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ALGEBRAIC GEOMETRY FOR MV-ALGEBRAS

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Abstract. In this paper we try to apply universal algebraic geometry to MV algebras, that is, we study "MV algebraic sets" given by zeros of MV polynomials, and their "coordinate MV algebras". We also relate algebraic and geometric objects with theories and models taken in Łukasiewicz many valued logic with constants. In particular we focus on the structure of MV polynomials and MV polynomial functions on a given MV algebra.

§1. Introduction.

The spirit of this paper can be summarized by the following quotation, taken from [12]:

In spite of its geometric origin, topos theory has in recent years some-times been perceived as a branch of logic, partly because of the contributions to the clarification of logic and set theory which it has made possible. However, the orientation of many topos theorists could perhaps be more accurately summarized by the observation that what is usually called mathematical logic can be viewed as a branch of algebraic geometry, and it is useful to make this branch explicit in itself.

This paper is a first attempt of applying the concepts of algebraic geometry over fields to the theory of MV-algebras. According to [15], rational polyhedra are the genuine algebraic varieties of the formulas of Łukasiewicz Logic, in a precise sense: zerosets of McNaughton functions coincide with rational polyhedra. Now, McNaughton functions are functions from $[0, 1]^n$ to $[0, 1]$, so that in the theory of [15], the MV algebra $[0, 1]$ plays a fundamental role. On the other hand, there are reasons to be interested in other MV algebras, because every MV algebra can be viewed as the Lindenbaum algebra of some many-valued logic, and as such, it has logical relevance. This is why we try in this paper to extend somewhat the theory of [15] to MV algebras as generally as possible.

In order to develop our theory we proceed along lines similar to Plotkin [18], Sela [20], and Kharlampovich–Myasnikov [8], where an algebraic geometry over varieties in universal algebra is developed (see also e.g. [1, 5, 6, 16, 17, 19]).

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The universal algebraic geometry has proven useful especially for groups, where it has allowed Sela and, independently, Kharlampovich–Myasnikov, to solve a long-standing problem of Tarski on the elementary equivalence of finitely generated free nonabelian groups and the decidability of their theory.

We note that, in classical algebraic geometry over fields, the central notion is that of polynomial. One has two possibilities:

- considering algebraic geometry without coefficients; this means taking polynomials with integer coefficients (e.g. $x^2 - 2y$) and evaluating them in arbitrary fields, possibly of fixed characteristic (provided that the difference, zero and one are admitted as primitive symbols, besides sum and product, so that polynomials form a ring with unity);
- considering algebraic geometry with coefficients, where polynomials take coefficients in a field K and are evaluated in field extensions of K.

It turns out that both possibilities can be extended to universal algebra, and this is done in [18]. Since universal algebra subsumes the equational theory of MV algebras, we can consider what happens in universal algebraic geometry (coefficient-free or not) over MV algebras. All what is needed for switching from one kind of algebraic geometry to the other is a suitable choice of the functional language with which polynomials are written.

1.1. Overview. This paper is mainly motivated by a desire of understanding MV polynomials and MV polynomial functions in view of applications to Łukasiewicz logic. There is an interesting interplay between MV polynomials and MV polynomial functions over arbitrary MV algebras. Moreover, it seems interesting to see what MV polynomial functions become when we move from $[0, 1]$ (where they are nicely characterized as continuous, piecewise affine functions by McNaughton Theorem) to other MV algebras, possibly lacking a natural notion of topology and continuity. More in detail, the main achievements of this paper are as follows:

- we give a form of Nullstellensatz for MV polynomial algebras $A[x_1, \dots, x_n]$, see Theorem 4.12;
- we give a universal algebraic duality between MV algebraic sets and their coordinate MV algebras (generalizing the duality of [15]), see Theorem 5.6;
- we introduce the definition of "polynomially complete" MV algebra (i.e., one where MV polynomials and MV polynomial functions coincide) and we give a characterization of polynomially complete MV chains, see Theorem 6.11;
- we introduce the definition of "strongly complete" MV algebra (i.e., an MV algebra A where every principal ideal in the polynomial MV algebra $A[x_1, \dots, x_n]$ coincides with its radical) and characterize these algebras as the simple divisible ones, see Theorem 6.18;
- we prove the folklore, but important, result that the MV algebraic subsets of $[0, 1]^n$ coincide with the usual closed sets, see Proposition 6.26;
- we identify MV polynomial functions over any MV algebra with a kind of truncated functions, (see Theorem 7.2) and we use these functions to represent coordinate MV algebras (see Corollary 7.7);
- we characterize MV polynomial functions on MV chains, see Corollary 8.5;
- we give a completeness criterion for Łukasiewicz logic with constants in terms of polynomial completeness, see Proposition 9.2.

1.2. Related work. Our main source of inspiration is [15], where the Galois connection between theories and models is fully described for infinite valued Łukasiewicz logic. Because of the completeness theorem, we can say that all information for this connection is already provided by the MV algebra $[0, 1]$. However, since we are interested in a Diophantine approach to MV algebraic geometry, we would like to go beyond $[0, 1]$ and consider any MV algebra A. This corresponds to adding to Łukasiewicz logic the complete (first order) diagram of A.

Of course in the generalization we lose something: for instance, we lose the tight connection between zeros of (single) polynomials and principal polynomial ideals given by Wójcicki's Theorem (see [4]) in the case of $A = [0, 1]$. However, many concepts of [15] still make sense, like the category of algebraic sets and Z-maps (here replaced by polynomial maps) and the category of MV algebras and homomorphisms, as well as the equivalence between them.

A predecessor of this work is [14], where the duality between finitely presented MV algebras and rational polyhedra is carefully described. The duality is obtained by specializing the duality between MV algebras and subspaces of cubes, or equivalently, by specializing the duality between semisimple MV algebras and closed subspaces of cubes.

§2. Preliminaries.

In this section we give some preliminary definitions and results on MV algebras, the Mundici equivalence, and the McNaughton Theorem.

2.1. MV algebras. In order to make this preliminary subsection not too long, we give only a quick review of MV algebras, referring to [4] for further details.

An *MV-algebra* is a structure $(A, \oplus, ^*, 0)$, where $\oplus$ is a binary operation (called truncated sum), * is a unary operation (the negation) and 0 is a constant, such that the following axioms are satisfied for any $a, b \in A$:

i) $(A, \oplus, 0)$ is an abelian monoid,
ii) $(a^*)^* = a$,
iii) $0^* \oplus a = 0^*$,
iv) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Note that all axioms of MV algebras are equations between terms, so MV algebras form a variety.

An example of an MV-algebra is given by the real interval $[0, 1]$ where $a \oplus b = \min\{a + b, 1\}$ and $a^* = 1 - a$. This MV-algebra is important because it generates the variety of all MV-algebras, and also because it is one of the basic structures in fuzzy logic and fuzzy set theory, see [21].

On an MV-algebra A we define auxiliary concepts as follows:

v) $1 := 0^*$. (the unit)
vi) $a \odot b := (a^* \oplus b^*)^*$. (the Łukasiewicz product)
vii) $a \ominus b := a \odot b^*$. (the truncated difference)
viii) $d(a, b) = (a \ominus b) \oplus (b \ominus a)$. (the Chang distance)

for any $a, b \in A$.

Sometimes we simply write ab for $a \odot b$.

Note that in $[0, 1]$ we have $a \odot b = max(0, a + b - 1)$, $a \ominus b = max(0, a - b)$, and $d(a, b) = |a - b|$.

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If n is a positive integer, it is customary to denote by na for the sum of a with itself taken n times. The *order* of an element $a \in A$, written $ord(a)$, is the smallest n such that $na = 1$, or ∞ if no such n exists.

In an MV algebra we define a partial order by letting $a \leq b$ if there is c such that $a \oplus c = b$. Every MV algebra is a distributive lattice with this partial order. The supremum is $a \vee b = (a^* \oplus b)^* \oplus b$ and the infimum is $a \wedge b = (a^* \vee b^*)^*$. An *MV chain* is a linearly ordered MV algebra.

There is a notion of divisible MV algebra quite close to the usual notion of divisible group. An element $a \in A$ is *n-divisible* if there is $b \in A$ such that $(n-1)b = a \ominus b$. Intuitively, b stands for a/n. An MV algebra is *divisible* if every element is n-divisible for every $n \geq 1$. For every MV algebra A there is a divisible MV algebra, denoted by $DH(A)$ (the *divisible hull* of A), such that $DH(A)$ extends A, and for every morphism $\mu : A \to E$ with E divisible, there is a unique morphism $v : DH(A) \to E$ extending μ. We use the notation $DH(G)$ also to denote the divisible hull of a group G.

Since MV-algebras form a variety, the notions of free MV algebra over a set of generators and MV-homomorphism are just the particular cases of the corresponding universal algebraic notions. We can also form quotients, but instead of congruences we use ideals. An *ideal* of an MV algebra A is a subset J of A such that if $a, b \in J$ then $a \oplus b \in J$, and if $c \in J$ and $d \leq c$ then $d \in J$.

Two elements $a, b \in A$ are called *congruent* modulo an ideal J, written $a \equiv_J b$, if $d(a, b) \in J$. Then the quotient set $A/\equiv_J$ has the structure of an MV algebra.

If S is a subset of an MV algebra A, we denote by $id(S)$ the ideal generated by S.

We define $J_1 + J_2 = id(J_1 \cup J_2)$. It results that $a \in J_1 + J_2$ if and only if there are $b_1 \in J_1, b_2 \in J_2$ such that $a \leq b_1 \oplus b_2$. Similarly we define the sum of an arbitrary set of ideals.

An ideal J is *prime* if for every $a, b \in A$, either $a \ominus b \in J$ or $b \ominus a \in J$. It turns out that an ideal J is prime if and only if $A/\equiv_J$ is linearly ordered.

The *radical* of an MV algebra A, denoted by $Rad(A)$, is the intersection of its maximal ideals. The *rank* of A is the cardinality of $A/Rad(A)$.

A *discrete* MV algebra is one with minimum nonzero element.

A *simple* MV algebra is an MV algebra where (0) is the only proper ideal. Equivalently, by [4], an MV algebra A is simple if and only if it is a subalgebra of $[0, 1]$. For instance, the simple finite algebras are given by $S_n = \{0, 1/n, 2/n, \dots, (n-1)/n, 1\}$ for every $n \geq 1$.

2.2. ℓu-groups, Mundici functors, and good sequences.

Definition 2.1. Recall that a *lattice ordered abelian group* (ℓ-group) is an abelian group G equipped with a translation-invariant, lattice partial order $\leq$. ℓ groups are often presented as algebraic structures in the language of groups, $+, -, 0$, plus the symbols $\wedge$ for infimum and $\vee$ for supremum.

An ℓ group with strong unit (ℓu-group) is a pair (G, u), where u (the strong unit) is a positive element of G, such that for every $g \in G$ there is a positive integer n such that $g \leq nu$.

Recall also from [4] that one can construct two functors Γ and Ξ, from the category of ℓu-groups and strong unit preserving ℓ-group homomorphisms to the category of MV algebras and MV algebra homomorphisms, and conversely, so that the pair (Γ, Ξ) is a categorial equivalence.

The construction (on objects) works as follows.

The functor Γ maps an ℓu-group G with strong unit u to the MV algebra $\Gamma(G, u)$ whose domain is the interval $\{x \in G | 0 \leq x \leq u\}$, the sum is $x \oplus y = (x + y) \wedge u$ and the negation is $x^* = u - x$.

Conversely, the construction of the functor Ξ relies on a technical notion known as good sequences.

Definition 2.2. A *good sequence* in an MV algebra A is a finite sequence $a = (a_1, \dots, a_n)$ of elements of A where the last element is zero, and $a_i \oplus a_{i+1} = a_i$ for every $i < n$.

There is a sum of good sequences, where $a + b$ is the sequence c such that

$$c_i = a_i \oplus (a_{i-1} \odot b_1) \oplus \cdots \oplus (a_1 \odot b_{i-1}) \oplus b_i.$$

So good sequences form a monoid M_A, and from this monoid the group $\Xi(A)$ can be defined via the usual Grothendieck construction: one takes pairs of elements of M_A, where (a, b) intuitively stands for $b - a$, and then identifies (a, b) with (c, d) whenever $a + d = b + c$. The order in $\Xi(A)$ is defined by $(a, b) \leq (c, d)$ whenever $b + c \leq a + d$. The strong unit of $\Xi(A)$ is the pair $((0), (1))$, where (0) and (1) are the good sequences consisting only of a 0 and a 1, respectively.

