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Schrödinger equations with asymptotically periodic terms

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We study the existence of non-trivial solutions for a class of asymptotically periodic semilinear Schrödinger equations in $\mathbb{R}^N$. By combining variational methods and the concentration-compactness principle, we obtain a non-trivial solution for the asymptotically periodic case and a ground state solution for the periodic one. In the proofs we apply the mountain pass theorem and its local version.

Keywords: semilinear elliptic equation; Schrödinger equation; mountain pass theorem; asymptotically periodic problems; concentration-compactness principle

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1. Introduction

In this paper we study the existence of non-trivial solutions for the semilinear Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $V\colon \mathbb{R}^N \to \mathbb{R}$ and $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions. In our main result we establish the existence of a solution for (1.1) under an asymptotic periodicity condition at infinity.

In order to precisely state our results, we denote by $\mathcal{F}$ the class of functions $h \in C(\mathbb{R}^N,\mathbb{R}) \cap L^\infty(\mathbb{R}^N,\mathbb{R})$ such that, for every $\varepsilon > 0$, the set $\{x \in \mathbb{R}^N : |h(x)| \geqslant \varepsilon\}$ has finite Lebesgue measure. We suppose that V is a perturbation of a periodic function at infinity in the following sense.

(V) There exist a constant $a_0 > 0$ and a function $V_0 \in C(\mathbb{R}^N,\mathbb{R})$, 1-periodic in x_i, $1 \leqslant i \leqslant N$, such that $V_0 - V \in \mathcal{F}$ and

$$V_0(x) \geqslant V(x) \geqslant a_0 > 0 \quad \text{for all } x \in \mathbb{R}^N.$$

Considering $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$ the primitive of $f \in C(\mathbb{R}^N \times \mathbb{R},\mathbb{R})$, we also suppose the following hypotheses:

(f_1) $F(x,t) \geqslant 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $f(x,t) = o(t)$ as $t \to 0$, uniformly for $x \in \mathbb{R}^N$;

(f_2) there exists a function $b \in C(\mathbb{R} \setminus \{0\}, \mathbb{R}^+)$ such that

$$\hat{F}(x,t) = \tfrac{1}{2} f(x,t)t - F(x,t) \geqslant b(t)t^2$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$;

(f_3) there exist $a_1 > 0$, $R_1 > 0$ and $\tau > \max\{1, N/2\}$ such that

$$|f(x,t)|^\tau \leqslant a_1 |t|^\tau \hat{F}(x,t)$$

for all (x,t) with $|t| > R_1$;

(f_4) uniformly in $x \in \mathbb{R}^N$, it holds that

$$\lim_{|t| \to +\infty} \frac{F(x,t)}{t^2} = +\infty;$$

(f_5) there exist $q \in (2, 2^*)$ and functions $h \in \mathcal{F}$, $f_0 \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, 1-periodic in x_i, $1 \leqslant i \leqslant N$, such that:

(i) $F(x,t) \geqslant F_0(x,t) = \int_0^t f_0(x,s)\,\mathrm{d}s$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$;

(ii) $|f(x,t) - f_0(x,t)| \leqslant h(x)|t|^{q-1}$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$;

(iii) $f_0(x,\cdot)/|\cdot|$ is increasing in $\mathbb{R} \setminus \{0\}$ for all $x \in \mathbb{R}^N$.

The main result of this paper can be stated as the following theorem.

THEOREM 1.1. *Suppose that V satisfies (V) and f satisfies (f_1)–(f_5). Then problem* (1.1) *possesses a solution.*

As a by-product of our calculations we can obtain a weak solution for the periodic problem. In this setting we can drop the condition (f_5) and we shall prove the following result.

THEOREM 1.2. *Suppose that $V(\cdot)$ and $f(\cdot,t)$ are 1-periodic in x_i, $1 \leqslant i \leqslant N$, and $V(x) \geqslant a_0 > 0$ for all $x \in \mathbb{R}^N$. If f satisfies (f_1), (f_3), (f_4) and*

$(f_2)'$ $\hat{F}(x,t) > 0$ *for all* $t \neq 0$,

then problem (1.1) *possesses a ground-state solution.*

Problems such as (1.1) have been the focus of intensive research in recent years. Initially, several authors dealt with the case in which f behaves like $q(x)|t|^{p-1}t$, $1 < p < 2^* - 1$, and V is constant (see [5, 6]). In the work of Rabinowitz [14] and Rabinowitz and Coti Zelati [8], the classical superlinear condition due to Ambrosetti and Rabinowitz was imposed:

(AR) there exists $\mu > 2$ such that

$$0 < \mu F(x,t) \leqslant f(x,t)t$$

for all $x \in \mathbb{R}^N$ and $t \neq 0$.

