PAPER

Quantum circuits generating four-qubit maximally entangled states

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Abstract

We describe quantum circuits generating four-qubit maximally entangled states, the amount of entanglement being quantified by using the absolute value of the Cayley hyperdeterminant as an entanglement monotone. More precisely we show that this type of four-qubit entangled states can be obtained by the action of a family of `CNOT` circuits on some special states of the LU orbit of the state $|0000\rangle$.

Keywords: quantum information; quantum circuits; entanglement; hyperdeterminant

1. Introduction

The original idea of using the hyperdeterminant to classify multipartite entanglement goes back to Miyake (Miyake, 2003; Miyake and Wadati, 2002). The hyperdeterminant (in the sense of Gelfand et al. 1992) is a generalization of the determinant to higher dimensions. Let $|\psi\rangle = \sum_{i_0,i_1,\ldots,i_{n-1}\in\{0,1\}} a_{i_0 i_1 \ldots i_{n-1}} |i_0 i_1 \ldots i_{n-1}\rangle$ be the state vector of an n-qubit system in the Hilbert space $\mathcal{H}^{\otimes n} = (\mathbb{C}^2)^{\otimes n}$; then, the hyperdeterminant of the format 2^n, denoted in this paper by Δ_n, is an homogenous multivariate polynomial in the 2^n variables $a_{i_0 i_1 \ldots i_{n-1}}$, with coefficients in $\mathbb{Z}$. It is invariant (up to a sign) by permutation of the qubits and also invariant by the action of the group SLOCC, the group of stochastic local operations assisted by classical communication, assimilated to the Cartesian product $SL(2,\mathbb{C})^n$. According to Miyake (Miyake, 2003; Miyake and Wadati, 2002), the more generic entanglement holds only for the states on which the hyperdeterminant does not vanish and the absolute value of Δ_n quantifies the amount of generic entanglement.

In this article, we focus on a four-qubit quantum system. In this case, the hyperdeterminant Δ_4 is a polynomial of degree 24 in 16 unknowns which has 2,894,276 terms and a rich geometric structure related with the triangulations of the 4-cube (Huggins et al., 2008). Despite of its huge size, it is still maniable and computable by several methods. Since the formula given by Schläfli in 1852 (Gelfand et al., 1992, Chapter 14, Section 4), other methods have been developed including an expression of Δ_4 in terms of fundamental SLOCC invariant polynomials of lower degree (Luque and Thibon 2003) and more recently an evaluation of Δ_4 via neural networks (Jaffali and Oeding 2020).

Following Miyake, we consider as Gour and Wallach (2012), that a four-qubit state with the highest amount of generic entanglement can be defined as a state maximizing the absolute value of Δ_4. In the rest of the paper, we refer to this type of state as a *maximum hyperdeterminant state* (sometimes abbreviated as MHS). Gour and Wallach (2012) conjectured that the state $|L\rangle$ (see Figure 1) is the unique maximum hyperdeterminant state, up to a local unitary operation. This

conjecture was proved in 2013 by Chen and Djokovic (2013) and the maximal value of $|\Delta_4|$ is $\frac{1}{2^8 3^9} = \frac{1}{5,038,848} \simeq 1.98 \times 10^{-7}$. As Δ_4 is a SLOCC-invariant polynomial, a simple consequence of this result is that the SLOCC orbit of $|L\rangle$ corresponds to its LU orbit (about the LU orbits and LU-invariant polynomials of four qubits, we refer the reader to Luque et al. 2007). Another interesting property of the state $|L\rangle$ is to be the only state (up to local unitary operations) maximizing the average Tsallis α-entropy of entanglement, for all $\alpha > 2$ (Gour and Wallach, 2010). Let us also mention two other maximum hyperdeterminant states, which have the property of having real coordinates (see Figure 1): $|\Phi_5\rangle$ (reported by Osterloh and Siewert 2006 and by Alsina 2017) and $|M_{2222}\rangle$ (reported by Jaffali 2020).

In Quantum Information and Computation, entangled states, and in particular maximally entangled states, play the role of an important physical resource (see e.g. the introduction of Chen and Djokovic 2013). Despite of that, to our knowledge, there is no proposal in the academic literature for quantum circuits capable of producing the state $|L\rangle$ or any other MHS. The goal of this work is merely to fill this gap by describing a family of quantum circuits that enable the generation of maximum hyperdeterminant states. We show that a MHS can be obtained by the action a certain type of CNOT gate circuits on a fully factorized state, namely a state of the LU orbit of $|0000\rangle$. As a consequence of this result, one can construct quantum circuits of relatively small depth generating the three states $|L\rangle$, $|\Phi_5\rangle$ and $|M_{2222}\rangle$.

The paper is structured as follows. Section 2 is a reminder on quantum circuits of CNOT gates and SWAP gates, where we introduce most of our notations and give some useful conjugation rules between these gates. In Section 3, we present the methodology and algorithms used in our numerical approach to find circuits generating maximum hyperdeterminant states. The two next Sections (4 and 5) are dedicated to the description of these circuits. Finally, in Section 6, we propose three simple quantum circuits generating the states $|L\rangle$, $|\Phi_5\rangle$ and M_{2222}, as well as an implementation of a circuit generating the state $|L\rangle$ into a quantum computer provided by the IBM quantum experience at https://quantum-computing.ibm.com/.

This article goes along with a Python module than can be downloaded at https://github.com/marcbataille/maximum-hyperdeterminant-states. Our module relies on the Python module SymPy (a Computer Algebra System), allowing us to perform symbolic and exact computation. It provides an implementation in the Python language of the different algorithms, quantum

$$|L\rangle = \frac{1}{\sqrt{3}}(|u_0\rangle + \omega |u_1\rangle + \omega^* |u_2\rangle) \tag{1}$$

where : $\omega = e^{\frac{i\pi}{3}}$

$$|u_0\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle) = |\Phi^+\rangle |\Phi^+\rangle$$

$$|u_1\rangle = \frac{1}{2}(|0000\rangle - |0011\rangle - |1100\rangle + |1111\rangle) = |\Phi^-\rangle |\Phi^-\rangle$$

$$|u_2\rangle = \frac{1}{2}(|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle) = |\Psi^+\rangle |\Psi^+\rangle$$

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

$$|\Phi_5\rangle = \frac{1}{\sqrt{6}}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle + \sqrt{2} |1111\rangle) \tag{2}$$

$$|M_{2222}\rangle = \frac{1}{\sqrt{8}} |v_1\rangle + \frac{\sqrt{6}}{4} |v_2\rangle + \frac{1}{\sqrt{2}} |v_3\rangle \tag{3}$$

where: $|v_1\rangle = \frac{1}{\sqrt{6}}(|0000\rangle + |0101\rangle - |0110\rangle - |1001\rangle + |1010\rangle + |1111\rangle)$

$|v_2\rangle = \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle)$

$|v_3\rangle = \frac{1}{\sqrt{8}}(-|0001\rangle + |0010\rangle - |0100\rangle + |0111\rangle + |1000\rangle - |1011\rangle + |1101\rangle - |1110\rangle)$

Figure 1. 4-qubits states for which $|\Delta_4|$ is maximal.

