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ON A 3-DIMENSIONAL ISOPERIMETRIC PROBLEM

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Let $L(P)$ denote the total edge length and $A(P)$ the total surface area of a three-dimensional convex polyhedron P. In [5] it was shown that if P belongs to the set $\mathscr{S}^3$ of all polyhedra with triangular faces then for all $P \in \mathscr{S}^3$

$$L(P)^2/A(P) \geq 12\sqrt{3} \approx 20{\cdot}785$$

with equality if and only if $P \in \mathscr{S}^3$ is a regular tetrahedron.

It is not difficult to establish the inequality

$$L(P)^2/A(P) \geq 2(12-\sqrt{3}) \approx 20{\cdot}536$$

for all $P \in \mathscr{P}^3$, where $\mathscr{P}^3$ denotes the set of all three-dimensional prisms. Equality holds only for the equilateral triangular right prism $P \in \mathscr{P}^3$ with base B and height $h=(6-\sqrt{3})\cdot p(B)/9$ where $p(B)$ is the perimeter of the base B.

Now, let P^3 be the set of all three-dimensional convex polyhedra and let $\alpha(P^3)$ denote the infimum of the quotients $L(P)^2/A(P)$ for all $P \in P^3$. A theorem of Aberth [1] implies:

$$\alpha(P^3) > 6\pi \approx 18{\cdot}850. \tag{1}$$

Here we will establish the following inequality:

$$\alpha(P^3) \leq 2\left(\sqrt{3}+\frac{8\pi}{3}\right) \approx 20{\cdot}219. \tag{2}$$

Proof of (2). Like Besicovitch [2], we construct a sequence of polyhedra $\{P[v]\}$, where $P[v]$ is a so-called "shell-polyhedron" with $2v$ vertices. We will show:

$$\lim_{v\to\infty} \frac{L(P[v])^2}{A(P[v])} = 2\left(\sqrt{3}+\frac{8\pi}{3}\right).$$

Let C be a circle of unit radius and MN a chord of length $\sqrt{3}$. We divide the longer one of the two arcs $\widehat{MN}$ into v ($v \geq 2$) equal arcs $\widehat{MX_1}, \widehat{X_1X_2}, \ldots, \widehat{X_{v-1}N}$. Evidently

$$U = 2v\sin\frac{2\pi}{3v}+\sqrt{3}$$

is the perimeter and

$$F = \frac{\sqrt{3}}{4}+\frac{v}{2}\sin\frac{4\pi}{3v}$$

is the area of the convex $(v+1)$-gon $MX_1X_2\ldots X_{v-1}N$.

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Let $P[v]$ be that polyhedron which is the convex hull of two such $(v+1)$-gons having the edge MN in common and lying in two different planes. Suppose, for instance, the angle between these planes is 2ϕ, $0<\phi<\pi/2$, with $\sin\phi=v^{-2}$. Let $MY_1Y_2\ldots Y_{v-1}N$ denote the $(v+1)$-gon congruent to $MX_1X_2\ldots X_{v-1}N$. Then the polyhedron $P[v]$ has edge length

$$L(P[v]) = 2U-\sqrt{3}+\sum_{j=1}^{v-1} X_jY_j$$

$$= \sqrt{3}+4v\sin\frac{2\pi}{3v}+\sum_{j=1}^{v-1} X_jY_j$$

and surface area

$$A(P[v]) = 2F+2\sin\frac{2\pi}{3v}\left[\frac{X_1Y_1}{2}\cos\phi_1+\frac{X_1Y_1+X_2Y_2}{2}\cos\phi_2 +\cdots+\frac{X_{v-2}Y_{v-2}+X_{v-1}Y_{v-1}}{2}\cos\phi_{v-1} +\frac{X_{v-1}Y_{v-1}}{2}\cos\phi_v\right].$$

Here $2\phi_j\,(j=1,2,\ldots,v)$ denotes the angle between the line $X_{j-1}X_j$ and the line $Y_{j-1}Y_j$ with $X_0\equiv Y_0\equiv M$, $X_v\equiv Y_v\equiv N$ and $\phi_j=0$ if $X_{j-1}X_j$ and $Y_{j-1}Y_j$ are parallel.

Since $\phi_j\le\phi$ for $j=1,2,\ldots,v$ and $\phi_j\ne\phi$ for at least some j, $1\le j\le v$, we have:

$$A(P[v]) > \frac{\sqrt{3}}{2}+v\sin\frac{4\pi}{3v}+2\sin\frac{2\pi}{3v}\cos\phi\sum_{j=1}^{v-1} X_jY_j.$$

On the other hand, since for $\phi>0$ not all ϕ_j can be zero:

$$A(P[v]) < \frac{\sqrt{3}}{2}+v\sin\frac{4\pi}{3v}+2\sin\frac{2\pi}{3v}\sum_{j=1}^{v-1} X_jY_j.$$

Now, for $j=1,2,\ldots,v-1$:

$$X_jY_j = 2\sin\phi\left[\frac{1}{2}+\sin\left(\frac{4\pi j}{3v}-\frac{\pi}{6}\right)\right]$$

$$= \sin\phi\left[1+\sqrt{3}\sin\frac{4\pi j}{3v}-\cos\frac{4\pi j}{3v}\right].$$

Hence

$$\sum_{j=1}^{v-1} X_jY_j = \sin\phi\left[v+\left(2\sin\frac{2\pi}{3v}\right)^{-1}\left\{3\sin\frac{2\pi(v-1)}{3v}-\sqrt{3}\cos\frac{2\pi(v-1)}{3v}\right\}\right].$$

Since $\sin\phi=v^{-2}$, it follows that

$$\sum_{j=1}^{v-1} X_jY_j = v^{-1}+\left(2v^2\sin\frac{2\pi}{3v}\right)^{-1}\left[3\sin\frac{2\pi(v-1)}{3v}-\sqrt{3}\cos\frac{2\pi(v-1)}{3v}\right].$$

Therefore

$$Q_1(v) < \frac{L(P[v])^2}{A(P[v])} < Q_2(v)$$

with

$$\lim_{v\to\infty} Q_1(v) = \lim_{v\to\infty} Q_2(v) = \frac{\left(\sqrt{3}+\frac{8\pi}{3}\right)^2}{\frac{\sqrt{3}}{2}+\frac{4\pi}{3}} = 2\left(\sqrt{3}+\frac{8\pi}{3}\right).$$

Consequently

$$\lim_{v\to\infty} \frac{L(P[v])}{A(P[v])} = 2\left(\sqrt{3}+\frac{8\pi}{3}\right)$$

and (2) is proved.

According to [3], the functional L can be defined on the set C^3 of all three-dimensional convex bodies. In [3] the set of convex bodies $C \in C^3$ with $L(C) < \infty$ is denoted by C_1^3. Now, suppose

$$\alpha(P^3) = 2\left(\sqrt{3}+\frac{8\pi}{3}\right) = \kappa.$$

Then the proof of (2) would imply that $\alpha(P^3)$ is attained by a convex body $\bar{C} \in C_1^3 \backslash P^3$. ($\bar{C}$ has evidently no inner points.) In [4, pp. 155–156], Fejes Tóth conjectured that $\alpha(P^3)$ is probably not attained by a convex polyhedron.

QUESTION. Does there exist a convex body $C \in C_1^3$ with $L(C)^2/A(C) < \kappa$?

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