This hypothesis has an important role in the proof that Palais–Smale sequences are bounded. In this work we assume condition (f_4), which is weaker than condition (AR). It has already appeared in the papers of Ding and Lee [9] and Ding and Szulkin [10].

We emphasize that in theorem 1.1 we do not suppose periodicity on V or $f(\cdot, t)$. Instead, we consider the asymptotically periodic case as in the paper of Lins and Silva [12]. Condition (f_5) means that f is a perturbation of the periodic function f_0. In this context, we refer the reader to [1–3, 12, 16, 17] for some related (but not comparable) results.

As an example of an application of our main theorem, we take $a \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N, \mathbb{R})$ 1-periodic in x_i, $1 \leqslant i \leqslant N$, with $a(x) \geqslant 2$. Define the functions

$$f(x,t) = a(x)t\ln(1+t) + \mathrm{e}^{-|x|^2} t(\ln(1+t) + 1 - \cos(t)), \quad t \geqslant 0,$$
$$f_0(x,t) = a(x)t\ln(1+t), \quad t \geqslant 0,$$

and $f(x,t) = -f(x,-t)$, $f_0(x,t) = -f_0(x,-t)$ for $t < 0$. This function satisfies (f_1)–(f_5) but not (AR). Moreover, $f(x,t)/t$ is oscillatory, and therefore the Nehari approach used in [18] is not applicable.

The paper is organized as follows. In § 2 we present some technical results that are used throughout the work while § 3 is devoted to the proofs of theorems 1.1 and 1.2.

2. Preliminary results

In this section we present some preliminaries for the proofs of our main theorems. We denote by $B_R(y)$ the open ball in $\mathbb{R}^N$ of radius $R > 0$ and centre at y. The Lebesgue measure of a set $A \subset \mathbb{R}^N$ will be denoted by $|A|$. For brevity we write $\int_A u$ instead of $\int_A u(x)\,\mathrm{d}x$. We also omit the set A whenever $A = \mathbb{R}^N$. Finally, $|\cdot|_p$ denotes the norm in $L^p(\mathbb{R}^N)$ for $1 \leqslant p \leqslant \infty$.

Throughout the paper we assume that the potential V satisfies assumption (V). This implies that the norm

$$\|u\|^2 = \int (|\nabla u|^2 + V(x)u^2), \quad u \in H^1(\mathbb{R}^N),$$

is equivalent to the usual one. In what follows we denote by H the space $H^1(\mathbb{R}^N)$ endowed with the above norm.

In our first lemma we obtain the basic estimates on the behaviour of the nonlinearity f.

LEMMA 2.1. *Suppose that f satisfies (f_1), (f_3) and part (ii) of (f_5). Then, for any given $\varepsilon > 0$, there exist $C_\varepsilon > 0$ and $p \in (2, 2^*)$ such that*

$$|f(x,t)| \leqslant \varepsilon|t| + C_\varepsilon|t|^{p-1}, \qquad |F(x,t)| \leqslant \varepsilon|t|^2 + C_\varepsilon|t|^p \tag{2.1}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Proof. Taking $\varepsilon > 0$ and using (f_1), we obtain $\delta > 0$ such that

$$|f(x,t)| \leqslant \varepsilon|t|, \quad x \in \mathbb{R}^N, \ |t| \leqslant \delta. \tag{2.2}$$

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By (f_3) there exists $R > 0$ satisfying

$$|f(x,t)|^\tau \leqslant a_1|t|^\tau \hat{F}(x,t) \leqslant \tfrac{1}{2}a_1|t|^{\tau+1}|f(x,t)|, \quad x \in \mathbb{R}^N,\ |t| \geqslant R.$$

Then, setting $p = 2\tau/(\tau-1)$, we can use $\tau > N/2$ to conclude that $2 < p < 2^*$. Moreover,

$$|f(x,t)| \leqslant C|t|^{(\tau+1)/(\tau-1)} = C|t|^{p-1}, \quad x \in \mathbb{R}^N,\ |t| \geqslant R. \tag{2.3}$$

From the continuity and periodicity of f_0 we obtain $M > 0$ such that

$$|f_0(x,t)| \leqslant M, \quad x \in \mathbb{R}^N,\ \delta \leqslant |t| \leqslant R.$$