Table 1. Classical single-qubit unitary gates

Name	Symbol	Matrix
Pauli-X	X	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Y	Y	$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z	Z	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Rotation around the $\hat{x}$ axis	$R_x(\theta)$	$e^{-i\theta X/2} = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$
Rotation around the $\hat{y}$ axis	$R_y(\theta)$	$e^{-i\theta Y/2} = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$
Rotation around the $\hat{z}$ axis	$R_z(\theta)$	$e^{-i\theta Z/2} = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$
Phase	P	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
T-gate	T	$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
Hadamard	H	$\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

gates and quantum states used in this work. As the proof of some assertions (mostly numerical equalities) consists only of basic linear algebra and calculus, we chose to refer the reader to the corresponding function of the module that does the job.

2. Quantum Circuits of CNOT and SWAP Gates

In this section, we introduce the main notations and conventions of the paper and we recall the definition of some classical quantum gates (Table 1) as well as some properties of the CNOT gates and SWAP gates often used in the rest of the article.

Let $n \geqslant 1$ be the number of qubits of the considered quantum register. We label each qubit from 0 to $n - 1$, thus following the usual convention. For coherence, we also number the lines and columns of a $n \times n$ matrix from 0 to $n - 1$, and we consider that a permutation in the symmetric group $\mathfrak{S}_n$ is a bijection of $\{0, \ldots, n - 1\}$.

If two normalized vectors $|\psi\rangle$ and $|\psi'\rangle$ of the Hilbert space $\mathcal{H}^{\otimes n}$ are equal up to a global phase, then they represent physically the same state and we write $|\psi\rangle \simeq |\psi'\rangle$. In the same way, we write $U \simeq U'$ for two unitary operators which are equal up to a global phase. In the design of quantum

circuits, we use the following correspondences between the classical gates :

$$R_z(\pi) \simeq Z, \; R_z(\pi/2) \simeq P, \; R_z(-\pi/2) \simeq P^\dagger, \; R_z(\pi/4) \simeq T \tag{4}$$

$$R_y(\pi) \simeq Y, \; R_y(\pi/2) = HZ = XH, \; R_y(-\pi/2) = ZH = HX \tag{5}$$

When one applies locally a single-qubit gate U to the qubit i of a n-qubit register, the corresponding action on the n-qubit system is that of the unitary operator

$$U_i = \underbrace{I \otimes \cdots \otimes I}_{i \text{ times}} \otimes U \otimes \underbrace{I \otimes \cdots \otimes I}_{n-i-1 \text{ times}} = I^{\otimes i} \otimes U \otimes I^{\otimes n-i-1}, \tag{6}$$

where $\otimes$ is the Kronecker product of matrices and I the identity matrix in dimension 2. As an example, if $n = 4$, $H_1 = I \otimes H \otimes I \otimes I$ and $H_0 H_3 = H \otimes I \otimes I \otimes H$. We also use vectors of $\mathbb{F}_2^n$ as labels to indicate the set of qubits to which the single-qubit unitary U is applied. Let $v = [v_0 \ldots v_{n-1}]^t$ be a (column) vector of $\mathbb{F}_2^n$, we denote by U_v the product $\prod_i U_i^{v_i}$.

A CNOT gate with target on qubit i and control on qubit j is denoted by $X_{[ij]}$ (not to be confused with X_i which denotes a Pauli-X gate applied on qubit i). The group generated by the CNOT gates acting on an n-qubit quantum system is denoted by $\langle \text{CNOT} \rangle_n$. Let us denote by $\text{GL}_n(\mathbb{F}_2)$ the general linear group over $\mathbb{F}_2$ in dimension n. A transvection matrix $[ij]$ ($i, j = 1 \ldots n - 1$ and $i \neq j$) is the matrix of $\text{GL}_n(\mathbb{F}_2)$ defined by $[ij] = I_n + E_{ij}$, where I_n is the identity in dimension n and E_{ij} is the matrix with all entries equal to zero but the entry (i, j) that is equal to 1. We recall that the transvection matrices generate the group $\text{GL}_n(\mathbb{F}_2)$ and that multiplying a matrix M to the left by a transvection matrix $[ij]$ is equivalent to adding the row j to the row i of M. From these facts, one can deduce that the group $\langle \text{CNOT} \rangle_n$ is isomorphic to $\text{GL}_n(\mathbb{F}_2)$, a possible isomorphism associating, to any gate $X_{[ij]}$, the transvection matrix $[ij]$ (see Bataille 2022 for more details). The order of $\langle \text{CNOT} \rangle_n$ is therefore equal to the order of $\text{GL}_n(\mathbb{F}_2)$:

$$|\langle \text{CNOT} \rangle_n| = 2^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} (2^i - 1). \tag{7}$$

Let A be any matrix in $\text{GL}_n(\mathbb{F}_2)$, we denote by X_A the element of $\langle \text{CNOT} \rangle_n$ associated with A, that is X_A is the product of any sequence of CNOT gates $X_{[i_1 j_1]}, \ldots, X_{[i_p j_p]}$ such that A can be decomposed in the product of the transvection matrices $[i_1 j_1], \ldots, [i_p j_p]$.

The SWAP gate that exchanges qubits i and j is denoted by $S_{(ij)}$. Let σ be a permutation of the symmetric group $\mathfrak{S}_n$. We also denote by σ the permutation matrix associated with the permutation σ. Consequently, (ij) denotes the transposition of $\mathfrak{S}_n$ that exchanges i and j as well as the corresponding transposition matrix in $\text{GL}_n(\mathbb{F}_2)$. The permutation matrix σ is defined as the matrix $A = (a_{ij})$ in $\text{GL}_n(\mathbb{F}_2)$ such that $a_{ij} = 1$ if and only if $i = \sigma(j)$. We recall that multiplying a matrix M to the left by a permutation matrix σ is equivalent to applying the permutation σ to the rows of M. In this case, each row R_i is replaced by the row $R_{\sigma^{-1}(i)}$. The group of permutation matrices is a subgroup of $\text{GL}_n(\mathbb{F}_2)$ which is isomorphic to the group generated by the SWAP gates acting on n qubits : to each SWAP gate $S_{(ij)}$ corresponds the transposition matrix (ij). We denote by S_σ the product of any sequence of SWAP gates $S_{(i_1 j_1)}, \ldots, S_{(i_p j_p)}$ such that $\sigma = (i_1 j_1) \ldots (i_p j_p)$.