The following lemma will be useful later:

Lemma 2.3. *Let A be any MV algebra. Let $g \in \Xi(A)$. Then g is a finite sum of elements of A with plus or minus signs.*

Proof. $g = (g \vee 0) + (g \wedge 0)$, so we can suppose $g \geq 0$. But by definition of the inverse Mundici functor Ξ, the positive cone of $\Xi(A)$ is given by the good sequences of elements of A, and every good sequence of elements of A is a finite sum of elements of A, see [4]. ⊣

2.3. Affine terms and polyhedra.

Definition 2.4. An *affine term* on an MV algebra A is an expression of the form

$$f(x_1, \dots, x_n) = m_1 x_1 + \cdots + m_n x_n + r,$$

where m_i are integers and $r \in \Xi(A)$.

A *polyhedron* on A is a finite union of subsets of A^n of the form

$$\bigcap_{i \in I} \{(x_1, \dots, x_n) \in A^n | f_i(x_1, \dots, x_n) \geq 0\},$$

where I is finite and each f_i is an affine term on A.

We call simply *polyhedra* the polyhedra on $[0, 1]$. Note that according to our definition, the vertices of polyhedra have real, not necessarily rational, coordinates, and a polyhedron is not necessarily connected or convex.

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2.4. The spectrum sheaf. We introduce some basic notions of sheaf theory applied to MV algebras. For more on sheaf theory see [13].

Recall from [7] that the *spectrum* of an MV algebra A is the set $Spec(A)$ of all prime ideals of A. The *maximal spectrum*, $Max(A)$, is the set of all maximal ideals.

Following [7], on the spectrum we can put the *co-Zariski topology*, i.e., the set of open subsets generated by the sets $W_a = \{P \in Spec(A) | a \in P\}$, where a ranges over A.

The *spectrum sheaf* of A, here denoted by $SH(A)$, is the second projection function from $E(A)$ to $Spec(A)$, where $E(A)$ is the set of all pairs $(a/P, P)$ such that $P \in Spec(A)$ and $a \in A$. Note that a/P is an element of the quotient MV algebra A/P. So, $SH(A)$ is a sheaf of MV algebras (actually MV chains) on the topological space $Spec(A)$, whose stalk at P is A/P for every $P \in Spec(A)$.

The topology we put on $E(A)$ has as a subbasis of opens the set of all sets of the form $W_{a,b} = \{(a/P, P) | P \in Spec(A), b \in P\}$ for some $a, b \in A$.

Every element $a \in A$ corresponds to a section $\hat{a}$ of the spectrum sheaf $SH(A)$, namely the map sending P to $(a/P, P)$. Conversely, in [7] it is shown that every section of $SH(A)$ is of the form $\hat{a}$ for some $a \in A$.

2.5. MV polynomial functions and a generalized McNaughton Theorem. We recall from [4] the notion of McNaughton function and McNaughton Theorem.

A function f from $[0,1]^n$ to $[0,1]$ is called a *McNaughton function* if it is continuous and there are k affine polynomials $p_1, \dots, p_k$ with integer coefficients (the constituents) such that for every $x \in [0,1]^n$ there is j such that $f(x) = p_j(x)$.

Then McNaughton Theorem says that McNaughton functions from $[0,1]^n$ to $[0,1]$ form an MV algebra isomorphic to the free MV algebra on n generators.

A slightly generalized version of McNaughton Theorem is in terms of what we call generalized McNaughton functions.

Definition 2.5. A function f from $[0,1]^n$ to $[0,1]$ is called a *generalized McNaughton function* if it is continuous and there are k affine polynomials with integer degree one coefficients and real degree zero coefficient (the constituents) such that for every $y \in [0,1]^n$ there is j such that $f(x) = p_j(x)$.

Let us specialize to MV algebras the usual notion of polynomial function (see Subsection 3.3 for more on polynomial functions). Then we can prove:

Proposition 2.6. *The set GM_n of generalized McNaughton functions from* $[0,1]^n$ *to* $[0,1]$, *equipped with pointwise operations, forms an MV algebra isomorphic to the MV algebra $PF_n([0,1])$ of MV polynomial functions from* $[0,1]^n$ *to* $[0,1]$.

Remark 2.7. The proof of the proposition is given in the paper [2] which, at the time of writing, is submitted. So we reproduce it here for completeness.

Proof. $PF_n([0,1])$ is included in GM_n like in [4], 3.1.8.

For the converse inclusion, given an affine function over $[0,1]$, say $f = m_0x_0 + \dots + m_nx_n + b$, we note that $f^\# = (f \vee 0) \wedge 1$ is an MV polynomial function. In fact, if b is integer, the result is proved in [4], 3.1.9; otherwise, it is enough first to write $b = b' + m$ with m integer and $0 \le b' < 1$, then replace b' with a variable y and consider the function $g = m_0x_0 + \dots + m_nx_n + y + m$, where the degree zero term is an integer, so g fits in the previous case.

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Note that the zerosets of MV polynomial functions on $[0,1]$ coincide with polyhedra (see Definition 2.4): in fact, for polynomials or polyhedra with only integer coefficients this follows from the decomposition of McNaughton functions given in [4], 3.3.1; and in the general case, once again we can take polynomials and polyhedra with only integer coefficients and then specialize some of its variables to elements of $[0,1]$.

The maximal spectrum of $PF_n([0,1])$ is a dense subspace of the spectrum of $PF_n([0,1])$ by the definition of co-Zariski topology, and it is a topological space homeomorphic to $[0,1]^n$ via the map sending $\bar{a} \in [0,1]^n$ to $\{f \in PF_n([0,1]) | f(\bar{a}) = 0\}$, see [4], 3.4.3 (it can be shown, however, that there are prime ideals which are not maximal, and to our knowledge, the spectrum of $PF_n([0,1])$ is far from being well understood).

Like in [7], Remark 9.5, the finite open covers of the spectrum of $PF_n([0,1])$ correspond to the finite covers of the maximal spectrum (i.e., the cube $[0,1]^n$) by polyhedra.

Given a function $\tau \in GM_n$ with constituents $h_1, \dots, h_k$, there are polyhedra $T_1, \dots, T_k$ whose union coincides with $[0,1]^n$ and such that for each i, h_i coincides with τ on T_i. Since $0 \leq \tau \leq 1$, it follows $0 \leq h_i \leq 1$ on T_i and h_i is affine on T_i, so h_i coincides with $h_i^{\#}$ on T_i and is an MV polynomial function on T_i.

So, every function $\tau \in GM_n$ gives a cover of $[0,1]^n$ by polyhedra, $T_i = Z(f_i)$, $f_1, \dots, f_k \in PF_n([0,1])$, $f_1 \wedge \cdots \wedge f_k = 0$, and a family $g_1, \dots g_k \in PF_n([0,1])$ such that $\tau = g_i$ on T_i and g_i and g_j coincide on $T_i \cap T_j$. These two families give a global section of the spectrum of $PF_n([0,1])$. By [7], every global section of the spectrum of $PF_n([0,1])$ corresponds to an element of $PF_n([0,1])$, so every element of GM_n corresponds to an element of $PF_n([0,1])$ as desired. ⊣

In this sense, McNaughton Theorem gives a characterization of MV polynomial functions on $[0,1]$.

§3. Terms and polynomials.

In this section we recall some notions of universal algebra which we will apply to the variety of MV algebras.

Often in the sequel we will use bars to denote tuples, for example, $\bar{x} = (x_1, \dots, x_n)$. The value of n will always be either irrelevant or clear from the context.

We refer to [3] for the concepts of function symbol, arity, functional language, algebra over a functional language.

3.1. Term algebras and congruences.

In this subsection we will give a sketchy treatment of the matter, for more details see [3].

Let X be a nonempty set of elements, called variables, and let F be a functional language.

The set of *terms over* X and F, denoted $T(X, F)$, is the least set of strings of symbols such that $X \subseteq T(X, F)$, and if $t_1, \dots, t_n \in T(X, F)$ and $f \in F$ has arity n, then $f(t_1, \dots, t_n) \in T(X, F)$.

$T(X, F)$ is called a *term algebra*.

An *equation* between terms is an expression $p = q$, where p, q are terms.

A *variety* on a type F is a class of algebras of type F defined by a set of equations between terms.

A *congruence* on a term algebra $T(X, F)$ is an equivalence relation θ on $T(X, F)$ such that, for every n-ary symbol $f \in F$, if $t_i \theta u_i$ for $i = 1, \dots, n$, then

$$f(t_1, \dots, t_n)\theta f(u_1, \dots, u_n).$$

3.2. Polynomials. We will use polynomials as is customary in universal algebra, where algebras of polynomials are free objects in certain varieties. Concerning polynomials, a couple of comments are in order.

First, it seems there is some mismatch between polynomials in usual algebraic geometry and polynomials in universal algebra. Polynomials in algebraic geometry are sums of monomials, and monomials are products of constants and variables. Instead, polynomials in universal algebra are something more abstract: they are equivalence class of terms. As pointed out by a referee, the point is that in algebraic geometry, polynomials are normal forms for terms, whereas in universal algebra, in general, there is no natural normal form for terms, and the variety of MV algebras is an example of this situation. For this reason, when dealing with MV algebras, we will be careful in defining polynomials according to universal algebra.

Second, in universal algebra (and also in classical algebraic geometry over fields) polynomials should not be confused with polynomial functions. Polynomials evaluate to functions, but the same function can come from different polynomials. We believe that the distinction between polynomials and polynomial functions is crucial, and we will come back to it several times in the paper.

So we introduce polynomials as follows.

Let A be an algebra of any type F. Let F_A be the type obtained from F by adding a constant symbol c_a for every $a \in A$. We call *atomic diagram* of A the set of all equalities of the form $c_{f(a_1,\dots,a_n)} = f(c_{a_1}, \dots, c_{a_n})$ where $f \in F$ is a n-ary symbol and $a_1, \dots, a_n \in A$.

Let V be a variety of a functional language F and let $A \in V$. Let $C_{V,A}$ be the congruence on terms of type F_A generated by the axioms for V and the atomic diagram of A.

We will denote by V_A the variety of V-algebras with constants in A, that is, the variety of type F_A axiomatized by $C_{V,A}$.

Definition 3.1. If $\bar{x}$ is a tuple of variables, we define the *algebra of polynomials with variables in* $\bar{x}$ *and constants in* A to be the quotient

$$A[\bar{x}] = T(\bar{x}, F_A)/C_{V,A}.$$

In universal algebra terms, it results that $A[\bar{x}]$ is the algebra freely generated by $\bar{x}$ in the variety V_A.

Note that the definition of polynomial algebra is relative to a variety: for instance, xy and yx are the same polynomial on commutative rings, but not on rings. However, in our case the variety will always be clear, so we omit it from the notation.

Note that it is not a problem to say what are the coefficients of a term (they are its constants), but it is more difficult to say what are the "coefficients" of a polynomial, because a polynomial is a congruence class of terms. For this reason we will be careful not to mention coefficients at all and to speak of constants rather than coefficients.

3.3. Polynomial functions. Given an algebra A of a functional language F and a term $p \in T(x_1, \ldots, x_n, F_A)$, we may define a function $p_A : A^n \to A$ "by substitution of variables" in a standard way. We call p_A the *polynomial function in n variables induced on A by p*.

Given two terms $p, q \in T(x_1, \ldots, x_n, F_A)$, we write $p \equiv_A q$ if $p_A = q_A$, that is, p and q induce the same function on A. Note that $\equiv_A$ is a congruence.

We note that polynomial functions in n variables on an algebra A form another algebra, which we will denote by $PF_n(A)$.

If two terms p, q over an algebra A are the same polynomial, then they induce the same function on A, that is $p \equiv_A q$, but not conversely. A criterion for equality of polynomials is given by the following lemma, whose proof is left to the reader:

Lemma 3.2. *Two terms p, q in n variables give the same polynomial on an algebra A if and only if there is an extension A' of $A[\bar{x}]$ such that p, q are congruent modulo $\equiv_{A'}$.*

In particular, p and q are the same polynomial on A if and only if $p \equiv_{A'} q$ for every algebra $A' \in V$ extending A.