Now, using part (ii) of (f_5), we obtain

$$|f(x,t)| \leqslant |h|_\infty |t|^{q-1} + M \leqslant \left(|h|_\infty + \frac{M}{\delta^{q-1}}\right)|t|^{q-1}, \quad x \in \mathbb{R}^N,\ \delta \leqslant |t| \leqslant R.$$

This, (2.2) and (2.3) proves the first inequality in (2.1). The second one follows directly by integration. □

In view of the above lemma, the functional $I\colon H \to \mathbb{R}$ given by

$$I(u) = \tfrac{1}{2}\|u\|^2 - \int F(x,u)$$

is well defined. Moreover, standard calculations show that $I \in C^1(H,\mathbb{R})$ and the Gateaux derivative of I is given by

$$I'(u)v = \int (\nabla u \nabla v + V(x)uv) - \int f(x,u)v$$

for any $u, v \in H$. Hence, the critical points of I are precisely the weak solutions of problem (1.1).

We recall that $(u_n) \subset H$ is called a Cerami sequence for the functional I at level $c \in \mathbb{R}$ if $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$. The following result is a version of the classical mountain pass theorem [4]. It says that the mountain pass geometry is sufficient for obtaining a Cerami sequence. We refer the reader to [15] for the proof.

Theorem 2.2. *Let E be a real Banach space. Suppose that $I \in C^1(E,\mathbb{R})$ satisfies $I(0) = 0$ and*

(I_1) there exist $\rho, \alpha > 0$ such that $I(u) \geqslant \alpha > 0$ for all $\|u\| = \rho$,

(I_2) there exists $e \in E$ with $\|e\| > \rho$ such that $I(e) \leqslant 0$.

Then I possesses a Cerami sequence at level

$$c = \inf_{\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1],E)\colon \gamma(0) = 0,\ \|\gamma(1)\| > \rho,\ I(\gamma(1)) \leqslant 0\}.$$

Since in our setting we are not able to prove compactness for I, we shall use the following local version of the above result (see [12, theorem 2.3]).

THEOREM 2.3 (local mountain pass theorem). *Let E be a real Banach space. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies $I(0) = 0$, (I_1) and (I_2). If there exists $\gamma_0 \in \Gamma$, Γ defined as in theorem 2.2, such that*

$$c = \max_{t \in [0,1]} I(\gamma_0(t)) > 0,$$

then I possesses a non-trivial critical point $u \in \gamma_0([0,1])$ at the level c.

In the next result we prove that the functional I verifies the geometric conditions of the mountain pass theorem.

LEMMA 2.4. *Suppose that f satisfies (f_1), (f_3), (f_4) and part (ii) of (f_5). Then I satisfies (I_1) and (I_2).*

Proof. By lemma 2.1 and Sobolev's inequality, we have

$$\int F(x, u) \leqslant \varepsilon |u|_2^2 + C_\varepsilon |u|_p^p \leqslant c_1 \varepsilon \|u\|^2 + C\|u\|^p$$

for some $c_1 > 0$. Since $p > 2$, we have

$$I(u) \geqslant (\tfrac{1}{2} - c_1\varepsilon)\|u\|^2 + o(\|u\|^2) \geqslant \alpha$$

for $\|u\| = \rho$ small enough. This proves (I_1).

In order to verify (I_2), we fix $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfying $\varphi(x) \geqslant 0$ in $\mathbb{R}^N$ and $\|\varphi\| = 1$. We claim that there is $R_0 > 0$ such that $I(R\varphi) < 0$ for any $R > R_0$. If this is true, it suffices to take $e = R\varphi$ with $R > 0$ large enough to get (I_2).

For the proof of the claim we set $k = 2/\int \varphi^2$ and use (f_4) to obtain $M > 0$ satisfying

$$F(x, t) \geqslant kt^2 \quad \text{for all } |t| \geqslant M.$$

Hence, setting $A_R = \{x \in \mathbb{R}^N : \varphi(x) \geqslant M/R\}$, we obtain

$$\int F(x, R\varphi) \geqslant \int_{A_R} F(x, R\varphi) \geqslant kR^2 \int_{A_R} \varphi^2. \tag{2.4}$$