Let τ be a transposition of $\mathfrak{S}_n$, it is easy to check that $S_\tau X_{[ij]} S_\tau = X_{[\tau(i) \tau(j)]}$. Hence, by induction, one has for any permutation σ:

$$S_\sigma X_{[ij]} S_\sigma^{-1} = X_{[\sigma(i) \sigma(j)]}. \tag{8}$$

Let U be a single-qubit unitary matrix and U_i the unitary corresponding to the action of U on qubit i (Identity (6)). One has, for any permutation σ

$$S_\sigma U_i S_\sigma^{-1} = U_{\sigma(i)} \tag{9}$$

and for any vector v in $\mathbb{F}_2^n$

$$S_\sigma U_v S_\sigma^{-1} = U_{\sigma v}, \tag{10}$$

where σv is the matrix product between σ and v.

The Pauli group for n qubits is the subgroup of the unitary group $U(2^n)$ generated by the Pauli gates X_i, Y_i and Z_i ($0 \leqslant i \leqslant n-1$). Since $Y = iXZ$ and $XZ = -ZX$, any element of this group can be written uniquely in the form

$$i^\lambda X_u Z_v, \tag{11}$$

where u and v are two vectors of the space $\mathbb{F}_2^n$ and $\lambda \in \{0, 1, 2, 3\}$. The CNOT gates normalize the Pauli group, and it is not difficult to prove the following conjugation rules of a Pauli gate by a CNOT gate: $X_{[ij]}Z_i X_{[ij]} = Z_i Z_j$, $X_{[ij]}Z_j X_{[ij]} = Z_j$, $X_{[ij]}X_i X_{[ij]} = X_i$ and $X_{[ij]}X_j X_{[ij]} = X_i X_j$. These rules can be generalized by induction as

$$X_A X_u Z_v X_A^{-1} = X_{Au} Z_{A^{-t}v}, \tag{12}$$

where u and v are vectors in $\mathbb{F}_2^n$, A is a matrix in $GL_n(\mathbb{F}_2)$ and A^{-t} a shorthand for $\left(A^{-1}\right)^t$.

3. Methodology used in the Numerical Exploration

We address the following problem, that we have already formulated in Bataille (2022) : is it possible to generate a state maximizing $|\Delta_4|$ by applying a CNOT gate circuit to a state of the LU orbit of $|0000\rangle$?

We use the classical Z-Y decomposition of a single-qubit unitary operator in the form $e^{i\varphi}R_z(\alpha)R_y(\beta)R_z(\alpha')$ (see e.g. Nielsen and Chuang 2011, Th. 4.1). Using this decomposition, any fully factorized unitary operator U depends, up to a global phase, on the 12 real parameters of the matrix

$$\mathcal{P} = \begin{bmatrix} \alpha_0 & \beta_0 & \alpha_0' \\ \alpha_1 & \beta_1 & \alpha_1' \\ \alpha_2 & \beta_2 & \alpha_2' \\ \alpha_3 & \beta_3 & \alpha_3' \end{bmatrix}. \tag{13}$$

Let us define the unitary $U(\mathcal{P})$ by

$$\begin{aligned} U(\mathcal{P}) = {}&R_z(\alpha_0)R_y(\beta_0)R_z(\alpha_0') \otimes R_z(\alpha_1)R_y(\beta_1)R_z(\alpha_1') \\ &\otimes R_z(\alpha_2)R_y(\beta_2)R_z(\alpha_2') \otimes R_z(\alpha_3)R_y(\beta_3)R_z(\alpha_3'). \end{aligned} \tag{14}$$

As a rotation around the $\hat{z}$ axis applied to $|0\rangle$ is just a change of phase, it is possible to write any state vector of the LU orbit of $|0000\rangle$ (up to a global phase), by using only two parameters for each qubit. So, any state in the LU orbit of $|0000\rangle$ is equal (up to a global phase) to the state $|\mathcal{P}\rangle$ defined by

$$|\mathcal{P}\rangle = R_z(\alpha_0)R_y(\beta_0) \otimes R_z(\alpha_1)R_y(\beta_1) \otimes R_z(\alpha_2)R_y(\beta_2) \otimes R_z(\alpha_3)R_y(\beta_3) |0000\rangle . \tag{15}$$

Using the definition of the rotation matrices around the $\hat{z}$ and $\hat{y}$ axes, one has

$$|\mathcal{P}\rangle = (a_0 |0\rangle + a_1 |1\rangle) \otimes (b_0 |0\rangle + b_1 |1\rangle) \otimes (c_0 |0\rangle + c_1 |1\rangle) \otimes (d_0 |0\rangle + d_1 |1\rangle), \tag{16}$$

$$\text{where } (a_0, a_1) = \left(e^{-i\alpha_0/2} \cos \frac{\beta_0}{2}, e^{i\alpha_0/2} \sin \frac{\beta_0}{2} \right),$$

$$(b_0, b_1) = \left(e^{-i\alpha_1/2} \cos \frac{\beta_1}{2}, e^{i\alpha_1/2} \sin \frac{\beta_1}{2} \right),$$

$$(c_0, c_1) = \left(e^{-i\alpha_2/2} \cos \frac{\beta_2}{2}, e^{i\alpha_2/2} \sin \frac{\beta_2}{2} \right),$$

$$(d_0, d_1) = \left(e^{-i\alpha_3/2} \cos \frac{\beta_3}{2}, e^{i\alpha_3/2} \sin \frac{\beta_3}{2} \right).$$

Any state resulting from the action of an unitary operator in $\langle \text{CNOT} \rangle_4$, on a state of the LU orbit of $|0000\rangle$ can be written (up to a global phase) in the form

$$X_A |\mathcal{P}\rangle, \tag{17}$$

where A is a matrix in $\text{GL}_4(\mathbb{F}_2)$ and $\mathcal{P}$ a matrix of parameters. In order to determine the states of type $X_A |\mathcal{P}\rangle$ capable of maximizing $|\Delta_4|$, it is sufficient to consider the right cosets of the subgroup of $\langle \text{CNOT} \rangle_4$ generated by the SWAP gates (group $\langle \text{SWAP} \rangle_4 \simeq \mathfrak{S}_4$), because $|\Delta_4|$ is invariant under permutation of the qubits, that is $|\Delta_4(X_A |\mathcal{P}\rangle)| = |\Delta_4(X_{\sigma A} |\mathcal{P}\rangle)|$ for any permutation matrix σ. The order of the group $\langle \text{CNOT} \rangle_4$ is 20,160 (see Identity (7)), so the number of right cosets of $\langle \text{SWAP} \rangle_4$ in $\langle \text{CNOT} \rangle_4$ is $20{,}160/24 = 840$. For each coset, we compute a representative of minimal length in the generators $X_{[ij]}$ (function `right_cosets_perm_GL4` of the Python module).