Other two useful corollaries are the following:

Corollary 3.3. *Let V be a variety and let $A \in V$ be an algebra. Let $p(\bar{x}) = q(\bar{x}) \in A[\bar{x}]$. Then $p(\bar{x}) = q(\bar{x}) \in A'[\bar{x}]$ for every extension A' of A.*

Proof. This is just because every extension of A' is also an extension of A. ⊣

Corollary 3.4. *Let V be a variety and let $A \in V$ be an algebra. Let $p(\bar{x}) = q(\bar{x}) \in A[\bar{x}]$. Then $p(\bar{x}) = q(\bar{x}) \in B[\bar{x}]$ for every subalgebra B of A containing the constants of p and q.*

Proof. This is just because the natural map from $B[\bar{x}]$ to $A[\bar{x}]$ is injective. ⊣

As an example of difference between polynomials and polynomial functions, consider the MV algebra $A = \{0, 1\}$ and the MV polynomial $p(x) = x \wedge x^*$. Then $p(x)$ induces the zero function on A, but it is not the zero polynomial in $A[x]$, because e.g. $[0, 1]$ is an extension of A, $[0, 1]$ contains $1/2$, and $p(1/2) = 1/2 \neq 0$.

§4. MV Algebraic Sets.

The previous section was devoted to arbitrary varieties of universal algebra. From now on we will work in the variety of MV algebras unless otherwise stated. In this section we focus on "Diophantine" algebraic geometry: that is, we take the same MV algebra A both to define constants in polynomials and to evaluate polynomials. Most of the results in this section are analogs of well-known facts of algebraic geometry over fields.

Definition 4.1. Let A be an MV-algebra. Let $S \subseteq A[x_1, \ldots, x_n]$. The set

$$V(S) = \{\bar{a} \in A^n \mid p(\bar{a}) = 0, \ \forall p \in S\}$$

is called the *MV algebraic set* determined by S.

Clearly if we let $J = id(S)$, the ideal of $A[\bar{x}]$ generated by S, then $V(J) = V(S)$. Thus algebraic sets are determined by ideals.

Note that for a given subset $S \subseteq A[\bar{x}]$ we may have $V(S) = \emptyset$. This would happen iff for each $\bar{a}$ there is $p \in S$ such that $p(\bar{a}) \neq 0$.

Definition 4.2. Call an ideal $J \subseteq A[\bar{x}]$ *singular* if $V(J) = \emptyset$. Otherwise call J *nonsingular*.

Notice that if $p(\bar{x}) \in A[\bar{x}]$ and has no zero, then $id(p(\bar{x}))$ is a singular ideal and conversely. For instance, every nonzero element $a \in A$, viewed as a constant polynomial, will have no zero in any extension of A.

Definition 4.3. Suppose we have a nonempty $X \subseteq A^n$. The set

$$I(X) = \{p \in A[x_1, \ldots, x_n] \mid p(\bar{y}) = 0, \ \forall \bar{y} \in X\}$$

is an ideal, called the *zero ideal* of X.

By convention we let $I(\emptyset) = A[x_1, \ldots, x_n]$.

Let us give some preliminary results on the operators I and V.

First, for every $S \subseteq A[x_1, \ldots, x_n]$ and $X \subseteq A^n$ we have

$$X \subseteq V(S) \iff S \subseteq I(X).$$

Hence, the pair (I, V) is a contravariant Galois connection between the powerset of $A[x_1, \ldots, x_n]$ and the powerset of A^n. Like in the classical Nullstellensatz, we can consider the objects of the form $I(V(J))$. We will derive a Nullstellensatz below.

In partial analogy with standard algebraic geometry over fields, the following is true:

Proposition 4.4. *If A is an MV chain then $V(J \cap K) = V(J) \cup V(K)$.*

Proof. Since $J \cap K \subseteq J, K$ we have that $V(J) \cup V(K) \subseteq V(J \cap K)$.

Conversely, suppose for an absurdity $\bar{a} \in V(J \cap K)$ and $\bar{a} \notin V(J) \cup V(K)$. Then there are $p \in J$, $q \in K$ such that $p(\bar{a}) \neq 0$ and $q(\bar{a}) \neq 0$. But $p \wedge q \in J \cap K$, so $(p \wedge q)(\bar{a}) = 0$, and since A is linearly ordered, we have $p(\bar{a}) = 0$ or $q(\bar{a}) = 0$. So we have a contradiction and the thesis holds. ⊣

Definition 4.5. Call an ideal $J \subseteq A[\bar{x}]$ a *point ideal* if for some $\bar{a} = (a_1, \ldots, a_n) \in A^n$ we have $J = I(\bar{a})$.

Note that $A[\bar{x}]/I(\bar{a})$ is isomorphic to A via the evaluation of polynomials at $\bar{a}$. Hence, if A is linearly ordered, then $I(\bar{a})$ is a prime ideal; and if A is simple, then $I(\bar{a})$ is a maximal ideal.

Proposition 4.6. *Each point ideal is nonsingular, nonzero and proper.*

Proof. Clearly, $\bar{a} \in V(I(\bar{a}))$ so $I(\bar{a})$ is nonsingular. Moreover $1 \notin I(\bar{a})$, thus $I(\bar{a})$ is proper.

Consider $\bar{a} = (a_1, \ldots, a_n)$. If $\bar{a} \neq \bar{0}$ let $p(\bar{x}) = a_1 x_1^* \oplus \cdots \oplus a_n x_n^*$. Then $p \neq 0$ and $p(\bar{a}) = 0$. If instead $\bar{a} = \bar{0}$ then just let $p(\bar{x}) = x_1$. Again $p \neq 0$ and $p(\bar{a}) = 0$. So in any case, the ideal $I(\bar{a})$ is nonzero. ⊣

Proposition 4.7. *For every nonsingular ideal $J \subseteq A[\bar{x}]$, we have $I(V(J)) = \bigcap_{\bar{a} \in V(J)} I(\bar{a})$.*

Proof. Suppose that $p \in I(V(J))$. Then for each $\bar{a} \in V(J)$ we have $p(\bar{a}) = 0$. Therefore, $p \in I(\bar{a})$ so $p \in \bigcap_{\bar{a} \in V(J)} I(\bar{a})$. Hence $I(V(J)) \subseteq \bigcap_{\bar{a} \in V(J)} I(\bar{a})$.

Conversely, suppose that $p \in \bigcap_{\bar{a} \in V(J)} I(\bar{a})$. Then $p(\bar{a}) = 0$ for all $\bar{a} \in V(J)$, that is $p \in I(V(J))$ and so $I(V(J)) = \bigcap_{\bar{a} \in V(J)} I(\bar{a})$. ⊣

PROPOSITION 4.8. *$I(\bar{a}) = I(\bar{b})$ iff $\bar{a} = \bar{b}$.*

PROOF. Let $\bar{a} = (a_1, \dots, a_n)$, $\bar{b} = (b_1, \dots, b_n)$. Assume $\bar{a} \neq \bar{b}$. Say $a_i \neq b_i$. Then $p(x_1, \dots, x_n) = d(x_i, a_i)$, where $d(x, y)$ is the Chang distance, is zero on $\bar{a}$ and nonzero on $\bar{b}$. ⊣

DEFINITION 4.9. For a nonsingular ideal $J \subseteq A[\bar{x}]$, the set

$$_{pt}\sqrt{J} = \bigcap\{I(\bar{a}) \mid J \subseteq I(\bar{a})\}$$

is an ideal, called the *point radical* of J. If J is singular we let $_{pt}\sqrt{J} = A[\bar{x}]$.

Observe that if J is nonsingular so that $V(J) \neq \emptyset$ then there is an $\bar{a} \in V(J)$. Thus for all $p \in J$ we have $p(\bar{a}) = 0$. Hence $J \subseteq I(\bar{a})$. Thus $J \subseteq {}_{pt}\sqrt{J}$.

As a consequence of Proposition 4.7 we have the following corollary, formally analogous to the Nullstellensatz of algebraic geometry:

COROLLARY 4.10. *For every ideal J, $I(V(J)) = {}_{pt}\sqrt{J}$.*

PROOF. We can suppose J is nonsingular. From Proposition 4.7 we have $I(V(J)) = \bigcap_{\bar{a} \in V(J)} I(\bar{a})$. Let, then $q \in I(V(J))$ and suppose for some $\bar{a}$ that $J \subseteq I(\bar{a})$. Then $p(\bar{a}) = 0$ for all $p \in J$. Thus $\bar{a} \in V(J)$ and it follows that $q(\bar{a}) = 0$ and $q \in I(\bar{a})$. Hence $q \in {}_{pt}\sqrt{J}$ by definition of point radical.

Conversely suppose $q \in {}_{pt}\sqrt{J}$ and that $\bar{a} \in V(J)$. Then $J \subseteq I(\bar{a})$. By definition of point radical we have $q \in I(\bar{a})$. So, $q \in I(V(J))$. ⊣

If $M \subseteq A[\bar{x}]$ is maximal and nonsingular, then either $I(V(M)) = M$ or $I(V(M)) = A[\bar{x}]$.

Suppose $I(V(M)) = A[\bar{x}]$. Then we have that $1 \in I(V(M))$. That is, $V(M) = \emptyset$.

PROPOSITION 4.11. *If J is an ideal, $Y \subseteq A^n$ and $J = \bigcap_{\bar{a} \in Y} I(\bar{a})$ then, $I(V(J)) = J$.*

PROOF. We claim that $Y \subseteq V(J)$. Let $\bar{a} \in Y$ and $p \in J$. Then $p \in I(\bar{a})$. Thus $p(\bar{a}) = 0$ and so $\bar{a} \in V(J)$. Now let $\bar{a} \in V(J)$. Then for all $p \in J$ we have $p(\bar{a}) = 0$. Hence $p \in I(\bar{a})$ and so $J \subseteq \bigcap_{\bar{a} \in V(J)} I(\bar{a})$. Therefore, we have $J \subseteq \bigcap_{\bar{a} \in V(J)} I(\bar{a}) \subseteq \bigcap_{\bar{a} \in Y} I(\bar{a}) = J$. So by Proposition 4.7 we have $I(V(J)) = J$. ⊣

We summarize the previous results in the following Nullstellensatz theorem:

THEOREM 4.12. *The ideals J such that $I(V(J)) = J$ are exactly the point-radical ideals.*

PROOF. This follows from Proposition 4.11 and Corollary 4.10. ⊣

PROPOSITION 4.13. $_{pt}\sqrt{_{pt}\sqrt{J}} = {}_{pt}\sqrt{J}$.

PROOF. $J \subseteq I(\bar{a})$ iff $_{pt}\sqrt{J} \subseteq I(\bar{a})$ iff $_{pt}\sqrt{_{pt}\sqrt{J}} \subseteq I(\bar{a})$. ⊣

COROLLARY 4.14. *For every ideal J, $I(V({}_{pt}\sqrt{J})) = {}_{pt}\sqrt{J}$.*

PROPOSITION 4.15. *If J_i is a family of point-radical ideals, then so is $\bigcap_i J_i$. that is, $_{pt}\sqrt{\bigcap_i J_i} = \bigcap_i J_i$.*

PROOF. We have $\bigcap_i J_i \subseteq {}_{pt}\sqrt{\bigcap_i J_i}$. Let $p \in {}_{pt}\sqrt{\bigcap_i J_i}$. Let $J_i \subseteq I(\bar{a})$ for some $\bar{a}$. Then $\bigcap_i J_i \subseteq I(\bar{a})$, thus $_{pt}\sqrt{\bigcap_i J_i} \subseteq I(\bar{a})$. Therefore, $p \in I(\bar{a})$ and so $p \in {}_{pt}\sqrt{J_i} = J_i$. Taking the intersection, $p \in \bigcap_i J_i$. ⊣

PROPOSITION 4.16. *If Z_i is a family of MV algebraic sets then* $I(\bigcap_i Z_i) = {}_{pt}\sqrt{\Sigma_i I(Z_i)}$, *where $\Sigma_i J_i$ is the ideal generated by $\bigcap_i J_i$.*

PROOF. Let $Z_i = V(J_i)$ for some ideals J_i.