Since $\varphi \geqslant 0$, we can choose $R_0 > 0$ such that, for any $R \geqslant R_0$, it holds that

$$\int_{A_R} \varphi^2 \geqslant \frac{1}{2} \int \varphi^2.$$

It follows from the definition of k and (2.4) that $\int F(x, R\varphi) \geqslant R^2$, and therefore

$$I(R\varphi) \leqslant \tfrac{1}{2}R^2 - R^2 = -\tfrac{1}{2}R^2 < 0$$

for any $R > R_0$. □

LEMMA 2.5. *Suppose that f satisfies (f_1)–(f_4) and part (ii) of (f_5). Then any Cerami sequence for I is bounded.*

Proof. We adapt here an argument from [9]. Let $(u_n) \subset H$ be such that

$$\lim_{n\to+\infty} I(u_n) = c \quad \text{and} \quad \lim_{n\to+\infty}(1+\|u_n\|)\|I'(u_n)\|_{H'} = 0.$$

It follows that

$$c + o_n(1) = I(u_n) - \tfrac{1}{2}I'(u_n)u_n = \int \hat{F}(x,u_n), \tag{2.5}$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Suppose by contradiction that, for some subsequence still denoted (u_n), we have that $\|u_n\| \to \infty$. By defining $v_n = u_n/\|u_n\|$, we obtain

$$o_n(1) = \frac{I'(u_n)u_n}{\|u_n\|^2} = 1 - \int \frac{f(x,u_n)v_n}{\|u_n\|},$$

and therefore

$$\lim_{n\to+\infty} \int \frac{f(x,u_n)v_n}{\|u_n\|} = 1. \tag{2.6}$$

For any $r \geqslant 0$ we set

$$g(r) = \inf\{\hat{F}(x,t) \colon x \in \mathbb{R}^N,\ |t| \geqslant r\}.$$

Let $R_1 > 0$ be given by (f_3). For any $|t| > R_1$ there holds

$$a_1\hat{F}(x,t) \geqslant \left(\frac{f(x,t)}{t}\right)^{\tau} \geqslant \left(\frac{2F(x,t)}{t^2}\right)^{\tau}.$$

Hence, it follows from (f_4) that $\hat{F}(x,t) \to \infty$ as $t \to \infty$ uniformly in $x \in \mathbb{R}^N$. This, (f_2) and the definition of g imply that $g(r) > 0$ for all $r > 0$ and $g(r) \to \infty$ as $r \to \infty$.

For $0 \leqslant a < b$ we define

$$\Omega_n(a,b) = \{x \in \mathbb{R}^N \colon a \leqslant |u_n(x)| < b\},$$

and for $a > 0$,

$$c_a^b = \inf\left\{\frac{\hat{F}(x,t)}{t^2} \colon x \in \mathbb{R}^N,\ a \leqslant |t| \leqslant b\right\}.$$

From (f_2) we have that $c_a^b > 0$. By using (2.5) and the above definitions, we obtain

$$\begin{aligned} c + o_n(1) &= \int_{\Omega_n(0,a)} \hat{F}(x,u_n) + \int_{\Omega_n(a,b)} \hat{F}(x,u_n) + \int_{\Omega_n(b,\infty)} \hat{F}(x,u_n) \\ &\geqslant \int_{\Omega_n(0,a)} \hat{F}(x,u_n) + c_a^b \int_{\Omega_n(a,b)} u_n^2 + g(b)|\Omega_n(b,\infty)|, \end{aligned}$$

and therefore, for some $C_1 > 0$, we have that

$$\max\left\{\int_{\Omega_n(0,a)} \hat{F}(x,u_n), c_a^b \int_{\Omega_n(a,b)} u_n^2, g(b)|\Omega_n(b,\infty)|\right\} \leqslant C_1. \tag{2.7}$$

The above inequality implies that $|\Omega_n(b,\infty)| \leqslant C/g(b)$. Recalling that $g(b) \to +\infty$ as $b \to +\infty$, we conclude that

$$\lim_{b\to+\infty} |\Omega_n(b,\infty)| = 0. \tag{2.8}$$

For fixed $\mu \in [2, 2^*)$, by Hölder's inequality and Sobolev embedding, we obtain, for some $C_2 > 0$,

$$\begin{aligned}\int_{\Omega_n(b,\infty)} |v_n|^\mu &\leqslant \biggl(\int_{\Omega_n(b,\infty)} |v_n|^{2^*}\biggr)^{\mu/2^*} |\Omega_n(b,\infty)|^{(2^*-\mu)/2^*} \\ &\leqslant C_2\|v_n\|^\mu |\Omega_n(b,\infty)|^{(2^*-\mu)/2^*} = C_2|\Omega_n(b,\infty)|^{(2^*-\mu)/2^*}.\end{aligned}$$