The computation of Δ_4 for a given state is performed using the algorithm proposed by Luque and Thibon in (2003, Section IV) (function `hyper_det` of the Python module). After eliminating all coset representatives X_A such that $|\Delta_4(X_A |\mathcal{P}\rangle)|$ vanishes for any $\mathcal{P}$, we obtain a list of 333 representatives (function `non_zero_HD_strings` of the Python module). For each of them, we use a random walk on the search space defined by the eight parameters α_i and β_i of $|\mathcal{P}\rangle$ in order to maximize the value of $|\Delta_4(X_A |\mathcal{P}\rangle)|$ (function `search_max_HD` of the Python module). We check that it is possible to reach the maximal value of $\frac{1}{2^8 3^9}$ for $|\Delta_4|$ (accuracy 10^{-22}) for only 12 coset representatives. These cosets are described in Section 5. Finally, from the approximate values of $\mathcal{P}$ computed by the random walk heuristic, we guess possible exact values for the parameters of $\mathcal{P}$ and we check (by CAS) that these exact values of $\mathcal{P}$ satisfy the equality $|\Delta_4(X_A |\mathcal{P}\rangle)| = \frac{1}{2^8 3^9}$.

4. A CNOT Circuit to Reach the Maximum of $|\Delta_4|$

Let i, j, k, ℓ be distinct integers in $\{0, 1, 2, 3\}$. We define as follows $M_k^{(i,j)}$, a product of CNOT gates, and $A_k^{(i,j)}$ the bit matrix of $\text{GL}_4(\mathbb{F}_2)$ associated with $M_k^{(i,j)}$.

$$M_k^{(i,j)} = X_{[ij]} X_{[jk]} X_{[ki]} X_{[i\ell]} X_{[\ell j]} \tag{18}$$

$$A_k^{(i,j)} = [ij][jk][ki][i\ell][\ell j] \tag{19}$$

In this section, we show how to reach the maximum of $|\Delta_4|$ using the operator

$$M_2^{(0,1)} = X_{[0\,1]} X_{[1\,2]} X_{[2\,0]} X_{[0\,3]} X_{[3\,1]}. \tag{20}$$

The results are extended to any operator of type $M_k^{(i,j)}$ in the next section. Since

$$A_2^{(0,1)} = [0\,1][1\,2][2\,0][0\,3][3\,1] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and

$$A_3^{(0,1)} = [0\ 1][1\ 3][3\ 0][0\ 2][2\ 1] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

we remark that $A_3^{(0,1)} = (0\ 2\ 3)A_2^{(0,1)}$. Hence,

$$M_3^{(0,1)} = S_{(0\ 2\ 3)}M_2^{(0,1)}, \tag{21}$$

which means that $M_3^{(0,1)}$ and $M_2^{(0,1)}$ represent the same coset. This coset is denoted by $\overline{(0,1)}$.

Proposition 1. *Let $\mathcal{P}_{\max}$ and $\mathcal{P}'_{\max}$ be the two matrices of parameters defined by*

$$\mathcal{P}_{\max} = \begin{bmatrix} \pi/2 & \pi/2 & 0 \\ \pi/2 & \pi/2 & 0 \\ \pi/4 & \cos^{-1}\frac{\sqrt{3}}{3} & 0 \\ \pi/4 & \cos^{-1}\frac{\sqrt{3}}{3} & 0 \end{bmatrix}, \quad \mathcal{P}'_{\max} = \begin{bmatrix} \pi/2 & \pi/2 & 0 \\ \pi/2 & \pi/2 & 0 \\ 3\pi/4 & \cos^{-1}\frac{\sqrt{3}}{3} & 0 \\ 3\pi/4 & \cos^{-1}\frac{\sqrt{3}}{3} & 0 \end{bmatrix}, \tag{22}$$

then the states

$$|\psi_{\max}\rangle = M_2^{(0,1)}|\mathcal{P}_{\max}\rangle \tag{23}$$

and

$$|\psi'_{\max}\rangle = M_2^{(0,1)}|\mathcal{P}'_{\max}\rangle \tag{24}$$

maximize the absolute value of the four-qubit hyperdeterminant. One has

$$\Delta_4(|\psi_{\max}\rangle) = \Delta_4(|\psi'_{\max}\rangle) = -\frac{1}{2^8 3^9}, \tag{25}$$

$$|\psi_{\max}\rangle = \frac{\sqrt{3}}{3}|w_1\rangle + \frac{3+\sqrt{3}}{6}e^{i\frac{\pi}{4}}|w_2\rangle + \frac{3-\sqrt{3}}{6}e^{i\frac{\pi}{4}}|w_3\rangle \tag{26}$$

and

$$|\psi'_{\max}\rangle = \frac{\sqrt{3}}{3}|w_1\rangle + \frac{3+\sqrt{3}}{6}e^{-i\frac{\pi}{4}}|w_2\rangle + \frac{3-\sqrt{3}}{6}e^{i\frac{3\pi}{4}}|w_3\rangle, \tag{27}$$

where

$$|w_1\rangle = \frac{1}{\sqrt{8}}(|0001\rangle + i|0011\rangle + |0101\rangle - i|0111\rangle$$
$$+ |1000\rangle + i|1010\rangle + |1100\rangle - i|1110\rangle),$$
$$|w_2\rangle = \frac{1}{2}(-|0000\rangle - i|0110\rangle - i|1011\rangle + |1101\rangle),$$
$$|w_3\rangle = \frac{1}{2}(|0010\rangle + i|0100\rangle - i|1001\rangle + |1111\rangle).$$

Proof. The different assertions can be checked using the function `check_psi_max_is_MHS` of the Python module. □

In our numerical search for matrices of parameters $\mathcal{P}$ such that $M_2^{(0,1)} |\mathcal{P}\rangle$ maximizes Δ_4, it appears that all values of $\mathcal{P}$ computed by the random walk heuristic are related to $\mathcal{P}_{\max}$ or to $\mathcal{P}'_{\max}$ by simple operations. These operations are described by the following lemma and its corollary. Numerical results suggest that these operations applied to the matrices $\mathcal{P}_{\max}$ or $\mathcal{P}'_{\max}$ are sufficient to describe all the possible matrices $\mathcal{P}$ such that $M_2^{(0,1)} |\mathcal{P}\rangle$ is a MHS (Conjecture 7).