Then, $\bar{a} \in \bigcap_i V(J_i)$ iff for all $q_i \in J_i$ we have $q_i(\bar{a}) = 0$.

Therefore, $I(\bigcap_i Z_i) = I(\bigcap_i V(J_i)) = I(V(\Sigma_i J_i)) = {}_{pt}\sqrt{\Sigma_i J_i}$. Since we can take the J_i to be point-radical ideals, we have $J_i = I(V(J_i)) = I(Z_i)$ we conclude the proof. ⊣

PROPOSITION 4.17. *There is a one-one correspondence between point-radical ideals and MV algebraic sets.*

PROOF. First we have a map ${}_{pt}\sqrt{J} \to V({}_{pt}\sqrt{J})$. This map is one-one since $I(V({}_{pt}\sqrt{J})) = {}_{pt}\sqrt{J}$.

Suppose W is an MV algebraic set; then for some ideal J we have $W = V(J)$. Moreover by Proposition 4.11 and Corollary 4.10 we can take J to be a point-radical. Thus the map ${}_{pt}\sqrt{J} \to V({}_{pt}\sqrt{J})$ is onto. ⊣

§5. Coordinate MV algebras.

Here again we are in Diophantine geometry.

DEFINITION 5.1. Let $Z \subseteq A^n$ be an MV algebraic set. By the *coordinate MV-algebra* of Z we mean the MV-algebra $A[x_1, \dots, x_n]/I(Z)$.

Now $Z = V(J)$ for an ideal J. Hence $I(Z) = I(V(J)) = {}_{pt}\sqrt{J}$. Thus,

PROPOSITION 5.2. *For every ideal J the coordinate MV-algebra of $V(J)$ is* $A[\bar{x}]/{}_{pt}\sqrt{J}$.

DEFINITION 5.3. Let $Z_1 \subseteq A^n$, $Z_2 \subseteq A^m$ be algebraic sets. A mapping $\phi : Z_1 \to Z_2$ is called a *MV polynomial map* iff there are MV polynomials $p_1, \dots, p_m \in A[x_1, \dots, x_n]$ such that

$$\phi(a_1, \dots, a_n) = (p_1(a_1, \dots, a_n), \dots, p_m(a_1, \dots, a_n))$$

for every $(a_1, \dots, a_n) \in Z_1$.

DEFINITION 5.4. Let $V(J_i)$ be algebraic sets. $i = 1, 2$. An MV polynomial map $\phi : V(J_1) \to V(J_2)$ is an *isomorphism* if there is an MV polynomial map $\psi : V(J_2) \to V(J_1)$ such that $\psi \circ \phi = 1_{V(J_1)}$ and $\phi \circ \psi = 1_{V(J_2)}$.

We shall see below that two MV algebraic sets are isomorphic iff their corresponding coordinate MV algebras are isomorphic.

Let A be an MV algebra and let Z be a MV algebraic subset of A^n; let $F(Z, A)$ be the MV algebra of MV polynomial maps from Z to A. Then:

PROPOSITION 5.5. *$A[\bar{x}]/I(Z)$ is isomorphic to $F(Z, A)$.*

PROOF. For $p, q \in A[\bar{x}]$, $p = q$ in $F(Z, A)$ iff $p(\bar{a}) = q(\bar{a})$ for all $\bar{a} \in Z$.

We then have a morphism $\psi : A[\bar{x}] \to F(Z, A)$ given by $\psi(p) = p|_Z$. Thus $\psi(p) = \psi(q)$ iff $p(\bar{a}) = q(\bar{a})$ for all $\bar{a} \in Z$ iff $d(p, q) = 0$ on Z iff $d(p, q) \in I(Z)$. ⊣

Note that for $Z = A^n$ we have $I(Z) = I(A^n) = {}_{pt}\sqrt{0}$ and $F(A^n, A) = PF_n(A)$. So, it is an interesting question, to be investigated below, which MV algebras enjoy the property ${}_{pt}\sqrt{0} = 0$ (a property we call polynomial completeness). Note that the

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analogous property $\sqrt{0} = 0$ in algebraic geometry corresponds to reduced rings, where $x^n = 0$ implies $x = 0$.

Now fix any MV algebra A. Let

$$MV(A) = \{A[x_1, \dots, x_n]/J \mid J = {}_{pt}\sqrt{J}, n = 1, 2, \dots\}.$$

Then $MV(A)$ is a category having as morphisms the MV-homomorphisms.

Likewise let

$$Z(A) = \{X \subseteq A^n | X \text{ is } MV \text{ algebraic}, n = 1, 2, \dots\}.$$

Then with MV polynomial maps as morphisms, $Z(A)$ becomes a category.

We have the following duality:

THEOREM 5.6. *For every MV algebra A, the categories $MV(A)$, and $Z(A)$ are dually isomorphic.*

PROOF. Let us adopt the abbreviations $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_m)$.

The following proposition is the first step towards Theorem 5.6: ⊣

PROPOSITION 5.7. *There is a contravariant functor F from $MV(A)$ to $Z(A)$ acting on objects as follows*: $F(A[\bar{x}]/J) = V(J)$.

PROOF. We have to define F on the morphisms of $MV(A)$.

Let $f : A[\bar{x}]/J_1 \to A[\bar{y}]/J_2$ be a morphism between the coordinate MV algebras of $V(J_1)$ and $V(J_2)$.

Let $p_1, \dots, p_n \in A[\bar{y}]$ be such that $p_i/J_2 = f(x_i/J_1)$. Then the map $\phi : V(J_2) \to V(J_1)$ given by

$$\phi(a_1, \dots, a_m) = (p_1(a_1, \dots, a_m), \dots, p_n(a_1, \dots, a_m))$$

is an MV polynomial map.

We claim ϕ is well defined. In fact, suppose we have $q_i \in A[\bar{y}]$ such that $q_i/J_2 = p_i/J_2$. Then $d(q_i, p_i) \in J_2$. Hence for $(a_1, \dots, a_m) \in V(J_2)$ we have $q_i(a_1, \dots, a_m) = p_i(a_1, \dots, a_m)$ and it follows that ϕ is well defined.

We define $F(f) = \phi$.

We claim now that if $(a_1, \dots, a_m) \in V(J_2)$, then $\phi(a_1, \dots, a_m) \in V(J_1)$.

To this end let $q(x_1, \dots, x_n) \in A[\bar{x}]$. Then $f : q/J_1 \to q(p_1, \dots, p_n)/J_2$ and if $q \in J_1$ we have $q(p_1, \dots, p_n)/J_2 = 0$, thus $q(p_1, \dots, p_n) \in J_2$. Therefore,

$$\begin{aligned} q(\phi(a_1, \dots, a_m)) &= q(p_1(a_1, \dots, a_m), \dots, p_n(a_1, \dots, a_m)) \\ &= q(p_1, \dots, p_n)(a_1, \dots, a_m) = 0 \end{aligned}$$

as $q(p_1, \dots, p_n) \in J_2$. The claim is proved and the proposition follows. ⊣

Conversely we have:

PROPOSITION 5.8. *There is a contravariant functor G from $Z(A)$ to $MV(A)$ acting on objects as follows*: $G(V(J)) = A[\overline{x}]/J$.

PROOF. Again we have to define G on morphisms of $Z(A)$.

Let $V(J_1) \subseteq A^n$, $V(J_2) \subseteq A^m$ be MV algebraic sets and suppose we have an MV polynomial map $\phi : V(J_1) \to V(J_2), \phi = (p_1, \dots, p_m)$ where the $p_i \in A[\bar{x}]$.

Define a map $f : A[\bar{y}]/J_2 \to A[\bar{x}]/J_1$ by

$$f(p/J_2) = p(p_1, \dots, p_m)/J_1.$$

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We claim that f is well defined. For let $p/J_2 = q/J_2$; then $d(p, q) \in J_2$. Let $\overline{a} \in V(J_1)$. Then

$$d(p,q)(p_1,\dots,p_m)(\overline{a}) = d(p(p_1,\dots,p_m),\ q(p_1,\dots,p_m))(\overline{a})$$
$$= d(p(p_1(\overline{a}),\dots,p_m(\overline{a})),\ q(p_1(\overline{a}),\dots,p_m(\overline{a}))).$$

Now, $(p_1(\overline{a}),\dots,p_m(\overline{a})) = \phi(\overline{a}) \in V(J_2)$. As $d(p, q) \in J_2$ we have $d(p, q)(\phi(\overline{a})) = 0$.

It follows that $p(p_1,\dots,p_m) = q(p_1,\dots,p_m)$ on $V(J_1)$, hence

$$p(p_1,\dots,p_m)/J_1 = q(p_1,\dots,p_m)/J_1,$$

and we have that f is well defined. We let $G(\phi) = f$.

It's clear that f is an $MV(A)$ morphism. This yields a contravariant functor $G : Z(A) \to MV(A)$. Hence the proposition follows. ⊣

Consider the composition $FG : Z(A) \to Z(A)$. Let $Z = V(J) \in Z(A)$. Then $G(Z) = A[\bar{x}]/J$. Hence we have, $FG(Z) = F(A[\bar{x}]/J) = V(J) = Z$. Since we can take J to be a point-radical ideal, the above is well defined and we see that $FG = id_{Z(A)}$.

To see that this is indeed the identity functor let's examine the action on morphisms.

Again, let $V(J_1) \subseteq A^n$, $V(J_2) \subseteq A^m$ be MV algebraic sets and suppose we have an MV polynomial map $\phi : V(J_1) \to V(J_2)$, $\phi = (p_1,\dots,p_m)$ where the $p_i \in A[\bar{x}]$.

Define $f_\phi : A[\bar{y}]/J_2 \to A[\bar{x}]/J_1$ by $f_\phi(p/J_2) = p(p_1,\dots,p_m)/J_1$. Letting $G(\phi) = f_\phi$ we have an MV_A morphism.

Now start with an $MV(A)$ morphism $f : A[\bar{y}]/J_2 \to A[\bar{x}]/J_1$ we let $p_i = f(x_i/J_2)$. Now set $\phi_f = (p_1,\dots,p_m)$. Then ϕ_f is a morphism $\phi_f : V(J_1) \to V(J_2)$ in the category $Z(A)$.

Starting with $\phi = (p_1,\dots,p_m) : V(J_1) \to V(J_2)$, consider now the composed morphism ϕ_{f_ϕ}. Then $f_\phi(p/J_2) = p(p_1,\dots,p_m)/J_1$. In particular, $f_\phi(x_i/J_2) = x_i(p_1,\dots,p_m) = p_i/J_1$ and we see that $\phi_{f_\phi} = \phi$. It now follows that FG is the identity functor on $Z(A)$.

Similarly, $GF = id_{MV(A)}$ is the identity functor on $MV(A)$ and the theorem follows.

As a consequence we see that two MV algebraic sets are isomorphic iff their corresponding coordinate MV algebras are isomorphic.

§6. Polynomial completeness. In this section we introduce a universal algebra concept related to the distinction between polynomials and polynomial functions. We have been careful in distinguishing between these two concepts. For some MV algebras they coincide, for some others they do not. When they coincide, we say that the MV algebra is polynomially complete.

Definition 6.1. An MV algebra A is *polynomially complete* if for every n, the only polynomial $p \in A[x_1,\dots,x_n]$ inducing the zero function on A is the zero polynomial. Equivalently by Lemma 3.2, A is polynomially complete if every polynomial which induces zero on A induces zero on every MV algebra A' extending A.

Note that a polynomial p may induce the zero polynomial in A without being the zero polynomial of A. For instance, $p(x) = x \wedge x^*$ induces the zero polynomial in

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S_1 but it is not the zero element of $S_1[x]$, because there are extensions of S_1 where p is not zero, for instance in $[0, 1]$ we have $p(1/3) \neq 0$.

The name suggests that polynomial functions over A describe completely the polynomials of A, because if A is polynomially complete, then the evaluation homomorphism from $A[x_1, \ldots, x_n]$ to $PF_n(A)$ is an MV algebra isomorphism.

We recall that an MV algebra A is defined to be algebraically closed (in the sense of Lacava, see [11]) if every polynomial $p \in A[x_1, \ldots, x_n]$ which has a zero in some extension of A has also a zero in A.