Since $\mu < 2^*$, we conclude that

$$\lim_{b\to+\infty} \int_{\Omega_n(b,\infty)} |v_n|^\mu = 0. \tag{2.9}$$

Again from (2.7), for $0 < a < b$ fixed, it follows that

$$\int_{\Omega_n(a,b)} |v_n|^2 = \frac{1}{\|u_n\|^2} \int_{\Omega_n(a,b)} u_n^2 \leqslant \frac{1}{\|u_n\|^2} \frac{C_1}{c_a^b} = o_n(1).$$

Let $C_3 > 0$ be such that $|u|_2 \leqslant C_3\|u\|$ for all $u \in H$ and consider $\varepsilon \in (0, \frac{1}{3})$. By (f_1), there exists $a_\varepsilon > 0$ such that

$$|f(x,t)| \leqslant \frac{\varepsilon|t|}{C_3^2} \quad \text{for all } |t| \leqslant a_\varepsilon.$$

Hence,

$$\int_{\Omega_n(0,a_\varepsilon)} \frac{f(x,u_n)v_n}{\|u_n\|} \leqslant \frac{\varepsilon}{C_3^2} \int_{\Omega_n(0,a_\varepsilon)} v_n^2 \leqslant \varepsilon. \tag{2.10}$$

By using (f_5) and recalling that $h \in L^\infty(\mathbb{R}^N, \mathbb{R})$, we obtain $C_4 > 0$ such that $|f(x,u_n)| \leqslant C_4|u_n|$ for every $x \in \Omega_n(a_\varepsilon, b_\varepsilon)$. Thus,

$$\int_{\Omega_n(a_\varepsilon,b_\varepsilon)} \frac{f(x,u_n)v_n}{\|u_n\|} \leqslant C_4 \int_{\Omega_n(a_\varepsilon,b_\varepsilon)} v_n^2 < \varepsilon \quad \text{for all } n \geqslant n_0. \tag{2.11}$$

If we set $2\tau' = 2\tau/(\tau-1) \in (2, 2^*)$, we can use condition (f_3), (2.7) and Hölder's inequality to obtain

$$\begin{aligned}\int_{\Omega_n(b_\varepsilon,\infty)} \frac{f(x,u_n)v_n}{\|u_n\|} &= \int_{\Omega_n(b_\varepsilon,\infty)} \frac{f(x,u_n)v_n^2}{|u_n|} \\ &\leqslant \biggl(\int_{\Omega_n(b_\varepsilon,\infty)} \frac{|f(x,u_n)|^\tau}{|u_n|^\tau}\biggr)^{1/\tau} \biggl(\int_{\Omega_n(b_\varepsilon,\infty)} |v_n|^{2\tau'}\biggr)^{1/\tau'} \\ &\leqslant a_1^{1/\tau} \biggl(\int_{\Omega_n(b_\varepsilon,\infty)} \hat{F}(x,u_n)\biggr)^{1/\tau} \biggl(\int_{\Omega_n(b_\varepsilon,\infty)} |v_n|^{2\tau'}\biggr)^{1/\tau'} \\ &\leqslant C_1 \biggl(\int_{\Omega_n(b_\varepsilon,\infty)} |v_n|^{2\tau'}\biggr)^{1/\tau'}.\end{aligned}$$

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This expression and (2.9) provides $b_\varepsilon > 0$ large in such way that

$$\int_{\Omega_n(b_\varepsilon,\infty)} \frac{f(x,u_n)v_n}{\|u_n\|} < \varepsilon \quad \text{for all } n \geqslant n_0. \tag{2.12}$$

Finally, inequalities (2.10)–(2.12) imply that

$$\int \frac{f(x,u_n)v_n}{\|u_n\|} \leqslant 3\varepsilon < 1,$$

which contradicts (2.6). Therefore, (u_n) is bounded in H. □

REMARK 2.6. If f is periodic, we can obtain (2.11) without condition (f_5). Moreover, in this case, it follows from periodicity and continuity of F_0 that $F_0(x,t)/t^2 \geqslant k = k(a,b) > 0$ for all $x \in \Omega_n(a,b)$, and therefore the above lemma holds under the setting of theorem 1.2.

The next result was inspired by [8, lemma 2.18]. See also a classical result due to Lieb [11, theorem 8.10].