Lemma 2. *Let $\mathcal{P}$ be a matrix of parameters and, for any k in $\{0, 1, 2, 3\}$, let us denote by*
$\mathcal{P}_{\alpha_k+\pi}$ *the matrix obtained from $\mathcal{P}$ by adding π to the parameter α_k,*
$\mathcal{P}_{-\beta_k}$, *the matrix obtained from $\mathcal{P}$ by taking the opposite of β_k,*
$\mathcal{P}_{-\alpha_k,\beta_k+\pi}$, *the matrix obtained from $\mathcal{P}$ by taking the opposite of α_k and adding π to β_k,*
then:

$$|\mathcal{P}_{\alpha_k+\pi}\rangle = -\mathrm{i}Z_k |\mathcal{P}\rangle \tag{28}$$
$$|\mathcal{P}_{-\beta_k}\rangle = Z_k |\mathcal{P}\rangle \tag{29}$$
$$|\mathcal{P}_{-\alpha_k,\beta_k+\pi}\rangle = -\mathrm{i}Y_k |\mathcal{P}\rangle \tag{30}$$

Proof. We prove only Identity (30), the proofs of Identities (28) and (29) being similar. Without loss of generality, we suppose that the last column of the matrix $\mathcal{P}$ is null and $k = 0$. On the one hand:

$$-\mathrm{i}Y_0 |\mathcal{P}\rangle = (-\mathrm{i}Y(a_0 |0\rangle + a_1 |1\rangle)) \otimes (b_0 |0\rangle + b_1 |1\rangle) \otimes (c_0 |0\rangle + c_1 |1\rangle) \otimes (d_0 |0\rangle + d_1 |1\rangle)$$
$$= (-a_1 |0\rangle + a_0 |1\rangle) \otimes (b_0 |0\rangle + b_1 |1\rangle) \otimes (c_0 |0\rangle + c_1 |1\rangle) \otimes (d_0 |0\rangle + d_1 |1\rangle),$$

where $(a_0, a_1) = \left(\mathrm{e}^{-\mathrm{i}\alpha_0/2} \cos\frac{\beta_0}{2}, \mathrm{e}^{\mathrm{i}\alpha_0/2} \sin\frac{\beta_0}{2}\right)$. On the other hand:

$$|\mathcal{P}_{-\alpha_0,\beta_0+\pi}\rangle = (a'_0 |0\rangle + a'_1 |1\rangle) \otimes (b_0 |0\rangle + b_1 |1\rangle) \otimes (c_0 |0\rangle + c_1 |1\rangle) \otimes (d_0 |0\rangle + d_1 |1\rangle),$$

where $(a'_0, a'_1) = \left(\mathrm{e}^{-\mathrm{i}(-\alpha_0/2)} \cos\frac{\beta_0+\pi}{2}, \mathrm{e}^{-\mathrm{i}\alpha_0/2} \sin\frac{\beta_0+\pi}{2}\right) = (-a_1, a_0)$.
Hence $-\mathrm{i}Y_0 |\mathcal{P}\rangle = |\mathcal{P}_{-\alpha_0,\beta_0+\pi}\rangle$. $\qquad\square$

Corollary 3. *Let A be a matrix in $\mathrm{GL}_4(\mathbb{F}_2)$ and $\mathcal{P}$ a matrix of parameters. If $|\Delta_4|$ is maximal for $X_A |\mathcal{P}\rangle$, then $|\Delta_4|$ is also maximal for $X_A |\mathcal{P}_{\alpha_k+\pi}\rangle$, $X_A |\mathcal{P}_{-\beta_k}\rangle$ and $X_A |\mathcal{P}_{-\alpha_k,\beta_k+\pi}\rangle$, for any k in $\{0, 1, 2, 3\}$.*

Proof. Suppose that $|\Delta_4|$ is maximal for $X_A |\mathcal{P}\rangle$. Let $\mathcal{P}' \in \{\mathcal{P}_{\alpha_k+\pi}, \mathcal{P}_{-\beta_k}, \mathcal{P}_{-\alpha_k, \beta_k+\pi}\}$. From Lemma 2 and Identity (11), there exists two vectors u and v in $\mathbb{F}_2^4$ such that $|\mathcal{P}'\rangle \simeq X_u Z_v |\mathcal{P}\rangle$. Hence, $X_A |\mathcal{P}'\rangle \simeq X_A X_u Z_v |\mathcal{P}\rangle \simeq X_A X_u Z_v X_A^{-1} X_A |\mathcal{P}\rangle$. So, using Identity (12), we deduce that $X_A |\mathcal{P}'\rangle \simeq X_{Au} Z_{A^{-t}v} X_A |\mathcal{P}\rangle$. Consequently, $X_A |\mathcal{P}'\rangle$ is in the LU orbit of $X_A |\mathcal{P}\rangle$, which implies that $|\Delta_4|$ is maximal for the state $X_A |\mathcal{P}'\rangle$. $\qquad\square$

Example 4. Let us apply the following sequence of operations on $\mathcal{P}_{\max}$:
$\alpha_0 \leftarrow \alpha_0 + \pi$, $\beta_2 \leftarrow -\beta_2$, $\alpha_3 \leftarrow -\alpha_3$, $\beta_3 \leftarrow \beta_3 + \pi$.

The resulting matrix of parameters is $\mathcal{P} = \begin{bmatrix} 3\pi/2 & \pi/2 & 0 \\ \pi/2 & \pi/2 & 0 \\ \pi/4 & -\cos^{-1}\frac{\sqrt{3}}{3} & 0 \\ -\pi/4 & \cos^{-1}\frac{\sqrt{3}}{3}+\pi & 0 \end{bmatrix}$ and

$$|\mathcal{P}\rangle = (-\mathrm{i}Z_0)Z_2(-\mathrm{i}Y_3) |\mathcal{P}_{\max}\rangle \simeq X_3 Z_0 Z_2 Z_3 |\mathcal{P}_{\max}\rangle.$$

Let $u = [0\ 0\ 0\ 1]^t$ and $v = [1\ 0\ 1\ 1]^t$. One has :

$$A_2^{(0,1)} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad A_2^{(0,1)} u = [0\ 1\ 1\ 1]^t$$

and

$$\left(A_2^{(0,1)}\right)^{-t} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \qquad \left(A_2^{(0,1)}\right)^{-t} v = [0\ 1\ 0\ 0]^t.$$

Hence $M_2^{(0,1)} |\mathcal{P}\rangle \simeq X_1 X_2 X_3 Z_1 |\psi_{\max}\rangle$.

Remark 5. We observe that the matrices $\mathcal{P}_{\max}$ and $\mathcal{P}'_{\max}$ are not related by the operations on parameters described in Lemma 2, that is there does not exist any gate $X_u Z_v$ in the four-qubit Pauli group such that $|\mathcal{P}'_{\max}\rangle \simeq X_u Z_v |\mathcal{P}_{\max}\rangle$. This implies that the state $|\psi_{\max}\rangle$ and the state $|\psi'_{\max}\rangle$ define distinct orbits by the action of the four-qubit Pauli group. Actually, from Identities (22) and (4), one has $|\mathcal{P}'_{\max}\rangle \simeq P_2 P_3 |\mathcal{P}_{\max}\rangle$. Using the method described in Section 3, we com-
pute a matrix of parameters $\mathcal{P}_{\psi \to \psi'} = \begin{bmatrix} -\pi/2 & -\pi/2 & -\pi/2 \\ \pi/2 & \pi & \pi \\ 0 & \pi/2 & \pi \\ -\pi/2 & -\pi/2 & -\pi/2 \end{bmatrix}$ and a phase $\varphi = -\frac{\pi}{3}$ such that

$|\psi'_{\max}\rangle = e^{i\varphi} U(\mathcal{P}_{\psi \to \psi'}) |\psi_{\max}\rangle$. Then, using Identities (4) and (5), we obtain

$$|\psi'_{\max}\rangle \simeq (PHP^\dagger \otimes PX \otimes H \otimes PHP^\dagger) |\psi_{\max}\rangle. \tag{31}$$

This last identity can be checked using the function *check_psi_to_psi_prime* of the Python module.