As in the classical case, the interplay between algebraic geometry and model theory MV algebras seems promising. For instance, in order to answer a question of a referee, we remark that algebraically closed MV algebras and existentially closed MV algebras do not coincide because:

- algebraically closed MV algebras can be axiomatized in first order logic, see [11], but
- existentially closed MV algebras cannot be axiomatized in first order logic, see [10].

Note also that polynomial completeness is a sort of algebraic closedness the other way round, in fact:

- A is polynomially complete when every identically zero polynomial in A is identically zero in every extension, whereas
- A is algebraically closed if every identically *nonzero* polynomial in A is identically nonzero in every extension.

In the notation of the previous sections, A is polynomially complete if and only if ${}_{pt}\sqrt{0} = 0$, or $I(V(0)) = 0$, in $A[x_1, \ldots, x_n]$ for every n. Moreover, polynomial completeness can be rephrased in universal algebra terms as follows:

PROPOSITION 6.2. *Let A be any MV algebra. The following are equivalent*:

- *A is polynomially complete*;
- *if two polynomials $p, q \in A[x_1, \ldots, x_n]$ induce the same function on A, then $p = q$*;
- *if two polynomials $p, q \in A[x_1, \ldots, x_n]$ induce the same function on A, then they induce the same function in every extension of A*;
- *A generates the variety MV_A of MV algebras with coefficients in A in the sense of Subsection* 3.2.

The first two points of the proposition are equivalent because $p = q$ holds if and only if $d(p, q) = 0$.

Despite our notion of polynomial completeness seems quite a natural universal algebra notion, we are not aware of previous treatments of this concept in universal algebra literature.

6.1. Polynomial completeness of divisible MV chains. The result announced in the title is performed in three steps.

PROPOSITION 6.3. $[0, 1]$ *is polynomially complete.*

PROOF. Let $\bar{x} = (x_1, \ldots, x_n)$ be variables and $\bar{d} = (d_1, \ldots, d_m) \in [0, 1]^m$. Let $p(\bar{x}, \bar{d}) \in [0, 1][\bar{x}]$ and suppose $p(\bar{x}, \bar{d}) = 0$ for every $\bar{x} \in [0, 1]^n$.

By the Di Nola representation theorem, see [4], 9.5.1, there is an embedding ψ of $[0,1][\bar{x}]$ in a Cartesian power D^I, where D is a divisible MV chain (more precisely, D is an ultrapower of $[0,1]$).

Denote by ϕ the canonical embedding of $[0,1]$ in $[0,1][\bar{x}]$ and, for $i \in I$, let π_i the i-th projection from D^I to D.

Let $\chi_i = \pi_i \circ \psi \circ \phi$. Then χ_i is an MV algebra morphism from $[0,1]$ to D.

Since D is a nontrivial MV algebra, χ_i is an embedding. Moreover, $[0,1]$ and D are divisible MV chains and the theory of divisible MV chains is model complete (it even has elimination of quantifiers, see [9]), so χ_i is an elementary embedding.

By definition of elementary embedding, since $p(\bar{x}, \bar{d}) = 0$ for every $\bar{x} \in [0,1]^n$, we have also $p(\bar{y}, \chi_i(d)) = 0$ for every $\bar{y} \in D^n$.

By taking the I-th Cartesian power we have $p(\bar{z}, \psi \circ \phi(\bar{d})) = 0$ for every $\bar{z} \in (D^I)^n$; since D^I is an extension of $[0,1][\bar{x}]$, by restriction $p(\bar{x}, \bar{d})$ induces 0 in $[0,1][\bar{x}]$. That is, $p(\bar{x}, \bar{d})$ is the zero polynomial of $[0,1][\bar{x}]$. ⊣

Proposition 6.4. *Every ultrapower of* $[0,1]$ *is polynomially complete.*

Proof. Let $[0,1]^* = [0,1]^I/U$ be an ultrapower of $[0,1]$, where U is an ultrafilter on the set I.

Let $\bar{x} = (x_1, \dots, x_n)$, $\bar{d} = (d_1, \dots, d_m) \in ([0,1]^*)^m$.

Let $p(\bar{x}, \bar{d}) \in [0,1]^*[\bar{x}]$ and assume p is not the zero polynomial.

Then there is an extension E of $[0,1]^*$ and a tuple $\bar{e} \in E^n$ such that $p(\bar{e}, \bar{d}) \neq 0$.

The components of $\bar{d}$ are elements of $[0,1]^*$, so they are classes modulo U of sequences indexed by I, let us denote them by $(\bar{d_i})_{i \in I}/U$.

Let $E^* = E^I/U$, and let δ be the canonical embedding of E in E^*. In particular, $\delta(\bar{e})$ is the constant tuple with components in $\bar{e}$.

From $p(\bar{e}, (\bar{d_i})_i) \neq 0$, since δ is an elementary embedding from E to E^*, we have $p(\delta(\bar{e}), (\bar{d_i})_i) \neq 0$. Hence, by Łoś Theorem on ultraproducts, we have $p(\bar{e}, \bar{d_i}) \neq 0$ where i ranges over a subset J of I belonging to U.

Since $[0,1]$ is polynomially complete, for every $i \in J$ there is $\bar{r}_i \in [0,1]^n$ such that $p(\bar{r}_i/U, \bar{d_i}/U) \neq 0$.

Taking the sequence $(\bar{r}_i)_i/U$ in $([0,1]^*)^n$, we have that p is nonzero in this element of $([0,1]^*)^n$. So p does not induce zero in $[0,1]^*$.

This means that $[0,1]^*$ is polynomially complete. ⊣

In the same way (as pointed out by a referee) one shows:

Corollary 6.5. *Every ultrapower of a polynomially complete MV algebra is polynomially complete.*

Proposition 6.6. *Every divisible MV chain is polynomially complete.*

Proof. Let D be a divisible MV chain.

Let $\bar{x} = (x_1, \dots, x_n)$ and $\bar{d} = (d_1, \dots, d_m)$, with $d_i \in D$.

Let $p(\bar{x}, \bar{d}) \in D[\bar{x}]$ and suppose $p(\bar{x}, \bar{d}) = 0$ for every $\bar{x} \in D^n$.

Embed D in an ultrapower $[0,1]^*$.

Since $[0,1]^*$ is polynomially complete, the polynomial MV algebra $[0,1]^*[\bar{x}]$ embeds in the algebra of functions $([0,1]^*)^I$, where $I = ([0,1]^*)^n$ and the elements of $[0,1]^*$ are mapped to constants.

A fortiori, there is an embedding ψ of $D[\bar{x}]$ in $([0,1]^*)^I$ and the elements of D are mapped to constants. That is, $\psi(d)_i = \psi(d)_j$ for every $i, j \in I$ and $d \in D$.

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So, calling ϕ the canonical embedding from D to $D[\bar{x}]$ and π_i the i-th projection from $([0,1]^*)^I$ to $[0,1]^*$, the compositions $\chi_i = \pi_i \circ \psi \circ \phi$ from D to $[0,1]^*$ are embeddings. In fact, suppose $\chi_i(d) = \chi_i(d')$ for some i. Then $\psi(d)_i = \psi(d')_i$. Then $\psi(d)_j = \psi(d')_j$ for every $j \in I$. So, $\psi(d) = \psi(d')$. But ψ is injective, so $d = d'$.

Now, since we assume $p(\bar{x}, \bar{d}) = 0$ for every $\bar{x} \in D^n$, we have $p(\bar{y}, \bar{d}) = 0$ for every $\bar{y} \in ([0,1]^*)^n$, and taking the Cartesian power, $p(\bar{z}, \bar{d}) = 0$ for every $\bar{z} \in (([0,1]^*)^I)^n$, so p induces zero in $([0,1]^*)^I$. By restriction, p induces zero in $D[\bar{x}]$; so p is the zero polynomial of $D[\bar{x}]$.

This means that D is polynomially complete. ⊣

6.2. A characterization of polynomially complete MV chains. We do not have an intrinsic characterization of polynomially complete MV algebras, however in this paper we give one for MV chains. We introduce a seemingly new concept.

Definition 6.7. A totally ordered group or MV algebra is *quasi-divisible* if for every $a < b$ and for every positive integer N there is c such that $a < Nc < b$.

Note that all divisible MV chains are quasidivisible, and all divisible totally ordered ℓ-groups are quasidivisible, but not the other way round. For instance, the MV chain B of the binary rationals of $[0,1]$, like every simple infinite MV algebra, is quasidivisible, but it is not divisible, because, e.g., it does not contain any x such that $2x = x^*$.

The following proposition is easy:

Proposition 6.8. *A totally ordered group G is order dense in $DH(G)$ if and only G it is quasidivisible.*

Proof. Assume G is order dense in $DH(G)$. Let $a < b \in G$. Let N be an integer. Then $a/N < b/N$ in $DH(G)$. So there is $c \in G$ with $a/N < c < b/N$, so $a < Nc < b$ and G is quasidivisible.

Conversely, let G be quasidivisible. Let $d < e \in DH(G)$. There is N such that $d = a/N$ and $e = b/N$, with $a < b \in G$. Taking $c \in G$ with $a < Nc < b$ and dividing by N we have $a/N < c < b/N$, that is, $d < c < e$. So, G is order dense in $DH(G)$. ⊣

Proposition 6.9. *Let A be an MV chain. Then A is quasidivisible if and only if $\Xi(A)$ is quasidivisible.*

Proof. Assume $\Xi(A)$ quasidivisible. Let $0 \leq a < b \in A$. Let N be a positive integer. Then there is $c \in \Xi(A)$ such that $a < Nc < b$. From $a \geq 0$ it follows $Nc \geq 0$ and $c \geq 0$; likewise, we have $0 \leq c \leq 1$ in $\Xi(A)$. So $c \in A$, and A is quasidivisible.

Conversely, suppose A quasidivisible. Let $g < h$ in $\Xi(A)$. Let N be an integer. Up to adding an integer multiple of N we can suppose $g > 0$. So we can write $g = n + a$ where n is a nonnegative multiple of the unit, $a \in A$ and $a < 1$. Since A is quasidivisible, A is a fortiori dense, so there is $a' \in A$ with $a < a' < 1$. Up to replacing h with $n + a'$, we can always suppose $h - g < 1$.

Now the idea is to approximate "sufficiently well" 1 and a with multiples of N.

More formally, by quasidivisibility we have $b \in A$ with

$$0 < 4nb < h - g,$$

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in particular $b < 1$ and

$$0 < 2b < (h - g)/2n.$$

We have then $c, d \in A$ with

$$1 - b < Nc < 1$$

and

$$b < Nd < 2b.$$

By adding, we have

$$1 < N(c + d) < 1 + 2b$$

and

$$1 < N(c + d) < 1 + (h - g)/2n.$$

Multiplying by n we have

$$n < nN(c + d) < n + (h - g)/2.$$

Likewise one finds $e \in A$ with

$$a < Ne < a + (h - g)/2,$$

and by adding, we have

$$g = n + a < N(nc + nd + e) < n + a + (h - g) = h.$$

So $\Xi(A)$ is quasidivisible. ⊣

In the same vein:

PROPOSITION 6.10. *Let A be an MV chain. Then A is order dense in $DH(A)$ if and only if $\Xi(A)$ is order dense in $DH(\Xi(A))$.*

PROOF. Assume A order dense in its divisible hull. Then A is quasidivisible. In fact, let $a < b \in A$ and let N be an integer. Then $a/N < b/N \in DH(A)$. Taking c such that $a/N < c < b/N$, we conclude $a < Nc < b$, and A is quasidivisible. Then $\Xi(A)$ is quasidivisible, and $\Xi(A)$ is order dense in its divisible hull.

Conversely, assume $\Xi(A)$ dense in its divisible hull. Let $a < b \in DH(A)$. Since $\Xi(A)$ is dense in $DH(\Xi(A))$ and $DH(A)$ is included in $DH(\Xi(A))$, there is $c \in \Xi(A)$ such that $a < c < b$. But $c \in A$, so A is dense in its divisible hull. ⊣

Now we have the following characterization of polynomially complete MV chains:

THEOREM 6.11. *For every MV chain A the following are equivalent:*

1. *A is polynomially complete;*
2. *A is order dense in its divisible hull;*
3. *A is quasidivisible.*

PROOF. 2 and 3 are equivalent because of the following chain of equivalences: A is quasidivisible if and only if $\Xi(A)$ is quasidivisible if and only if $\Xi(A)$ is order dense in its divisible hull if and only if A is order dense in its divisible hull.