LEMMA 2.7. *Suppose that f satisfies (f_1) and (f_2). Let $(u_n) \subset H$ be a Cerami sequence for I at level $c > 0$. If $u_n \rightharpoonup 0$ weakly in H, then there exist a sequence $(y_n) \subset \mathbb{R}^N$ and $R > 0$, $\alpha > 0$ such that $|y_n| \to \infty$ and*

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \geqslant \alpha > 0.$$

Proof. Suppose, by contradiction, that the lemma is false. Then, for any $R > 0$, we have that

$$\limsup_{n\to\infty} \int_{B_R(y)} |u_n|^2 = 0 \quad \text{for all } R > 0.$$

Hence, we can use a result of Lions (see [13, lemma I.1] or [8, lemma 2.18]) to conclude that $|u_n|_s \to 0$ for any $s \in (2, 2^*)$. It follows from the second inequality in (2.1) that

$$\limsup_{n\to+\infty} \int F(x,u_n) \leqslant \limsup_{n\to\infty} \left(\varepsilon \int |u_n|^2 + C_\varepsilon \int |u_n|^p \right) \leqslant C\varepsilon,$$

where we have used the boundedness of (u_n) in $L^2(\mathbb{R}^N)$. Since ε is arbitrary, we conclude that $\int F(x,u_n) \to 0$ as $n \to +\infty$. The same argument and the first inequality in (2.1) imply that $\int f(x,u_n)u_n \to 0$ as $n \to +\infty$.

Since (u_n) is a Cerami sequence, we obtain

$$c = \lim_{n\to\infty} [I(u_n) - \tfrac{1}{2} I'(u_n)u_n] = \lim_{n\to\infty} \int (\tfrac{1}{2} f(x,u_n)u_n - F(x,u_n)) = 0,$$

which contradicts $c > 0$. The lemma is proved. □

We finish the section by stating two technical convergence results. The proofs can be found in lemmas 5.1 and 5.2 of [12], respectively.

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LEMMA 2.8. *Suppose that V satisfies (V) and f satisfies (f_5). Let $(u_n) \subset H$ be a bounded sequence and let $v_n(x) = v(x - y_n)$, where $v \in H$ and $(y_n) \subset \mathbb{R}^N$. If $|y_n| \to \infty$, then we have*

$$(V_0(x) - V(x))u_n v_n \to 0, \qquad (f_0(x, u_n) - f(x, u_n))v_n \to 0$$

strongly in $L^1(\mathbb{R}^N)$ as $n \to \infty$.

LEMMA 2.9. *Suppose that $h \in \mathcal{F}$ and $s \in [2, 2^*]$. If $(v_n) \subseteq H^1(\mathbb{R}^N)$ is such that $v_n \rightharpoonup v$ weakly in H, then*

$$\lim_{n\to+\infty} \int h|v_n|^s = \int h|v|^s.$$

3. Proofs of the main results

In this section we denote by $I_0 \colon H \to \mathbb{R}$ the functional associated with the periodic problem, namely,

$$I_0(u) = \frac{1}{2}\int (|\nabla u|^2 + V_0(x)u^2) - \int F_0(x, u).$$

We also consider the following norm in $H^1(\mathbb{R}^N)$:

$$\|u\|_0 = \left(\int |\nabla u|^2 + V_0(x)u^2\right)^{1/2},$$

which is equivalent to the usual norm of this space.

We are ready to prove our main theorem.

Proof of theorem 1.1. By lemma 2.4 and theorem 2.2, there exists a sequence $(u_n) \subset H$ such that

$$I(u_n) \to c \geqslant \alpha > 0 \quad \text{and} \quad (1 + \|u_n\|)I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

Applying lemma 2.5 we may assume, without loss of generality, that $u_n \rightharpoonup u$ weakly in H. We claim that $I'(u) = 0$. Indeed, since $C_0^\infty(\mathbb{R}^N)$ is dense in H, it suffices to show that $I'(u)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. We have

$$I'(u_n)\varphi - I'(u)\varphi = o_n(1) - \int [f(x, u_n) - f(x, u)]\varphi. \tag{3.2}$$

Using the Sobolev embedding theorem we can assume that, up to a subsequence, $u_n \to u$ in $L^s_{\text{loc}}(\mathbb{R}^N)$ for each $s \in [1, 2^*)$ and[1]

$$u_n(x) \to u(x) \quad \text{a.e. on } K \text{ as } n \to \infty,$$
$$|u_n(x)| \leqslant w_s(x) \in L^s(K) \quad \text{for every } n \in \mathbb{N} \text{ and a.e. on } K,$$

where K denotes the support of the function φ. Therefore,

$$f(x, u_n) \to f(x, u) \quad \text{a.e. on } K \text{ as } n \to \infty,$$

[1]Here, 'a.e.' is a short form of 'almost everywhere on K or almost every point x in K in sense of measure'.

