0.0058. \end{aligned} \tag{18}$$

The actual value in this case is $\bar{F}(2) - \exp(-2|\mu) = 0.0004$. Thus, we can use this method to bound the accuracy of approximating T by an exponential distribution with mean ET. Of course, we would need to perform (18) for various values of t.

This method can be used for a general ergodic birth and death process to approximate the probability that the first passage time to a state j, starting in steady state restricted to $\{j\}^c = \{i \neq j\}$, exceeds t. The most difficult aspect is to compute the lower bound a, the smallest eigenvalue of $-Q^*$, where Q^* is the restriction of the infinitesimal matrix Q to $\{j\}^c x\{j\}^c$. There has been a great deal of fairly recent work relevant to this problem by Diaconis and various co-workers. In contrast, the computation of the exact distribution of T is considerably more difficult.

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