To prove that 1 implies 3, suppose by contradiction that A is a polynomially complete MV chain but is not quasidivisible. Then, since A is an MV chain, there are $a < b \in A$ and N such that for every $x \in A$, $Nx \leq a$ or $b \leq Nx$. Then the polynomial

$$p(x) = (Nx \ominus a) \wedge (b \ominus Nx)$$

is zero on A; and since $DH(A)$ is quasidivisible, we can find a point $d \in DH(A)$ with $p(d) \neq 0$. So p induces zero on A but not on its extension $DH(A)$, in contrast with the assumption 1.

To prove that 3 implies 1, suppose A is quasidivisible. By condition 2, A is order dense in $DH(A)$.

Let $p \in A[x_1, \dots, x_n]$ and assume $p(d) \neq 0$ for some $d \in DH(A)^n$. Since $DH(A)$ is a divisible MV chain, it enjoys the same first order properties of $[0, 1]$; in particular, since $p(d) \neq 0$, there is a product P of n nontrivial intervals in $DH(A)^n$ where p has no zeros. By order density, P contains a point $a \in A^n$, so $p(a) \neq 0$, hence p does not induce the function 0 in A.

Summing up, every polynomial p inducing 0 in A induces 0 in $DH(A)$ as well.

To prove A is polynomially complete, suppose a polynomial p induces zero in A. We have seen that p induces zero in $DH(A)$. Take any extension E of A. There is an MV algebra F containing both E and $DH(A)$. Since $DH(A)$ is polynomially complete, p induces zero on F. Since F extends E, p induces zero on E as well. So, A is polynomially complete, and 3 implies 1. ⊣

Corollary 6.12. *Every MV chain can be embedded in a polynomially complete MV chain.*

Proof. The divisible hull of any MV chain is polynomially complete by the previous Theorem. ⊣

Corollary 6.13. *Every simple infinite MV chain is polynomially complete.*

Proof. Any such chain is order dense in $[0, 1]$ which is divisible, so it is order dense in its divisible hull. ⊣

Corollary 6.14. *Let A be an MV chain.*

1. *If A is discrete, then A is not polynomially complete.*
2. *If A has finite rank, then it is not is polynomially complete.*

Proof. This can be seen directly as follows.

1. Let a be the atom of A. Then every element of A satisfies $x = 0$ or $a \leq x$, so it is a zero of $p(x) = x \wedge (a \ominus x)$. However, by the Di Nola Representation Theorem, A embeds in an ultrapower of $[0, 1]$ where $a/2$ exists, and $a/2$ does not annihilate p. So, by Lemma 3.2, p is not the zero polynomial of A.
2. Embed A in an ultrapower U of $[0, 1]$. Let $1/N$ be a rational which is not the standard part of any element of A. Then for every $x \in A$, either $x \leq 1/4N$ or $x \geq 1/2N$ (where the inequalities are taken in U). So every element of A, either x verifies $(4N-1)x \leq x^*$ or $x^* \leq (2N-1)x$. In any case, x annihilates
$$p(x) = ((4N-1)x \ominus x^*) \wedge (x^* \ominus (2N-1)x).$$
But U is an extension of A where $3/8N$ exists, and $3/8N$ does not annihilate p. So by Lemma 3.2, p is not the zero polynomial of A. ⊣

6.3. Polynomial completeness for general MV algebras. The previous theorem gives a quite satisfactory characterization of polynomially complete MV chains. It remains to investigate polynomially complete MV algebras in general. A first step is the following:

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LEMMA 6.15. *Let I be a set. If for every $i \in I$ an MV algebra A_i is polynomially complete, then the Cartesian product $\Pi_{i\in I} A_i$ is polynomially complete as well.*

PROOF. Let $A = \Pi_{i\in I} A_i$. Let $p(\bar{x}, \bar{d}) \in A[\bar{x}]$ induce zero on A. Then for every $i \in I$, $p(\bar{x}, \bar{d_i}) \in A_i[\bar{x}]$ induces zero on A_i. Let E be an extension of A. By the joint embedding property of MV algebras we can suppose that E extends A_i for every i.

Since A_i is polynomially complete and E is an extension of A_i, we have that, for every $i \in I$, $p(\bar{x}, \bar{d_i})$ induces zero in E. By taking the I-th cartesian power, $p(\bar{x}, \bar{d})$ induces zero in E^I and also in E, which is embedded in E^I via constant maps. So, A is polynomially complete. ⊣

COROLLARY 6.16. *Every power of* $[0, 1]$ *is polynomially complete.*

COROLLARY 6.17. *Every MV algebra is embedded in a polynomially complete MV algebra.*

PROOF. Every MV algebra is embedded into an MV algebra of the form $([0, 1]^*)^I$. Now $[0, 1]^*$ is polynomially complete by Proposition 6.4. So the result follows from the previous lemma. ⊣

6.4. Beyond polynomial completeness. The MV algebra $[0, 1]$ enjoys a strong form of completeness: for every polynomial p, we have $I(V(p)) = id(p)$, the ideal generated by p. This is essentially Wójcicki's Theorem, see [4] and Proposition 6.22 below. We generalize this situation as follows.

We say that an MV algebra A is *strongly complete* if for every polynomial $p \in A[x_1, \dots, x_n]$, we have $I(V(p)) = id(p)$.

Note that every strongly complete MV algebra is also polynomially complete (take $p = 0$ in the definition). Moreover we have a sharp characterization of strongly complete MV algebras:

THEOREM 6.18. *An MV algebra is strongly complete if and only if it is simple and divisible.*

We divide the proof of the theorem in some propositions.

PROPOSITION 6.19. *Every strongly complete MV algebra A is simple.*

PROOF. Suppose A is not simple. Then A has a nonzero element a of infinite order. Let p be the constant polynomial a. Then $V(p) = \emptyset$ and $I(V(p)) = I(\emptyset) = A[x_1, \dots, x_n]$, which is different from $id(p)$ since this last ideal is proper. ⊣

PROPOSITION 6.20. *Every strongly complete MV algebra A is algebraically closed* (*in the sense of* [11]).

PROOF. Suppose there is a polynomial $p \in A[x_1, \dots, x_n]$ with no zeros in A but with some zero in an extension A' of A. Then p has infinite order, so $V(p) = \emptyset$ and $I(V(p)) = I(\emptyset) = A[x_1, \dots, x_n]$, which is different from $id(p)$ since this last ideal is proper. ⊣

COROLLARY 6.21. *Every strongly complete MV algebra is simple and divisible.*

PROOF. This follows from the previous proposition because every algebraically closed MV algebra is divisible. ⊣

Now recall the definition of generalized McNaughton functions in Subsection 2.5.

PROPOSITION 6.22. $[0, 1]$ *is strongly complete.*

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PROOF. By Proposition 2.6, every polynomial in $[0,1][\bar{x}]$ defines a generalized McNaughton function.

Now let $p, q \in [0,1][\bar{x}]$ be such that $q \in I(V(p))$. Equivalently, $V(p) \subseteq V(q)$.

Since p and q are piecewise affine functions, we can write $[0,1]^n = P_1 \cup \cdots \cup P_m$, where P_i are polyhedra and p and q are affine on each P_i. Consider the vertices of these polyhedra. Since $q \neq 0$ implies $p \neq 0$, and since vertices are finitely many, there will be k such that $q \leq kp$ on every vertex. Since the function $q - kp$ is convex, the inequality $q \leq kp$ extends to the entire $[0,1]^n$. So, $q \ominus kp$ induces zero in $[0,1]^n$, and by polynomial completeness of $[0,1]$, $q \ominus kp$ is the zero polynomial of $[0,1][\bar{x}]$ and $q \in id(p)$ in $[0,1][\bar{x}]$. ⊣

PROPOSITION 6.23. *Every simple divisible MV chain A is strongly complete.*

PROOF. We use the fact that the theory of divisible MV chains is model complete (it even has the elimination of quantifiers), see [9].

Suppose A is a simple and divisible MV chain. Let $p(\bar{x}, \bar{s}), q(\bar{x}, \bar{t})$ be two polynomials in $A[\bar{x}]$ such that every zero of p in A is also a zero of q. Since the theory of divisible MV chains is model complete, the unique embedding of A in $[0,1]$ is elementary. So, every zero of p in $[0,1]$ is also a zero of q.

Since $[0,1]$ is strongly complete, p is in the ideal generated by q as polynomials in $[0,1]$. So, for some n, $p \leq nq$ as polynomials in $[0,1]$, that is, $p \ominus nq$ is the zero polynomial in $[0,1]$. By restriction, $p \ominus nq$ is the zero polynomial in A, and p is in the ideal generated by q as polynomials in A. ⊣

Since every simple MV algebra is a chain, as we have seen, this concludes the proof of Theorem 6.18.

The situation of radicals of principal ideals in strongly complete MV algebras is clear: the radical of a principal ideal is the ideal itself. However we have other algebras, where the radical of a principal ideal is principal. For example:

PROPOSITION 6.24. *Let $n \geq 1$ be an integer. Let S_n be the MV chain with $n+1$ elements. In S_n, every ideal of the form $I(V(J))$ is principal.*

PROOF. Let J be an ideal of $S_n[x_1, \dots, x_k]$. Let $V(J) = S$. S is a subset of S_n^k.

By the one dimensional McNaughton theorem, for every $a \in S_n$ there is an MV polynomial $p_a(x)$ in one variable whose only root in $[0,1]$ is a.

For every $(b_1, \dots, b_k) \in S_n^k$ let $p_b(x_1, \dots, x_k) = p_{b_1}(x_1) \oplus \cdots \oplus p_{b_k}(x_k)$.

There is an MV polynomial $p_S(x_1, \dots, x_k)$ whose roots in $[0,1]^k$ are exactly the elements of S. This is $p_S = \bigwedge_{b \in S} p_b(x_1, \dots, x_k)$.

Suppose $q \in I(V(J))$ is a polynomial in $S_n[x_1, \dots, x_k]$. Then in $[0,1]$ we have $V(q) \supseteq V(p_S)$.

Since $[0,1]$ is strongly complete, in $[0,1]$ we have that q is in the ideal of p_S, that is, $q \leq m p_S$ for some integer m.

But polynomial inequalities are preserved under passing to subalgebras, so $q \leq m p_S$ in $S_n[x_1, \dots, x_k]$, and $I(V(J))$ is generated by p_S. ⊣

An example where radicals are not principal is $[0,1]$ itself:

PROPOSITION 6.25. *In $[0,1][x]$ there is an ideal J such that $I(V(J))$ is not principal.*

PROOF. Let $C \subseteq [0,1]$ be the Cantor set. C is closed, so it is MV algebraic, say $C = V(J)$. Suppose the ideal $I(C)$ is principal. Let g be a generator.

Then $C = V(g)$. But the zeroset of an MV polynomial in $[0, 1]$ is a finite union of intervals and points. C does not have this shape. ⊣

6.5. An important remark. At this point, a crucial remark has to be made on the relations between MV algebras and algebraic geometry. We have the following, somewhat disappointing, collapse result:

PROPOSITION 6.26. *The MV algebraic subsets of* $[0, 1]^n$ *coincide with the usual closed sets.*

PROOF. MV algebraic subsets are zeros of sets of MV polynomials and every MV polynomial is continuous, so every MV algebraic set is closed.

The converse implication follows from the facts below:

- every rational interval of $[0, 1]$ is MV algebraic (this follows from McNaughton Theorem);
- so every product of n rational intervals in $[0, 1]^n$ is also MV algebraic;
- MV algebraic subsets of $[0, 1]^n$ form the closed sets of a topology;
- the usual topology of $[0, 1]^n$ (as a set of closed sets) is generated by products of n rational intervals;
- hence, every closed subset of $[0, 1]^n$ is MV algebraic. ⊣

So, as pointed out by a referee, we can say that MV algebraic geometry over $[0, 1]$ collapses to usual general topology. This does not mean that we cannot find any sense in which $[0, 1]$ is interesting from the point of view of algebraic geometry. For instance, polynomial completeness is a nontrivial property of $[0, 1]$ expressed in purely algebro-geometric terms.