Remark 6. Since $M_3^{(0,1)} = S_{(0\ 2\ 3)} M_2^{(0,1)}$, it is easy to see that the set of all matrices of parameters $\mathcal{P}$ having their last column null such that $M_3^{(0,1)} |\mathcal{P}\rangle$ maximizes $|\Delta_4|$ is equal to the set of all matrices of parameters $\mathcal{P}$ having their last column null such that $M_2^{(0,1)} |\mathcal{P}\rangle$ maximizes $|\Delta_4|$. We denote this set by PMAX$^{(0,1)}$.

Conjecture 7. *Any matrix in* PMAX$^{(0,1)}$ *can be obtain from* $\mathcal{P}_{\max}$ *or from* $\mathcal{P}'_{\max}$ *by a sequence of the operations on parameters described by Lemma 2.*

5. All `CNOT` Circuits to Reach the Maximum of $|\Delta_4|$

We generalize the results of the previous section by describing all the four-qubit `CNOT` circuits that enable to produce a state maximizing $|\Delta_4|$ when they act on the LU orbit of $|0000\rangle$.

Proposition 8. *Let i, j, k, ℓ be distinct integers in $\{0, 1, 2, 3\}$, then $M_k^{(i,j)}$ and $M_\ell^{(i,j)}$ define the same right coset of the subgroup $\langle \mathtt{SWAP} \rangle_4$ in $\langle \mathtt{CNOT} \rangle_4$. This right coset is denoted by $\overline{(i,j)}$:*

$$\overline{(i,j)} = \{S_\sigma X_{[ij]} X_{[jk]} X_{[ki]} X_{[i\ell]} X_{[\ell j]} \mid \sigma \in \mathfrak{S}_4\}. \tag{32}$$

Proof. Let σ be the permutation $\begin{pmatrix} 0 & 1 & 2 & 3 \\ i & j & k & \ell \end{pmatrix}$. Using Identity (8), we conjugate each member of the

equality $M_3^{(0,1)} = S_{(0\,2\,3)} M_2^{(0,1)}$ (Identity (21)) by S_σ and obtain $M_\ell^{(i,j)} = S_{(ik\ell)} M_k^{(i,j)}$. $\square$

Proposition 9. *Let i, j, i', j' in $\{0, 1, 2, 3\}$ such that $i \neq j$ and $i' \neq j'$. If (i, j) and (i', j') are distinct couples, then $\overline{(i,j)}$ and $\overline{(i', j')}$ are distinct cosets.*

Proof. We check that for any i, j, k, i', j', k' in $\{0, 1, 2, 3\}$ (i, j, k distinct and i', j', k' distinct), if $A_k^{(i,j)} = \sigma A_{k'}^{(i', j')}$ for some permutation matrix σ, then $(i, j) = (i', j')$ (function `check_distinct_cosets` of the Python module). $\square$

Proposition 10. *For any permutation σ, one has : $S_\sigma \overline{(i,j)} S_\sigma^{-1} = \overline{(\sigma(i), \sigma(j))}$.*

Proof. Using Identity (8), one has: $S_\sigma M_k^{(i,j)} S_\sigma^{-1} = M_{\sigma(k)}^{(\sigma(i), \sigma(j))}$. The result follows from Proposition 8. $\square$

Remark 6 can be generalized to any coset $\overline{(i,j)}$, and one can define $\mathrm{PMAX}^{(i,j)}$ as being the set of all matrices $\mathcal{P}$ having their last column such that the state $M_k^{(i,j)} |\mathcal{P}\rangle$ maximizes $|\Delta_4|$.

Proposition 11. *Let i, j be distinct integers in $\{0, 1, 2, 3\}$ and σ be a permutation such that $\sigma(0) = i$ and $\sigma(1) = j$, then :*

$$\mathrm{PMAX}^{(i,j)} = \sigma \mathrm{PMAX}^{(0,1)}. \tag{33}$$

Proof. We prove that $\mathrm{PMAX}^{(i,j)} \subset \sigma \mathrm{PMAX}^{(0,1)}$, the other inclusion being similar. Let $\mathcal{P}$ be a matrix of parameters in $\mathrm{PMAX}^{(i,j)}$ and let $|\psi\rangle = M_k^{(i,j)} |\mathcal{P}\rangle$. Then $|\psi\rangle$ maximizes $|\Delta_4|$ and so does $S_\sigma^{-1} |\psi\rangle$. One has :

$$S_\sigma^{-1} |\psi\rangle = S_\sigma^{-1} M_k^{(i,j)} |\mathcal{P}\rangle = S_\sigma^{-1} M_k^{(i,j)} S_\sigma S_\sigma^{-1} |\mathcal{P}\rangle = M_{k'}^{(0,1)} S_\sigma^{-1} |\mathcal{P}\rangle,$$

where $k' \in \{2, 3\}$. Moreover, $S_\sigma^{-1} |\mathcal{P}\rangle = S_\sigma^{-1} U(\mathcal{P}) |0000\rangle = S_\sigma^{-1} U(\mathcal{P}) S_\sigma |0000\rangle$ and, by using Identity (9), one obtains $S_\sigma^{-1} |\mathcal{P}\rangle = |\sigma^{-1} \mathcal{P}\rangle$, hence

$$S_\sigma^{-1} |\psi\rangle = M_{k'}^{(0,1)} |\sigma^{-1} \mathcal{P}\rangle.$$

As $S_\sigma^{-1} |\psi\rangle$ maximizes $|\Delta_4|$, then $\sigma^{-1} \mathcal{P} \in \mathrm{PMAX}^{(0,1)}$, so $\mathcal{P} \in \sigma \mathrm{PMAX}^{(0,1)}$ and we deduce that $\mathrm{PMAX}^{(i,j)} \subset \sigma \mathrm{PMAX}^{(0,1)}$. $\square$

Our numerical results based on the use of the random walk heuristic suggest the following conjecture.