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and using (2.1) we obtain

$$|f(x,u_n)\varphi| \leqslant \varepsilon|w_2|\,|\varphi| + C_\varepsilon|w_{p-1}|\,|\varphi| \in L^1(K).$$

Thus, taking the limit in (3.2) and using the Lebesgue dominated convergence theorem, we obtain

$$I'(u)\varphi = \lim_{n\to\infty} I'(u_n)\varphi = 0,$$

which implies that $I'(u) = 0$.

If $u \neq 0$, the theorem is proved. So, we deal in the following with the case in which $u = 0$. By lemma 2.7, we recall that there exist a sequence $(y_n) \subset \mathbb{R}^N$, $R > 0$ and $\alpha > 0$ such that $|y_n| \to \infty$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \geqslant \alpha > 0. \tag{3.3}$$

Without loss of generality, we may assume that $(y_n) \subset \mathbb{Z}^N$ (see [7, p. 7]). Setting $\tilde{u}_n(x) = u_n(x+y_n)$ and observing that $\|\tilde{u}_n\| = \|u_n\|_0$, up to a subsequence we have that $\tilde{u}_n \rightharpoonup \tilde{u}$ in H, $\tilde{u}_n \to \tilde{u}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and for almost every $x \in \mathbb{R}^N$. From (3.3), we have $\tilde{u} \neq 0$.

Claim 3.1. $I_0'(\tilde{u}) = 0$.

To prove the claim we take $\varphi \in C_0^\infty(\mathbb{R}^N)$ and define, for each $n \in \mathbb{N}$, $\varphi_n(x) = \varphi(x - y_n)$. Arguing as in the beginning of the proof and using the periodicity of f_0, we obtain

$$I_0'(\tilde{u})\varphi = I_0'(\tilde{u}_n)\varphi + o_n(1) = I_0'(u_n)\varphi_n + o_n(1),$$

and therefore it suffices to check that $I_0'(u_n)\varphi_n = o_n(1)$. To achieve this objective we notice that, by lemma 2.8,

$$\begin{aligned} I_0'(u_n)\varphi_n &= I'(u_n)\varphi_n + \int [V_0(x) - V(x)]u_n\varphi_n - \int [f_0(x,u_n) - f(x,u)]\varphi_n \\ &= I'(u_n)\varphi_n + o_n(1). \end{aligned}$$

So, by (3.1), the claim is verified.

Claim 3.2. $\displaystyle\liminf_{n\to\infty} \int \hat{F}(x,u_n) \geqslant \int \hat{F}_0(x,\tilde{u})$.

By using part (ii) of (f_5) and a straightforward calculation, we obtain

$$|\hat{F}(x,t) - \hat{F}_0(x,t)| \leqslant \left(\frac{1}{2} + \frac{1}{q}\right) h(x)|t|^q.$$

Since $u_n \rightharpoonup 0$ weakly in H, it follows from the above inequality and lemma 2.9 that

$$\lim_{n\to\infty} \int \hat{F}(x,u_n) = \lim_{n\to\infty} \int \hat{F}_0(x,u_n) = \liminf_{n\to\infty} \int \hat{F}_0(x,\tilde{u}_n) \geqslant \int \hat{F}_0(x,\tilde{u}),$$

where we have also used the periodicity of $\hat{F}_0$.

The above claim and (3.1) provide

$$c = \lim_{n\to\infty}[I(u_n) - \tfrac{1}{2}I'(u_n)u_n] = \liminf_{n\to\infty}\int \hat{F}(x,u_n) \geqslant \int \hat{F}_0(x,\tilde{u})$$
$$= I_0(\tilde{u}) - \tfrac{1}{2}I_0'(\tilde{u})\tilde{u} = I_0(\tilde{u}),$$

and therefore $I_0(\tilde{u}) \leqslant c$. It follows from part (iii) of (f_5) that $\max_{t\geqslant 0} I_0(t\tilde{u}) = I_0(\tilde{u})$. Hence, by the definition of c, (V) and part (i) of (f_5), we have that

$$c \leqslant \max_{t\geqslant 0} I(t\tilde{u}) \leqslant \max_{t\geqslant 0} I_0(t\tilde{u}) = I_0(\tilde{u}) \leqslant c.$$

We can now invoke theorem 2.3 to conclude that I possesses a critical point at level $c > 0$. This finishes the proof. □

We proceed now with the proof of the periodic result.