When $[0, 1]$ is replaced by any other MV chain, algebraic sets still form the closed sets of a topology, but this topology is not well understood yet (we have studied the one dimensional topology of MV chains in a paper submitted). Surely the analogy with classical algebraic geometry can be helpful. However we cannot expect a complete parallelism. For instance, classical Zariski topology on algebraic varieties is Noetherian, whereas our MV topologies usually are not.

§7. A characterization of polynomial functions on arbitrary MV algebras.

The aim of this section is to identify polynomial functions on arbitrary MV algebras with a kind of truncated functions. These functions do not depend on any topology on the algebra, unlike McNaughton functions, which are continuous in the natural topology of $[0, 1]$. What we obtain is Theorem 7.2.

The idea is that, for any MV algebra A, we can relate truncated infima of suprema of affine functions from A^n to $\Xi(A)$ (recall that (Γ, Ξ) is the Mundici functorial equivalence), with MV polynomial functions on A.

DEFINITION 7.1. Let A be an MV algebra and $(G, u) = \Xi(A)$. For an element $g \in G$, we define $\rho(g) = (g \vee 0) \wedge u$. This defines a function $\rho : G \to A$ called the *squashing function*.

A *minimax term* over A is a term (in the language of ℓ-groups) of the form

$$t(x_1, \dots, x_n) = \bigwedge_i \bigvee_j f_{ij}(x_1, \dots, x_n),$$

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where each f_{ij} is affine; that is, $t(x_1, \ldots, x_n)$ is a finite infimum of finite suprema of affine terms.

A *truncated term* over A is an expression of the form $\rho \circ t$, where t is a minimax term over A.

A *truncated function in n variables* over A is a function from A^n to A defined by a truncated term over A.

We let $TF_n(A)$ be the set of all truncated functions in n variables over A.

We note that the set $TF_n(A)$ is an MV algebra. In fact, we can define $t \oplus u = \rho \circ (t + u)$ and $\neg t = u - t$.

Theorem 7.2. *Let A be an MV algebra. Then the MV algebras $TF_n(A)$ and $PF_n(A)$ coincide.*

Proof. One direction is well known, that is:

Lemma 7.3. *Let A be any MV algebra. Every MV polynomial function from A^n to A can be defined by a truncated term.*

Proof. Let f be a polynomial function from A^n to A defined by a polynomial p. Then there is a truncated term t which defines f. The proof is by induction on p. The case of variables or constants is easy. The case of a MV algebraic sum holds because if f is defined by t and g is defined by t', then $f \oplus g$ is defined by $\rho \circ (t + t')$. The case of the negation holds because if f is defined by t, then $\neg f$ is defined by $\rho \circ (u - t)$. ⊣

Now the following lemma gives a partial converse, in the case of affine functions:

Lemma 7.4. *Let A be an MV algebra, (G, u) the associated ℓu-group and let n be a positive integer. For every affine function $f : A^n \to G$ there is an MV polynomial ϕ such that $\rho \circ f$ is the function defined by ϕ on A.*

Proof. We begin with a claim:

Claim 7.5. *Let A be an MV algebra. Let X be a set. Let $g : X \to \Xi(A)$ and let $h : X \to A$ be functions. Then:*

- $\rho \circ (g + h) = ((\rho \circ g) \oplus h) \odot (\rho \circ (g + u))$.
- $\rho \circ (u - g) = u - (\rho \circ g)$.

Proof. If A is an MV chain, a direct inspection is enough.

Now let A be any MV algebra. By the Di Nola Representation Theorem we have an embedding $A \subseteq \Pi_{i \in I} C_i$, where I is a possibly infinite set of indices, and C_i are MV chains. Moreover, by construction of the inverse Mundici functor Ξ, we have the embedding

$$\Xi(\Pi_i C_i) \subseteq \Pi_i \Xi(C_i).$$

Hence we have also the embedding

$$\Xi(A) \subseteq \Pi_i \Xi(C_i).$$

Note that both embeddings are morphisms of ℓ groups, but not of ℓu-groups; actually the product $\Pi_i \Xi(C_i)$ is an ℓ group, but it will not be in general an ℓu group because it may have no strong unit. Note that the strong unit u of the group $\Xi(\Pi_i C_i)$ is the sequence $(u_i)_{i \in I}$, where u_i is the unit of C_i; hence for every $x \in \Xi(\Pi_i C_i)$, $\rho(x)$ is the sequence $(\rho(x_i))_{i \in I}$; that is, squashing can be made componentwise.

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So we have $g(x) = (g_i(x))_{i\in I}$ and $h(x) = (h_i(x))_{i\in I}$, with $g_i : X \to C_i$ and $h_i : X \to \Xi(C_i)$. Write $\Xi(C_i) = (G_i, u_i)$. Now for every i we have $g_i : X \to C_i$, $h_i : X \to G_i$, and the claim can be proved componentwise by reducing it to the MV chain case. ⊣

Now let A be an MV algebra. Let $G = \Xi(A)$ be the corresponding ℓu-group (all intervals in the proof are taken in G).

Let $f(x_1,\dots,x_n)$ be an affine term. So $f(x_1\dots,x_n) = g_0 + m_1x_1 + \cdots + m_nx_n$, where $m_i \in \mathbb{Z}$ and $g_0 \in G$.

Since $g_0 \in G$, by Lemma 2.3, g_0 is a finite sum of elements of A with plus or minus signs. Likewise, m_ix_i is a sum of variables with plus or minus signs.

So, we can write

$$f(x_1,\dots,x_n) = r_1 + \cdots r_p + r_{p+1}y_{p+1}\cdots + r_my_m,$$

where $r_i \in [-u,u]$ for $1 \le i \le p$, $r_i = \pm 1$ for $p+1 \le i \le m$, and y_i are variables. We go by induction on $m \ge 1$.

Assume $m = 1$. If f is a constant then $\rho \circ f$ is a constant belonging to A. If $f(x) = x_i$, then $\rho \circ f$ is a projection. If $f(x) = -x_i$, then every value of f is negative, so $\rho \circ f = 0$.

Assume $m > 1$. Then $f = g + h$, where g has less summands than f and $h(\overline{x}) = rx_i$ with $r = \pm 1$, or $h(\overline{x}) = s$, with $s \in [-u,u]$. By inductive hypothesis, $\rho \circ g$ is defined by a polynomial ϕ. Now we distinguish two cases according to the values of r and s.

Case 1: if $r = 1$ or $0 < s \le u$ then $h : A^n \to A$, so h is defined by a polynomial ψ, and $\rho\circ f = ((\rho\circ g)\oplus h)\odot(\rho\circ(u+g))$ as in Claim 7.5. Since $-g$ has less summands than f, $\rho \circ (-g)$ is defined by a polynomial χ, and $\rho \circ (u+g)$ is defined by $\neg\chi$. So, $\rho \circ f$ is defined by $(\phi \oplus \psi) \odot \neg\chi$.

Case 2: if $r = -1$ or $-u \le s < 0$ then $g+h = (g-u)+(u+h)$ and $u+h : A^n \to A$, so $\rho\circ f = ((\rho\circ(g-u))\oplus(u+h))\odot(\rho\circ g)$. Now $-h$ is defined by a polynomial ψ, so $u+h$ is defined by $\neg\psi$. We have to find a polynomial χ which defines $\rho\circ(g-u)$.

Now $g(x) = r_1 + \cdots + r_p + r_{p+1}y_{p+1} + \cdots + r_my_m$, where $r_i \in [-u,u]\setminus\{0\}$ for $1 \le i \le p$, $r_i = \pm 1$ for $p+1 \le i \le m$ and y_i are variables.

Case 2.1. If all coefficients r_i are ≤ 0 then $\rho\circ(g-u) = 0$ and $\chi = 0$.

Case 2.2. If there is $j_0 \le p$ with $r_{j_0} > 0$ then

$$(g-u)x = r_1 + \cdots + (r_{j_0} - u) + r_p + r_{p+1}y_{p+1} + \cdots + r_my_m,$$

so $-u < r_{j_0} - u \le 0$ and the inductive hypothesis applies to $g-u$, so $\rho\circ(g-u)$ is defined by a polynomial χ.

Case 2.3. If there is $j_0 \in \{p+1,\dots,m\}$ with $r_{j_0} > 0$, then $r_{j_0} = 1$. We set $h_0(x) = y_{j_0}$ and $g_0(x) = g(x) - y_{j_0} - u$. So $g - u = g_0 + h_0$ where g_0 satisfies the induction hypothesis and $h_0 : A^n \to A$. We are in the hypothesis of Case 1, so $\rho\circ(g-u)$ is defined by a polynomial χ.

Now $\rho\circ(g+h)$ is defined by the polynomial $((\chi\oplus\neg\psi)\odot\phi)$.

This proves the lemma. ⊣

The previous lemma can be generalized as follows:

Lemma 7.6. *Let A be an MV algebra, (G,u) the associated ℓu-group and let n be a positive integer. For every minimax term $f : A^n \to G$ there is an MV polynomial ϕ such that $\rho \circ f$ is the function defined by ϕ on A.*

PROOF. The lemma follows from the previous one, since every minimax function is a finite infimum of finite suprema of affine functions, and MV polynomials are closed under finite infima and finite suprema. ⊣

Now Theorem 7.2 follows from Lemma 7.3 and Lemma 7.6. ⊣

Using the notation $f|_Z$ to denote a function f restricted to a set Z, we have:

COROLLARY 7.7. *Every coordinate MV algebra is an MV algebra of truncated functions restricted to an MV algebraic set.*

PROOF. Every coordinate MV algebra has the form $A[\bar{x}]/I(Z)$ where Z is an MV algebraic set. By Proposition 5.5, we have $A[\bar{x}]/I(Z) = F(Z, A) = PF_n(A)|_Z$. By Theorem 7.2, $PF_n(A)|_Z$ and $TF_n(A)|_Z$ coincide. ⊣

§8. Generalized McNaughton theorems for MV chains. Theorem 7.2 is a topology free characterization of MV polynomials with truncated functions, valid for arbitrary MV algebras. In this section we give other results in the same vein involving MV chains.

We can exploit McNaughton Theorem, and the previously defined notion of affine function, to give the following characterization of zerosets of polynomials in MV chains.

PROPOSITION 8.1. (*McNaughton Theorem for zerosets in MV chains*) *Let A be an MV chain. The zerosets in A^n of polynomials in n variables with coefficients in A coincide with polyhedra on A.*

PROOF. Consider a polynomial $p(\bar{x}, \bar{a})$, where $\bar{a}$ is a tuple of elements of A.

By McNaughton Theorem, [0, 1] verifies the following first order sentence:

$$\forall \bar{x}, \bar{y}. p(\bar{x}, \bar{y}) = 0 \iff (\bar{x}, \bar{y}) \in P_1 \cup \cdots \cup P_k, \tag{1}$$

where $P_i = \bigwedge_{j \in J_i} \{(\bar{x}, \bar{y}) | \Sigma_j m_{ij} x_j + m'_{ij} y_j + n_j \geq 0\}$, J_i is finite and m_{ij}, m'_{ij}, n_j are integers.

By elementary equivalence, the formula holds also in every divisible MV chain D.

Specializing $\bar{y}$ to any vector $\bar{a} \in D^m$ we have in D:

$$\forall \bar{x}. p(\bar{x}, \bar{a}) = 0$$
$$\iff \bar{x} \in Q_1 \cup \cdots \cup Q_k,$$

where $Q_i = \bigwedge_{j \in J_i} \{\bar{x} | \Sigma_j m_{ij} x_j + \Sigma_l m'_{il} a_l + n_i \geq 0\}$ and m_{ij}, m'_{il}, n_i are integers.

But $\Sigma_j m'_{ij} a_j + n_j$ belongs to $\Xi(A)$, so the zeroset has the required form when A is divisible.

If A is a chain which is not divisible, we can just apply the result to the divisible hull of A and note that the sentence (1) is universal, so by restriction it holds also in A.

For the converse, it is enough to show that, in every MV chain A, every polyhedron $\bigwedge_{i \in I} \{\bar{x} | \Sigma_j m_{ij} x_{ij} + r_i \geq 0\}$ is the zeroset of a polynomial.