Conjecture 12. *Any four-qubit $\mathtt{CNOT}$ circuit capable of maximizing $|\Delta_4|$ by acting on the LU orbit of $|0000\rangle$ belongs to one of the 12 cosets $\overline{(i,j)}$.*

$$|L\rangle = \mathrm{e}^{-\mathrm{i}\frac{11\pi}{12}} \, U(\mathcal{P}_{\psi \to L}) \, |\psi_{\max}\rangle, \text{ where } \mathcal{P}_{\psi \to L} = \begin{bmatrix} \pi & \pi/2 & -\pi/2 \\ \pi/2 & -\pi/2 & \pi \\ 0 & \pi & \pi \\ 0 & -\pi/2 & \pi/2 \end{bmatrix}$$

$$|\Phi_5\rangle = \mathrm{e}^{-\mathrm{i}\frac{7\pi}{12}} \, U(\mathcal{P}_{\psi \to \Phi_5}) \, |\psi_{\max}\rangle, \text{ where } \mathcal{P}_{\psi \to \Phi_5} = \begin{bmatrix} -\pi/3 & \theta - \pi & -3\pi/4 \\ \pi/3 & \theta & 3\pi/4 \\ \pi & \theta & 3\pi/4 \\ 2\pi/3 & \pi - \theta & \pi/4 \end{bmatrix}$$

and $\theta = \cos^{-1} \frac{\sqrt{3}}{3}$

$$|M_{2222}\rangle = \mathrm{e}^{\mathrm{i}\frac{5\pi}{12}} \, U(\mathcal{P}_{\psi \to M_{2222}}) \, |\psi_{\max}\rangle, \text{ where } \mathcal{P}_{\psi \to M_{2222}} = \begin{bmatrix} \pi/2 & \pi/4 & 0 \\ -\pi/2 & -\pi/4 & \pi/2 \\ 0 & -\pi/2 & \pi/4 \\ \pi/2 & 3\pi/4 & \pi \end{bmatrix}$$

Figure 2. LU operators generating the states $|L\rangle$, $|\Phi_5\rangle$ and $|M_{2222}\rangle$, from the state $|\psi_{\max}\rangle$.

Example 13. Let $\sigma = (013)$. The unitary $S_\sigma M_2^{(0,1)}$ acting on the state $|\mathcal{P}_{\max}\rangle$ generates the state $S_{(013)} |\psi_{\max}\rangle$ that maximizes $|\Delta_4|$. This state can be produced by the unitary $M_{\sigma(2)}^{(\sigma(0),\sigma(1))} = M_2^{(1,3)}$ acting on the state $|\sigma \mathcal{P}_{\max}\rangle$, where

$$\sigma \mathcal{P}_{\max} = \sigma \begin{bmatrix} \pi/2 & \pi/2 & 0 \\ \pi/2 & \pi/2 & 0 \\ \pi/4 & \cos^{-1}\frac{\sqrt{3}}{3} & 0 \\ \pi/4 & \cos^{-1}\frac{\sqrt{3}}{3} & 0 \end{bmatrix} = \begin{bmatrix} \pi/4 & \cos^{-1}\frac{\sqrt{3}}{3} & 0 \\ \pi/2 & \pi/2 & 0 \\ \pi/4 & \cos^{-1}\frac{\sqrt{3}}{3} & 0 \\ \pi/2 & \pi/2 & 0 \end{bmatrix}.$$

6. Circuits Generating the States $|L\rangle$, $|\Phi_5\rangle$ and $|M_{2222}\rangle$

Let $|\psi\rangle$ be a state in the set $\{|L\rangle, |\Phi_5\rangle, |M_{2222}\rangle\}$. Following the Gour-Wallach conjecture (Gour and Wallach, 2012) proved by Chen and Djokovic (2013), there exists a matrix $\mathcal{P}$ of parameters and a phase φ such that

$$|\psi\rangle = \mathrm{e}^{\mathrm{i}\varphi} U(\mathcal{P}) |\psi_{\max}\rangle. \tag{34}$$

In order to compute $\mathcal{P}$ and φ, one can try to solve a 16 equations non-linear system but its resolution seems to be out of reach of current equation solvers (we used Maple and Python SymPy solvers). However, one can turn the problem of finding a solution of (34) into an optimization problem thanks to this simple remark : $|\psi\rangle = \mathrm{e}^{\mathrm{i}\varphi} U(\mathcal{P}) |\psi_{\max}\rangle$ if and only if the sum of the absolute values of the 16 coordinates of $|\psi\rangle - \mathrm{e}^{\mathrm{i}\varphi} U(\mathcal{P}) |L\rangle$ vanishes. Again, we use a random walk on a search space of 13 parameters (the 12 parameters of $\mathcal{P}$ plus the phase φ) to minimize this sum and obtain an approximate solution of the system (function `search_LU_from_state1_to_state2` of the Python module). From this approximate solution, we guess a possible form of an exact solution. Then, we check by CAS (using the Python module SymPy) that this form is indeed an exact solution. The results are summarized in Figure 2. Finally, using Identities (4) and (5), and

$$\theta = \cos^{-1}\frac{\sqrt{3}}{3}$$
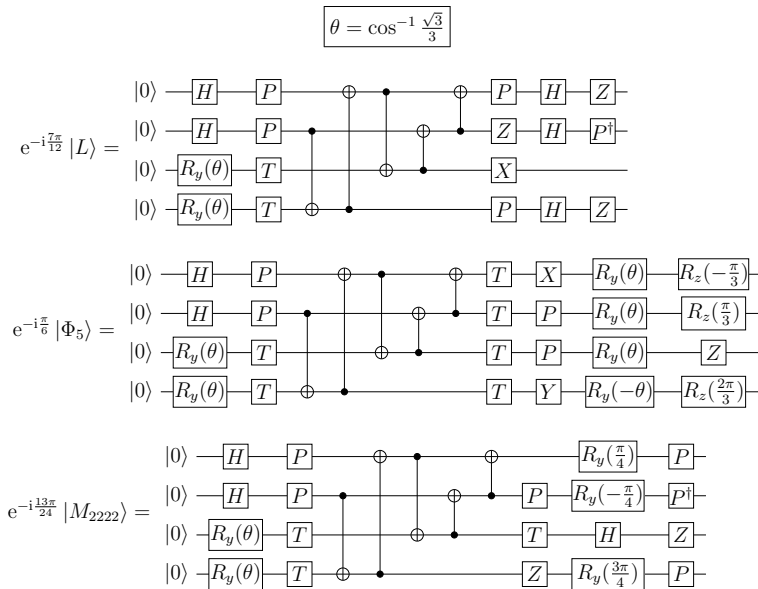
Figure 3. Quantum circuits generating the states $|L\rangle$, $|\Phi_5\rangle$ and $|M_{2222}\rangle$ up to a global phase. For better readability, most of the rotations around the $\hat{y}$ and $\hat{z}$ axes defined by the matrices of parameters are written using the universal single-qubit gates H, P, T (see Identity (4)).
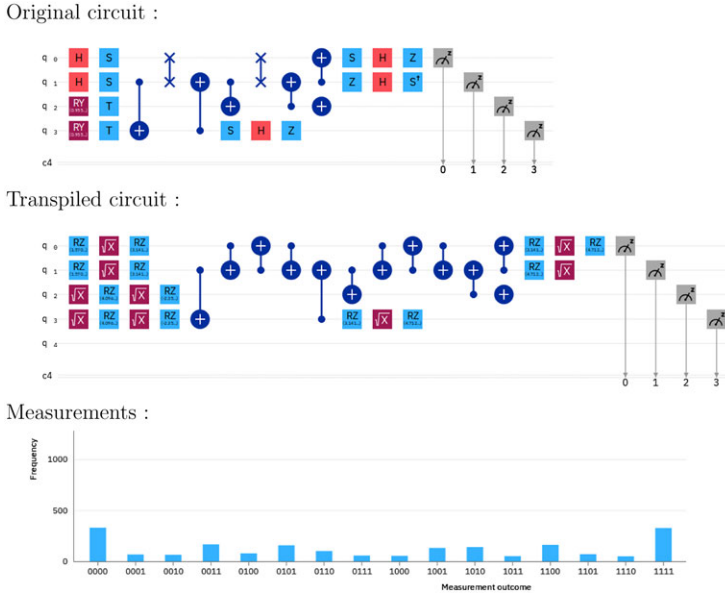
combining the results in Figure 2 with those of Proposition 1 , it is possible to build simple quantum circuits generating the states $|L\rangle$, $|\Phi_5\rangle$ and $|M_{2222}\rangle$. These circuits are depicted in Figure 3, and one can check that they are correct by using the function `check_circuits` of the Python module.