Proof of theorem 1.2. We first notice that lemmas 2.1, 2.4 and 2.5 are still valid under the assumptions of theorem 1.2. Hence, by lemma 2.4, we can use theorem 2.2 to obtain a sequence $(u_n) \subset H$ such that

$$\lim_{n\to+\infty} I_0(u_n) = c_0 \quad \text{and} \quad \lim_{n\to+\infty}(1 + \|u_n\|_0)\|I_0'(u_n)\| = 0,$$

where c_0 is the mountain pass level of I_0. Arguing as in the proof of theorem 1.1, we conclude that $u_n \rightharpoonup u$ weakly in H with $I_0'(u) = 0$.

As before, we only need to consider the case in which $u = 0$. By lemma 2.7, there are a sequence $(y_n) \subset \mathbb{Z}^N$, an $R > 0$ and an $\alpha > 0$ such that $|y_n| \to \infty$ as $n \to \infty$ and

$$\limsup_{n\to\infty}\int_{B_R(y_n)} |u_n|^2 \geqslant \alpha > 0. \tag{3.4}$$

Writing $\tilde{u}_n(x) = u_n(x+y_n)$ and observing that $\|\tilde{u}_n\|_0 = \|u_n\|_0$, up to a subsequence, we have $\tilde{u}_n \rightharpoonup \tilde{u}$ weakly in H, $\tilde{u}_n \to \tilde{u}$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $\tilde{u}_n(x) \to \tilde{u}(x)$ almost everywhere in $\mathbb{R}^N$. The local convergence and (3.4) imply that $\tilde{u} \neq 0$. Arguing as in the first claim of the proof of theorem 1.1, we conclude that $I_0'(\tilde{u}) = 0$, and therefore we obtain a non-zero weak solution.

In view of the above existence result, the number

$$m = \inf\{I_0(u); u \in E \text{ and } I'(u) = 0\} > 0$$

is well defined. We claim that m is achieved. Indeed, let $(u_n) \subset H$ be a minimizing sequence for m, namely,

$$I_0(u_n) \to m, \quad I_0'(u_n) = 0 \quad \text{and} \quad u_n \neq 0.$$

Since (u_n) is a Cerami sequence for I_0, it follows from lemma 2.5 that it is bounded. Moreover, using $I_0'(u_n)u_n = 0$ and (2.1) with ε small, we can obtain $k > 0$ satisfying $\|u_n\|_0 \geqslant k$. Thus, arguing as in the preceding paragraph, we obtain a translated subsequence $(\tilde{u}_n)$, which has a non-zero weak limit u_0 such that $I_0'(u_0) = 0$ and $\tilde{u}_n(x) \to u_0(x)$ a.e. in $\mathbb{R}^N$. By Fatou's lemma,

$$m = \lim_{n\to\infty} I_0(u_n) = \lim_{n\to\infty} I_0(\tilde{u}_n) = \liminf_{n\to\infty}\int \hat{F}_0(x,\tilde{u}_n) \geqslant \int \hat{F}_0(x,u_0) = I_0(u_0).$$

Consequently, $I_0(u_0) = m$, and therefore $u_0 \neq 0$ is a ground-state solution. □

We finish the paper by pointing out that, with a little more regularity on f, we are able to obtain signed solutions. Indeed, following a standard process we may define

$$f^+(x,t) = \begin{cases} f(x,t), & t \geqslant 0, \\ 0, & t < 0. \end{cases}$$

Then f^+ satisfies (f_1)–(f_5) and we can consider the truncated functional

$$I^+(u) = \tfrac{1}{2}\|u\|^2 - \int F^+(x,u),$$

where F^+ is the primitive of f^+, to obtain a non-zero critical point u of I^+. Testing the derivative at u with the function $\min\{u(x),0\}$, we conclude that $u \geqslant 0$. If we suppose that f is Lipschitz continuous, we can apply standard elliptic regularity theory and the maximum principle to prove that $u > 0$ in $\mathbb{R}^N$ (see [14]). If we truncate the right-hand side of the nonlinearity f we can also obtain a negative solution. We omit the details.

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