Now, in the case $r_i = 0$, the result follows again from McNaughton Theorem. If instead $r \neq 0$, we can write $r = \Sigma_l n_l a_l$, where $a_l \in A$ and n_l are integers.

So, we can introduce one variable y_l for each l, apply the previous case $r = 0$ to the polynomial $\{\bar{x} | \Sigma_j m_j x_j + \Sigma_l n_l y_l \geq 0\}$ (which has no term of degree zero) and then specialize each variable y_l to a_l. ⊣

With the same kind of argument one can prove the following characterization of polynomial functions for MV-chains.

Definition 8.2. Let A be an MV chain. Call *topology free McNaughton function on A* a function $f : A^n \to A$ for which there is a covering of A^n by finitely many polyhedra $P_1, \dots, P_k$ such that f is affine on each polyhedron.

Note that the definition of topology free McNaughton function on an MV chain A does not involve any topology on A.

Let $TFM_n(A)$ be the MV algebra of all topology free McNaughton functions from A^n to A.

First we observe:

Proposition 8.3. $TFM_n([0, 1]) = GM_n$.

Proof. The topology free McNaughton functions on $[0, 1]^n$ are continuous because of the following standard fact in topology: if a topological space X is covered by finitely many closed sets $C_1, \dots, C_n$ and a function f with domain X is continuous on each C_i, then f is continuous on X. In our case, the C_i are polyhedra. Hence every topology free McNaughton function is continuous, and so it is a generalized McNaughton function.

Conversely, given a generalized McNaughton function, we can suppose that the domain of all its constituents are polyhedra as is proved in [4], Subsection 3.3, for McNaughton functions. Hence every generalized McNaughton function is a topology free McNaughton function. ⊣

Corollary 8.4. $PF_n([0, 1]) = TFM_n([0, 1])$.

Proof. By Proposition 2.6, $PF_n([0, 1]) = GM_n$. ⊣

Corollary 8.5. (*McNaughton Theorem for polynomials in chains*) *Let A be an MV chain. Then $PF_n(A) = TFM_n(A)$.*

Proof. If A is divisible, then it is elementarily equivalent to $[0, 1]$ and the thesis follows from the previous corollary. If A is not divisible, then the thesis holds for $DH(A)$ and by restriction it transfers to A. ⊣

We note that the previous result implies a characterization of *polynomial functions* on arbitrary MV chains, but on polynomially complete MV chains, this gives immediately (by definition) a characterization of *polynomials*, which is closer to the spirit of McNaughton Theorem.

Corollary 8.6. *In every simple divisible MV chain A, for every ideal $J \subseteq A[x_1, \dots, x_n]$, the ideal $I(V(J))$ is principal if and only if $V(J)$ is a polyhedron on A.*

Proof. If $I(V(J)) = id(g)$ then applying the operator V we have $V(J) = V(g)$ and $V(J)$ is a polyhedron by Proposition 8.1.

Conversely, if $V(J)$ is a polyhedron, then $V(J) = V(g)$ for some polynomial g, hence $I(V(J)) = I(V(g))$, and since A is strongly complete, $I(V(J)) = I(V(g)) = id(g)$ is principal. ⊣

Corollary 8.7. *In every simple divisible MV chain A, the operator I is a bijection between polyhedra on A and principal ideals of $A[x_1, \dots, x_n]$.*

The previous corollary generalizes Theorem 3.20 of [15].

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§9. Łukasiewicz logic with constants. In this section, after the algebraic results of the previous sections, we turn to logic.

Like classical algebraic geometry, MV algebraic geometry can be studied from three different viewpoints: geometric (the algebraic sets), algebraic (coordinate algebras) and logical (theories and models). While the first two approaches are studied in the previous sections of this paper, we are left with giving the basics of logic for Diophantine MV algebraic geometry. Like [4] defines Łukasiewicz logic, we must define Łukasiewicz logic with constants in a fixed MV algebra A, which, according to the Diophantine approach, will be both the MV algebra where the constants of polynomials are taken and the MV algebra where polynomials are evaluated.

In order to begin the study of Łukasiewicz logic with constants in a fixed MV algebra A, denoted by $Ł_\infty(A)$, we start from the approach of [4], chapter 4, and we modify it by adding constants denoting elements of A.

Like any other logic we must specify the syntax and semantics of $Ł_\infty(A)$. First, formulas are defined inductively as follows:

- variables $X_1, X_2, \dots$ are formulas;
- constants c_a for every $a \in A$ are formulas;
- if α is a formula, then $\neg\alpha$ is a formula;
- if α, β is a formula, then $\alpha \to \beta$ is a formula.

The semantics of $Ł_\infty(A)$ is given in terms of valuation functions v from variables to elements of A. The value of a formula α in a valuation v is an element $v(\alpha)$ of A defined by:

- $v(X_i)$ when X_i is a variable;
- a when the formula is the constant c_a;
- $v(\neg\alpha) = v(\alpha)^*$;
- $v(\alpha \to \beta) = (v(\alpha)^*) \oplus v(\beta)$.

Now the notions of satisfaction, model, tautology, semantic consequence are like [4].

In particular, a model of a formula α is a valuation v such that $v(\alpha) = 1$. A formula α is a tautology if $v(\alpha) = 1$ for every valuation v.

A formula α is a semantic consequence of a set of formulas Θ if every model of Θ is also a model of α.

Note that Theorem 4.1.4 of [4] says that a tautology on $[0, 1]$ is a tautology everywhere, and it is not clear how to reformulate this theorem in our setting, for at least two reasons. First, if we change the MV algebra A, the language of $Ł_\infty(A)$ changes. Second, the theorem depends on the fact that $[0, 1]$ generates the variety of MV algebras, so with respect to an arbitrary MV algebra A, $[0, 1]$ has a particular status.

In $Ł_\infty(A)$ we give also a deductive system, extending the one of [4], Section 4.3 with axioms for constants. The axioms are:

- $\alpha \to (\beta \to \alpha)$;
- $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$;
- $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$;
- $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$;
- $c_{a^*\oplus b} \to (c_a \to c_b)$;

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- $(c_a \to c_b) \to c_{a^* \oplus b}$;
- $c_{a^*} \to \neg c_a$;
- $\neg c_a \to c_{a^*}$.

The only rule is Modus ponens, defined as usual: from α and $\alpha \to \beta$ derive β.

The notions of provable formula, proof, possibly with hypotheses, and theory are standard. The same holds for Lindenbaum MV algebra. We denote by $Lind(A)$ the Lindenbaum algebra of $Ł_\infty(A)$: that is, the set of all formulas of $Ł_\infty(A)$ modulo mutual provability. However, $Lind(A)$ is simply the polynomial MV algebra in countably many variables:

Proposition 9.1. *For every MV algebra A, the MV algebras $Lind(A)$ and $A[x_1, x_2, \ldots]$ are isomorphic.*

Proof. First, up to rewriting every MV algebra term by means of negation and implication, we can suppose that MV algebra terms with constants in A and formulas of $Ł_\infty(A)$ coincide.

Moreover, up to rewriting every equality $\alpha = \beta$ as a pair of formulas $\alpha \to \beta$ and $\beta \to \alpha$, the MV algebra axioms and the diagram of A are provable in $Ł_\infty(A)$, and mutual provability is a congruence.

Even more, every congruence between MV terms which includes the MV algebra axioms and the diagram of A contains all axioms of $Ł_\infty(A)$ and is closed under modus ponens.

Summing up, mutual provability in $Ł_\infty(A)$ is the *smallest* congruence between MV-terms which includes the MV algebra axioms and the diagram of A.

So, recalling the definitions of Lindenbaum MV algebra and MV polynomial algebra as quotients, $Lind(A)$ and $A[x_1, x_2, \ldots]$ are the same set of terms modulo the same congruence, and so they are equal. ⊣

We will say that a logic is *complete* if tautologies coincide with provable formulas (by logic here we mean any set of strings equipped with a deductive system and a set of valuation functions taking values in one or more MV algebras).

Clearly, for every A, every provable formula of $Ł_\infty(A)$ is a tautology. The converse implication does not hold in general, but we have a characterization in terms of polynomial completeness:

Proposition 9.2. *For every MV algebra A, the logic $Ł_\infty(A)$ is complete if and only if A is polynomially complete.*

Proof. Let A be polynomially complete. Let α be a tautology of $Ł_\infty(A)$. Let $X_1, \ldots, X_n$ be the letters occurring in α. Then α can be considered as a polynomial p_α in n variables, which induces the constant function 1 in A. Since A is polynomially complete, p_α induces 1 in every extension of A, including the Lindenbaum MV algebra of $Ł_\infty(A)$. So, α is provable.

Conversely, assume $Ł_\infty(A)$ is complete. Let p be a polynomial in n variables which induces the constant 1 in A. Then p corresponds to a formula α_p in n letters $X_1, \ldots, X_n$ and α_p is a tautology in $Ł_\infty(A)$. By completeness, α_p is provable in $Ł_\infty(A)$. But all extensions of A satisfy the axioms of $Ł_\infty(A)$. So, α_p is valid in every extension of A, and p induces 1 in every extension of A. Hence, A is polynomially complete. ⊣

Finally we mention that one can also consider a "non Diophantine" logic $Ł'_\infty(A)$, which is identical to $Ł_\infty(A)$, except that formulas are evaluated in an arbitrary extension of A, rather than A itself. This time we have:

PROPOSITION 9.3. *For every MV algebra A, $Ł'_\infty(A)$ is complete.*

PROOF. Clearly every provable formula is a tautology.

To prove the converse, note that provability in $Ł_\infty(A)$ and $Ł'_\infty(A)$ coincide, so $Lind(A)$ is also the Lindenbaum algebra of $Ł'_\infty(A)$. Moreover, $Lind(A)$ is an extension of A. Now let α be a tautology of $Ł'_\infty(A)$. Then α induces the function 1 in every extension of A, so also in $Lind(A)$. Hence α is provable. $\dashv$

§10. Conclusions and further research. The results of this paper lend to several generalizations.

For instance, we believe that polynomial completeness and related concepts are worth investigating. In particular, we do not have yet a structural characterization of polynomial completeness on MV algebras (we have one only for MV chains).

In [15] a study of finitely presented MV algebras is exposed, based on rational polyhedra in $[0, 1]^n$. It would be interesting to extend the results of [15] as far as possible to general MV algebras. To this aim one could translate the framework of [15] in our more general situation, where:

- formulas ϕ correspond to polynomials p,
- polynomials evaluating to zero correspond to formulas evaluating to one (this convention is somewhat of a mismatch between algebraic geometry and logic),
- theories Φ correspond to ideals J,
- finitely axiomatizable theories correspond to principal ideals,
- polynomials may have constants taken from an arbitrary MV algebra A,
- the function Mod on theories corresponds to the function V on ideals of polynomials,
- the function Th on MV algebraic subsets of $[0, 1]^n$ corresponds to the function I on MV algebraic subsets of A^n.

Now possible directions for future research are the following.

1) We can ask questions related to composed functions like $Th(Mod(T))$. Wójcicki's Theorem implies that if T is a finitely axiomatized theory in Łukasiewicz logic, then $Th(Mod(T))$ coincides with T. In algebraic terms, this corresponds to $I(V(p)) = id(p)$ for every polynomial p, which we called strong completeness. Actually this property can be transferred only to very few MV algebras: in fact, we have seen that it holds only for simple divisible MV algebras.

2) Since Wójcicki's Theorem does not help when polynomials may have constants, we could consider weakenings of strong completeness. For instance, the fact that the ideal $I(V(p))$ is principal for every polynomial p corresponds to stating that for every finitely axiomatizable theory T, the theory $Th(Mod(T))$ is finitely axiomatized. MV algebras with this weakened form of Wójcicki's Theorem could be investigated. More generally, one can investigate what are the ideals J such that $I(V(J))$ is principal. This corresponds to considering the theories T such that $Th(Mod(T))$ is finitely axiomatizable.

3) As a first step in this program, in this paper we have briefly introduced Łukasiewicz logic with constants in an MV algebra A, denoted by $Ł_\infty(A)$, and we have derived Proposition 9.2 as a first example of transfer of information between the structure A and the logical properties of $Ł_\infty(A)$. We hope that many other examples will be discovered.

4) Finally, the results obtained so far suggest that also non Diophantine algebraic geometry for MV algebras deserves to be studied. This will be done in future papers.

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