   We implemented the circuit generating the state $|L\rangle$ in one of the quantum computers publicly available at https://quantum-computing.ibm.com/. In those computers, full connectivity between the qubits is not achieved and the direct connections allowed between two qubits are given by a graph. Moreover, due to the noise in the gates, and particularly in the 2-qubit gates, it is of crucial importance to use as few CNOT gates as possible. We chose the 5-qubit ibmq_quito computer because its graph is $\{\{1, 0\}, \{1, 2\}, \{1, 3\}, \{3, 4\}\}$; hence, the gates $X_{[0\,1]}, X_{[1\,2]}$ and $X_{[3\,1]}$ of the CNOT subcircuit that implements the operator $M_2^{(0,1)} = X_{[0\,1]}X_{[1\,2]}X_{[2\,0]}X_{[0\,3]}X_{[3\,1]}$ are already native gates. The two other gates of the CNOT subcircuit, namely $X_{[2\,0]}$ and $X_{[0\,3]}$, are not native CNOT gates and can be simulated thanks to the use of SWAP gates. Finally, the operator $M_2^{(0,1)}$ can be implemented using only 11 native CNOT gates:

$$M_2^{(0,1)} = X_{[0\,1]}X_{[1\,2]}\underbrace{X_{[0\,1]}X_{[1\,0]}X_{[0\,1]}}_{S_{(0\,1)}} X_{[2\,1]}X_{[1\,3]}\underbrace{X_{[0\,1]}X_{[1\,0]}X_{[0\,1]}}_{S_{(0\,1)}} X_{[3\,1]}. \tag{35}$$

   After compilation by the IBM algorithm (the process is called *transpilation* on the website), the quantum circuit implementing the state $|L\rangle$ uses 22 single-qubit native gates, 11 CNOT gates and has a total depth of 18 (see Figure 4). However, despite of this moderate length, we observed after measurement the apparition of a large quantity of scorias (see the bar chart in Figure 4). Indeed, from Identity (1), one has

$$|L\rangle = \tfrac{1}{2}e^{i\frac{\pi}{6}}(|0000\rangle + |1111\rangle) + \tfrac{\sqrt{3}}{6}e^{-i\frac{\pi}{3}}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle), \tag{36}$$

Original circuit :

Transpiled circuit :

Measurements :

Figure 4. Implementation in the ibmq_quito quantum computer of a circuit generating the state $|L\rangle$. The bar chart is based on 2000 measurements.

so, ideally, the states $|0001\rangle$, $|0010\rangle$, $|0100\rangle$, $|1000\rangle$, $|0111\rangle$ $|1011\rangle$, $|1101\rangle$, or $|1110\rangle$ should not appear after measurement. The main causes of this problem are, on one hand, the measurement errors (average readout error is about 3 percent on this device), and on the other hand, the noise in the gates (CNOT gate average error is about 1.3 percent). This suggests that there are still significant technological challenges to overcome before we can implement the state $|L\rangle$ in a reliable fashion.

7. Conclusion and Perspectives

In this work, we described how a CNOT circuit acting on a factorized state can produce four-qubit maximum hyperdeterminant states and we proposed a quantum circuit generating the state $|L\rangle$, whose interesting properties where described by Gour and Wallach (2012) and by Chen and Djokovic (2013). It would be interesting to know whether it is possible to generalize this result when the number of qubits n is greater than 4. Is it still possible to reach a MHS by a CNOT circuit acting on a factorized state? What would be in this case the generalization of the unitary $M_k^{(i,j)}$ to higher dimensions? However, answering these questions seems to be currently out of reach because an explicit polynomial expression of the hyperdeterminant is known only up to 4 qubits. A first approach would be to know if a generically entangled state (i.e. a state $|\psi\rangle$ such that $\Delta_n(|\psi\rangle) \neq 0$) can be produced by a CNOT circuit acting on a factorized state in the case of any n-qubit system. Indeed, the vanishing of Δ_n can be tested using the following criterion (Gelfand et al., 1992, p. 445): let $A = \sum_{0 \leqslant i_0,\dots,i_{n-1} \leqslant 1} a_{i_0 \dots i_{n-1}} x_{i_0}^{(0)} \dots x_{i_{n-1}}^{(n-1)}$ be the multilinear form associated with the n-qubit state $|\psi\rangle = \sum_{0 \leqslant i_0,\dots,i_{n-1} \leqslant 1} a_{i_0 \dots i_{n-1}} |i_0 \dots i_{n-1}\rangle$, then the condition $\Delta_n(|\psi\rangle) = 0$ means that the system

$$\left\{ A = \frac{\partial A}{\partial x_0^{(0)}} = \frac{\partial A}{\partial x_1^{(0)}} = \cdots = \frac{\partial A}{\partial x_0^{(n-1)}} = \frac{\partial A}{\partial x_1^{(n-1)}} = 0 \right\} \qquad (37)$$

has a solution $\left(x_0^{(0)}, x_1^{(0)}, \ldots, x_0^{(n-1)}, x_1^{(n-1)} \right)$ such that $\left(x_0^{(i)}, x_1^{(i)} \right) \neq (0,0)$ for any $i = 0 \ldots n-1$.

Such a solution is called non-trivial. Therefore, to show that a state $|\psi\rangle$ is generically entangled, it is sufficient to prove that the system corresponding to $|\psi\rangle$ has no solutions apart from the trivial solutions. We will go back to these questions in future works.

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